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Two-loop five-point all-plus helicity Yang-Mills amplitude

David C. Dunbar and Warren B. Perkins

College of Science, Swansea University, Swansea SA2 8PP, United Kingdom

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We recompute the recently derived two-loop five-point all-plus Yang-Mills amplitude using unitarity and recursion. Recursion requires augmented recursion to determine the subleading pole. Using these methods, the simplicity of this amplitude is understood.

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I. INTRODUCTION

Computing perturbative scattering amplitudes is a key challenge in quantum field theory both for comparing theories with experiment and for understanding the symmetries and consistency of theories. Explicit analytic expressions for scattering amplitudes have proved to be useful windows into the behavior of the underlying theory. Technical developments have been crucial to computing these amplitudes. Two key methods based upon unitarity [1,2] and on-shell recursion [3] have produced a great many spectacular results particularly for maximally supersymmetric field theories.

Recently the two-loop all-plus five-point amplitude has been computed in QCD [4,5] using d-dimensional unitarity techniques. Subsequently this amplitude was presented in a very elegant and compact form [6]. In this form, the amplitude consists of a piece driven by the infrared (IR) structure of the amplitude and a "remainder" piece. In this article, we demonstrate how this form can be generated using a combination of four-dimensional unitarity and (augmented) recursion which provides an understanding of the simplicity of the amplitude.

Following Gehrmann *et al.* [6], the all-plus amplitude at leading color may be written[1]

$$\begin{aligned}\mathcal{A}_5(1^+,2^+,3^+,4^+,5^+)|_{\text{leading color}}\\ =g^3\sum_{L\geq 1}(g^2Nc_\Gamma)^L\times\sum_{\sigma\in S_5/Z_5}\mathrm{tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}})\\ \times A_5^{(L)}(\sigma(1)^+,\sigma(2)^+,\sigma(3)^+,\sigma(4)^+,\sigma(5)^+)\end{aligned}\tag{1.1}$$

and the object we wish to compute is the color-stripped two-loop amplitude $A_5^{(2)}(1^+,2^+,3^+,4^+,5^+)$.

The IR and UV behavior of the amplitude are well specified [7] and motivate a partition of the amplitude:

$$A_5^{(2)}=A_5^{(1)}\left[-\sum_{i=1}^{5}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}+\frac{5\pi^2}{12}\right]+F_5^{(2)}+\mathcal{O}(\epsilon).\tag{1.2}$$

The leading term in Eq. (1.2) contains the necessary IR and UV terms. In this equation, $A_5^{(1)}$ is the all-ϵ form of the one-loop amplitude. The remainder function $F_5^{(2)}$ is to be determined. We further organize $F_5^{(2)}$ into cut-constructible and rational pieces,

$$F_5^{(2)}=F_5^{cc}+R_5^{(2)}.\tag{1.3}$$

II. CUT-CONSTRUCTIBLE PIECES

In [4], d-dimensional unitarity was used to compute a master integral representation of the full two-loop five-point all-plus amplitude $A_5^{(2)}(1^+,2^+,3^+,4^+,5^+)$. When using d-dimensional unitarity the cuts of the amplitude have cut legs defined in $d=4-2\epsilon$ dimensions. Given a Feynman diagram expansion of an amplitude, polynomial reduction [8–14] can be used to obtain a corresponding set of master integrals. The reduction process involves cutting each diagram and repeatedly isolating the irreducible contribution on each cut. For example, the pentabox diagram has all eight propagators in loops and has a nonvanishing eightfold cut. The first step of the division is to evaluate the numerator on the eightfold cut, thus determining the nonvanishing contribution when all eight propagators vanish. The remainder is then evaluated on all possible sevenfold cuts and so on. This approach can also be used in a similar manner to the one-loop unitarity method. Each set of cuts determines a partition of the full set of Feynman diagrams into blocks which must be of lower loop order, in this case tree or one-loop blocks. Summing over all diagrams yields an on-shell amplitude for each block. The contribution from each cut is then determined using the product of these amplitudes for each block.

Here, alternatively, four-dimensional amplitudes will be used to determine the cut-constructible pieces of the remainder function and then the remaining rational pieces

[1]The factor c_Γ is defined as $\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)/(4\pi)^{2-\epsilon}$. Note this gives a factor of $1/(16\pi^2)$ relative to other normalizations in the literature.

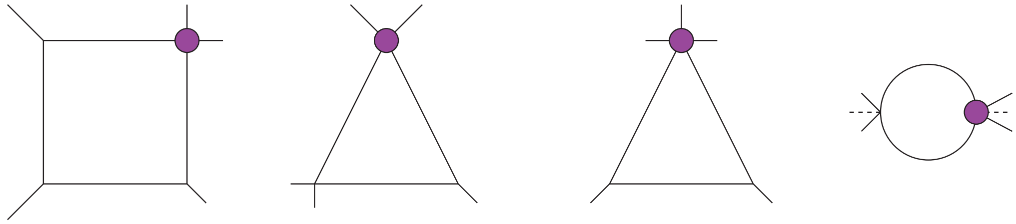

FIG. 1. Contributions to the two-loop amplitudes involving an all-plus loop (indicated by the solid disc).

will be calculated recursively. For the all-plus amplitude, considerable simplification arises when we restrict ourselves to four-dimensional cuts because all four-dimensional cuts of the one-loop all-plus amplitude vanish. After discarding scale free cuts, the reduction process only receives contributions from structures of the forms shown in Fig. 1, where the solid disc denotes an uncut one-loop all-plus amplitude. These contributions involving the all-plus one-loop amplitude can be evaluated using one-loop techniques with the one-loop subamplitude as a vertex. The n-point all-plus one-loop amplitude is [15]

$$A^{(1)}(1^+,2^+,\ldots,n^+) = -\frac{i}{3}\sum_{1\le k_1<k_2<k_3<k_4\le n}\frac{\langle k_1k_2\rangle[k_2k_3]\langle k_3k_4\rangle[k_4k_1]}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n1\rangle}+O(\epsilon). \tag{2.1}$$

Note that for the four-point amplitude there are no box functions with nonvanishing coefficients and the remainder function for the four-point amplitude is purely rational [16].

