PHYSICAL REVIEW D **93,** 085021 (2016)

Noncommutative Maxwell-Chern-Simons theory: One-loop dispersion relation analysis

M. Ghasemkhani[1,*] and R. Bufalo[2,†]

[1]*Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran*
[2]*Instituto de Física Teórica, Universidade Estadual Paulista Rua Dr. Bento Teobaldo Ferraz 271, Bloco II Barra Funda, São Paulo 01140-070, SP, Brazil*
(Received 16 December 2015; published 18 April 2016)

In this paper, we study the three-dimensional noncommutative Maxwell-Chern-Simons theory. In the present analysis, a complete account for the gauge field two-point function renormalizability is presented and physical significant quantities are carefully established. The respective form factor expressions from the gauge field self-energy are computed at one-loop order. More importantly, an analysis of the gauge field dispersion relation, in search of possible noncommutative anomalies and infrared finiteness, is performed for three special cases, with particular interest in the highly noncommutative limit.

DOI: 10.1103/PhysRevD.93.085021

I. INTRODUCTION

In our search for a better understanding of physical phenomena in nature, we have faced many drawbacks in explaining well-recognized four-dimensional problems; in particular, it is of higher importance to understand why physical nature dwells in four dimensions. In our attempts to come close to answering these and other questions, we have employed means that are sufficiently intricate so that it has proven useful to wander into lower-dimensional spacetime models [1]. Although our initial hope was solely motivated by the wishful thought that we could learn useful things in a simpler setting, the wandering into lower-dimensional models has proven to be very fertile and has stimulated significantly the development of our knowledge in such a way that we are now able to explain statistical systems and condensed matter physics by means of planar physics—in two dimensions and three dimensions.

In recent years, we have witnessed a major advance in the description of so many important condensed matter phenomena by means of connections with high-energy theories [2]. We can cite BCS superconductivity [3] and novel materials such as graphene [4] and Weyl semimetals [5] as some of the most remarkable examples where a partial or full description is obtained by means of the use of effective proposals of gauge field theory models [6]. In particular, within the broad class of effective models, one can find proposals in which the violation of Lorentz symmetry is analyzed in some materials by means of effective low-energy theories; e.g., a Lorentz-violating effective version of QED is used to describe Weyl semimetals [7].

In addition to the technological advance motivated by the development of new material and matter states, we do also live nowadays in a thrilling era of rich high-precision experiments in particle physics, testing long-dated gauge theories, where structural pillars of theoretical gauge theories are scrutinized, in particular the CPT theorem and Lorentz symmetry [8], whose violation would be a sensitive signal for unconventional underlying physics. Furthermore, if we enlarge our scope and add to our interest the description of the nature behavior at shortest distances [9,10], i.e., a quantum theory of gravity, or even the so-called minimal length scale physics, one inexorably finds that noncommutative geometry is one of the highly motivated and richer frameworks [11], including phenomenological inspirations [12,13]. In attempts to accommodate quantum mechanics and general relativity within a common framework [14], one finds uncertainty principles that are compatible with noncommuting coordinates, showing that the spacetime noncommutativity naturally emerges at Plank scale.

It is well known that noncommutative geometry is a self-sufficient theory, which over the past decade has found motivation in several theoretical frameworks [15–19], but one should highlight its prominent role in the many phenomenological attempts to detect sensitive deviations originated from physics at the Planck scale [9,10,12]. In summary, in the noncommutative scenario, it is supposed that the spacetime coordinate operators do not commute with each other and satisfy the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

in which $\theta^{\mu\nu}$ is a constant antisymmetric matrix of dimension of length squared. To construct a noncommutative field theory, using the Weyl-Moyal (symbol) correspondence [20], the ordinary product is replaced by the Moyal star product defined as

$$f(x)\star g(x) = f(x)\exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \vec{\partial}_\nu\right)g(x). \tag{1.2}$$

*m_ghasemkhani@sbu.ac.ir
†rbufalo@ift.unesp.br

Inserting the above star product into the Lagrangian density of the ordinary field theory yields a highly nonlocal theory, including higher-derivative terms which are not present in commutative theory. Furthermore, the study of NC gauge theories have uncovered several interesting properties. In particular, a common feature in these theories is that the high-momentum modes (UV) affect the physics at large distances (IR) leading to the appearance of the so-called UV/IR mixing [21], even in theories with massive particles. Contrary to the initial expectation (which was that non-commutativity could render UV finite field theories), this mixing complicates the renormalization of the theory. Despite the many attempts to cure it [22,23], with no complete success, the problem has not yet been fully understood.

In the exact same way as we have discussed above, the noncommutative three-dimensional field theory, in particular gauge theory, can find application in the study of planar physics in condensed matter and statistical physics [24–28]. Despite the fact that some perturbative aspects of the noncommutative three-dimensional field theory have been studied in the context of the Chern-Simons theory [29–33] and QED3 [34], to the best of our knowledge, none of the aforementioned studies were concerned with the analysis of the anomalies that the noncommutativity can cause in the physical content of the field, e.g., UV/IR mixing that can be present in the physical dispersion relation of the gauge field due to radiative corrections [35] and, therefore, modify significantly the behavior of the quantum field in the description of a given phenomenon [36]. In addition, this calculation allows also an analysis regarding the infrared finiteness of the given cases [37].

The Maxwell-Chern-Simons theory consists of an important model with the striking feature of allowing a massive gauge field theory without any gauge symmetry breaking [38], in this case we have the so-called topologically massive electrodynamics (it describes a helicity ± 1 mode). The presence of commutative (noncommutative) Chern-Simons action can also be seen as resulting from quantum effects, arising from integrating out the fermionic fields in commutative (noncommutative) massive QED_3 [39,40]. Moreover, we can refer to some analysis, with a different scope than ours, in regard to the noncommutative Maxwell-Chern-Simons theory [41,42] and its supersymmetric extension [43,44], as well as to the higher-derivative extensions of the Chern-Simons action (in both commutative and noncommutative space) [45–47].

In this paper, we discuss the gauge field two-point function renormalizability and physically significant quantities on the one-loop order polarization tensor of the three-dimensional noncommutative Maxwell-Chern-Simons theory, with particular interest in analyzing the gauge field dispersion relation in search of possible noncommutative anomalies and infrared finiteness. We begin, at Sec. II, by reviewing the general properties of the gauge-invariant noncommutative Maxwell-Chern-Simons theory as well as its discrete symmetries. We determine the one-loop $1PI$ self-energy function, and by considering a general tensor form for it, we are able to find relations for the respective form factors. Moreover, in Sec. III, the renormalizability for the gauge field two-point function in this model is carefully established and afterwards analyzed, since it can be jeopardized by the UV/IR mixing [48]. Within this context, a multiplicative renormalization holds, and quantities of physical significance are readily defined. In Sec. IV, we compute explicitly the planar and nonplanar contributions for the form factor expressions, where the commutative limit of the given outcome is investigated. Finally, in Sec. V, we establish three particular physical cases of interest. In particular, we examine the highly noncommutative limit, where its physical dispersion relation is discussed. In Sec. VI, we summarize the results and present our final remarks.

II. GENERAL REMARKS

We start our analysis by considering the gauge-invariant Lagrangian density of the noncommutative Maxwell-Chern-Simons theory in a Minkowski spacetime,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2e}{3}A_\mu\star A_\nu\star A_\lambda\right) + \mathcal{L}_{g.f} + \mathcal{L}_{gh}, \tag{2.1}$$

where, the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]_\star$. The gauge-fixing term is chosen as to the usual Lorentz condition

$$\mathcal{L}_{g.f} = \frac{\xi}{2}B\star B + B\star(\partial_\mu A^\mu),$$

where B is the Nakanishi-Lautrup auxiliary field and the ghost term reads

$$\mathcal{L}_{gh} = \partial^\mu \bar{c}\star D^\star_\mu c,$$

where the covariant derivative is defined such as $D^\star_\mu c = \partial_\mu c - ie[A_\mu, c]_\star$. The full theory (2.1) is invariant under the BRST Slavnov transformations:

$$sA_\mu = D^\star_\mu c, \quad sc = ie\, c\star c, \quad s\bar{c} = -B, \quad sB = 0. \tag{2.2}$$

The auxiliary field B can be integrated out, since it plays no part on the theory's dynamics. Now, the tree-level propagator for the gauge field can be readily obtained, in the Landau gauge $\xi = 0$, as

$$D_{\mu\nu}(p) = \frac{-i}{p^2(p^2 - m^2)}(p^2\eta_{\mu\nu} - p_\mu p_\nu + im\epsilon_{\mu\nu\lambda}p^\lambda), \tag{2.3}$$

where m^2 is the gauge field mass originating from the Chern-Simons term.

By completeness, in order to discuss the one-loop structure of the polarization tensor, it is useful to review on the discrete symmetries of parity (**P**), charge conjugation (**C**) and time reversal (**T**), for a three-dimensional noncommutative spacetime [38,49]:

(i) *Parity*

Parity transformation in $2+1$ dimensions is indeed a reflection described by $x_1 \to -x_1$ and $x_2 \to x_2$. Under parity, the gauge field transforms as

$$A^0 \to A^0, \quad A^1 \to -A^1, \quad A^2 \to A^2, \tag{2.4}$$

which leads to a **P**-invariant noncommutative Maxwell term if we consider that the parameter θ is not changed under a parity transformation. However, the Chern-Simons kinetic term changes sign under **P**,

$$\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \to -\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{2.5}$$

whereas, for the interaction term of the Chern-Simons part, we obtain

$$\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda \to -\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda. \tag{2.6}$$

It is thus concluded that the total noncommutative Chern-Simons terms are **P**-odd.

(ii) *Charge conjugation*

Under a charge conjugation transformation, the gauge field changes as $A_\mu \to -A_\mu$ and consequently the noncommutative Maxwell term is not **C**-invariant unless we consider $\theta \to -\theta$, which has an intuitive explanation discussed in [50]. Furthermore, the Chern-Simons kinetic term transforms as

$$\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \to \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{2.7}$$

To study the **C** transformation of the Chern-Simons interaction part, it is useful to rewrite it as

$$\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda = \frac{1}{2} \epsilon^{\mu\nu\lambda} A_\mu \star [A_\nu, A_\lambda]_\star, \tag{2.8}$$

and therefore we have that under a charge conjugation transformation

$$\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda \to \epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda. \tag{2.9}$$

Accordingly, under the above consideration, we see that the noncommutative Chern-Simons term is **C**-even.

