

Kinematic dynamo, supersymmetry breaking, and chaos

Igor V. Ovchinnikov

*Electrical Engineering Department, University of California at Los Angeles,
Los Angeles, 90095 California, USA*

Torsten A. EnBlin

*Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Straße 1, 85748 Garching, Germany,
Ludwig-Maximilians-Universität München, Geschwister-Scholl-Platz 1, 80539 Munich, Germany,
and Technische Universität München, Exzellenzcluster Universe,
Boltzmannstraße 2, 85748 Garching, Germany*

(Received 5 December 2015; published 20 April 2016)

The kinematic dynamo (KD) describes the growth of magnetic fields generated by the flow of a conducting medium in the limit of vanishing backaction of the fields onto the flow. The KD is therefore an important model system for understanding astrophysical magnetism. Here, the mathematical correspondence between the KD and a specific stochastic differential equation (SDE) viewed from the perspective of the supersymmetric theory of stochastics (STS) is discussed. The STS is a novel, approximation-free framework to investigate SDEs. The correspondence reported here permits insights from the STS to be applied to the theory of KD and vice versa. It was previously known that the fast KD in the idealistic limit of no magnetic diffusion requires chaotic flows. The KD-STS correspondence shows that this is also true for the diffusive KD. From the STS perspective, the KD possesses a topological supersymmetry, and the dynamo effect can be viewed as its spontaneous breakdown. This supersymmetry breaking can be regarded as the stochastic generalization of the concept of dynamical chaos. As this supersymmetry breaking happens in both the diffusive and the nondiffusive cases, the necessity of the underlying SDE being chaotic is given in either case. The observed exponentially growing and oscillating KD modes prove physically that dynamical spectra of the STS evolution operator that break the topological supersymmetry exist with both real and complex ground state eigenvalues. Finally, we comment on the nonexistence of dynamos for scalar quantities.

DOI: [10.1103/PhysRevD.93.085023](https://doi.org/10.1103/PhysRevD.93.085023)**I. INTRODUCTION**

The magnetohydrodynamical dynamo has a long scientific history, and many important theoretical insights on it have already been provided (for example, Refs. [1–6] and others). Roughly speaking, the magnetic dynamo phenomenon is the ability of a moving conducting medium to generate and/or to sustain a magnetic field. This phenomenon is widespread in astrophysical objects like galaxies (Refs. [7–10], for example), galaxy clusters [11–14], stars [15–21], and planets including Earth [1–3,17,22,23]. System-sized ordered fields can be observed in objects with large-scale ordered flows such as those occurring in rotating galaxies, stars, and planets and are usually attributed to the action of the so-called large-scale, mean-field, or $\alpha - \omega$ dynamo. Systems without a large-scale flow pattern can also harbor dynamos driven by turbulent fluid motions. These are called small-scale, fluctuating, or turbulent dynamos and are believed to maintain the magnetic fields in galaxy clusters. Although the theory of dynamos is relatively mature, it still provides a field of active and interesting research.

Dynamo theory addresses two different regimes. The first one is the kinematic dynamo (KD) [3,24–26]. In this regime, the amplified magnetic field is too weak to affect

the flow of the conducting medium. This regime is realized, for example, in the early stages of galaxy formation. The other regime is the nonlinear dynamo. Here, the backaction of a sufficiently strong field on the flow is not negligible anymore. Such backactions lead eventually to a saturated dynamo state, with an average stationary magnetic energy linked to the kinetic energy of the flow. This regime seems to be realized in the magnetic fields in developed galaxies and galaxy clusters. This paper, however, focuses solely on the KD limit.

It is known that the KD in the idealistic diffusionless case requires chaotic underlying flows of matter (see, e.g., Ref. [27] and references therein). It was not known until this work whether this relation also holds if magnetic diffusivity is present. Establishing the chaos-KD relation under diffusivity requires a rigorous mathematical generalization of the concept of deterministic chaos to stochastic flows, which has been missing so far. The recently found supersymmetric theory of stochastics (STS) closes this gap [28,29].

The STS is an approximation-free theory of stochastic differential equations (SDEs). Instead of investigating the ensemble of stochastic trajectories generated by an SDE,

the STS analyzes the actions (or pullbacks) induced by these trajectories on the elements of the exterior algebra of the phase space—the differential forms of various degrees, in the following called wave functions. These wave functions can be averaged over the stochastic noise configurations, as the exterior algebra, together with the SDE-induced actions on it, is a linear space in which the stochastic averaging is legitimate. In contrast, the concept of the stochastically averaged trajectories themselves cannot be straightforwardly defined for nonlinear phase spaces. The stochastically averaged SDE-defined pullback, i.e., the stochastic evolution operator, provides a complete picture of all aspects of the stochastic dynamics. For example, the temporal evolution of an initial δ -functional probability distribution in the phase space can be explicitly constructed from it.

The KD-STs correspondence that we want to highlight here emerges from the identification of the fluid flow and the magnetic fields of the KDs with the flow vector field of an SDE and the (nonsupersymmetric) 2-forms (or 1-forms in case one described the magnetic field in terms of the magnetic vector potential), respectively. This connection permits the transfer of insights obtained within the STS to the KD theory and vice versa.

In the STS, all SDEs possess a topological (or De Rahm) supersymmetry. This means that the stochastic evolution operator commutes¹ with the so-called exterior derivative or De Rahm operator, \hat{d} , of the exterior algebra (see Sec. II). As a mathematical consequence of this (see Sec. III B), all eigenstates of a time-independent stochastic evolution operator of a stationary SDE are divided into the finite number (one for each De Rahm cohomology class) of the zero-eigenvalue (or steady-state) supersymmetric singlets and the infinite number of the nonsupersymmetric doublets with arbitrary but bounded from below eigenvalues.

If there are growing eigenstates, the fastest growing eigenstate must be identified as the ground state of the model. Indeed, if an initial wave function has a contribution by such an eigenstate, this fastest growing eigenstate will dominate over the initial wave function after a sufficiently long period. The supersymmetry is said to be broken spontaneously in such a situation because this ground state has nonzero eigenvalue and thus is nonsupersymmetric.

KDs are known to show growing magnetic fields [27]. Thus, the supersymmetry of their corresponding SDEs is spontaneously broken. In other words, the existence of the growing modes of the KD is the physical proof that the supersymmetry breaking spectra of the STS evolution operator are realizable. Moreover, the well-established existence of the oscillating fastest growing KD modes [30–32] demonstrate that the STS spectra with complex ground state eigenvalues are also realizable.

¹The infinitesimal stochastic evolution operator is actually \hat{d} exact, which is stricter than a mere commutativity with \hat{d} .

The concept of spontaneous supersymmetry breaking in the STS, required for the existence of the growing KD modes, can actually be identified with the stochastic generalization of the concept of deterministic chaos [28]. For spontaneously broken symmetries, the Goldstone theorem predicts² the existence of gapless Goldstone-Nambu excitations, which have infinite characteristic time scales. This long-term dynamical behavior can be regarded as the manifestation of the ubiquitously observed “chaotic or dynamical long-range order” that reveals itself in such phenomena as the butterfly effect [33], $1/f$ noise [34], and the power-law statistics of various sudden (or instantonic) processes like solar flares [35], earthquakes [36], and neuronal avalanches [37]. Therefore, the long-range phenomena associated with the presence of the dynamical long-range order must also be present in the KD systems, which is one of the interesting conclusions that can be drawn from the KD-STs correspondence discussed in this paper.