The box contribution is readily evaluated using a quadruple cut [17]. With the labeling of Fig. 2, the cut momenta are

$$\ell_1=\frac{\langle cd\rangle}{\langle ec\rangle}\bar\lambda_d\lambda_e,\qquad \ell_2=\frac{\langle c|P_{de}|}{\langle ec\rangle}\lambda_e,$$
$$\ell_3=\frac{\langle e|P_{cd}|}{\langle ec\rangle}\lambda_c,\qquad \ell_4=\frac{\langle ed\rangle}{\langle ec\rangle}\bar\lambda_d\lambda_c, \tag{2.2}$$

giving the coefficient of the box function[2]

$$\mathcal{C}_{\{a,b\},c,d,e}=M_4^{(1)}(a^+,b^+,l_3^+,l_2^+)\times M_3^{\rm tree}(l_3^-,c^+,l_4^+)\times M_3^{\rm tree}(l_4^-,d^+,l_1^-)\times M_3^{\rm tree}(l_1^+,e^+,l_2^-) = \frac{i}{6}\frac{[ab]^2[cd][de]}{\langle ce\rangle}. \tag{2.3}$$

This is the coefficient of the integral function $I_4^{1\rm m}(s_{cd},s_{de},s_{ab})$, where [18]

$$I_4^{1\rm m}(S,T,M^2)=-\frac{2}{ST}\Big[\tfrac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-M^2)^{-\epsilon}] +{\rm Li}_2\left(1-\frac{M^2}{S}\right)+{\rm Li}_2\left(1-\frac{M^2}{T}\right) +\frac{1}{2}\ln^2\left(\frac{S}{T}\right)+\frac{\pi^2}{6}\Big] \tag{2.4}$$

and overall factors of c_Γ have been removed according to the normalization of Eq. (1.2). This integral function splits into singular terms plus a remainder $I_4^{1\rm m}=I_4^{1\rm m:IR}+I_4^{1\rm m:F}$, where

$$I_4^{1\rm m:IR}(S,T,M^2) \equiv -\frac{2}{ST}\left[-\frac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-M^2)^{-\epsilon}]\right]. \tag{2.5}$$

The IR infinite terms, $I_4^{1\rm m:IR}$, in this combine with the IR infinite terms in the triangle integral functions to produce the correct IR infinite terms in the two-loop amplitude while the finite pieces, $I_4^{1\rm m:F}$, contribute to the remainder function.

The triangle contributions can be evaluated using triple cuts [19–22] and a canonical basis [23]. Each one-mass triangle $I_3^{1\rm m}(s_{ed})$ has two helicity configurations which give identical coefficients,

$$\mathcal{C}_{\{a,b,c\},d,e}=\frac{i}{6}\frac{s_{de}}{\langle ed\rangle\langle ab\rangle\langle bc\rangle}\left(s_{ba}\left(\frac{[ea]}{\langle dc\rangle}-\frac{[da]}{\langle ec\rangle}\right) - s_{bc}\left(\frac{[ec]}{\langle da\rangle}-\frac{[dc]}{\langle ea\rangle}\right)-2[de][ac]\right) \tag{2.6}$$

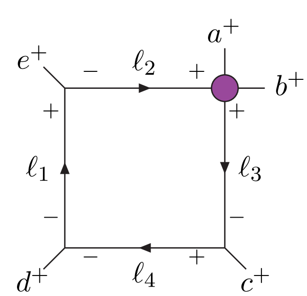

FIG. 2. The labeling and internal helicities of the quadruple cut.

[2] External legs attached to the one-loop corner are enclosed in brackets thus $\{\cdots\}$.

and the integral function is

$$I_3^{1\mathrm{m}}(K^2) = \frac{1}{\epsilon^2}(-K^2)^{-1-\epsilon}. \tag{2.7}$$

Similarly, the two mass triangle contributions are

$$\mathcal{C}_{\{a,b\},c,(d,e)} = \frac{i}{6}\frac{[ab]^2}{\langle cd\rangle\langle de\rangle\langle ec\rangle}[c|P_{de}|c\rangle I_3^{2\mathrm{m}}(s_{ab}, s_{de}), \tag{2.8}$$

where the two-mass triangle function is,

$$I_3^{2\mathrm{m}}(K_1^2, K_2^2) = \frac{1}{\epsilon^2}\frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)}. \tag{2.9}$$

The bubble contributions can be evaluated using double cuts and a canonical basis [23]. The product of amplitudes in each double cut is order ℓ^{-2} and hence the bubble coefficients vanish. This is consistent with the absence of ϵ^{-1} singularities in the amplitude.

The boxes, one-mass and two-mass triangles all have IR infinite terms of the form

$$\frac{1}{\epsilon^2}(-K^2)^{-\epsilon}.$$

A specific choice of $K^2 = s_{ab}$ arises from three box functions,

$$I_4^{1\mathrm{m}}(\{a,b\},c,d,e) : I_4^{1\mathrm{m}}(\{c,d\},e,a,b) : I_4^{1\mathrm{m}}(\{d,e\},a,b,c),$$

four two-mass triangle functions,

$$I_3^{2\mathrm{m}}(\{a,b\},c,(d,e)) : I_3^{2\mathrm{m}}(\{a,b\},(c,d),e) : I_3^{2\mathrm{m}}(\{c,d\},e,(a,b)) : I_3^{2\mathrm{m}}(\{d,e\},(a,b),c)$$

and a single one-mass triangle function $I_3^{1\mathrm{m}}(\{c,d,e\},a,b)$. Summation over the box and triangle contributions gives an overall coefficient of $A_5^{(1),\epsilon^0}(a^+, b^+, c^+, d^+, e^+)$,

$$\left(\sum \mathcal{C}_{\{a,b\},c,d,e} I_4^{1-\mathrm{mass}} + \sum \mathcal{C}_{\{a,b,c\},d,e} I_3^{1\mathrm{m}} + \sum \mathcal{C}_{\{a,b\},c,(d,e)} I_3^{2\mathrm{m}}\right)_{\mathrm{IR}} = A_5^{(1),\epsilon^0}(a^+, b^+, c^+, d^+, e^+) \times \sum_{i=1}^{5}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}, \tag{2.10}$$

where $A_5^{(1),\epsilon^0}(a^+, b^+, c^+, d^+, e^+)$ is the order ϵ^0 truncation of the one-loop amplitude. A key step is to promote the coefficient of these terms to be the all-ϵ form of the one-loop amplitude which then gives the correct singular structure of the amplitude.

The finite part of the one-mass boxes, $I_4^{1\mathrm{m:F}}$, then gives the cut-constructible part of the remainder function,

$$F_5^{cc} = \sum \frac{i}{6}\frac{[ab]^2[cd][de]}{\langle ce\rangle} \times \left(-\frac{2}{s_{cd}s_{de}}\right)\left[\mathrm{Li}_2\left(1 - \frac{s_{ab}}{s_{cd}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{ab}}{s_{de}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{cd}}{s_{de}}\right) + \frac{\pi^2}{6}\right], \tag{2.11}$$

in agreement with Ref. [6]. This combination of dilogarithms can either be viewed as a truncated box or, as recognized in Ref. [6], the $D = 8$ dimensional box. This combination arises in one-loop amplitudes without ϵ^{-2} IR singularities [17,19].