(iii) *Time reversal*

Under a time reversal transformation, the gauge field now changes as

$$A^0 \to A^0, \quad A^1 \to -A^1, \quad A^2 \to -A^2, \tag{2.10}$$

which yields a **T**-invariant noncommutative Maxwell term, with the condition $\theta \to -\theta$. For the Chern-Simons free part, we obtain

$$\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \to -\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{2.11}$$

as well as

$$\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda \to -\epsilon^{\mu\nu\lambda} A_\mu \star A_\nu \star A_\lambda. \tag{2.12}$$

Therefore, the noncommutative Chern-Simons term is also **T**-odd. In view of the above arguments on discrete symmetries, we conclude that the noncommutative Maxwell action is even under **CP** and **PT**, while the noncommutative Chern-Simons action is **CP**-odd and **PT**-even, although both of these actions are separately **CPT** invariant. We expect that the tensor structure of the photon polarization tensor, which is induced by quantum effects, inherits (respects) the properties, gauge and discrete symmetries, from the classical theory.

The one-loop contributions for the gauge field self-energy are those from the cubic, tadpole self-interaction and ghost loop, and these diagrams are depicted in Fig. 1. A detailed account for each one of them can be found at Appendix A. These contributions can be conveniently written in the following form (A6),

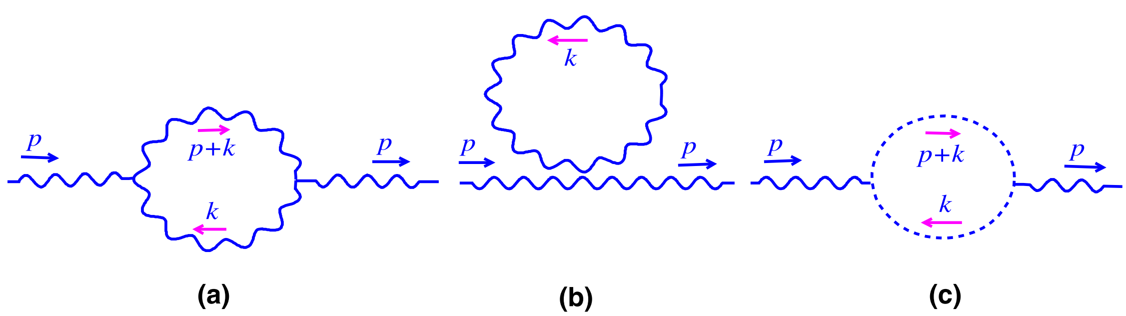

FIG. 1. One-loop Feynman diagrams in noncommutative Maxwell-Chern-Simons theory: (a) gauge loop, (b) tadpole loop, (c) ghost loop.

$$\Pi_{\mu\nu}(p) = e^2 \int \frac{d^3k}{(2\pi)^3} \sin^2\left(\frac{p\wedge k}{2}\right) \frac{\mathcal{N}^{\rm g}_{\mu\nu} + 2\mathcal{N}^{\rm gh}_{\mu\nu} + 2\mathcal{N}^{\rm t}_{\mu\nu}}{k^2(k^2-m^2)(p+k)^2((p+k)^2-m^2)}, \tag{2.13}$$

where we have used the notation $p\wedge k = p_\mu \theta^{\mu\nu} k_\nu$. Also, the tensor quantities at the numerator are defined, respectively, by Eq. (A5),

$$\mathcal{N}^{\rm g}_{\mu\nu} = (im\epsilon_{\mu\alpha\beta} + (p+2k)_\mu \eta_{\alpha\beta} + (p-k)_\beta \eta_{\mu\alpha} - (2p+k)_\alpha \eta_{\mu\beta})(im\epsilon_{\nu\rho\sigma} - (p+2k)_\nu \eta_{\rho\sigma} + (k-p)_\sigma \eta_{\rho\nu} + (2p+k)_\rho \eta_{\nu\sigma}) \\ \times (k^2\eta^{\alpha\rho} - k^\alpha k^\rho + im\epsilon^{\alpha\rho\lambda}k_\lambda)((p+k)^2\eta^{\beta\sigma} - (p+k)^\beta (p+k)^\sigma - im\epsilon^{\beta\sigma\xi}(p+k)_\xi), \tag{2.14}$$

and Eqs. (A7) and (A8),

$$\mathcal{N}^{\rm gh}_{\mu\nu} = m^4(k_\mu k_\nu + k_\mu p_\nu) - m^2(2k^2 + p^2 + 2p.k)(k_\mu k_\nu + k_\mu p_\nu) + k^2(k^2 + 2p.k + p^2)(k_\mu k_\nu + k_\mu p_\nu), \tag{2.15}$$

$$\mathcal{N}^{\rm t}_{\mu\nu} = -m^2(k^2 + 2p.k + p^2)(k^2\eta_{\mu\nu} + k_\mu k_\nu) + (k^4 + 2k^2p^2 + p^4 + 4k^2(p.k) + 4p^2(p.k) + 4(p.k)^2)(k^2\eta_{\mu\nu} + k_\mu k_\nu). \tag{2.16}$$

As a check for Eq. (2.13), we see that, apart from the trigonometric factor $\sin^2(\frac{p\wedge k}{2})$, the remaining of the expression is exactly the same as the one appearing in [37], where a detailed one-loop analysis of the Yang-Mills-Chern-Simons theory is presented. The most general tensor structure of the photon self-energy in a noncommutative three-dimensional spacetime is given as

$$\Pi^{\mu\nu} = \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi^\star_{\rm e} + \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2}\tilde{\Pi}^\star_{\rm e} + i\Pi^{\rm A}_{\rm o}\epsilon^{\mu\nu\lambda}p_\lambda \\ + \Pi^{\rm S}_{\rm o}(\tilde{p}^\mu u^\nu + \tilde{p}^\nu u^\mu), \tag{2.17}$$

where we had chosen $u_\mu = \epsilon_{\mu\alpha\beta}p^\alpha \tilde{p}^\beta$, with $p^\mu u_\mu = \tilde{p}^\mu u_\mu = 0$ as our orthonormal basis. We notice, however, that $\Pi^{\mu\nu}$ in this basis has nine terms that reduce to four terms due to the Ward identity[1] (for further details, see Appendix B). It is notable that the tensor structure of the first and the third term in (2.17) is analogous to that of the tree-level counterparts in commutative Maxwell-Chern-Simons model, which is a free theory. Indeed, the first term is even under **CP** and **PT**, while the third term is **CP**-odd and **PT**-even, they are similar to the commutative Maxwell and Chern-Simons actions, respectively.

On the other hand, the second and the fourth terms have no tree-level counterparts in commutative Maxwell-Chern-Simons model, arising fully from quantum effects. Furthermore, the second term in (2.17), similar to the first term, is even under **CP** and **PT**, while the fourth term, similar to the third term, is **CP**-odd and **PT**-even. Consequently, the behavior of the quantum effect terms is the same as that of the tree-level terms, as we expected. We observe hence that the specific decomposition appearing in the tensor structure of the photon self-energy in (2.17) is physically justified, using the aforementioned discussion on discrete symmetries.

Besides, we see that the one-loop photon self-energy (2.13) is invariant under $\theta \to -\theta$, which is in agreement with its tensor structure described by (2.17). Consequently, all of the form factor coefficients $\Pi^\star_{\rm e}$, $\tilde{\Pi}^\star_{\rm e}$, $\Pi^{\rm A}_{\rm o}$ and $\Pi^{\rm S}_{\rm o}$ in (2.17) are expected to be even in θ, at least at one-loop level, as we see in the following identities, Eqs. (B10)–(B13),

$$\Pi^\star_{\rm e} = \eta_{\mu\nu}\Pi^{\mu\nu} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\Pi^{\mu\nu}, \tag{2.18}$$

$$\tilde{\Pi}^\star_{\rm e} = -\eta_{\mu\nu}\Pi^{\mu\nu} + 2\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\Pi^{\mu\nu}, \tag{2.19}$$

$$\Pi^{\rm A}_{\rm o} = \frac{i}{2p^2}\epsilon_{\mu\nu\alpha}p^\alpha \Pi^{\mu\nu}, \tag{2.20}$$

$$\Pi^{\rm S}_{\rm o} = -\frac{1}{2\tilde{p}^4 p^2}(u_\mu \tilde{p}_\nu + u_\nu \tilde{p}_\mu)\Pi^{\mu\nu}. \tag{2.21}$$

With these quantities we have introduced all necessary information on regard to our analysis. Next, we shall proceed and write explicitly the one-loop expressions for the form factors of the gauge field self-energy.