The structure of the paper is as follows. In Sec. II, it is demonstrated that the dynamical (stochastic) equations of the stationary KD possess topological supersymmetry. In Sec. III, the SDE corresponding to the KD effect is established, and the relation of STS to growing KD modes, chaos, and supersymmetry breaking is elaborated. Section IV concludes the paper.

II. SUPERSYMMETRY OF THE KINEMATIC DYNAMO EQUATION

Temporal evolution of the magnetic field within the KD effect is governed by the induction equation:

$$\partial_t B = \partial \times (v \times B) + \eta \Delta B. \quad (1)$$

Here, $\partial_t \equiv \partial/\partial t$, $\partial = \{\partial_i \equiv \partial/\partial x^i, x^i = x, y, z, i = 1, 2, 3\}$, is the gradient operator; $\Delta = \partial^2 = \partial_i \partial_i$,³ is the Laplace operator; \times denotes the vector product of two vectors so that $\partial \times$ is the curl of a vector; $B = \{B^i, i = 1, 2, 3\}$ is the magnetic field; $v = \{v^i, i = 1, 2, 3\}$ is the vector field of the underlying flow velocity of the conducting medium; and $\eta = 1/(\sigma\mu)$ is the magnetic diffusivity with σ and μ being the electrical conductivity and permeability. The first term on the rhs of Eq. (1) represents the well-known magnetohydrodynamical phenomenon of magnetic fields being frozen into the conducting medium, whereas the second term describes the magnetic field diffusion.

²This version of the Goldstone theorem applies to “spatially extended” models with infinite-dimensional phase spaces. For finite-dimensional phase spaces, the theorem reduces to the statement of the protected degeneracy of the ground state so that there is a “zero-energy” excitation that can remember perturbations forever, i.e., the butterfly effect.

³The summation over repeated indices is assumed throughout the paper.

Our first goal is to translate Eq. (1) into the coordinate-free language of exterior algebra used by the STS. Instead of the vector B , we use differential forms of second degree to describe magnetic fields. Such 2-forms provide the coordinate-free representation of the same object,

$$F = \frac{1}{2!} F_{ij} dx^i \wedge dx^j = \hat{\Delta} A. \quad (2)$$

Here, $A = A_i dx^i$ is the 1-form of the magnetic vector potential, $\hat{\Delta} = dx^i \wedge \partial / \partial x^i$ is the exterior derivative or the De Rahm operator, and \wedge is the wedge or antisymmetric product of differentials. In components, the antisymmetric contravariant tensor, F_{ij} , called the magnetic field tensor, is given as

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B^k = \begin{pmatrix} 0 & B^z & -B^y \\ -B^z & 0 & B^x \\ B^y & -B^x & 0 \end{pmatrix},$$

where ϵ_{ijk} is the antisymmetric Levi-Cevitá tensor.

Equation (1) in components is

$$\partial_t B^i = \epsilon^{ipq} \partial_p \epsilon_{qkl} v^k B^l + \eta \Delta B^i.$$

We work in the Euclidian metric. Thus, lowering and raising the indices has no effect on the values of the components of the antisymmetric tensor, e.g., $\epsilon_{ijk} = \epsilon^{ijk}$. With the use of the identity

$$\epsilon_{qkl} \epsilon^{ipq} = \det \begin{pmatrix} \delta_k^i & \delta_k^p \\ \delta_l^i & \delta_l^p \end{pmatrix}, \quad (3)$$

where δ_j^i is the Kronecker delta and with $\partial_i B^i = 0$, Eq. (1) can be rewritten as

$$\partial_t B^i = -(\partial_j v^j) B^i + B^j v_{ij}^i + \eta \Delta B^i,$$

where $v_{ij}^i = \partial_j v^i$. Using now

$$B^i = \frac{1}{2} \epsilon^{ikl} F_{kl},$$

the induction equation can be further rewritten as

$$\begin{aligned} \partial_t \frac{1}{2} \epsilon^{ikl} F_{kl} &= -\partial_j v^j \frac{1}{2} \epsilon^{ikl} F_{kl} + \frac{1}{2} \epsilon^{jkl} F_{kl} v_{ij}^i \\ &+ \eta \Delta \frac{1}{2} \epsilon^{ikl} F_{kl}. \end{aligned}$$

Multiplying both sides of this equation by ϵ_{iab} and summing over index i , we get

$$\partial_t F_{ab} = -\partial_j v^j F_{ab} + \frac{1}{2} \epsilon_{iab} \epsilon^{jkl} F_{kl} v_{ij}^i + \eta \Delta F_{ab}.$$

With the help of the identity

$$\epsilon_{iab} \epsilon^{jkl} = \det \begin{pmatrix} \delta_i^j & \delta_a^j & \delta_b^j \\ \delta_i^k & \delta_a^k & \delta_b^k \\ \delta_i^l & \delta_a^l & \delta_b^l \end{pmatrix},$$

one arrives at

$$\begin{aligned} \partial_t F_{ab} &= -\partial_j v^j F_{ab} + \frac{1}{2} (2F_{ab} v_{ij}^j - 2F_{jb} v_{ia}^j - 2F_{aj} v_{ib}^j) \\ &+ \eta \Delta F_{ab} \end{aligned}$$

or

$$\partial_t F_{ab} = -(v^j \partial_j F_{ab} + v_{ia}^j F_{jb} + F_{aj} v_{ib}^j) + \eta \Delta F_{ab}.$$

Turning now to the coordinate-free object F in Eq. (2), Eq. (1) takes the form

$$\partial_t F = -\hat{H}_{\text{KD}} F, \quad \hat{H}_{\text{KD}} = \hat{\mathcal{L}}_v - \eta \hat{\Delta}, \quad (4)$$

where the Lie derivative along v is defined as

$$\hat{\mathcal{L}}_v = v^i \partial_i + v_{ij}^i dx^j \wedge \hat{i}_i.$$

Here, \hat{i}_i is the interior multiplication acting on a differential form $\psi = \frac{1}{k!} \psi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ as

$$\hat{i}_i \psi = \frac{1}{(k-1)!} \psi_{ii_2 \dots i_k} dx^{i_2} \wedge \dots \wedge dx^{i_k}.$$

The resulting Eq. (4) is rather natural. As we already mentioned, the first term on the rhs of Eq. (1) describes the infinitesimal temporal evolution of the magnetic field “frozen” into the conducting medium. This freezing of the magnetic field is the well-known magnetohydrodynamical effect. In the coordinate-free setting, the frozen field corresponds to the evolution solely due to the flow along v , and this evolution is given by the Lie derivative, which is also known as the physical derivative.

The next step toward STS is to establish the supersymmetric structure of the KD evolution operator, \hat{H}_{KD} .