III. RATIONAL PIECES

We obtain $R_5^{(2)}$ using the on-shell recursion techniques introduced by Britto-Cachazo-Feng and Witten (BCFW) to compute tree amplitudes [3]. In this technique, the amplitude is found by introducing a shift that transforms the amplitude into an analytic function of a complex parameter, z, then using Cauchy's theorem to reconstruct the rational part from its poles:

$$\frac{1}{2\pi i}\oint\frac{A(z)}{z} = A(0) + \sum_{z_j\neq 0}\mathrm{Res}\left[\frac{A(z)}{z}\right]\bigg|_{z_j}. \tag{3.1}$$

Taking the contour to be the circle at infinity, the left hand side of Eq. (3.1) vanishes provided the shifted amplitude vanishes for large values of z. As the poles in the amplitude are determined by its factorizations, the unshifted amplitude is obtained in terms of lower point on-shell tree amplitudes:

$$A_n^{\mathrm{tree}}(0) = \sum_{i,\lambda} A_{r_i+1}^{\mathrm{tree},\lambda}(z_i)\frac{i}{K^2}A_{n-r_i+1}^{\mathrm{tree},-\lambda}(z_i). \tag{3.2}$$

The usual shift involves a pair of spinors:

$$\bar\lambda_a \to \bar\lambda_{\hat a} = \bar\lambda_a - z\bar\lambda_b, \qquad \lambda_b \to \lambda_{\hat b} = \lambda_b + z\lambda_a. \tag{3.3}$$

We wish to apply on-shell recursion to $R_5^{(2)}$; however, there are some obstacles. Firstly the shift of Eq. (3.3) does not produce an expression which has the correct cyclic symmetry. This is usually a signature that the expression does not vanish at infinity as may also be inferred from the behavior of the cut-constructible terms. (This can be checked *a posteriori* from the expressions in ref. [6].)

Instead, we use the shift [24,25]

$$\begin{aligned}\lambda_c &\to \lambda_{\hat c} = \lambda_c + z[de]\lambda_\eta,\\ \lambda_d &\to \lambda_{\hat d} = \lambda_d + z[ec]\lambda_\eta,\\ \lambda_e &\to \lambda_{\hat e} = \lambda_e + z[cd]\lambda_\eta,\end{aligned} \tag{3.4}$$

where λ_η is an arbitrary spinor. Under this shift the cut-constructible terms vanish as $z \to \infty$, an indication that the rational part will also have well behaved asymptotics.

A further issue is the existence of double poles in the amplitude. These arise beyond tree level. In principle, these are not a barrier to computation since, if we have a function whose expansion about z_i is

$$f(z) = \frac{a_{-2}}{(z-z_i)^2} + \frac{a_{-1}}{(z-z_i)} + \text{finite}, \tag{3.5}$$

then

$$\text{Residue}\left(\frac{f(z)}{z}, z_i\right) = -\frac{a_{-2}}{z_i^2} + \frac{a_{-1}}{z_i}. \tag{3.6}$$

However, for loop amplitudes, only the leading singularities have been determined in general and there are no general theorems for the subleading terms. We overcome this barrier by using axial gauge techniques to determine the extra information required to perform recursion. This is termed "augmented recursion".

There are two contributions to the factorization:

$$A_3^{\text{tree}} \times \frac{1}{K^2} \times A_4^{2\,\text{loop}} \quad \text{and} \quad A_3^{1\,\text{loop}} \times \frac{1}{K^2} \times A_4^{1\,\text{loop}}. \tag{3.7}$$

The full rational term is the sum of contributions from these two channels,

$$R_5^{(2)} = R_5^{\text{t}-2} + R_5^{1-1}. \tag{3.8}$$

$R_5^{\text{t}-2}$ involves only single poles and is directly evaluated using the rational part of the four-point two-loop amplitude [16],

$$R_4^{(2)}(K^+, b^+, c^+, d^+) = \frac{i}{6}\frac{[Kb]}{\langle Kb\rangle}\frac{[cd]}{\langle cd\rangle}\left(\frac{s_{bd}^2}{s_{cd}s_{bc}} + 8\right). \tag{3.9}$$

Setting $\eta = b$, the shift excites this factorization channel three times, giving

$$\begin{aligned}R_5^{\text{t}-2} = &\left[A_3^t(c^+, d^+, K^-)\frac{1}{s_{cd}}R_4^{(2)}(K^+, \hat e^+, a^+, b^+)\right]\Bigg|_{\langle\hat c\,\hat d\rangle=0}\\ &+ \left[A_3^t(d^+, e^+, K^-)\frac{1}{s_{de}}R_4^{(2)}(K^+, a^+, b^+, \hat c^+)\right]\Bigg|_{\langle\hat d\,\hat e\rangle=0}\\ &+ \left[A_3^t(e^+, a^+, K^-)\frac{1}{s_{ea}}R_4^{(2)}(K^+, b^+, \hat c^+, \hat d^+)\right]\Bigg|_{\langle\hat e a\rangle=0}.\end{aligned} \tag{3.10}$$

The second channel, R_5^{1-1}, has double poles associated with the diagram shown in Fig. 3. The existence of double poles means we must determine the subleading contributions which are not captured by the naive factorization. These "pole under the pole" contributions have been determined for a number of one-loop amplitudes using augmented recursion [26–29]. The contribution from this channel can be computed using axial gauge techniques [30–32] by considering diagrams of the form shown in Fig. 4, where τ^1 represents an approximation to the doubly massive current. A key feature of the axial gauge is that the internal legs have helicity assignments and vertices only involve nullified momenta as defined in Eq. (A2). Using the axial gauge three-point vertices, the contribution from Fig. 4 with the indicated helicity assignment is

$$\begin{aligned}C^{\alpha^+\beta^-} = \int &\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{\langle\alpha b\rangle^2}{\langle\beta b\rangle^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\\ &\times \tau^1(\beta^-, \alpha^+, a^+, b^+, c^+),\end{aligned} \tag{3.11}$$

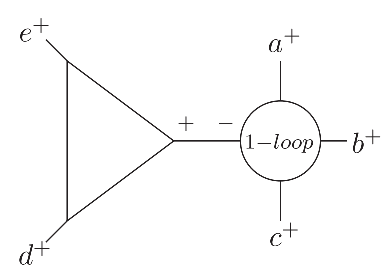

FIG. 3. The origin of the double poles in s_{de}. The diagram has an explicit pole and an additional pole can arise from the triangle integral.