A. Form factors

In order to evaluate the above relations, Eqs. (2.18)–(2.21), we shall take the respective tensor contraction with the one-loop expression (2.13). First, the relation (2.18) yields the following result,

[1]Since the tensor $\Pi^{\mu\nu}$ is not totally symmetric, the Ward identity must hold for both $p_\mu \Pi^{\mu\nu} = 0$ and $p_\nu \Pi^{\mu\nu} = 0$.

$$\Pi_{\mathrm{e}}^{\star}(p^2) = ie^2\int\frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{1}{(p+k)^2((p+k)^2-m^2)}\Bigg\{6(p.k)+4\frac{(k.\tilde{p})^2}{\tilde{p}^2}-16m^2$$
$$+\frac{1}{(k^2-m^2)}\left(16(p.k)^2+10p^2(p.k)+p^4+\frac{(k.\tilde{p})^2}{\tilde{p}^2}(4(p.k)-2p^2+16m^2)-26m^2(p.k)-12m^2p^2-16m^4\right)$$
$$+\frac{1}{k^2(k^2-m^2)}\Bigg(\frac{(k.\tilde{p})^2}{\tilde{p}^2}(m^2(8p^2+14(p.k))-4(p.k)^2-2p^4-8p^2(p.k))$$
$$-4m^2p^2(p.k)-8m^2(p.k)^2+3p^2(p.k)^2+4(p.k)^3\Bigg)\Bigg\}. \tag{2.22}$$

Besides, from the relation (2.19), we obtain

$$\tilde{\Pi}_{\mathrm{e}}^{\star}(p^2) = ie^2\int\frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{1}{(p+k)^2((p+k)^2-m^2)}\Bigg\{2k^2+2(p.k)-4p^2+16m^2-8\frac{(k.\tilde{p})^2}{\tilde{p}^2}$$
$$+\frac{1}{(k^2-m^2)}\left(m^2(30(p.k)+7p^2+14m^2)+2m^4-3p^4-10p^2(p.k)+\frac{(k.\tilde{p})^2}{\tilde{p}^2}(4p^2-8(p.k)-32m^2)\right)$$
$$+\frac{1}{k^2(k^2-m^2)}\Bigg[\frac{(k.\tilde{p})^2}{\tilde{p}^2}(8(p.k)^2+4p^4+16p^2(p.k)-2m^2[8p^2+14(p.k)])$$
$$+(p.k)(m^2(4p^2+7(p.k))+p^2(p.k)+4(p.k)^2)\Bigg]\Bigg\}. \tag{2.23}$$

Moreover, from the relation (2.20), we find the odd form factor $\Pi_{\mathrm{o}}^{\mathrm{A}}$ as

$$\Pi_{\mathrm{o}}^{\mathrm{A}}(p^2) = \frac{2mie^2}{p^2}\int\frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{1}{(p+k)^2((p+k)^2-m^2)}$$
$$\times\left\{5p^2+\frac{((5(p.k)+4p^2+7m^2)p^2-5(p.k)^2)}{(k^2-m^2)}-\frac{(p.k)^2(2m^2+5(p.k)+4p^2)}{k^2(k^2-m^2)}\right\}. \tag{2.24}$$

Finally, the form factor $\Pi_{\mathrm{o}}^{\mathrm{S}}$ follows (2.21) as

$$\Pi_{\mathrm{o}}^{\mathrm{S}}(p^2) = \frac{i}{\tilde{p}^4p^2}e^2\int\frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{(u.k)(\tilde{p}.k)}{(p+k)^2((p+k)^2-m^2)}\Bigg\{4+\frac{1}{(k^2-m^2)}(16m^2-2p^2+4(p.k))$$
$$+\frac{1}{k^2(k^2-m^2)}(14m^2(p.k)+8m^2p^2-4(p.k)^2-2p^4-8p^2(p.k))\Bigg\}. \tag{2.25}$$

In order to evaluate the above momentum integration, we can make use of the standard rules for Feynman integrals. We shall next continue with our formal development by providing now a detailed account for the renormalizability of the photon two-point function, which analysis will allow us to define properly the one-loop dispersion relation.

III. RENORMALIZED GAUGE PROPAGATOR AND MASS

We shall now formally establish the gauge field two-point function renormalizability. In particular, we want to determine the renormalized gauge propagator and mass, which allow us to define the physical pole and, therefore, the dispersion relation of the gauge field. We start by writing the complete propagator expression (B9),[2]

$$i\mathcal{D}_{\mu\nu} = \frac{p^2-\Pi_{\mathrm{e}}^{\star}-\tilde{\Pi}_{\mathrm{e}}^{\star}}{\mathcal{R}}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}-\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right)$$
$$+\frac{p^2-\Pi_{\mathrm{e}}^{\star}}{\mathcal{R}}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}+\frac{m+\Pi_{\mathrm{o}}^{\mathrm{A}}}{\mathcal{R}}i\varepsilon_{\mu\nu\lambda}p^\lambda+\frac{\xi}{p^4}p_\mu p_\nu, \tag{3.1}$$

[2]For a complete account of this discussion, see Appendix B. Moreover, we take $\Pi_{\mathrm{o}}^{\mathrm{S}}=0$ in Eq. (B9). This is explicitly shown in Eqs. (4.14) and (4.15), which is expected to be true to all orders.

where the quantity $\mathcal{R}$ at the denominator is given by

$$\mathcal{R} = (p^2 - \Pi_e^\star)(p^2 - \Pi_e^\star - \tilde{\Pi}_e^\star) + p^2[(\tilde{p}^2\Pi_o^S)^2 - (m + \Pi_o^A)^2].$$

It proves to be convenient for our development to make a few replacements,

$$\{\Pi_e^\star, \tilde{\Pi}_e^\star\} \to p^2\{\Pi_e, \tilde{\Pi}_e\}, \tag{3.2}$$

where Π_e and $\tilde{\Pi}_e$ are dimensionless form factors. With this new definition, and also introducing the notation $\Pi'_e = \Pi_e + \tilde{\Pi}_e$, the exact propagator (3.1) is conveniently rewritten as

$$\begin{aligned} i\mathcal{D}_{\mu\nu} = {} & \frac{1}{(1-\Pi_e)\left[p^2 - \frac{(m+\Pi_o^A)^2}{(1-\Pi_e)(1-\Pi'_e)}\right]}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\right) \\ & + \frac{1}{(1-\Pi'_e)\left[p^2 - \frac{(m+\Pi_o^A)^2}{(1-\Pi_e)(1-\Pi'_e)}\right]}\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \\ & + \frac{(m+\Pi_o^A)}{(1-\Pi_e)(1-\Pi'_e)\left[p^2 - \frac{(m+\Pi_o^A)^2}{(1-\Pi_e)(1-\Pi'_e)}\right]}\frac{i\varepsilon_{\mu\nu\lambda}p^\lambda}{p^2} \\ & + \frac{\xi}{p^4}p_\mu p_\nu. \end{aligned} \tag{3.3}$$

From this expression, we can readily identify the respective renormalization functions and renormalized mass. Thus, we introduce the wave function and mass renormalization constants as the following,

$$\mathcal{Z} = 1 - \Pi_e, \qquad \tilde{\mathcal{Z}} = 1 - \Pi'_e, \qquad \mathcal{Z}_m = 1 + m^{-1}\Pi_o^A, \tag{3.4}$$

and we also define the renormalized mass as

$$m_{ren}^2 = \frac{(m+\Pi_o^A)^2}{(1-\Pi_e)(1-\Pi'_e)} = \frac{\mathcal{Z}_m^2}{\mathcal{Z}\tilde{\mathcal{Z}}}m^2. \tag{3.5}$$

The most significant consequence arising from the multiplicative property of (3.5) is that the gauge symmetry is exactly preserved at classical (tree) and quantum (loop) levels. Mainly because $m = 0$ corresponds to the noncommutative Maxwell theory which is gauge invariant at any order, without any mass generation. Hence, by taking into account the above definitions, Eqs. (3.4) and (3.5), we rewrite the propagator (3.3) in the form

$$\begin{aligned} i\mathcal{D}_{\mu\nu} = {} & \frac{1}{\mathcal{Z}[p^2 - m_{ren}^2]}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\right) \\ & + \frac{1}{\tilde{\mathcal{Z}}[p^2 - m_{ren}^2]}\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \\ & + \frac{m_{ren}}{\sqrt{\mathcal{Z}\tilde{\mathcal{Z}}}[p^2 - m_{ren}^2]}\frac{i\varepsilon_{\mu\nu\lambda}p^\lambda}{p^2} + \frac{\xi}{p^4}p_\mu p_\nu. \end{aligned} \tag{3.6}$$

It should be emphasized that, in view of Eq. (3.6), the multiplicative renormalization holds for the gauge-invariant noncommutative Maxwell-Chern-Simons theory, which is a remarkable result. Furthermore, we should stress the fact that the physical massive pole $p^2 = m_{ren}^2$ is found to be naturally present at all physical terms of the complete propagator.

Since we are working within a perturbation theory approach, we can express the physically significant renormalized mass m_{ren} (3.5) at the lowest-order in terms of the form factors as

$$\begin{aligned} m_{ren} &= \frac{(m+\Pi_o^A)}{\sqrt{(1-\Pi_e)(1-\Pi'_e)}} \\ &\simeq m\left(1 + \frac{1}{m}\Pi_o^A + \Pi_e + \frac{1}{2}\tilde{\Pi}_e + \mathcal{O}(\alpha^2)\right). \end{aligned} \tag{3.7}$$

whereas the dispersion relation $p^2 = m_{ren}^2$, the renormalized mass at the lowest order is given by (3.7) and, hence, written in a convenient form:

$$\omega^2 = |\vec{p}|^2 + m^2\left(1 + \frac{2}{m}\Pi_o^A + 2\Pi_e + \tilde{\Pi}_e + \mathcal{O}(\alpha^2)\right). \tag{3.8}$$

With this renormalizability discussion, we conclude our formal development for NC Maxwell-Chern-Simons theory. We will proceed now to compute explicitly the one-loop self-energy function, in particular its form factors. Afterwards, we shall particularize the results by considering some physical relevant limits where analytical expressions for the dispersion relation are found. In addition to these discussions on the dispersion relation, we will scrutinize it for an analysis of the UV/IR mixing in order to verify whether or not it jeopardizes the theory's renormalizability [48].

IV. ONE-LOOP RADIATIVE CORRECTION

In order to compute the momentum integration on the form factors we shall make use of the standard Feynman parametrization and dimensional regularization method. Some relevant results for the nonplanar integration can be found at Appendix C, also we present the complete expression of some lengthy expression of the form factors in Appendix D.