First, we recall the Cartan formula,

$$\hat{\mathcal{L}}_v = [\hat{\Delta}, \hat{i}_v], \quad (5)$$

where $\hat{i}_v = v^i \hat{i}_i$ denotes the interior multiplication with v and the square brackets denote the bigraded commutator. This is defined by

$$[\hat{X}, \hat{Y}] = \hat{X} \hat{Y} - (-1)^{\deg \hat{X} \cdot \deg \hat{Y}} \hat{Y} \hat{X},$$

where deg is the degree of the operator, i.e., the number of dx 's minus the number of \hat{v} 's. For example, $\text{deg } \hat{d} = 1$ and $\text{deg } \hat{v} = -1$ so that the bigraded commutator in Eq. (5) is actually an anticommutator.

Second, the Laplace operator can be given by

$$\hat{\Delta} = -[\hat{d}, \hat{d}^\dagger], \quad (6)$$

where $\hat{d}^\dagger = -\hat{v}_i \delta^{ij} \partial_j$ is the Hodge conjugate of the exterior derivative with respect to the Euclidian metric.

Substituting Eqs. (6) and (5) into Eq. (4), one finds the explicitly supersymmetric form of the KD evolution operator,

$$\hat{H}_{\text{KD}} = [\hat{d}, \hat{\tilde{d}}], \quad (7)$$

where

$$\hat{\tilde{d}} = \hat{v}_i + \eta \hat{d}^\dagger.$$

The notational similarity of the exterior derivative \hat{d} and the evolution-defining $\hat{\tilde{d}}$ is motivated by the fact that for a purely diffusive dynamics, with $\hat{H} = -\eta \hat{\Delta}$, these operators are the Hodge conjugates to each other (up to a diffusion constant) with respect to the Euclidian metric, $\hat{\tilde{d}} = \eta \hat{d}^\dagger$.

We now show the supersymmetry of \hat{H}_{KD} . We recall that due to the nilpotency of the exterior derivative, $\hat{d}^2 = 0$, one has

$$[\hat{d}, [\hat{d}, \hat{X}]] = 0 \quad (8)$$

for any \hat{X} . Thus, an immediate consequence of Eq. (8) and the “ \hat{d} -exactness” of the KD evolution operator in Eq. (7) is that it commutes with \hat{d} :

$$[\hat{d}, \hat{H}_{\text{KD}}] = 0. \quad (9)$$

This suggests that \hat{d} is a symmetry of the KD dynamics. Since $\hat{d} = dx^i \wedge \partial/\partial x^i$ removes a bosonic (commuting) variable (an x^i is removed by $\partial/\partial x^i$) and replaces it by a fermionic one (an anticommuting $dx^i \wedge$ is added), it converts bosonic variables into fermionic ones. Therefore, the symmetry of the evolution operator \hat{H}_{KD} with respect to \hat{d} can be regarded as a supersymmetry since an exchange of a boson by a fermion in the wave function does not change its dynamics.

As a consequence of its supersymmetry conserving dynamics, the KD is described equivalently in terms of the magnetic field tensor and in terms of the vector potential in Eq. (2),

$$\begin{aligned} \partial_t A &= -\hat{H}_{\text{KD}} A \\ \Rightarrow \partial_t \hat{d} A &= -\hat{d} \hat{H}_{\text{KD}} A \\ \Rightarrow \partial_t \hat{d} A &= -\hat{H}_{\text{KD}} \hat{d} A \\ \Rightarrow \partial_t F &= -\hat{H}_{\text{KD}} F. \end{aligned} \quad (10)$$

From the traditional KD perspective, it may look surprising that the magnetic field tensor and magnetic vector potential obey exactly the same equation, despite being mathematically different, although related objects. From a STS perspective, this equivalence just shows the consistence of our calculations. In fact, the descriptions of the magnetic field evolution in terms of A and F must be equivalent. This suggests on its own that the exterior derivative (connecting A and $F = \hat{d}A$) must be commutative with the evolution operator. Thus, we could just as well have guessed the existence of this supersymmetry from the very outset.

Note, however, that Eq. (7) not only implies the commutativity of the evolution operator with the exterior derivative, Eq. (9), but it also implies that all the supersymmetric eigenstates have zero eigenvalue, as we discuss in the next section.

The close relation between supersymmetry, invariance under the action of \hat{d} , and algebraic topology, relations of topological sets with their boundaries, which we mentioned in the introduction, was first established in Ref. [38]. This relation resulted in the discovery of Witten-type topological or cohomological field theories (see, e.g., Refs. [39–42] and references therein as well as Ref. [43] for a review).⁴ The STS that we are going to discuss next can be looked at as a member of this class of theories.⁵

III. CONNECTION BETWEEN THE KD THEORY AND THE STS

A. Corresponding stochastic system

In this section, we will show that the stationary KD equation including magnetic diffusivity is the STS stochastic evolution operator of a certain SDE. To find this corresponding SDE, let us address here a more general SDE, establish its stochastic evolution operators, and compare it with \hat{H}_{KD} . This general SDE can be thought of as one describing the trajectory of a test particle

⁴In the path integral representation of cohomological field theories, the \hat{d} supersymmetry is denoted as Q and is identified as the topological supersymmetry or as the gauge-fixing Becchi-Rouet-Stora-Tyutin symmetry.

⁵In a full-fledged cohomological field theory, one is interested only in the supersymmetric ground states of the model. In the STS, on the other hand, one is primarily interested in the ground states, which for chaotic systems are nonsupersymmetric as we will argue. From this point of view, the STS can be recognized as a cohomological theory only in a generalized sense.

propagating along the fluid flow while its velocity is also subject to Gaussian white noise,

$$\dot{x}^i(t) = v^i(x(t)) + (2\Theta)^{1/2} e_a^i(x(t)) \xi^a(t). \quad (11)$$

Here, x is the particle (or phase-space) position; $v = \{v^i\}$ is the stationary part of the flow vector field; $e_a(x) = \{e_a^i(x); i, a = 1, 2, 3\}$ is a set of vector fields that can be characterized as the ‘‘prime vectors’’ of flow fluctuations at x and that are assumed position dependent for now; $\xi^a(t) \in \mathbb{R}^1$ with $a \in \{1, 2, 3\}$ are the Gaussian white fluctuations with the standard stochastic averages,

$$\langle \xi^a(t) \rangle_{\text{noise}} = 0 \quad \text{and} \quad (12)$$

$$\langle \xi^a(t) \xi^b(t') \rangle_{\text{noise}} = \delta^{ab} \delta(t - t'); \quad (13)$$

and Θ is the intensity or temperature of these fluctuations. Getting a bit ahead, the flow vector field of the SDE corresponding to the KD equation will turn out to be the same as in the original KD problem. This is the reason why we do not introduce a new notation for the flow in Eq. (11).