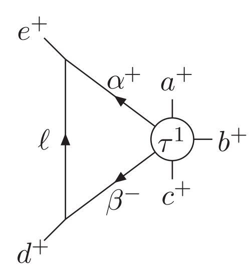

FIG. 4. The nonfactorizing contribution to the pole. We must also include the case with the helicities on α and β reversed.

where α and β are the momenta

$$\beta = \ell + d \quad \text{and} \quad \alpha = -\ell + e. \tag{3.12}$$

Within τ, β and α are loop-momenta dependent; however, the combination $\beta + \alpha$ is not.

As discussed in [29], τ^1 does not need to capture the full off-shell behavior of the current, but it must satisfy two conditions: it must reproduce the leading singularity as $s_{\alpha\beta} \to 0$ with $\alpha^2, \beta^2 \neq 0$ (C1) and it must reproduce the amplitude in the limit $\alpha^2, \beta^2 \to 0$, $s_{\alpha\beta} \neq 0$ (C2). The current, as detailed in Appendix A, is

$$\begin{aligned}\tau^1(\beta^-,\alpha^+,a^+,b^+,c^+) = \frac{i}{3}\frac{1}{\langle ab\rangle^2}\Bigg[&-\frac{[\alpha c]^2[qc]}{[c\beta][\beta q]} + \mathcal{F} + [c|q|\beta\rangle\frac{([c\alpha][\alpha q][kq]+[\alpha q]^2[ck])}{[\beta q][kq]2k.q}\\ &+\frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\left(\frac{\langle b|\beta\alpha|b\rangle}{s_{\beta\alpha}}\frac{[q|\beta+\alpha|b\rangle}{[q|\beta+\alpha|a\rangle}\right)\\ &+\left(\frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\frac{\langle b\alpha\rangle\langle ba\rangle}{\langle\alpha a\rangle}\frac{[q|\beta|b\rangle}{[q|\beta+\alpha|a\rangle} - \frac{\langle\beta a\rangle^3[a\alpha]\langle b\alpha\rangle}{\langle\beta c\rangle\langle cb\rangle\langle a\alpha\rangle^2}\right)\Bigg].\end{aligned} \tag{3.13}$$

$C^{\alpha^+\beta^-}$ is split up into five pieces: sl, sf, sk, dp and ap corresponding to the terms in τ given in (3.13),

$$C^{\alpha^+\beta^-} = C^{\alpha^+\beta^-:\mathrm{sl}} + C^{\alpha^+\beta^-:\mathrm{sf}} + C^{\alpha^+\beta^-:\mathrm{sk}} + C^{\alpha^+\beta^-:\mathrm{dp}} + C^{\alpha^+\beta^-:\mathrm{ap}}. \tag{3.14}$$

The term $C^{\alpha^+\beta^-:\mathrm{dp}}$ contains the double pole and is

$$\begin{aligned}C^{\alpha^+\beta^-:\mathrm{dp}} &= \int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\\ &\quad\times\frac{\langle b|\beta\alpha|b\rangle}{s_{\beta\alpha}}\frac{[q|\beta+\alpha|b\rangle}{[q|\beta+\alpha|a\rangle}\\ &= \frac{i}{9}\frac{[bc]\langle ac\rangle\langle b|de|b\rangle}{\langle ab\rangle^2\langle bc\rangle^2}\frac{1}{\langle de\rangle^2}\frac{[q|d+e|b\rangle}{[q|d+e|a\rangle}.\end{aligned} \tag{3.15}$$

The final term does not contain $[\beta q]$ and is labeled $C^{\alpha^+\beta^-:\mathrm{ap}}$:

$$\begin{aligned}C^{\alpha^+\beta^-:\mathrm{ap}} &= \int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{i}{3}\frac{1}{\langle ab\rangle^2}\\ &\quad\times\left(\frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\frac{\langle b\alpha\rangle\langle ba\rangle}{\langle\alpha a\rangle}\frac{[q|\beta|b\rangle}{[q|\beta+\alpha|a\rangle} + \frac{\langle\beta a\rangle[a|\alpha|b\rangle}{\langle\beta c\rangle\langle cb\rangle}\right).\end{aligned} \tag{3.16}$$

As this term contains only a single pole, the approximation

$$\frac{\langle X\alpha\rangle}{\langle Y\alpha\rangle} = \frac{\langle X\alpha\rangle}{\langle Y\alpha\rangle}\frac{\langle Yd\rangle}{\langle Yd\rangle} = \frac{\langle Xd\rangle}{\langle Yd\rangle} + \mathcal{O}(\langle\alpha d\rangle) \tag{3.17}$$

can be used to leading order, leaving cubic triangle integrals:

$$\begin{aligned}C^{\alpha^+\beta^-:\mathrm{ap}} &= \frac{i}{9}\frac{[de]}{\langle de\rangle}\frac{1}{\langle ab\rangle^2}\left(\frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\frac{\langle bd\rangle\langle ba\rangle}{\langle da\rangle}\frac{[q|2d+e|b\rangle}{[q|d+e|a\rangle}\right.\\ &\quad\left.+\frac{\langle da\rangle[a|d+2e|b\rangle}{\langle dc\rangle\langle cb\rangle}\right).\end{aligned} \tag{3.18}$$

The term $C^{\alpha^+\beta^-:\mathrm{sl}}$ is

$$\begin{aligned}C^{\alpha^+\beta^-:\mathrm{sl}} &= \int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{\langle\alpha b\rangle^2}{\langle\beta b\rangle^2}\\ &\quad\times\frac{i}{3}\frac{1}{\langle ab\rangle^2}\left(-\frac{[\alpha c]^2[qc]}{[\beta q][c\beta]}\right)\\ &= \frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[qc]}{\langle db\rangle\langle eb\rangle}\sum_{n=0,2}\int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}[d|\ell|b\rangle[e|\ell|b\rangle\\ &\quad\times\frac{[c|\beta|b\rangle^{1-n}[c|P_{de}|b\rangle^n\kappa_n}{(\beta+q)^2}+\cdots\end{aligned} \tag{3.19}$$

where $\kappa_2 = \kappa_0 = 1$, $\kappa_1 = -2$ and the $+\ldots$ reflects the use of a leading-order approximation based on

$$\frac{1}{2\beta\cdot X} - \frac{1}{(\beta+X)^2} = \frac{\beta^2}{2\beta\cdot X(\beta+X)^2}. \tag{3.20}$$

For $n = 0$, 1, this is readily reduced to triangles using

$$[e|\ell|b\rangle[d|\ell|b\rangle = \frac{\beta^2\langle b|\ell e|b\rangle + \alpha^2\langle b|\ell d|b\rangle - \ell^2\langle b|(\ell-e)P_{de}|b\rangle}{\langle ed\rangle}. \tag{3.21}$$

As all of the numerator factors have ℓ contracted with b, only the scalar part of the shifted Feynman parameter integral survives. This removes two of the triangles completely. Quadratic numerators in the surviving triangle give rational contributions, while linear numerators do not.