A. Transverse part Π_e

In this first study we will perform the calculation by reviewing some relevant detail. We start by making use of the Feynman parametrization to write the denominator of Eq. (2.22) in the form

$$\begin{aligned}\Pi_{e}(p^2) = \frac{ie^2}{p^2}\int\frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\Bigg\{\Gamma(2)\int d\Phi\frac{1}{[(p+k)^2-\Delta_1^2]^2}\left(6(p.k)+4\frac{(k.\tilde{p})^2}{\tilde{p}^2}-16m^2\right)\\ +\Gamma(3)\int d\Upsilon\frac{1}{[(k+(y+z)p)^2-\Delta_2^2]^3}\bigg(\frac{(k.\tilde{p})^2}{\tilde{p}^2}(4(p.k)-2p^2+16m^2)\\ +16(p.k)^2+10p^2(p.k)+p^4-26m^2(p.k)-12m^2p^2-16m^4\bigg)\\ +\Gamma(4)\int d\Xi\frac{1}{[(k+(z+w)p)^2-\Delta_3^2]^4}\bigg(4(p.k)^3-4m^2p^2(p.k)-8m^2(p.k)^2+3p^2(p.k)^2\\ +\frac{(k.\tilde{p})^2}{\tilde{p}^2}((8p^2+14(p.k))m^2-4(p.k)^2-2p^4-8p^2(p.k))\bigg)\Bigg\},\end{aligned} \tag{4.1}$$

where we have introduced the following notation for the integration measures,

$$\int d\Phi=\int dxdy\delta(x+y-1),\quad \int d\Upsilon=\int dxdydz\delta(x+y+z-1),\quad \int d\Xi=\int dxdydzdw\delta(x+y+z+w-1), \tag{4.2}$$

and defined the following quantities as well:

$$\Delta_1^2 = ym^2, \tag{4.3}$$

$$\Delta_2^2=(x+z)m^2-(y+z)(1-y-z)p^2, \tag{4.4}$$

$$\Delta_3^2=(y+w)m^2-(z+w)(1-z-w)p^2. \tag{4.5}$$

Next, by making a suitable change of variables on the momentum integration on the terms of (4.1), we find the expression

$$\begin{aligned}\Pi_{e}(p^2) = \frac{i\mu^{2(3-\omega)}e^2}{p^2}\Bigg\{\Gamma(2)\int d\Phi\left((-6p^2-16m^2)\Omega_2(\Delta_1)+4\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\Omega_2^{\mu\nu}(\Delta_1)\right)\\ +\Gamma(3)\int d\Upsilon\bigg([16(y+z)^2p^4-10(y+z)p^4+(26(y+z)-12)m^2p^2+p^4-16m^4]\Omega_3(\Delta_2)\\ +\left[16p_\mu p_\nu+\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}(-(4(y+z)+2)p^2+16m^2)\right]\Omega_3^{\mu\nu}(\Delta_2)\bigg)\\ +\Gamma(4)\int d\Xi\bigg(p^4[4(z+w)(1-2(z+w))m^2+(z+w)^2(3-4(z+w))p^2)]\Omega_4(\Delta_3)\\ +\bigg[(m^2(8-14(z+w))p^2-4(z+w)^2p^4-2p^4+8(z+w)p^4)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\\ +(-8m^2+3p^2-12(z+w)p^2)p_\mu p_\nu\bigg]\Omega_4^{\mu\nu}(\Delta_3)-\frac{4}{\tilde{p}^2}p_\mu p_\nu\tilde{p}_\lambda\tilde{p}_\beta\Omega_4^{\mu\nu\lambda\beta}(\Delta_3)\bigg)\Bigg\},\end{aligned} \tag{4.6}$$

where, by convenience, we have introduced the following notation for the momentum integration:

$$\{\Omega_a, \Omega_a^{\mu\nu}, \Omega_a^{\mu\nu\lambda\beta}\}(\Delta_i) = \int \frac{d^\omega Q}{(2\pi)^\omega} \sin^2\left(\frac{p\wedge Q}{2}\right) \frac{\{\mathbf{1}, Q^\mu Q^\nu, Q^\mu Q^\nu Q^\lambda Q^\beta\}}{[Q^2 - \Delta_i^2]^a}. \tag{4.7}$$

The integration from (4.7) can be readily calculated (see Appendix C). In particular, we can separate the planar and nonplanar contributions by using the trigonometric relation $2\sin^2(\frac{p\wedge Q}{2}) = 1 - \cos(p\wedge Q)$. The expressions for the planar and nonplanar contributions are explicitly given by Eqs. (D1) and (D2), respectively.

A remark is in place for the expressions Eqs. (D1) and (D2) (as well for the next ones). Since the remaining integrals on the Feynman parameters of these expressions are rather difficult to compute exactly (and no substantial information would be obtained), we shall leave it only indicated and evaluate them in some particular cases, which imply some simplification on the integrand, and we can hence discuss some interesting physical implications. We will also do this for the remaining contributions.

In particular, one can realize that, at the commutative limit, i.e., $\theta \to 0$, the planar and nonplanar contributions, Eqs. (D1) and (D2), result as expected in

$$(\Pi_\mathrm{e})_\mathrm{p}(p^2) + \lim_{\theta\to 0}(\Pi_\mathrm{e})_\mathrm{n-p}(p^2) = 0. \tag{4.8}$$

Furthermore, this vanishing result is in agreement with the tree-level (propagator) structure of the commutative Maxwell-Chern-Simons which is a free theory.

B. NC transverse part $\tilde{\Pi}_\mathrm{e}$

Next, in order to evaluate the integration of the contribution (2.23), we follow the aforementioned steps and write it conveniently in terms of the quantities $\Omega_a^{\mu\nu\cdots}(\Delta_i)$

$$\begin{aligned}\tilde{\Pi}_\mathrm{e}(p^2) = \frac{i\mu^{2(3-\omega)}e^2}{p^2}\Bigg\{&\Gamma(2)\int d\Phi\left[(-4p^2+16m^2)\Omega_2(\Delta_1) + \left(2\eta_{\mu\nu} - 8\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right)\Omega_2^{\mu\nu}(\Delta_1)\right]\\ &+\Gamma(3)\int d\Upsilon\bigg((m^2(-30(y+z)p^2+7p^2+16m^2) - 3p^4 + 10(y+z)p^4)\Omega_3(\Delta_2)\\ &+(4p^2+8(y+z)p^2-32m^2)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\Omega_3^{\mu\nu}(\Delta_2)\bigg)\\ &+\Gamma(4)\int d\Xi\bigg(p^4[-4m^2(z+w)+7m^2(z+w)^2+(z+w)^2p^2-4(z+w)^3p^2]\Omega_4(\Delta_3)\\ &+\bigg[8((z+w)^2p^2+4p^2-16(z+w)p^2-2m^2[8-14(z+w)])p^2\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\\ &+(7m^2+p^2-12(z+w)p^2)p_\mu p_\nu\bigg]\Omega_4^{\mu\nu}(\Delta_3) + \frac{8}{\tilde{p}^2}p_\mu p_\nu\tilde{p}_\lambda\tilde{p}_\beta\Omega_4^{\mu\nu\lambda\beta}(\Delta_3)\bigg)\Bigg\}.\end{aligned} \tag{4.9}$$

Once again, the planar and nonplanar contributions can be computed separately. For convenience, the expressions for the planar and nonplanar contributions are written in Appendix D, given by Eqs. (D3) and (D4), respectively.

In contrast with the previous case, we see from Eqs. (D3) and (D4), that the planar and nonplanar parts do not sum to zero at the commutative limit. This equation is a manifestation of UV/IR mixing[3] [21]

$$(\tilde{\Pi}_\mathrm{e})_\mathrm{p}(p^2) + \lim_{\theta\to 0}(\tilde{\Pi}_\mathrm{e})_\mathrm{n-p}(p^2) = -\frac{e^2}{4\pi}\frac{1}{p^2}\left[\frac{1}{|\tilde{p}|} + \frac{2m}{3}\right] \neq 0. \tag{4.10}$$

As it will be discussed later, one should already notice that the presence of an UV/IR mixing term here might render the theory to be inconsistent, spoiling hence the renormalizability of the theory. Besides, it is worth notice that the UV/IR mixing in $2+1$ dimensions appears in a less severe degree as $\frac{1}{|\tilde{p}|}$, while in $3+1$ dimensions it is given as $\frac{1}{\tilde{p}^2}$. Furthermore, in comparison to the form factor Π_e outcome, we see that the commutative limit for the contribution $\tilde{\Pi}_\mathrm{e}$ is related to the fact that it did not have a tree-level counterpart. So this can be traced back to a purely noncommutative (quantum) effect.

C. *CP* odd part $\Pi_\mathrm{o}^\mathrm{A}$

Moreover, we rewrite Eq. (2.24) conveniently in terms of the quantities $\Omega_a^{\mu\nu\cdots}(\Delta_i)$ such as

[3]Although, the noncommutative Maxwell-Chern-Simons theory in three dimensions is UV finite, we observe explicitly a UV/IR mixing in our one-loop results.

$$\Pi_{\mathrm{o}}^{\mathrm{A}}(p^2) = \frac{2mie^2}{p^2}\mu^{2(3-\omega)}\left\{5p^2\Gamma(2)\int d\Phi\Omega_2(\Delta_1) + \Gamma(3)\int d\Upsilon[((-6+15x-5x^2)p^2+7m^2)p^2\Omega_3(\Delta_2) - 5p_\mu p_\nu\Omega_3^{\mu\nu}(\Delta_2)]\right.$$
$$- p^2\Gamma(4)\int d\Xi(z+w)^2\{2m^2-5(z+w)p^2+4p^2\}\Omega_4(\Delta_3)$$
$$\left.-\Gamma(4)\int d\Xi(2m^2-5(z+w)p^2+4p^2-10(z+w)p^2)p_\mu p_\nu\Omega_4^{\mu\nu}(\Delta_3)\right\}. \tag{4.11}$$

Once again, we write down the expressions for the planar and nonplanar contributions in Appendix D, explicitly written in Eqs. (D5) and (D6), respectively.

In agreement with our expectations, the sum of the planar and nonplanar contributions, Eqs. (D5) and (D6), at the commutative limit, vanishes

$$(\Pi_{\mathrm{o}}^{\mathrm{A}})_{\mathrm{p}}(p^2) + \lim_{\theta\to 0}(\Pi_{\mathrm{o}}^{\mathrm{A}})_{\mathrm{n-p}}(p^2) = 0. \tag{4.12}$$

Exactly as it did happened with the form factor Π_{e} in (4.8), this vanishing result is compatible with the free nature of the commutative Maxwell-Chern-Simons theory, where the form factor $\Pi_{\mathrm{o}}^{\mathrm{A}}$ has a tree-level counterpart.