First, an evolution operator $M_{t't}^*$ is defined, which describes how the dynamics of the system acts on elements of the exterior algebra of the position (or phase) space. The elements of the exterior algebra form a Hilbert space, the space of differential forms of all degrees, $\Omega(X)$. We recall that our magnetic field tensor and magnetic vector potential are among such elements; they are 2- and 1-forms, respectively. Mathematically, $M_{t't}^*$ is the pullback of the inverse, finite-time evolution diffeomorphism. For a fixed noise configuration, this evolution operator is

$$M_{t't}^* = \mathcal{T} e^{-\int_{t'}^t d\tau \hat{\mathcal{L}}_{v(\tau)}}, \quad (14)$$

where \mathcal{T} expresses chronological ordering of the terms to its right (see below). The time-dependent flow vector field is the entire rhs of Eq. (11),

$$v(t) = v^i + (2\Theta)^{1/2} e_a^i \xi^a(t),$$

and the corresponding time-dependent Lie derivative of this flow is

$$\hat{\mathcal{L}}_{v(t)} = \hat{\mathcal{L}}_v + (2\Theta)^{1/2} \xi^a(t) \hat{\mathcal{L}}_{e_a}. \quad (15)$$

Since the Lie derivative is linear in its argument, a stochastic term is just added to the time-independent Lie derivative given by Eq. (5). The operator of the chronological ordering \mathcal{T} is needed in Eq. (14) because in general instances of $\hat{\mathcal{L}}_{v(\tau)}$ at different times do not commute with each other. Equation (14) follows immediately from one of

the definitions of the Lie derivative, which is the infinitesimal pullback along a vector field.

Secondly, the finite-time evolution operator $M_{t't}^*$ is averaged over all the configurations of the stochastic noise:

$$\hat{M}_{t't'} = \langle M_{t't}^* \rangle_{\text{noise}}. \quad (16)$$

As $M_{t't}^*$ is a linear operator acting on $\Omega(X)$, the Hilbert space of all differential forms in which superpositions are possible, $\hat{M}_{t't'}$ represents a mathematically meaningful average. This averaged finite-time evolution operator describes how wave functions evolve in time under the action of a dynamical system:

$$\begin{aligned} \psi(t) &= \hat{M}_{t't'} \psi(t'), \\ \psi &\in \Omega(X). \end{aligned} \quad (17)$$

The infinitesimal stochastic evolution of the wave function can be given via the stochastic evolution equation

$$\partial_t \psi(t) = -\hat{H} \psi(t), \quad (18)$$

where the stochastic evolution operator (SEO) is defined by

$$\hat{H} = \lim_{\delta t \rightarrow 0} \frac{\hat{1}_\Omega - \hat{M}_{(t+\delta t)t}}{\delta t}. \quad (19)$$

STS investigates the evolution of the complete set of forms $\psi \in \Omega(X)$, whereas in the KD theory usually only the 2-forms $F \in \Omega^2(X) \subset \Omega(X)$ are of interest [or 1-forms $A \in \Omega^1(X) \subset \Omega(X)$ if one chooses to work with the vector potential].⁶ For example, the total probability density 3-form $\psi(t) = \rho(x, t) dx^1 \wedge dx^2 \wedge dx^3 \in \Omega^3(X)$ to find the system in a specific phase-space volume around location x at time t evolves according to Eq. (18) restricted to $\Omega^3(X)$. The correspondingly restricted SEO is the conventional Fokker-Planck operator and will be denoted here as $\hat{H}^{(3)}: \Omega^3(X) \rightarrow \Omega^3(X)$.

We now work out the SEO of our SDE (11). Using Eqs. (14), (15), (16), and (19); the formal definition of the chronologically ordered exponentiation of the operator in Eq. (14),

⁶In the STS, the wave functions have the meaning of generalized probability distributions in the coordinate-free setting; e.g., the top differential forms can be viewed as the total probability distributions (when strictly positive and normalized to 1), whereas the lower-degree differential forms can be looked upon as the conditional probability distributions (under the same conditions).

$$\begin{aligned} \mathcal{T} e^{-\int_{t'}^t d\tau \hat{\mathcal{L}}_{v(\tau)}} &= \hat{1} - \int_{t'}^t d\tau \hat{\mathcal{L}}_{v(\tau)} \\ &+ \int_{t'}^t d\tau_1 \hat{\mathcal{L}}_{v(\tau_1)} \int_{t'}^{\tau_1} d\tau_2 \hat{\mathcal{L}}_{v(\tau_2)} - \cdots; \end{aligned} \quad (20)$$

and the stochastic averages of the Gaussian white noise from Eqs. (12) and (13), one readily finds⁷

$$\hat{H} = \hat{\mathcal{L}}_v - \Theta \hat{\mathcal{L}}_{e_a} \hat{\mathcal{L}}_{e_a}. \quad (21)$$

In order to establish the supersymmetric structure of the SEO, we recall that the bigraded commutator with the exterior derivative is a bigraded differentiation,

$$[\hat{d}, \hat{X} \hat{Y}] = [\hat{d}, \hat{X}] \hat{Y} + (-1)^{\deg \hat{X}} \hat{X} [\hat{d}, \hat{Y}]. \quad (22)$$

Thus, the bigraded commutator with \hat{d} and the Cartan formula (5) imply that

$$\hat{\mathcal{L}}_{e_a} \hat{\mathcal{L}}_{e_a} = [\hat{d}, \mathbf{1}_{e_a} \hat{\mathcal{L}}_{e_a}]. \quad (23)$$

Thus,

$$\hat{H} = [\hat{d}, \hat{\mathcal{A}}], \quad \hat{\mathcal{A}} = \hat{v} - \Theta \mathbf{1}_{e_a} \hat{\mathcal{L}}_{e_a}. \quad (24)$$

The SEO \hat{H} of our SDE (11) is therefore the KD evolution operator \hat{H}_{KD} from the previous section,

$$\hat{H} = \hat{H}_{\text{KD}}, \quad \text{for } e_a^i(x) = \delta_a^i \quad \text{and} \quad \Theta = \eta. \quad (25)$$

In other words, the KD evolution operator is the SEO of the following SDE,

$$\dot{x}^i = v^i(x) + (2\eta)^{1/2} \xi^i(t), \quad (26)$$

where $\xi^i(t) = \delta_a^i \xi^a(t)$ is additive Gaussian white noise.

B. Dynamical spectra in STS

STS investigates the eigenspectrum of wave functions of the exterior algebra under the action of a SDE. As we have seen, the KD modes are a subset of these wave functions for

⁷Note that on derivation of Eq. (21), the last term acquires the factor

$$\begin{aligned} \lim_{\delta t \rightarrow 0} 2\delta t^{-1} \int_t^{t+\delta t} d\tau_1 \int_t^{\tau_1} d\tau_2 \delta(\tau_1 - \tau_2) \\ = \lim_{\delta t \rightarrow 0} \delta t^{-1} \int_t^{t+\delta t} d\tau_1 = 1. \end{aligned}$$

The first equality here follows from $\int_t^{\tau_1} d\tau_2 \delta(\tau_1 - \tau_2) = 1/2$, which is a consequence of defining the δ distribution to be the limit of a narrowing sequence of symmetric functions.

a suitably constructed SDE, and therefore the STS classification of dynamical systems directly applies to KDs.