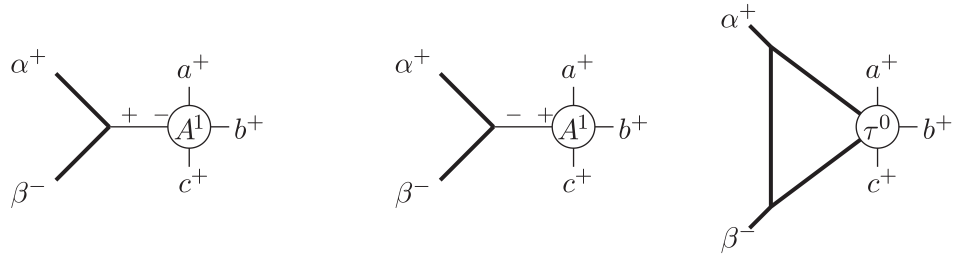

FIG. 5. Sources of $s_{\alpha\beta}$ poles in τ^1.

As the $n = 2$ case involves a linear box, rational contributions are not expected from this term. Overall,

$$C^{\alpha^+\beta^-:\mathrm{sl}} = \frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[qc][de]}{\langle de\rangle}\frac{[c|e|b\rangle}{2s_{eq}}. \tag{3.22}$$

The third term in (3.13) involves the terms with a $k^2/s_{\alpha\beta}$ factor from $\mathcal{F}$. These give the $C^{\alpha^+\beta^-:\mathrm{sk}}$ contribution:

$$C^{\alpha^+\beta^-:\mathrm{sk}} = \frac{i}{3}\int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{\langle\alpha b\rangle^2}{\langle\beta b\rangle^2} \times [c|q|\beta\rangle\frac{([c\alpha][\alpha q][kq]+[\alpha q]^2[ck])}{[\beta q][kq]2k\cdot q}\frac{1}{\langle ab\rangle^2}. \tag{3.23}$$

Using the same leading-order approximations as in the previous case,

$$C^{\alpha^+\beta^-:\mathrm{sk}} = -\frac{i}{18}\Bigg[\frac{[cq][ed]}{\langle ab\rangle^2\langle ed\rangle 2k\cdot q}\frac{1}{s_{eq}}[5[q|e|b\rangle[c|e|b\rangle + 3[q|d|b\rangle[c|e|b\rangle + [q|e|b\rangle[c|d|b\rangle] + \frac{[cq][ed][c|k|b\rangle}{\langle ab\rangle^2\langle ed\rangle(2k\cdot q)^2}[5[q|e|b\rangle + 4[q|d|b\rangle]\Bigg]. \tag{3.24}$$

Finally there is the contribution from the second term in (3.13). This term reproduces the factorizing contribution shown in the second part of Fig. 5. The corresponding integral

$$C^{\alpha^+\beta^-:\mathrm{sf}} = \int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{\langle\alpha b\rangle^2}{\langle\beta b\rangle^2}\frac{\langle\beta k\rangle[\alpha q]^2}{[\beta q][kq]} \times\frac{1}{s_{\alpha\beta}}A^{(1)}(k^+,a^+,b^+,c^+) = C^{-+:\mathrm{tri}}\times\frac{1}{s_{\alpha\beta}}A^{(1)}(k^+,a^+,b^+,c^+), \tag{3.25}$$

where the triangle integral,

$$C^{-+:\mathrm{tri}} = \int\frac{d^d\ell}{\ell^2\alpha^2\beta^2}\frac{[d|\ell|b\rangle[e|\ell|b\rangle}{\langle db\rangle\langle eb\rangle}\frac{\langle\alpha b\rangle^2}{\langle\beta b\rangle^2}\frac{\langle\beta k\rangle[\alpha q]^2}{[\beta q][kq]}, \tag{3.26}$$

is closely related to the $(+,+,-)$ one-loop splitting function. Comparing with the one-loop splitting function leads to

$$C^{\alpha^+\beta^-:\mathrm{sf}} + C^{\alpha^+\beta^-:\mathrm{sf}} = \frac{1}{3}\frac{[qd][qe][ed]}{[kq]^2}\times\frac{1}{s_{\alpha\beta}}A^{(1)}(k^+,a^+,b^+,c^+). \tag{3.27}$$

Having determined the rational contributions arising from Fig. 4, the corresponding residues can be obtained by applying the shift (3.4) and extracting the coefficient of the $(z-z_0)^{-1}$ term in the Laurent expansion. The process can be repeated for the other internal helicity configuration of the triangle. A similar procedure can be applied to the other two factorization channels: $\langle\hat{c}\,\hat{d}\rangle\to 0$ and $\langle\hat{e}a\rangle\to 0$. As $\lambda_q = \lambda_b$ the five-point single-minus amplitudes in these cases need to written in a form where the terms containing the $\langle\alpha\beta\rangle\to 0$ pole reproduce the axial gauge factorization.

Summing over the various contributions yields a rational term that has the correct cyclic symmetry and is independent of $\bar{\lambda}_q$. These are highly nontrivial checks since these symmetries are not manifest during the recursive calculation and are only restored at the final stage (provided all terms have been correctly computed).

After some considerable algebra, these terms can be reduced to match the form given in Ref. [6][3]

$$R_5^{(2)} = \frac{i}{6\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}\times(R_5^a + R_5^b), \tag{3.28}$$

where

[3]We find a perfect match provided we replace tr_- of Ref. [6] by tr_+ in term R_5^b. The tr_+ of R_5^b correctly gives the collinear limit as demonstrated in Appendix B.

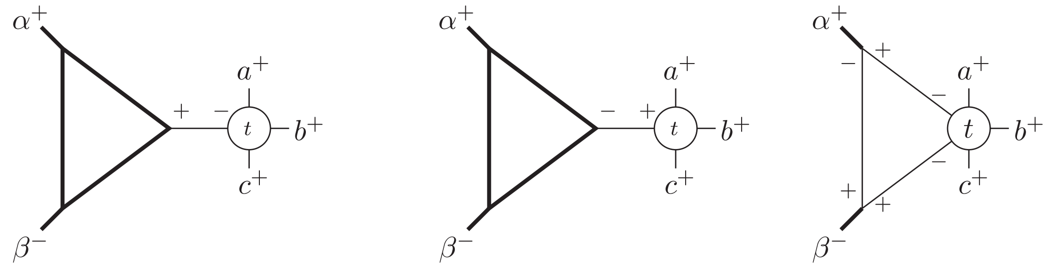

FIG. 6. Sources of $s_{\alpha\beta}$ poles in contributions from τ^0.

$$R_5^a = \frac{2}{3}\sum \frac{\mathrm{tr}_+^2(4512)}{s_{45}s_{12}},$$
$$R_5^b = \sum\left(\frac{10}{3}s_{12}s_{23} + \frac{2}{3}s_{12}s_{34}\right), \qquad (3.29)$$

and the sum cycles the five indices.