D. NC odd part $\Pi_{\mathrm{o}}^{\mathrm{S}}$

Finally, we shall now analyze the NC odd part $\Pi_{\mathrm{o}}^{\mathrm{S}}$; however, due to its vanishing results, we present its final expressions here in order to discuss its conclusion. We rewrite Eq. (2.25) in terms of the quantities $\Omega_a^{\mu\nu\cdots}(\Delta_i)$ such as

$$\Pi_{\mathrm{o}}^{\mathrm{S}}(p^2) = \frac{i}{\tilde{p}^4p^2}\mu^{2(3-\omega)}e^2u_\mu\tilde{p}_\nu\left\{4\int d\Phi\Omega_2^{\mu\nu}(\Delta_1) + 2\Gamma(3)\int d\Upsilon(8m^2-p^2-2(y+z)p^2)\Omega_3^{\mu\nu}(\Delta_2)\right.$$
$$\left.+2\Gamma(4)\int d\Xi(-2p_\lambda p_\beta\Omega_4^{\mu\nu\lambda\beta}(\Delta_3) + p^2[4m^2-2(z+w)^2p^2-p^2-(z+w)(7m^2-4p^2)]\Omega_4^{\mu\nu}(\Delta_3))\right\}. \tag{4.13}$$

Based on the results for the momentum integration, Eqs. (C2) and (C6), it is easy to conclude that the planar contribution vanishes

$$(\Pi_{\mathrm{o}}^{\mathrm{S}})_{\mathrm{p}}(p^2) = \frac{i}{2\tilde{p}^4p^2}e^2u_\mu\tilde{p}_\nu\{\eta^{\mu\nu}A(\Delta_1,\Delta_2,\Delta_3) + p_\lambda p_\beta(\eta^{\mu\nu}\eta^{\lambda\beta}+\eta^{\mu\lambda}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\lambda})B(\Delta_1,\Delta_2,\Delta_3)\}, = 0. \tag{4.14}$$

The last equality follows since $u.p=0$ and $u.\tilde{p}=0$. Now, proceeding in the same way, we have for the nonplanar part that

$$(\Pi_{\mathrm{o}}^{\mathrm{S}})_{\mathrm{n-p}}(p^2) = -\frac{i}{2\tilde{p}^4p^2}e^2u_\mu\tilde{p}_\nu\left\{\frac{\eta^{\mu\nu}}{\omega}C(\Delta_1,\Delta_2) + \frac{\tilde{p}^\mu\tilde{p}^\nu}{\tilde{p}^2}D(\Delta_1,\Delta_2) + p_\lambda p_\beta\left[(\eta^{\mu\nu}\eta^{\lambda\beta}+\eta^{\nu\lambda}\eta^{\mu\beta}+\eta^{\nu\beta}\eta^{\lambda\mu})E(\Delta_3)\right.\right.$$
$$\left.\left.+\frac{\tilde{p}^\lambda\tilde{p}^\beta\tilde{p}^\nu\tilde{p}^\mu}{\tilde{p}^4}G(\Delta_3) + \left(\eta^{\lambda\beta}\frac{\tilde{p}^\nu\tilde{p}^\mu}{\tilde{p}^2} + \text{sym permutations}\right)F(\Delta_3)\right]\right\} = 0. \tag{4.15}$$

Again we have that the resulting expression is proportional to $u.\tilde{p}=0$ and $u.p=0$. These vanishing results are in agreement with the literature, since we can think about the Bose-Einstein symmetry on the $\Pi^{\mu\nu}$, i.e., $\mu\leftrightarrow\nu$ and $p\to -p$, in addition to its accidental symmetry $\theta\to-\theta$, these combined facts show that the term $\Pi_{\mathrm{o}}^{\mathrm{S}}$ will not be radiatively generated at higher order as well.

V. DISPERSION RELATION AND LIMITING CASES

In order to establish some limits of special interest, we consider the scaling $eA_\mu \to \mathcal{A}_\mu$ on the Lagrangian (2.1), this implies into the following change[4]

$$\mathcal{L} = -\frac{1}{4e^2}\mathcal{F}_{\mu\nu}\star\mathcal{F}^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}\left(\mathcal{A}_\mu\star\partial_\nu\mathcal{A}_\lambda + \frac{2}{3}\mathcal{A}_\mu\star\mathcal{A}_\nu\star\mathcal{A}_\lambda\right), \tag{5.1}$$

[4]It is notable that the mass dimension of the gauge field and the coupling constant in $2+1$ dimensions is equal to $\frac{1}{2}$, hence the mass dimension of the new gauge field $\mathcal{A}_\mu$ and κ is equal to 1 and 0, respectively.

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]_\star$ and also we have introduced a new parameter $\kappa e^2 \equiv m$. Therefore, from such Lagrangian we can immediately read three limits of interest:

(i) The NC Chern-Simons model is obtained when $e^2 \to \infty$, i.e., $m^2 \to \infty$, so that the ratio $\kappa = m/e^2$ is kept finite;

(ii) The NC Maxwell model is obtained when $\kappa \to 0$, i.e., $m^2 \to 0$, so that e^2 is kept finite;

(iii) We can consider the low-momenta limit, also known as highly noncommutative limit, i.e., $p^2/m^2 \to 0$ while $\tilde{p}$ is kept finite.

These three cases will be analyzed accordingly from our previous results in Sec. IV, allowing us to obtain closed expression for the resulting form factors.

A. NC Chern-Simons model

This first limit is somehow laborious, and demands some careful analysis. We can analyze the NC Chern-Simons theory by taking directly the limit $m^2 \to \infty$ in the expression (2.13), we obtain the following result:

$$\Pi_{\mu\nu}(p) = \frac{m}{\kappa} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \frac{1}{(p+k)^2} \sin^2\left(\frac{p \wedge k}{2}\right) [k_\mu p_\nu - k_\nu p_\mu] \tag{5.2}$$

As usual, one can put the different poles under the same denominator, which result into a change of the type: $Q_\mu = k_\mu + c p_\mu$, where c is some function of the Feynman parameter(s), for instance. After this manipulation, one can easily find the expression

$$\Pi_{\mu\nu}(p) = \frac{m}{\kappa} \int \frac{d^3Q}{(2\pi)^3} \frac{1}{(Q^2 - \Delta^2)^2} \sin^2\left(\frac{p \wedge Q}{2}\right) \times [Q_\mu p_\nu - Q_\nu p_\mu] = 0. \tag{5.3}$$

This vanishing result shows that no radiative correction to the gauge field propagator is generated and accordingly its exhibits the free nature of the noncommutative Chern-Simons theory, which is in agreement with a previous analysis [32].

B. NC Maxwell model

From either definition (2.13) or form factors, Eqs. (2.22), (2.23), and (2.24), the limit $m \to 0$ follows straightforwardly. In this case, the form factor $\Pi_e(p^2)$ is obtained from the sum of the planar and nonplanar contributions, Eqs. (D1) and (D2), respectively, and it yields to

$$\begin{aligned}
\Pi_e(p^2) = {}& \frac{3e^2}{8\pi}|\tilde{p}| + \frac{e^2}{16\pi}\frac{1}{|p|}\left\{\frac{1}{2}\int \frac{d\Upsilon}{(\Delta_2^2)^{\frac{3}{2}}}((4(y+z)+2)\Delta_2^2 \Sigma^{(-)}(p,\tilde{p},\Delta_2) \right. \\
&+ (16(y+z)^2 - 10(y+z) + 1)\Sigma^{(+)}(p,\tilde{p},\Delta_2) - 16\Delta_2^2\Sigma(p,\tilde{p},\Delta_2)) \\
&- \frac{1}{4}\int \frac{d\Xi}{(\Delta_3^2)^{\frac{5}{2}}}((z+w)^2(3-4(z+w))\Sigma^{(1)}(p,\tilde{p},\Delta_3) - (3-12(z+w))\Delta_3^2\Sigma^{(+)}(p,\tilde{p},\Delta_3) \\
&\left. + (2 + 4(z+w)^2 - 8(z+w))\Delta_3^2\Sigma^{(2)}(p,\tilde{p},\Delta_3) - 4\Delta_3^4\Sigma^{(-)}(p,\tilde{p},\Delta_3))\right\},
\end{aligned} \tag{5.4}$$

where in order to simplify the notation we have introduced the quantities

$$\Sigma(p,\tilde{p},\Delta_i) = 1 - e^{-\Delta_i|p||\tilde{p}|}, \tag{5.5}$$

$$\Sigma^{(\pm)}(p,\tilde{p},\Delta_i) = 1 - (1 \pm \Delta_i|p||\tilde{p}|)e^{-\Delta_i|p||\tilde{p}|}, \tag{5.6}$$

$$\Sigma^{(1)}(p,\tilde{p},\Delta_i) = 3 - (3 + 3\Delta_i|p||\tilde{p}| + \Delta_i^2 p^2 \tilde{p}^2)e^{-\Delta_i|p||\tilde{p}|}, \tag{5.7}$$

$$\Sigma^{(2)}(p,\tilde{p},\Delta_i) = 1 - [1 + \Delta_i|p||\tilde{p}| - \Delta_i^2 p^2 \tilde{p}^2]e^{-\Delta_i|p||\tilde{p}|}, \tag{5.8}$$

here the quantities Δ_2 and Δ_3 are given as

$$\Delta_2^2 = -(y+z)(1-y-z), \tag{5.9}$$

$$\Delta_3^2 = -(z+w)(1-z-w). \tag{5.10}$$

Besides, from Eqs. (D3) and (D4), we find for the NC transverse part $\tilde{\Pi}_e$ the following

$$\tilde{\Pi}_{\mathrm{e}}(p^2) = \frac{e^2}{4\pi}\left[|\tilde{p}| - \frac{1}{|\tilde{p}|p^2}\right] + \frac{e^2}{32\pi}\frac{1}{|p|}\left\{\int \frac{d\Upsilon}{(\Delta_2^2)^{\frac{3}{2}}}((10(y+z)-3)\Sigma^{(+)}(p,\tilde{p},\Delta_2) - 4(1+2(y+z))\Delta_2^2\Sigma^{(-)}(p,\tilde{p},\Delta_2))\right.$$
$$-\frac{1}{2}\int\frac{d\Xi}{(\Delta_3^2)^{\frac{5}{2}}}((z+w)^2[1-4(z+w)]\Sigma^{(1)}(p,\tilde{p},\Delta_3) + (12(z+w)-1)\Delta_3^2\Sigma^{(+)}(p,\tilde{p},\Delta_3)$$
$$\left. - 4(1+2(z+w)^2 - 4(z+w))\Delta_3^2\Sigma^{(2)}(p,\tilde{p},\Delta_3) + 8\Delta_3^4\Sigma^{(-)}(p,\tilde{p},\Delta_3))\right\}. \tag{5.11}$$