The stochastic evolution equation (18) is linear. Thus, the time evolution of any wave function, ψ , can be constructed from the complete eigensystem of the SEO \hat{H} ,

$$\begin{aligned} \psi(t) &= \sum_{\alpha} a_{\alpha} \alpha e^{-\mathcal{E}_{\alpha} t}, \quad \text{with} \\ \hat{H} \alpha &= \mathcal{E}_{\alpha} \alpha \quad \text{and} \\ a_{\alpha} &= \langle \alpha | \psi \rangle = \int_X \psi \wedge \bar{\alpha}. \end{aligned} \quad (27)$$

The set of eigenvectors $\{\alpha\}$ of \hat{H} forms a complete biorthogonal basis (see below), such that $\hat{1}_{\Omega} = \sum_{\alpha} |\alpha\rangle \langle \alpha|$ permits the decomposition of any wave function in $\Omega(X)$ into the eigenmodes. The eigenvalues determine whether eigenmodes are growing ($\text{Re} \mathcal{E}_{\alpha} < 0$), decaying ($\text{Re} \mathcal{E}_{\alpha} > 0$), oscillating ($\text{Im} \mathcal{E}_{\alpha} \neq 0$), or stationary ($\mathcal{E}_{\alpha} = 0$). As the evolution of any wave function is fully determined by the behavior of the eigenmodes it is composed of, it is sufficient to study the eigensystem of the SEO to understand the properties of the corresponding dynamical system.

Analogously, KD theory concentrates on the spectrum of dynamo eigenmodes, which are just a subset of $\{\alpha\}$.

The eigensystem of the SEO has the following properties.⁸ First of all, for nonzero temperatures, $\Theta \geq 0$, the SEO is elliptic, and thus (the real part of) its spectrum is bounded from below.⁹ Second, the SEO is real, and hence its eigenvalues are either real or come in complex conjugate pairs known in the dynamical systems theory as the Ruelle-Pollicott resonances. This property of its spectra implies that the SEO is pseudo-Hermitian [44]. As a pseudo-Hermitian operator, the SEO's eigensystem is complete and biorthogonal:

$$\begin{aligned} \hat{H} \alpha &= \mathcal{E}_{\alpha} \alpha \quad \text{and} \quad \bar{\alpha} \hat{H} = \bar{\alpha} \mathcal{E}_{\alpha}, \quad \text{with} \\ \langle \alpha | \beta \rangle &= \int_X \beta \wedge \bar{\alpha} = \delta_{\alpha\beta}. \end{aligned}$$

Here, the kets $|\alpha\rangle \equiv \alpha \in \Omega = \Omega(X)$ and bras $\langle \alpha| \equiv \bar{\alpha} = \sum_{\beta} \eta_{\beta\alpha} \star(\beta^*) \in \Omega$, with $\star: \Omega^k \rightarrow \Omega^{D-k}$ being the Hodge

⁸We discuss the properties of the SEO under the assumption of a compact phase space. This assumption is made for simplicity in order to avoid complications of purely mathematical origin that appear in the non-compact setting, in which the eigensystem will depend on the choice of the class of functions that we believe constitute the Hilbert space. From the physical point of view, this assumption is not a restriction. For example, a classical model related to the KD phenomenon is the ABC flow defined on a 3-torus and planetary, stellar, and galactic dynamos act within finite volumes without flows reaching infinity.

⁹In fact, this must be true even for zero temperatures as follows from the analysis of the spectra of transfer operators in the dynamical systems theory.

star operator, are the differential forms of the corresponding right and left eigenfunctions of the SEO. The bras and kets are related through the nontrivial Hilbert space metric $\eta_{\beta\alpha}$, which is the inverse of the overlap matrix of the kets, $\sum_{\beta} \eta_{\beta\alpha} \int_X \gamma \wedge \star(\beta^*) = 1_{\alpha\gamma}$.

The operator of the degree of a differential form $\hat{k} = dx^i \wedge \hat{i}_i$, with $\hat{k}\psi = k\psi$ and $\psi \in \Omega^k(X)$, commutes with \hat{H} ,¹⁰ and thus the degree of an eigenstate of \hat{H} is a good quantum number, $\hat{k}|\alpha\rangle = k_\alpha|\alpha\rangle$. In other words, the dynamics specified by \hat{H} do not convert between different k -forms but evolve separately within each Ω^k subspace of Ω . We denote as $\hat{H}^{(k)}$ the projection of \hat{H} on Ω^k so that the block diagonal structure of the SEO can be expressed as $\hat{H} = \text{diag}(\hat{H}^{(D)}, \dots, \hat{H}^{(0)})$.

Corresponding bras and kets have complimentary degrees: if $|\alpha\rangle \equiv \alpha \in \Omega^k$, then $\langle\alpha| \equiv \bar{\alpha} \in \Omega^{D-k}$ if D is the dimension of the phase space. Otherwise, the norm of this eigenstate, $\langle\alpha|\alpha\rangle$, would vanish. Much like in quantum theory, the bra-ket combination $P_\alpha = \bar{\alpha} \wedge \alpha \in \Omega^D$ has the meaning of the total probability distribution associated with this eigenstate, and its norm $\int_X P_\alpha = \int_X \alpha \wedge \bar{\alpha} = \langle\alpha|\alpha\rangle$ should be strictly positive for nontrivial states. Consequently, $\langle\alpha| \equiv \bar{\alpha} \in \Omega^{D-k}$.

The supersymmetric structure of the SEO separates all eigenstates into two groups:

- (i) Almost all eigenstates come in nonsupersymmetric “bosonic-fermionic” pairs, $|\alpha\rangle$ and $|\alpha'\rangle = \hat{d}|\alpha\rangle$,¹¹ that have the same eigenvalue because \hat{H} commutes with \hat{d} .¹²
- (ii) Some of the eigenstates are the supersymmetric singlets $|\theta\rangle$ that are nontrivial in the De Rahm cohomology. Supersymmetry of a state $|\theta\rangle$ means that it is \hat{d} closed, $\hat{d}|\theta\rangle = 0$, but not \hat{d} exact, meaning that no θ' exists such that $|\theta\rangle = \hat{d}|\theta'\rangle$.

The \hat{d} -exact form of the SEO, i.e., $\hat{H} = [\hat{d}, \hat{d}]$, implies that all eigenstates with nonzero eigenvalue are nonsupersymmetric. Indeed, for $\hat{H}|\alpha\rangle = \mathcal{E}_\alpha|\alpha\rangle$ and $\mathcal{E}_\alpha \neq 0$, we have two possibilities:

- (1) The first possibility is that $\hat{d}|\alpha\rangle \neq 0$. In this case, the statement is trivial because \hat{H} commutes with \hat{d} , and consequently the state $|\alpha'\rangle = \hat{d}|\alpha\rangle \neq 0$ has the same eigenvalue, $\hat{H}|\alpha'\rangle = \mathcal{E}_\alpha|\alpha'\rangle$, and we have a non-supersymmetric boson-fermion pair, $|\alpha\rangle$ and $|\alpha'\rangle$.

¹⁰This is the trivial consequence of the fact that the degree of the stochastic evolution operator is zero, $\text{deg } \hat{H} = 0$.

¹¹The attempt to continue this construction of eigenstates of the form $\hat{d}^n|\alpha\rangle$ terminates because of the nilpotency of the exterior derivative, $\hat{d}^2 = 0$.

¹²In the KD context, the vector potential A and the magnetic field tensor $F = \hat{d}A$ are an example of the bosonic-fermionic relation between pairs of nonsupersymmetric eigenstates.