IV. CONCLUSIONS

Using four-dimensional unitarity and recursion, we have been able to reproduce the two-loop five-point all-plus Yang-Mills amplitude. Key to this is the observation that four-dimensional unitarity can be used to generate the IR singular terms whose coefficient, the one-loop amplitude, can be promoted to its all-ϵ form. With this identification the finite remainder terms follow. Computation of the cut-constructible terms is straightforward while computing the rational terms is fairly complicated but only involves one-loop integrals and avoids genuine two-loop integration. We intend to apply these techniques to further "pseudo-one-loop" amplitudes [33].

ACKNOWLEDGMENTS

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APPENDIX A: OFF-SHELL CURRENT

In this appendix, we compute an effective current $\tau^1(\alpha^+, \beta^-, c^+, d^+, e^+)$ where α and β are the off-shell legs. We will not generate the exact current but one which is sufficient to determine the poles in the amplitude. Specifically, as shown in [29], τ^1 must satisfy two conditions: (C1) it must reproduce the leading singularity as $s_{\alpha\beta} \to 0$ with α^2, $\beta^2 \neq 0$ and (C2) it must reproduce the amplitude in the limit α^2, $\beta^2 \to 0$, $s_{\alpha\beta} \neq 0$.

We use an axial gauge formalism [30–32] in which helicity labels can be used for internal lines and off-shell internal legs in the vertices are nullified using a reference spinor: given a reference null momentum η, any off-shell leg with momentum K can be nullified using

$$K^\flat = K - \frac{K^2}{[\eta|K|\eta\rangle}\eta, \qquad (A1)$$

which gives spinors

$$\lambda_K = \alpha K|\eta], \qquad \bar{\lambda}_K = \alpha^{-1}\frac{K|\eta\rangle}{[\eta|K|\eta\rangle}. \qquad (A2)$$

For convenience, we will choose the reference spinor to be $q = \bar{\lambda}_q \lambda_b$ leaving $\bar{\lambda}_q$ arbitrary.

Our task is to identify the part of the current which will generate $s_{\alpha\beta}^{-1}$ poles. The diagrams which lead to these poles are shown in Fig. 5.

The first diagram of Fig. 5 contains a $\langle\alpha\beta\rangle^{-1}$ factor and hence, after the integration within the diagram as in Fig. 4 generates the double pole piece of the rational terms. The second diagram contains a $[\alpha\beta]^{-1}$ factor and so does not enhance the order of the $\langle de\rangle$ pole.

The possible sources of $s_{\alpha\beta}$ poles in the third structure are illustrated in Fig. 6. With this helicity configuration, the (triangle) $\times$ (tree) factorizations with β^- in the triangle are absent as there are insufficient negative helicity legs to form a nonvanishing tree. Also, any triangles involving β^- and α^+ must be *mixed* (i.e. contain both $(++-)$ and $(--+)$ corners) and are therefore finite. This removes contributions of the form (singular triangle) $\times$ (on-shell propagator) $\times$ (current with vanishing amplitude). As there are no contributions with a $1/s_{\alpha\beta}$ propagator, any poles in $s_{\alpha\beta}$ must comes from the loop integration. Such singularities arise from the integration region with the loop momenta all proportional to $\alpha + \beta$, i.e. a specific null momentum. For these contributions, the loop momenta can be taken to be on-shell (hence the of thin lines for the propagators in the third part of Fig. 6). While there is a helicity configuration which gives a nonvanishing tree amplitude for the third corner, this amplitude vanishes when the propagators are collinear, i.e. the tree vanishes in the region of interest and the contribution is finite as $s_{\alpha\beta} \to 0$. Thus, there are no poles in $s_{\alpha\beta}$ arising from the third structure in Fig. 5, and it can be neglected when considering condition C1 (the finite contributions, of course, are relevant for condition C2).

τ^1 can be constructed from the five-point one-loop amplitude [34]:

$$A_5^{(1)}(\beta^-,\alpha^+,a^+,b^+,c^+) = \frac{i}{3}\frac{1}{\langle ab\rangle^2}\left[-\frac{[\alpha c]^3}{[\beta\alpha][c\beta]}+\frac{\langle\beta b\rangle^3[bc]\langle ac\rangle}{\langle\beta\alpha\rangle\langle\alpha a\rangle\langle bc\rangle^2}-\frac{\langle\beta a\rangle^3[a\alpha]\langle b\alpha\rangle}{\langle\beta c\rangle\langle cb\rangle\langle\alpha a\rangle^2}\right]. \tag{A3}$$

To satisfy C1 without compromising C2, corrections of order α^2 and β^2 are introduced to reproduce the factorization channels in Fig. 5. Using axial gauge rules and the one-loop amplitude [16]

$$A_4^{(1)}(d^-,a^+,b^+,c^+) = -\frac{i}{3}\frac{[ac]^2 s_{ac}}{[da]\langle ab\rangle\langle bc\rangle[cd]}, \tag{A4}$$

the pole arising from the first structure is

$$\frac{[\alpha k]\langle\beta q\rangle^2}{\langle\alpha q\rangle\langle kq\rangle}\frac{1}{s_{\alpha\beta}}A^{(1)}(k^-,a^+,b^+,c^+) = -\frac{i}{3}\frac{\langle\beta q\rangle^2}{\langle\alpha q\rangle^2}\frac{\langle q|\alpha\beta|q\rangle}{s_{\alpha\beta}}\frac{[ac]^2 s_{ac}}{[ka]\langle ab\rangle\langle bc\rangle[ck]\langle kq\rangle^2}, \tag{A5}$$

where $k=-k_a-k_b-k_c$ which is null on the pole. With $\lambda_q\to\lambda_b$ the four-point kinematics on the loop amplitude allow this to be written as

$$-\frac{i}{3}\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\frac{\langle b|\alpha\beta|b\rangle}{s_{\alpha\beta}}\frac{[ac]^2 s_{ac}}{[ka]\langle ab\rangle\langle bc\rangle[ck]\langle kb\rangle^2} = -\frac{i}{3}\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\frac{\langle b|\alpha\beta|b\rangle}{s_{\alpha\beta}}\frac{\langle ac\rangle[bc]\langle kb\rangle}{\langle ab\rangle^2\langle bc\rangle^2\langle ka\rangle}. \tag{A6}$$

This factor can be built into τ^1 by taking (A3) and making the substitution

$$\frac{\langle\beta b\rangle^3}{\langle\beta\alpha\rangle\langle\alpha a\rangle}\to\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\left(\frac{\langle b|\beta\alpha|b\rangle}{s_{\beta\alpha}}\frac{[q|\beta+\alpha|b\rangle}{[q|\beta+\alpha|a\rangle}+\frac{\langle b\alpha\rangle\langle ba\rangle}{\langle\alpha a\rangle}\frac{[q|\beta|b\rangle}{[q|\beta+\alpha|a\rangle}\right) \tag{A7}$$

in the second term. Equation (A7) is an identity in the limit $\alpha^2,\beta^2\to 0$, and so condition C2 is not compromised. The leading term as $s_{\alpha\beta}\to 0$ then exactly reproduces the contribution from (A6).