Finally, for the CP odd form factor $\Pi_{\mathrm{o}}^{\mathrm{A}}$, Eqs. (D5) and (D6), we find as expected a vanishing result:

$$\Pi_{\mathrm{o}}^{\mathrm{A}}(p^2) = (\Pi_{\mathrm{o}}^{\mathrm{A}})_{\mathrm{p}}(p^2) + (\Pi_{\mathrm{o}}^{\mathrm{A}})_{\mathrm{n-p}}(p^2) = 0. \tag{5.12}$$

1. Dispersion relation

In the NC Maxwell theory we take the limit $m \to 0$. Thus, we can write the complete propagator (3.6) as the following

$$i\mathcal{D}_{\mu\nu} = \frac{1}{[p^2 - |p|\Pi_{\mathrm{e}}^{(1)}(p^2)]}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right) + \frac{1}{[p^2 - |p|\tilde{\Pi}_{\mathrm{e}}^{(1)}(p^2)]}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2} + \frac{\xi}{p^4}p_\mu p_\nu, \tag{5.13}$$

where the form factor expressions are defined so that $\{\Pi_{\mathrm{e}}^{(1)}, \tilde{\Pi}_{\mathrm{e}}^{(1)}\} = |p|\{\Pi_{\mathrm{e}}, \tilde{\Pi}_{\mathrm{e}}\}$, with Π_{e} and $\tilde{\Pi}_{\mathrm{e}}$ given by Eqs. (5.4) and (5.11), respectively.

In particular, the poles obtained above in Eq. (5.13), i.e., $p^2 - |p|\Pi_e^{(1)}(p^2)$, reproduce a similar profile as those found on the three-dimensional Yang-Mills theory [37,51], which allow us to (partially) discuss the infrared finiteness of the model.

In order to illustrate the pole behavior, we can take small perturbations around $p^2\tilde{p}^2$, so that at the leading-order the form factor expressions are reduced to

$$\Pi_{\mathrm{e}}^{(1)}(p^2) = \frac{7e^2}{24\pi}|p||\tilde{p}| - \frac{13ie^2}{512}p^2\tilde{p}^2, \tag{5.14}$$

and

$$\tilde{\Pi}_{\mathrm{e}}^{(1)}(p^2) = -\frac{e^2}{12\pi}|p||\tilde{p}| - \frac{e^2}{4\pi}\frac{1}{|p||\tilde{p}|} + \frac{3ie^2}{256}p^2\tilde{p}^2. \tag{5.15}$$

These equations at the commutative limit change to the following:

$$\lim_{\theta\to 0}\Pi_{\mathrm{e}}^{(1)}(p^2) = 0, \tag{5.16}$$

$$\lim_{\theta\to 0}\tilde{\Pi}_{\mathrm{e}}^{(1)}(p^2) = -\frac{e^2}{4\pi}\frac{1}{|p||\tilde{p}|}. \tag{5.17}$$

We note that (5.16) is consistent with the commutative Maxwell action, which is a free theory, however (5.17), similar to (4.10), exhibits the UV/IR mixing effect in $2+1$ dimensions and does not correspond to any counterpart term in commutative Maxwell theory. The presence of the UV/IR mixing in the NC Maxwell theory emphasize the fact that this theory is not infrared finite.

C. Highly noncommutative Maxwell-Chern-Simons model

We now study the third limiting case, which describes the low-momentum (or highly noncommutative) behavior of NC Maxwell–Chern-Simons model. For the highly noncommutative case, i.e., considering the limit $p^2/m^2 \to 0$ while $\tilde{p}^2$ is kept finite, we can proceed in the exactly same way as before. In this scenario, the complete form factor $\Pi_{\mathrm{e}}(p^2)$ is obtained from Eqs. (D1) and (D2), this results into

$$\Pi_{\mathrm{e}}(p^2) \simeq \frac{1}{16\pi\kappa}\left\{\int\frac{d\Phi}{\sqrt{\Delta_1^2}}\left[6 + 4(4-y)\frac{m^2}{p^2}\right]\Sigma(m,\tilde{p},\Delta_1) + \frac{1}{2}\int\frac{d\Upsilon}{(\Delta_2^2)^{\frac{3}{2}}}\left(\left[2(13(y+z)-6) - 16\frac{m^2}{p^2}\right]\Sigma^{(+)}(m,\tilde{p},\Delta_2)\right.\right.$$
$$\left. + 2\left((2(y+z)+1)\Delta_2^2 - 8\frac{m^2}{p^2}\Delta_2^2 + 8(y+z)(1-y-z)\right)\Sigma^{(-)}(m,\tilde{p},\Delta_2)\right)$$
$$\left. -\frac{1}{4}\int\frac{d\Xi}{(\Delta_3^2)^{\frac{3}{2}}}\left(8\Sigma^{(+)}(m,\tilde{p},\Delta_3) - (8-14(z+w))\Sigma^{(2)}(m,\tilde{p},\Delta_3) - 4\Delta_3^2\Sigma^{(-)}(m,\tilde{p},\Delta_3)\right)\right\} + \mathcal{O}\left(\frac{p^2}{m^2}\right), \tag{5.18}$$

where the functions $\Sigma^{(i)}(m,\tilde{p},\Delta_j)$ are those defined before, Eqs. (5.5)–(5.8), but now the quantities Δ_i are given by

$$\Delta_1^2 = y, \qquad \Delta_2^2 \simeq (x+z) + \mathcal{O}\left(\frac{p^2}{m^2}\right), \qquad \Delta_3^2 \simeq (y+w) + \mathcal{O}\left(\frac{p^2}{m^2}\right). \tag{5.19}$$

Moreover, we find for the NC transverse part $\tilde{\Pi}_{\mathrm{e}}$, the sum of the Eqs. (D3) and (D4), the following expression

$$\begin{aligned}\tilde{\Pi}_{\mathrm{e}}(p^2) \simeq \frac{1}{16\pi\kappa}\Bigg\{&\int \frac{d\Phi}{\sqrt{\Delta_1^2}}\left(2\left[2+(y-8)\frac{m^2}{p^2}\right]\Sigma(m,\tilde{p},\Delta_1) - \frac{4m^2}{p^2}\left[\frac{\sqrt{y}}{m|\tilde{p}|}+y\right]e^{-\sqrt{y}m|\tilde{p}|}\right)\\ &+\frac{1}{2}\int\frac{d\Upsilon}{(\Delta_2^2)^{\frac{3}{2}}}\left(\left[7+16\frac{m^2}{p^2}-30(y+z)\right]\Sigma^{(+)}(m,\tilde{p},\Delta_2)\right.\\ &\left.-4\left[\left(1+2(y+z)-8\frac{m^2}{p^2}\right)\Delta_2^2+8(y+z)(1-y-z)\right]\Sigma^{(-)}(m,\tilde{p},\Delta_2)\right)\\ &-\frac{1}{4}\int\frac{d\Xi}{(\Delta_3^2)^{\frac{3}{2}}}\left(4(4-7(z+w))\Sigma^{(2)}(m,\tilde{p},\Delta_3) - 7\Sigma^{(+)}(m,\tilde{p},\Delta_3)+8\Delta_3^2\Sigma^{(-)}(m,\tilde{p},\Delta_3)\right)\Bigg\} + \mathcal{O}\left(\frac{p^2}{m^2}\right),\end{aligned} \tag{5.20}$$

At last, for the CP odd form factor $\Pi_{\mathrm{o}}^{\mathrm{A}}$, Eqs. (D5) and (D6), we get

$$\begin{aligned}\Pi_{\mathrm{o}}^{\mathrm{A}}(p^2) \simeq -\frac{1}{8\pi}\frac{m}{\kappa}\Bigg\{&5\int\frac{d\Phi}{\sqrt{\Delta_1^2}}\Sigma(m,\tilde{p},\Delta_1) - \frac{1}{2}\int\frac{d\Upsilon}{(\Delta_2^2)^{\frac{3}{2}}}(7\Sigma^{(+)}(m,\tilde{p},\Delta_2)+5\Delta_2^2\Sigma(m,\tilde{p},\Delta_2))\\ &-\frac{1}{2}\int\frac{d\Xi}{(\Delta_3^2)^{\frac{3}{2}}}\Sigma^{(+)}(m,\tilde{p},\Delta_3)\Bigg\} + \mathcal{O}\left(\frac{p^2}{m^2}\right).\end{aligned} \tag{5.21}$$

An important check of our results for the low-momenta limit is needed. On one hand, in the commutative limit, Eqs. (4.8), (4.10), and (4.12), we have considered $\tilde{p}\to 0$, but this is explicitly read as $\theta\to 0$ when p is kept finite. On the other hand, however, we can equally consider $\tilde{p}\to 0$ as given by $p\to 0$ with $\theta =$ finite. We can immediately conclude from the low-momenta limit expressions, Eqs. (5.18), (5.20), and (5.21), that the latter limit is in agreement with the (former) commutative limit. Moreover, in order to understand this point consider the scale $\theta = \frac{1}{\Lambda^2}$, hence, the commutative limit can be interpreted as $\frac{p}{\Lambda}\ll 1$, where p is the external momentum. The condition $\frac{p}{\Lambda}\ll 1$ can then happen in two distinct cases:

$$\begin{cases}\Lambda\to\infty, & p = \text{finite}\\ p\to 0, & \theta = \text{finite}\end{cases} \tag{5.22}$$

Hence, we see that these two limits are indeed the same and our results are correct. In possess of the above explicit results we can proceed to the analysis the dispersion relation behavior for this case, and discuss the UV/IR mixing issue.