- (2) The other possibility is that $\hat{d}|\alpha\rangle = 0$. Then, using $\hat{H} = \hat{d}\hat{d} + \hat{d}\hat{d}$ and the fact that $\hat{d}|\alpha\rangle = 0$, we have $\mathcal{E}_\alpha|\alpha\rangle = (\hat{d}\hat{d} + \hat{d}\hat{d})|\alpha\rangle = \hat{d}\hat{d}|\alpha\rangle$, and thus $|\alpha\rangle = \hat{d}|\tilde{\alpha}\rangle$, with $|\tilde{\alpha}\rangle = \hat{d}|\alpha\rangle/\mathcal{E}_\alpha$. Up to a “gauge,” i.e., up to a \hat{d} -closed piece, $|\tilde{\alpha}\rangle = |\tilde{\alpha}'\rangle + |\tilde{\alpha}''\rangle$ with $\hat{d}|\tilde{\alpha}''\rangle = 0$ is an eigenstate of the SEO with the same eigenvalue as $|\alpha\rangle$.¹³ Again, we have a boson-fermion pair of eigenstates $|\alpha\rangle = \hat{d}|\tilde{\alpha}'\rangle \in \Omega^k$ and $|\tilde{\alpha}'\rangle \in \Omega^{k-1}$ of the same eigenvalue.

In this manner, all eigenstates with nonzero eigenvalues are non- \hat{d} -symmetric boson-fermion pairs of states. Thus, we just came to an important conclusion—all supersymmetric singlet states have vanishing eigenvalues.

This implies that any growing mode of a KD cannot be a supersymmetric state of the corresponding SEO. Operational KDs must exhibit a broken supersymmetry.

C. Nonmagnetic modes

Let us now investigate whether the dynamo phenomenon must be intrinsic to magnetic fields or whether other quantities described by 0- or 3-forms could be amplified exponentially by the Lie action of a stationary flow field.

Any physically meaningful stochastic model must have the steady-state (zero-eigenvalue) total probability distribution,¹⁴ $\psi_{\text{TE}} = P_{\text{TE}} d^3x \in \Omega^3$ with $\int_X \psi_{\text{TE}} = 1$. This state can be recognized as the state of thermodynamic equilibrium (TE). This eigenstate is supersymmetric because $\hat{d}\psi_{\text{TE}} = 0$, which is true for all 3-forms, and it is not \hat{d} exact because otherwise its integral over X would vanish. This state is always the “ground state” for Ω^3 . In other words, among all the eigenstates in Ω^3 , the state of thermodynamic equilibrium has the least real part of its eigenvalue, and this is zero. This can be seen from the following qualitative, yet robust argument.

¹³To see this, one resolves $|\tilde{\alpha}\rangle$ in the eigenstates of $\hat{H}^{(k-1)}$, with k being the degree of $|\alpha\rangle$, $|\tilde{\alpha}\rangle = \sum_{\tilde{\alpha}_1} C_{\tilde{\alpha}_1} |\tilde{\alpha}_1\rangle + \sum_{\tilde{\alpha}_2} C_{\tilde{\alpha}_2} |\tilde{\alpha}_2\rangle$, where the labels $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ run over the eigenstates such that $\hat{d}|\tilde{\alpha}_1\rangle \neq 0$ and $\hat{d}|\tilde{\alpha}_2\rangle = 0$, respectively. Applying \hat{d} to the above resolution of $|\tilde{\alpha}\rangle$ yields $|\alpha\rangle = \sum_{\tilde{\alpha}_1} C_{\tilde{\alpha}_1} \hat{d}|\tilde{\alpha}_1\rangle$. By the supersymmetry of \hat{H} , any $\hat{d}|\tilde{\alpha}_1\rangle$ is an eigenstate of $\hat{H}^{(k)}$ with the same eigenvalue as $|\tilde{\alpha}_1\rangle$. Furthermore, in the above resolution of $|\alpha\rangle$, only eigenstates with the same eigenvalue as $|\alpha\rangle$ has itself can appear. This is because each eigenstate of a complete basis is linearly independent of the other eigenstates and in particular cannot be given as a linear combination of other eigenstates with different eigenvalues. Thus, in the most general situation, when there are no additional degeneracies, one eigenstate $|\tilde{\alpha}'\rangle$ of $\hat{H}^{(k-1)}$ must exist such that $|\alpha\rangle = \hat{d}|\tilde{\alpha}'\rangle$ and $|\tilde{\alpha}\rangle = |\tilde{\alpha}'\rangle + |\tilde{\alpha}''\rangle$, with $\hat{d}|\tilde{\alpha}''\rangle = 0$.

¹⁴In the theory of deterministic dynamics, the counterpart of this state is known as invariant measure, and the Krylov–Bogolyubov theorem states that it always exists.

All non- \hat{d} -symmetric eigenstates α from Ω^3 are \hat{d} exact, i.e., of the form $|\alpha\rangle = \hat{d}|\alpha'\rangle$.¹⁵ The integral of the kets of these states is zero, $\int_X \alpha = \int_X \hat{d}\alpha' = 0$, which means that their wave functions must be negative at least somewhere on X .

Imagine now that the ground state on Ω^3 is a non- \hat{d} -symmetric eigenstate with a real negative eigenvalue. An arbitrary total probability distribution will have nonzero contribution from such an eigenstate. Its temporal evolution according to Eq. (27) will lead to a state dominated by this eigenstate after a sufficiently long time of evolution as this state is an exponentially growing mode. Consequently, the total probability distribution will become negative somewhere on the phase space, since this state must be negative in some regions of the phase space as discussed in the previous paragraph.

A negative probability distribution is of course illogical and therefore in contradiction to the above assumption that a non- \hat{d} -symmetric state with negative eigenvalue exists in Ω^3 . Using similar reasoning, one can also rule out the possibility of a pair of Ruelle-Pollicott resonances with a negative real part of their eigenvalues being the ground states on Ω^3 . Thus, one arrives at the conclusion that the supersymmetric state of thermodynamic equilibrium is always the ground state of Ω^3 . It can be said that the conventional Focker-Planck operator $\hat{H}^{(3)}$ never breaks supersymmetry spontaneously; i.e., its ground state is always supersymmetric.

This means that density modes do not grow infinitely under a stationary flow but saturate in this supersymmetric state.¹⁶

It can also be shown that $\hat{H}^{(0)}$ is related by a similarity transformation to $\hat{H}_T^{(3)}$, where \hat{H}_T is the SEO of the time-reversed SDE, i.e., the SDE with the opposite flow vector field, and the above argumentation also applies to $\hat{H}_T^{(3)}$. Operators related by a similarity transformation are isospectral so that $\hat{H}^{(0)}$ must also never break the supersymmetry on its own, just as $\hat{H}^{(3)}$.¹⁷ In conclusion, in three dimensions, only nonsupersymmetric 1-forms and 2-forms can spontaneously break the overall supersymmetry of the

¹⁵All non- \hat{d} -symmetric pairs of states are of the form $|\alpha'\rangle$ and $|\alpha\rangle = \hat{d}|\alpha'\rangle \neq 0$. Since all 3-forms are \hat{d} closed, the non- \hat{d} -symmetric 3-forms can only be of the type $|\alpha\rangle = \hat{d}|\alpha'\rangle$.

¹⁶The cosmologically educated reader might wonder whether the growing modes of cosmic structure formation are not a counterexample. They are not, as these appear in a linearized description of the transport of cosmic matter. Once all matter is swept into cosmic structures (in a nonexpanding universe, to make the mathematical analogy fit), it will stay there. These structures cannot grow further once their supply regions are empty. The merging of structures would require a temporal change of the velocity field, which is outside the class of systems investigated in this work.