Similarly, the second structure gives

$$\mathcal{F} = \frac{\langle\beta k\rangle[\alpha q]^2}{[\beta q][kq]}\frac{1}{s_{\alpha\beta}}A^{1-l}(k^+,a^+,b^+,c^+) = -\frac{i}{3}\frac{\langle\beta k\rangle[\alpha q]^2}{[\beta q][kq]}\frac{1}{s_{\alpha\beta}}\frac{[ck]^2}{\langle ab\rangle^2}. \tag{A8}$$

Away from the pole, k is interpreted as its nullified form, so that

$$\begin{aligned}\mathcal{F} &= \frac{i}{3}\frac{[\alpha q]^2}{[\beta q][kq]}\frac{1}{s_{\alpha\beta}}\frac{[ck]([c\alpha]\langle\alpha\beta\rangle+\delta[c|q|\beta\rangle)}{\langle ab\rangle^2}\\ &= \frac{i}{3}\frac{1}{s_{\alpha\beta}}\frac{([\alpha q][kq][ck][c\alpha]\langle k\beta\rangle+\delta[c|q|\beta\rangle[\alpha q]^2[ck])}{[\beta q][kq]\langle ab\rangle^2}\\ &= \frac{i}{3}\frac{1}{s_{\alpha\beta}}\left[\frac{[c\alpha]^2[\alpha q]\langle\alpha\beta\rangle}{[\beta q]\langle ab\rangle^2}+\delta[c|q|\beta\rangle\frac{([c\alpha][\alpha q][kq]+[\alpha q]^2[ck])}{[\beta q][kq]\langle ab\rangle^2}\right],\end{aligned} \tag{A9}$$

where

$$\delta = \frac{\alpha^2}{2\alpha\cdot q}+\frac{\beta^2}{2\beta\cdot q}-\frac{k^2}{2k\cdot q}. \tag{A10}$$

Now,

$$\begin{aligned}\frac{\langle\alpha\beta\rangle}{s_{\alpha\beta}}-\frac{1}{[\beta\alpha]} &= \frac{\langle\alpha\beta\rangle[\beta\alpha]-s_{\alpha\beta}}{s_{\alpha\beta}[\beta\alpha]}\\ &= \frac{(\alpha^\flat+\beta^\flat)^2-s_{\alpha\beta}}{s_{\alpha\beta}[\beta\alpha]}\\ &= -\left(\frac{\alpha^2}{2\alpha\cdot q}+\frac{\beta^2}{2\beta\cdot q}\right)\frac{2k\cdot q}{s_{\alpha\beta}[\beta\alpha]}\end{aligned} \tag{A11}$$

and the first term in the amplitude is

$$\begin{aligned}-\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[\alpha c]^3}{[\beta\alpha][c\beta]} &= -\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[\alpha c]^3}{[\beta\alpha][c\beta]}\frac{[\beta q]}{[\beta q]}\\ &= -\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[\alpha c]^2}{[\beta\alpha][c\beta][\beta q]}[\beta q][\alpha c]\\ &= -\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[\alpha c]^2[qc]}{[c\beta][\beta q]}-\frac{i}{3}\frac{1}{\langle ab\rangle^2}\frac{[\alpha c]^2[q\alpha]}{[\beta\alpha][\beta q]}.\end{aligned} \tag{A12}$$

Using (A11), the second term of (A12) matches the first term of (A9) up to corrections of order α^2 and β^2. However, in addition to terms of order α^2 and β^2, the second term of (A9) contains a term of order $k^2/s_{\alpha\beta}$. This term does not contribute to the $s_{\alpha\beta}^{-1}$ pole in τ^1 and is not present in the amplitude when α^2, $\beta^2\to 0$. The current is therefore obtained by replacing the second term of (A12) by $\mathcal{F}$ with the order $k^2/s_{\alpha\beta}$ term removed. The current is then

$$\tau^1(\beta^-,\alpha^+,a^+,b^+,c^+) = \frac{i}{3}\frac{1}{\langle ab\rangle^2}\Bigg[-\frac{[\alpha c]^2[qc]}{[c\beta][\beta q]} + \mathcal{F} + [c|q|\beta\rangle\frac{([c\alpha][\alpha q][kq]+[\alpha q]^2[ck])}{[\beta q][kq]2k\cdot q} + \frac{[bc]\langle ac\rangle}{\langle bc\rangle^2}\frac{\langle\beta b\rangle^2}{\langle\alpha b\rangle^2}\left(\frac{\langle b|\beta\alpha|b\rangle}{s_{\beta\alpha}}\frac{[q|\beta+\alpha|b\rangle}{[q|\beta+\alpha|a\rangle} + \frac{\langle b\alpha\rangle\langle ba\rangle}{\langle\alpha a\rangle}\frac{[q|\beta|b\rangle}{[q|\beta+\alpha|a\rangle}\right) - \frac{\langle\beta a\rangle^3[a\alpha]\langle b\alpha\rangle}{\langle\beta c\rangle\langle cb\rangle\langle\alpha a\rangle^2}\Bigg], \tag{A13}$$

where, by construction, the third term exactly reproduces the first structure in Fig. 5 and the second term gives the $s_{\alpha\beta}^{-1}$ pole in the second structure in Fig. 5. This expression therefore satisfies condition C1. The modifications to the amplitude are all $\mathcal{O}(\alpha^2,\beta^2)$ and, therefore, do not compromise condition C2.