1. Dispersion relation

The remaining integration on the Eqs. (5.18), (5.20), and (5.21) can be computed analytically without further complication thanks to the simplification due to the limit $p^2/m^2\to 0$. We, thus, obtain for the transverse form factor the explicit expression

$$\begin{aligned}\Pi_{\mathrm{e}}(p^2) \simeq{}& \frac{1}{2\pi\kappa}\frac{1}{m^2p^2\tilde{p}^4}\left[-48+6\sqrt{m^2\tilde{p}^2}+(m^2\tilde{p}^2)^2+(48+42\sqrt{m^2\tilde{p}^2}+18m^2\tilde{p}^2+5(m^2\tilde{p}^2)^{\frac{3}{2}})e^{-\sqrt{m^2\tilde{p}^2}}\right]\\ &+\frac{1}{120\pi\kappa}\frac{1}{(m^2\tilde{p}^2)^3}\left[39600+39720\sqrt{m^2\tilde{p}^2}+250m^2\tilde{p}^2+570(m^2\tilde{p}^2)^{\frac{3}{2}}\right.\\ &-240(m^2\tilde{p}^2)^{\frac{5}{2}}+131(m^2\tilde{p}^2)^3-5(7920+24\sqrt{m^2\tilde{p}^2}-2800m^2\tilde{p}^2\\ &\left.-714(m^2\tilde{p}^2)^{\frac{3}{2}}+32(m^2\tilde{p}^2)^2+26(m^2\tilde{p}^2)^{\frac{5}{2}}+7(m^2\tilde{p}^2)^3)e^{-\sqrt{m^2\tilde{p}^2}}\right]+\mathcal{O}\left(\frac{p^2}{m^2}\right),\end{aligned} \tag{5.23}$$

and next, for the NC transverse form factor, we find the result

$$
\begin{aligned}
\tilde{\Pi}_{\mathrm{e}}(p^2) \simeq &-\frac{1}{4\pi\kappa}\frac{1}{m p^2 \tilde{p}^3}\left[-24 + (m^2\tilde{p}^2)^{\frac{3}{2}} + (24 + 24\sqrt{m^2\tilde{p}^2} + 13m^2\tilde{p}^2 + 4(m^2\tilde{p}^2)^{\frac{3}{2}})e^{-\sqrt{m^2\tilde{p}^2}}\right] \\
&-\frac{1}{240\pi\kappa}\frac{1}{(m^2\tilde{p}^2)^{\frac{5}{2}}}\Big[-26880 + 3060m^2\tilde{p}^2 + 229(m^2\tilde{p}^2)^{\frac{3}{2}} - 465(m^2\tilde{p}^2)^2 \\
&+ 5\Big(5376\Big(1+\sqrt{m^2\tilde{p}^2}\Big) + 2076m^2\tilde{p}^2 + 284(m^2\tilde{p}^2)^{\frac{3}{2}} \\
&+ 11(m^2\tilde{p}^2)^2 - 10(m^2\tilde{p}^2)^{\frac{5}{2}}\Big)e^{-\sqrt{m^2\tilde{p}^2}}\Big] + \mathcal{O}\left(\frac{p^2}{m^2}\right).
\end{aligned} \tag{5.24}
$$

Finally, for the CP odd form factor, we get

$$
\Pi_{\mathrm{o}}^{\mathrm{A}}(p^2) \simeq -\frac{m}{24\pi\kappa}\frac{1}{(m^2\tilde{p}^2)^{\frac{3}{2}}}\left[2(3 + 9m^2\tilde{p}^2 + (m^2\tilde{p}^2)^{\frac{3}{2}}) - 3\Big(2 + 2\sqrt{m^2\tilde{p}^2} + 7m^2\tilde{p}^2 + 7(m^2\tilde{p}^2)^{\frac{3}{2}}\Big)e^{-\sqrt{m^2\tilde{p}^2}}\right] + \mathcal{O}\left(\frac{p^2}{m^2}\right). \tag{5.25}
$$

We can analyze the modifications caused into the dispersion relation by the noncommutativity from examining first the formula (3.7). By simplicity, we shall consider the contribution up to the lowest order in $\alpha = e^2/4\pi$. This implies into the following expression for the renormalized mass

$$
m_{\mathrm{ren}} \simeq m + \Pi^{(1)} \tag{5.26}
$$

where we have defined $\Pi^{(1)} = \Pi_{\mathrm{o}}^{\mathrm{A}} + m\Pi_{\mathrm{e}} + \frac{m}{2}\tilde{\Pi}_{\mathrm{e}}$, so that its expression reads

$$
\begin{aligned}
\Pi^{(1)} = &\frac{m}{8\pi\kappa}\frac{1}{m^2 p^2 \tilde{p}^4}\Big[3\Big(-64 + 16\sqrt{m^2\tilde{p}^2} + (m^2\tilde{p}^2)^2\Big) \\
&+ \Big(192 + 144\sqrt{m^2\tilde{p}^2} + 48(m^2\tilde{p}^2) + 7(m^2\tilde{p}^2)^{\frac{3}{2}} - 4(m^2\tilde{p}^2)^2\Big)e^{-\sqrt{m^2\tilde{p}^2}}\Big] \\
&+ \frac{m}{96\pi\kappa}\frac{1}{(m^2\tilde{p}^2)^3}\Big[-31680 + 5280\sqrt{m^2\tilde{p}^2} + 2016m^2\tilde{p}^2 - 180(m^2\tilde{p}^2)^{\frac{3}{2}} \\
&- 171(m^2\tilde{p}^2)^{\frac{5}{2}} + 51(m^2\tilde{p}^2)^3 + \Big(31680 + 26400\sqrt{m^2\tilde{p}^2} + 8544m^2\tilde{p}^2 + 804(m^2\tilde{p}^2)^{\frac{3}{2}} \\
&- 388(m^2\tilde{p}^2)^2 - 31(m^2\tilde{p}^2)^{\frac{5}{2}} + 66(m^2\tilde{p}^2)^3\Big)e^{-\sqrt{m^2\tilde{p}^2}}\Big] + \mathcal{O}\left(\frac{|p|}{m\kappa}\right) + \mathcal{O}\left(\frac{m|\tilde{p}|}{\kappa}\right).
\end{aligned} \tag{5.27}
$$

In order to illustrate our result, we can consider small perturbation in powers of $m^2\tilde{p}^2$, and we find at the leading-order that

$$
\Pi^{(1)} = -\frac{1}{8\pi\kappa}\frac{m^2}{p^2|\tilde{p}|} + \frac{m}{\kappa}\left[-\frac{317}{1260\pi}\sqrt{m^2\tilde{p}^2} + \cdots\right] + \mathcal{O}\left(\frac{|p|}{m\kappa}\right) + \mathcal{O}\left(\frac{m^2\tilde{p}^2}{\kappa^2}\right) + \mathcal{O}\left(\frac{m^3|\tilde{p}|}{\kappa p^2}\right). \tag{5.28}
$$

Finally, substituting (5.28) into (3.8), we obtain the following dispersion relation

$$
\begin{aligned}
\omega^2 = &\vec{p}^2 + m^2 - \frac{1}{4\pi\kappa}\frac{m^3}{p^2|\tilde{p}|} + \mathcal{O}\left(\frac{|p|}{m\kappa}\right) + \mathcal{O}\left(\frac{m|\tilde{p}|}{\kappa}\right) \\
&+ \mathcal{O}\left(\frac{m^3|\tilde{p}|}{\kappa p^2}\right).
\end{aligned} \tag{5.29}
$$

In particular, we see from this expression that the highly noncommutative behavior has an UV/IR instability in the NC momentum and consequently the theory is not infrared finite. These facts can be seen as originating from the general result in Eq. (4.10).

Furthermore, the insertion of the form factor expressions (5.23), (5.24), and (5.25) into the relation (3.6) gives us the one-loop photon propagator in the low-energy limit of the noncommutative Maxwell-Chern-Simons theory. One of the main physical consequences of this corrected propagator is that we can determine the noncommutative one-loop corrections to the electrostatic potential energy of the Maxwell-Chern-Simons model. By taking into account the time-like photons from the above result, we find the expression[5]

$$V^{\text{1-loop}}(r) = e^2 \int dx_0 \times \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 - m_{\text{ren}}^2} e^{-i\vec{p}.\vec{r}+ip_0x_0}[1+\Pi_e], \tag{5.30}$$

that by setting $\Pi_e = 0$, the free part of the potential is given by $V^{\text{free}}(r) = -\frac{e^2}{2\pi}K_0(mr) \to \frac{e^2}{2\pi}\ln(mr)$ as $mr \ll 1$. Now, considering only the noncommutative contribution, i.e., by taking into account the leading contribution in Eqs. (5.23) and (5.28), we obtain the following deviation for the potential

$$\begin{aligned}\delta V^{\text{1-loop}}(r) &= -\frac{|\theta|e^4}{20160\pi^2}\int p^3 dp \frac{1}{p^3(p^2+m^2) - \frac{e^2}{4\pi}\frac{m^2}{|\theta|}} \times [4009p^2 + 4284m^2]J_0(pr), \\ &= -\frac{a|\theta|e^4}{\pi^2}\frac{1}{r}\int_0^\infty dp \frac{p^3(p^2+bm^2r^2)}{p^3(p^2+m^2r^2) - \frac{e^2}{4\pi}\frac{m^2r^5}{|\theta|}}J_0(p),\end{aligned} \tag{5.31}$$

in which $a = 0.198$ and $b = 1.068$. By means of a simple analysis, we see that the denominator of the above integrand has one real positive root and four complex roots, named here as p_0 and p_j, respectively. Due to its pole structure, the integrand can be written as a pseudo function plus a delta function. Hence, after some computation [52], we finally arrive at

$$\delta V^{\text{1-loop}}(r) = -\frac{a|\theta|e^4}{\pi^2}\frac{1}{r}\left\{-i\pi\frac{A(p_0)}{B(p_0)}J_0(p_0) + 1 + \sum_{j=1}^{4} c_j\left[K_0(-ip_j) + \frac{\pi}{2}(iJ_0(p_j) - H_0(p_j))\right]\right\}, \tag{5.32}$$

where $A = p_0^3(p_0^2 + bm^2r^2)$, $B = p_0^3(p_0^2 + m^2r^2) - \frac{e^2}{4\pi}\frac{m^2r^5}{|\theta|}$ and H is the Struve function. Also, the coefficient c_j arises from the fraction decomposition which is given by

$$\frac{A(p)}{B(p)} = 1 + \sum_{j=o}^{4}\frac{c_j}{p - p_j} \tag{5.33}$$

with $c_j = \frac{A(p_j)}{B'(p_j)}$. Furthermore, by means of illustration, we consider the behavior of the above expression again at $mr \ll 1$, so that we can compare it with the usual free result. Hence, by using the asymptotic expansion of the Bessel and Struve functions [52], we find

$$\delta V^{\text{1-loop}}(r) = -\frac{a|\theta|e^4}{\pi^2}\frac{1}{r}\left(1 - i\pi\frac{A(p_0)}{B(p_0)}\right). \tag{5.34}$$