¹⁷The supersymmetric ground state of $\hat{H}^{(0)}$ is a constant function on X .

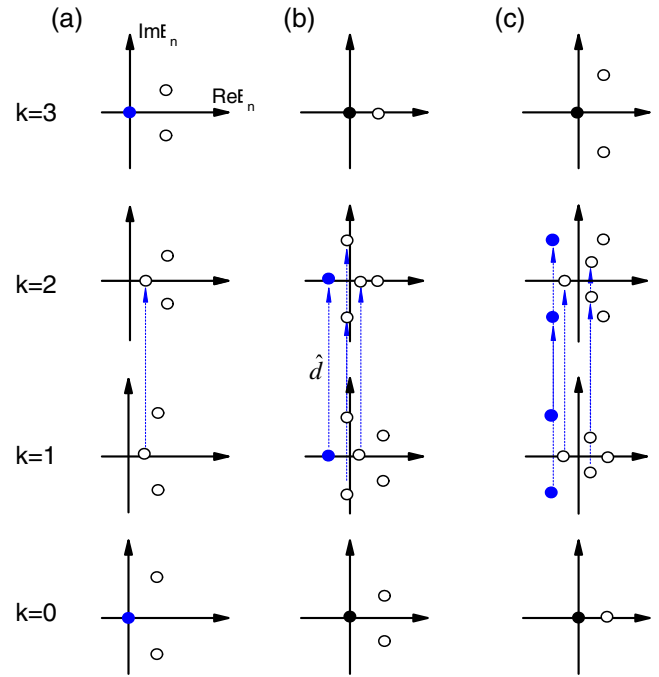


FIG. 1. The three possible spectra of the stochastic evolution operator of the STS of a model with the phase space being a three-dimensional compact sphere. Such models have only two supersymmetric eigenstates (big dots at the origin) in one-to-one correspondence with the two De Rahm cohomology classes of the 3D sphere in the zeroth ($k=0$) and the third ($k=3$) degrees. Spectrum (a) corresponds to the unbroken topological supersymmetry because both ground states are supersymmetric with one of them (for $k=3$) being the state of thermodynamic equilibrium. The other two spectra (b and c) correspond to the “chaotic” dynamics, i.e., to the situations when the ground states (big leftmost dots) are nonsupersymmetric because they have nonzero eigenvalues. The eigenstates of the stochastic evolution operator that represent the modes of the magnetic field in the KD theory are the non- \hat{d} -symmetric pairs of 1- and 2-forms connected by the dashed lines representing the action of the d operator. Note that the SEO spectra for 1- and 2-forms may also have other eigenstates (with positive real parts of their eigenvalues) that cannot be associated with the magnetic field. Those from Ω^0 can be thought of as the eigenstates of the gauge potential $\varphi \in \Omega^0$ with which one can perform the gauge-invariance transformation of the vector potential, $A \rightarrow A + \hat{d}\varphi$, without changing the magnetic field. The additional non- \hat{d} -closed eigenmodes in Ω^2 are unphysical because they correspond to the magnetic field configurations such that $\hat{d}F = (\text{div}B)d^3x \neq 0$; i.e., they contain magnetic monopoles.

SEO. For this reason, a dynamo mechanism is only known for the vector potential of the magnetic fields, but not for the scalar potential of the electric field.

The above discussion leads to the conclusion that there are only three possible forms of spectra of SEO (see Fig. 1), with the 1- and 2-forms being the eigenstates of the KD operator. The first type in Fig. 1(a) corresponds to the unbroken topological supersymmetry because the ground

states of the model are supersymmetric.¹⁸ The other two types of spectra correspond to the spontaneously broken topological supersymmetry because the ground states have nonzero eigenvalues and therefore are nonsupersymmetric. These correspond to growing KD modes, either just exponentially growing, or with a superimposed oscillation.

D. Stochastic chaos

It is known that nondiffusive KDs require some non-integrability of the underlying flow, which is a signature of (deterministic) chaos [27]. The centerpiece of the theory of deterministic chaos is the butterfly effect (BE), which is a high sensitivity of the subsequent evolution of a system to perturbations and/or variations in initial conditions. The BE is often regarded as the defining property of chaos [33,45].

$$Z_t|_{t \rightarrow \infty} = \text{Tre}^{-t\hat{H}}|_{t \rightarrow \infty} \approx \sum_g e^{-t\mathcal{E}_g} = \begin{cases} \text{const} & \text{Fig. 1a,} \\ 2e^{t|\mathcal{E}_g|} & \text{Fig. 1b,} \\ 4 \cos(t\text{Im}\mathcal{E}_g)e^{t|\text{Re}\mathcal{E}_g|} & \text{Fig. 1c,} \end{cases}$$

where the label g runs over the ground states. This equation shows that for spectra as displayed in Fig. 1(b) the stochastically averaged number of periodic solutions must grow exponentially. The broken topological supersymmetry therefore seems to be a prerequisite for deterministic chaos in deterministic as well as in stochastic systems. Therefore, this spectral classification of dynamical chaos seems to be the natural generalization of the concept of deterministic chaos to the stochastic regime.

In the case of a spectrum as displayed in Fig. 1(c), the dynamical partition function can become negative. This implies that in this case the dynamical partition function cannot represent the stochastically averaged number of periodic solutions. Furthermore, there are theorems in the dynamical system theory stating that for a certain class of (expanding) models that mimic chaotic behavior the eigenvalue of the ground state must be real.²⁰ In light of these theorems, the spectrum in Fig. 1(c) looks suspicious.

This raises the question of whether such STS spectra are realizable. Here, the KD-STs correspondence proves useful for the STS. As flow configurations which exhibit growing oscillating KD modes are known [30–32], the existence of spectra like shown in Fig. 1(c) is proven.

The supersymmetry breaking picture of stochastic chaos provides a theoretical explanation of the BE via the Goldstone theorem [29].

In order to identify spontaneous topological supersymmetry breaking with the emergence of deterministic chaos, we investigate the number of periodic trajectories of a dynamical system. It is well known that in many deterministic chaotic systems the number of periodic trajectories grows exponentially as a function of their periods in the long-time limit.¹⁹ This exponential growth comes from the infinite number of unstable periodic orbits with unlimited periods that constitute strange or chaotic attractors [47]. In Ref. [29], it was shown that for some classes of models the stochastically averaged number of periodic solutions must be represented by the dynamical partition function,

E. Limit of the KD-STs correspondence

The reported KD-STs correspondence is in a sense accidental. It is possible due to the linearity of the induction equation and its specific form, i.e., the Lie derivative along the velocity field representing the evolution of magnetic field lines frozen into the media. In a more general situation as given by the nonlinear dynamo, this picture breaks down. On the other hand, the STS is applicable to all stochastic and deterministic differential equations, and the dynamical equations of the nonlinear dynamo is one of them. If the STS theory of the nonlinear dynamo would be constructed, it would be a full-fledged “field theory” with an infinite-dimensional phase space of magnetic field configurations as compared to the finite-dimensional phase space of test particle trajectories considered here for the KD. Instead of Eq. (26), the underlying equation of motion will be the induction equation itself. The magnetic field (and perhaps other functions/fields over the real space) would no longer be the “wave function,” as in case of the KD, but rather the coordinates of the model, whereas the wave functions would depend on them in a functional manner.