APPENDIX B: COLLINEAR LIMITS

We consider the collinear limit of the amplitude as an important consistency test and to illustrate some key features. The collinear limit occurs when adjacent momenta k_a and k_b become collinear,

$$k_a \to z\times K, \qquad k_b \to (1-z)\times K = \bar{z}K. \tag{B1}$$

In this limit, amplitudes factorize as

$$A_n^{(L)}(\cdots,k_a^h,k_b^{h'},\cdots) \to \sum_{L_s,h''} S_{-h''}^{hh',(L_s)} \times A_{n-1}^{(L-L_s)}(\cdots K^{h''},\cdots), \tag{B2}$$

where $S_{-h''}^{hh',(L_s)}$ are the various splitting functions. For our amplitude, the tree amplitude vanishes for both choices of h'' and

$$A_5^{(2)}(\cdots k_a^+,k_b^+\cdots) \to S_-^{++,\mathrm{tree}}\times A_4^{(2)}(\cdots K^+,\cdots) + \sum_{h''=\pm} S_\pm^{++,(1)}\times A_4^{(1)}(\cdots K^\mp,\cdots). \tag{B3}$$

The first important result is that to all orders in ϵ,

$$A_5^{(1)} \to S_-^{++,\mathrm{tree}}\times A_4^{(1)}. \tag{B4}$$

The all-ϵ forms of these amplitudes are [35]

$$A_4^{(1)}(1^+,2^+,3^+,4^+) = \frac{2i\epsilon(1-\epsilon)}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\times s_{12}s_{23}I_4^{D=8-2\epsilon},$$

$$A_5^{(1)}(1^+,2^+,3^+,4^+,5^+) = \frac{i\epsilon(1-\epsilon)}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle} \times [s_{23}s_{34}I_4^{(1),D=8-2\epsilon} + s_{34}s_{45}I_4^{(2),D=8-2\epsilon} + s_{45}s_{51}I_4^{(3),D=8-2\epsilon} + s_{51}s_{12}I_4^{(4),D=8-2\epsilon} + s_{12}s_{23}I_4^{(5),D=8-2\epsilon} + (4-2\epsilon)\varepsilon(1,2,3,4)I_5^{D=10-2\epsilon}]. \tag{B5}$$

In the collinear limit, the pentagon $I_5^{D=10-2\epsilon}$ does not contribute since its coefficient vanishes for four-point kinematics. The one-mass boxes do not individually become the massless box; however, by examining the hypergeometric representation of these functions [18], we see that they combine to all orders in ϵ to yield the massless box. This is quite important because the expansion in ϵ of the boxes, e.g. the massless box

$$I_4^{D=8-2\epsilon} = \frac{-1}{2\epsilon(3-2\epsilon)(1-2\epsilon)}\left(\frac{st}{u^2}\right)\left(\frac{u^2}{st} + \epsilon\left(-\frac{u^2}{st} + \mathrm{Log}^2[s/t]/2\right)\right. \tag{B6}$$

$$\left. + \epsilon^2\left(-\frac{u^2}{st} + \mathrm{Li}_3(1+s/t) + \mathrm{Li}_3(1+t/s)\right)\right), \tag{B7}$$

involves more complex functions including polylogarithms. These, when multiplied by the IR singular terms, contribute to the amplitude.

Next, consider the IR singular factor,

$$F_n^0 = \sum_i -\frac{1}{\epsilon^2}(-s_{ii+1})^{-\epsilon}. \tag{B8}$$

In the collinear limit,

$$F_n^0 \to F_{n-1}^0 + r_-^{++} + \Delta, \tag{B9}$$

where [1]

$$r_-^{++} = -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s_{ab})}\right)^\epsilon + 2\ln z\ln(1-z) + \frac{1}{3}z(1-z) - \frac{\pi^2}{6} \tag{B10}$$

and

$$\Delta = \log(s_{ab})\log(z\bar{z}) - \log(s_{a-1,a})\log(z) - \log(s_{b,b+1})\log(\bar{z}) - \log(z)\log(\bar{z}) - \frac{1}{3}z\bar{z} + \frac{\pi^2}{4}. \tag{B11}$$

The combination $S_-^{++,\mathrm{tree}} \times r_-^{++}$ is the one-loop splitting function.

Consequently,

$$A_5^{(1)} \times F_5^0 \to S_-^{++,\mathrm{tree}} A_4^{(1)}(F_4^0 + r_-^{++} + \Delta) = S_-^{++,\mathrm{tree}}(A_4^{(1)}F_4^0) + (S_-^{++,\mathrm{tree}} r_-^{++})A_4^{(1)} + S_-^{++,\mathrm{tree}} A_4^{(1)}\Delta. \tag{B12}$$

In the last term, $S_-^{++,\mathrm{tree}} A_4^{(1)}\Delta$, we need only keep the one-loop amplitude to order ϵ^0.

When we consider the remainder function of Eq. (2.11) in the collinear limit, we find

$$F_5^{cc} \to -S_-^{++,\mathrm{tree}} A_4^{(1)}\Delta + \text{rational terms}. \tag{B13}$$

This is consistent with the absence of a F_4^{cc} term in the four-point amplitude.

The rational terms $R_5^{(2)}$ must satisfy

$$R_5^{(2)} \to S_-^{++,\mathrm{tree}} \times R_4^{(2)}(++++) + S_+^{++,(1)} \times A_4^{(1)}(+++-) + S_-^{++,(1)}|_{\mathrm{rat}} \times A_4^{(1)}(++++), \tag{B14}$$

where $S_-^{++,(1)}|_{\mathrm{rat}}$ is the rational part of the splitting function.

$S_+^{++}A_4^{(1)}(+++-)$ arises as a $[ab]/\langle ab\rangle^2$ pole which is a double pole for complex momenta. If we consider $a = 4$, $b = 5$, for example, two of the terms in R_5^a contribute. These terms are (using the terms of Eq. (3.29) rather than the form in Ref. [6])

$$\frac{i}{9}\frac{[1\,2][4\,5]}{\langle 1\,2\rangle^2\langle 4\,5\rangle^2}\frac{\langle 2\,4\rangle^2\langle 5\,1\rangle}{\langle 2\,3\rangle\langle 3\,4\rangle} + \frac{i}{9}\frac{[4\,5][2\,3]}{\langle 4\,5\rangle^2\langle 2\,3\rangle^2}\frac{\langle 5\,2\rangle^2\langle 3\,4\rangle}{\langle 5\,1\rangle\langle 1\,2\rangle}. \tag{B15}$$

The 45 collinear limit of this is then (with some algebraic manipulation)

$$\frac{i}{9}\times\frac{\sqrt{z\bar{z}}[45]}{\langle 45\rangle^2}\times\left(\frac{[12]}{\langle 12\rangle^2}\frac{\langle 24\rangle^2\langle 41\rangle}{\langle 23\rangle\langle 34\rangle} + \frac{[23]}{\langle 23\rangle^2}\frac{\langle 42\rangle^2\langle 34\rangle}{\langle 41\rangle\langle 12\rangle}\right) = \frac{-\sqrt{z\bar{z}}[4\,5]}{3\langle 4\,5\rangle^2}\times\left(\frac{-i[1\,3]^2 u}{3[K1]\langle 1\,2\rangle\langle 2\,3\rangle[3K]}\right) = S_+^{++,(1)} \times A_4^{(1)}(+++-) \tag{B16}$$

as required.

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