IV. CONCLUSION

We established a correspondence between the theory of KD and the recently found approximation-free STS. We showed that the KD equation is essentially the stochastic evolution operator of a related SDE and thus it possesses topological supersymmetry. This connection allowed us to identify the KD effect as the supersymmetry breaking phenomenon in the corresponding SDE. We further argued that the SDEs related to KDs with growing magnetic modes are chaotic in a generalized stochastic sense. We showed

¹⁸By coincidence, nonsupersymmetric eigenstates with zero eigenvalues can exist. Such degeneracy can be removed by a deformation of the model.

¹⁹The exponential rate of this growth is related to various versions of entropy (such as topological entropy, see, e.g., Ref. [46] and references therein) introduced in dynamical system theory.

²⁰In terminology of Ref. [48], the finite-time SEO is the generalized transfer operator (GTO), and it is the spectrum of the GTO which is addressed there.

that the flow pattern of stationary KDs can only amplify quantities associated with vector potentials (like magnetic fields) but not those associated with scalar potentials or which are densities.

Furthermore, the existence of the growing modes of the magnetic field in the KDs can be viewed as a proof that the supersymmetry breaking spectra of the stochastic evolution operator with both real and complex conjugate eigenvalues of the ground states are realizable. This finding is valuable for the STS. We believe that further work may reveal other important insights on both sides of the KD-STs correspondence established in this paper.

ACKNOWLEDGMENTS

This work was supported partly by the Excellence Cluster Universe. I. V. O. would like to thank the Max Planck Institute for Astrophysics for the hospitality during his visit in the summer 2015. T. A. E. thanks the International Space Science Institute (ISSI) in Bern for its hospitality and the ISSI International Team 323 for a stimulating atmosphere. We acknowledge valuable comments on the manuscript by David Butler, Sebastian Dorn, Maksim Greiner, Reimar Leike, Daniel Pumpe, and Theo Steininger.

-
- [1] E. Bullard and H. Gellman, *Phil. Trans. R. Soc. A* **247**, 213 (1954).
- [2] E. N. Parker, *Astrophys. J.* **122**, 293 (1955).
- [3] S. I. Braginskii, *Geomagn. Aeron.* **4**, 898 (1964).
- [4] E. N. Parker, *Astrophys. J.* **162**, 665 (1970).
- [5] F. Krause and K.-H. Raedler, *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Pergamon Press, Oxford, 1980).
- [6] R. Rüdiger and G. Hollerbach, *The Magnetic Universe: Geophysical and Astrophysical Dynamo Theory* (Wiley-VCH Verlag GmbH & Co, KGaA, Weinheim, 2004).
- [7] E. N. Parker, *Astrophys. J.* **401**, 137 (1992).
- [8] R. Beck, A. D. Poezd, A. Shukurov, and D. D. Sokoloff, *Astron. Astrophys.* **289**, 94 (1994).
- [9] K. Subramanian, *Mon. Not. R. Astron. Soc.* **294**, 718 (1998).
- [10] R. Beck, *Astron. Astrophys.* **470**, 539 (2007).
- [11] A. Ruzmaikin, D. Sokolov, and A. Shukurov, *Mon. Not. R. Astron. Soc.* **241**, 1 (1989).
- [12] K. Dolag, M. Bartelmann, and H. Lesch, *Astron. Astrophys.* **387**, 383 (2002).
- [13] T. A. Enßlin and C. Vogt, *Astron. Astrophys.* **453**, 447 (2006).
- [14] F. Vazza, M. Brüggem, C. Gheller, and P. Wang, *Mon. Not. R. Astron. Soc.* **445**, 3706 (2014).
- [15] E. N. Parker, *Astrophys. J.* **198**, 205 (1975).
- [16] G. A. Glatzmaier, *J. Comput. Phys.* **55**, 461 (1984).
- [17] K.-H. Raedler, *Astron. Nachr.* **307**, 89 (1986).
- [18] P. Charbonneau and K. B. MacGregor, *Astrophys. J.* **559**, 1094 (2001).
- [19] H. C. Spruit, *Astron. Astrophys.* **381**, 923 (2002).
- [20] J. Braithwaite, *Astron. Astrophys.* **449**, 451 (2006).
- [21] M. K. Browning, *Astrophys. J.* **676**, 1262 (2008).
- [22] J. B. Taylor, *Proc. R. Soc. A* **274**, 274 (1963).
- [23] W. Kuang and J. Bloxham, *Nature (London)* **389**, 371 (1997).
- [24] I. Lerche, *Astrophys. J.* **166**, 627 (1971).
- [25] P. H. Roberts, *Phil. Trans. R. Soc. A* **272**, 663 (1972).
- [26] K. Li, P. W. Livermore, and A. Jackson, *J. Comput. Phys.* **229**, 8666 (2010).
- [27] E. Ott, *Phys. Plasmas* **5**, 1636 (1998).
- [28] I. V. Ovchinnikov, *Chaos* **23**, 013108 (2013).
- [29] I. V. Ovchinnikov, *Entropy* **18**, 108 (2016).
- [30] I. Baryshnikova and A. Shukurov, *Astron. Nachr.* **308**, 89 (1987).
- [31] A. Ruzmaikin, D. Sokolov, and A. Shukurov, *J. Fluid Mech.* **197**, 39 (1988).
- [32] K. Li, P. W. Livermore, and A. Jackson, *J. Comput. Phys.* **229**, 8666 (2010).
- [33] A. E. Motter, and D. K. Campbell, *Phys. Today* **66**, 27 (2013).
- [34] P. Dutta and P. M. Horn, *Rev. Mod. Phys.* **53**, 497 (1981).
- [35] M. Aschwanden, *Self-Organized Criticality in Astrophysics: Statistics of Nonlinear Processes in the Universe* (Springer-Verlag, Berlin, 2011).
- [36] B. Gutenberg and C. F. Richter, *Nature (London)* **176**, 795 (1955).
- [37] J. M. Beggs and D. Plenz, *J. Neurosci.* **23**, 11167 (2003).
- [38] E. Witten, *J. Diff. Geom.* **17**, 661 (1982).
- [39] E. Frenkel, A. Losev, and N. Nekrasov, *Nucl. Phys. B Proc. Suppl.* **171**, 215 (2007).
- [40] J. M. F. Labastida, *Commun. Math. Phys.* **123**, 641 (1989).
- [41] E. Witten, *Commun. Math. Phys.* **118**, 411 (1988).
- [42] E. Witten, *Commun. Math. Phys.* **117**, 353 (1988).
- [43] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, *Phys. Rep.* **209**, 129 (1991).
- [44] A. Mostafazadeh, *Nucl. Phys.* **B640**, 419 (2002).
- [45] D. Ruelle, *Phys. Today* **67**, 9 (2014).
- [46] A. Manning, *Dynamical Systems—Warwick 1974*, Lecture Notes in Mathematics Vol. 468 (Springer-Verlag, Berlin, 1975), p. 185.
- [47] R. Gilmore, *Rev. Mod. Phys.* **70**, 1455 (1998).
- [48] D. Ruelle, *Not. Am. Math. Soc.* **49**, 887 (2002).