It is important to emphasize the strong departure due to the noncommutativity of the leading radial dependence of the above expression, $\delta V^{\text{1-loop}} \propto \frac{1}{r}$, when compared to the usual Maxwell-Chern-Simons (confining) static potential energy $V^{\text{free}} \propto \ln r$, at $mr \ll 1$.

It is worth noticing that an investigation of the Lamb shift effect in noncommutative QED_4 was carried out in [53]. Since a complete discussion for this physical process requires us to take into account the charge renormalization, this will be considered further in the forthcoming paper [54].

VI. CONCLUDING REMARKS

In this paper, we have studied in complete detail the gauge field complete propagator at one-loop order in the NC Maxwell-Chern-Simons theory. A careful account covering all the renormalizability aspects of this two-point function has been presented, in particular by establishing the respective renormalization constants and subsequently the gauge field renormalized mass. It is worth mentioning that, as expected from a gauge theory, a multiplicative renormalization holds for the theory.

We first discussed in detail the tensor structure of the gauge field self-energy at one-loop order. This has been supplemented by a full account on the discrete symmetries for a three-dimensional noncommutative spacetime. The explicit expressions of the form factor were calculated by following the standard rules of Feynman integration. In particular, we found that the commutative limit of the complete form factor $\tilde{\Pi}_e$ displays a manifestation of IR/UV mixing, since the planar and nonplanar contributions sum to a nonvanishing result. Besides, we explicitly showed that the NC **CP**-odd form factor Π_o^S identically vanishes.

In order to discuss some physical consequences of the considered model, we have scrutinized some particular limits: (i) the NC Chern-Simons theory; (ii) the NC Maxwell theory; and (iii) the low-momenta limit, highly noncommutative Maxwell-Chern-Simons theory. First, we found that, as expected, the NC Chern-Simons theory is actually a free theory. Next, we showed that the massless limit, $m = 0$, is well behaved in this context, and that the

[5]Here, by means of clarity, we have restored the usual notation in terms of $e^2 = m/\kappa$.

dispersion relation for the NC Maxwell theory displays the same profile as in the three-dimensional Yang-Mills theory, with no radiative mass generation, but with an UV/IR mixing instability.

Finally, the highly noncommutative limit was considered, and analytical expressions have been obtained for the form factors. Within this context, we examined its dispersion relation and found that it is not infrared finite, more precisely an UV/IR mixing instability due to the NC momentum. Besides, using the one-loop expressions of the form factors, we have determined the noncommutative corrections to the photon propagator in the low-momenta limit. As a physical outcome of the one-loop gauge field propagator, we have discussed the noncommutative corrections to the electrostatic potential. In particular, the low-momenta limit of the Maxwell-Chern-Simons theory (when coupled to matter fields) is of major interest for physical application in planar materials, in particular to the description of new materials in the framework of condensed matter physics [6,55], which allows the use of effective low-energy models.

It is worth mentioning that a complete account of the noncommutative Maxwell-Chern-Simons theory renormalizability must contain an analysis of the vertex functions. This complementary study is now under scrutiny [54]. The analysis takes into account the renormalization of the 3-point vertex function and ghost sector, as well other vertex functions, allowing us then to fully discuss the renormalization of the gauge coupling, determining the theory's beta function, as well as the presence of the UV/IR mixing and infrared finiteness.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for his/her comments and suggestions to improve this paper. We would like to express our special gratitude to M. M. Sheikh-Jabbari for his valuable comments and enlightening discussions. Also, the useful remarks from M. Khorrami are greatly appreciated. R. B. thankfully acknowledges FAPESP for support, Project No. 2011/20653-3.

APPENDIX A: ONE-LOOP ANALYSIS OF THE PHOTON SELF-ENERGY

We shall now write down the full contribution one-loop expression for the photon self-energy; see Fig. 1. This contribution has the following form,

$$\Pi_{\mu\nu} = \Pi^{\rm g}_{\mu\nu} + \Pi^{\rm gh}_{\mu\nu} + \Pi^{\rm t}_{\mu\nu}, \tag{A1}$$

where the explicit expression for the ghost, cubic, and quartic self-interacting diagrams are given by

$$\Pi^{\rm gh}_{\mu\nu}(p) = 2e^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\frac{1}{(p+k)^2}\sin^2\left(\frac{p\wedge k}{2}\right)k_\mu(k+p)_\nu, \tag{A2}$$

$$\Pi^{\rm g}_{\mu\nu}(p) = e^2 \int \frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{1}{k^2(k^2-m^2)} \times \frac{1}{(p+k)^2[(p+k)^2-m^2]}\mathcal{N}^{\rm g}_{\mu\nu}, \tag{A3}$$

$$\Pi^{\rm t}_{\mu\nu}(p) = 2e^2 \int \frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right)\frac{\eta_{\mu\nu}k^2 + k_\mu k_\nu}{k^2(k^2-m^2)}, \tag{A4}$$

and the tensor at the numerator of Eq. (A3) is defined as

$$\begin{aligned}\mathcal{N}^{\rm g}_{\mu\nu} = {}& (im\epsilon_{\mu\alpha\beta} + (p+2k)_\mu\eta_{\alpha\beta} + (p-k)_\beta\eta_{\mu\alpha} - (2p+k)_\alpha\eta_{\mu\beta})(im\epsilon_{\nu\rho\sigma} - (p+2k)_\nu\eta_{\rho\sigma} + (k-p)_\sigma\eta_{\rho\nu} + (2p+k)_\rho\eta_{\nu\sigma}) \\ & \times (k^2\eta^{\alpha\rho} - k^\alpha k^\rho + im\epsilon^{\alpha\rho\lambda}k_\lambda)((p+k)^2\eta^{\beta\sigma} - (p+k)^\beta(p+k)^\sigma - im\epsilon^{\beta\sigma\xi}(p+k)_\xi).\end{aligned} \tag{A5}$$

We note that the denominator on these three contributions is different. Hence, we can write the complete contribution in the following way,

$$\Pi_{\mu\nu}(p) = e^2 \int \frac{d^3k}{(2\pi)^3}\sin^2\left(\frac{p\wedge k}{2}\right) \times \frac{\mathcal{N}^{\rm g}_{\mu\nu} + 2\mathcal{N}^{\rm gh}_{\mu\nu} + 2\mathcal{N}^{\rm t}_{\mu\nu}}{k^2(k^2-m^2)(p+k)^2((p+k)^2-m^2)}, \tag{A6}$$

where we have conveniently introduced the new tensor quantities:

$$\begin{aligned}\mathcal{N}^{\rm gh}_{\mu\nu} = {}& m^4(k_\mu k_\nu + k_\mu p_\nu) \\ & - m^2(2k^2 + p^2 + 2p.k)(k_\mu k_\nu + k_\mu p_\nu) \\ & + k^2(k^2 + 2p.k + p^2)(k_\mu k_\nu + k_\mu p_\nu),\end{aligned} \tag{A7}$$

$$\begin{aligned}\mathcal{N}^{\rm t}_{\mu\nu} = {}& -m^2(k^2 + 2p.k + p^2)(k^2\eta_{\mu\nu} + k_\mu k_\nu) \\ & + (k^4 + 2k^2p^2 + p^4 + 4k^2(p.k) + 4p^2(p.k) \\ & + 4(p.k)^2)(k^2\eta_{\mu\nu} + k_\mu k_\nu).\end{aligned} \tag{A8}$$

APPENDIX B: TENSOR STRUCTURE OF THE PHOTON SELF-ENERGY

In order to discuss the tensor structure of the complete photon propagator, we shall introduce and consider the following vectors, p_μ, $\tilde{p}_\mu = \theta_{\mu\nu} p^\nu$ and $u_\mu = \epsilon_{\mu\alpha\beta} p^\alpha \tilde{p}^\beta$, as an orthogonal basis. In particular, it is easy to see that

$$p_\mu \tilde{p}^\mu = 0, \qquad p^\mu u_\mu = 0, \qquad \tilde{p}^\mu u_\mu = 0 \tag{B1}$$

and that the completeness relation is also satisfied:

$$\frac{u_\mu u_\nu}{u^2} + \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \frac{p_\mu p_\nu}{p^2} = \eta_{\mu\nu}. \tag{B2}$$

The polarization tensor in this basis is written as

$$\begin{aligned} \Pi_{\mu\nu} = {} & (a_1 p_\mu + a_2 \tilde{p}_\mu + a_3 u_\mu) p_\nu \\ & + (b_1 p_\mu + b_2 \tilde{p}_\mu + b_3 u_\mu) \tilde{p}_\nu \\ & + (c_1 p_\mu + c_2 \tilde{p}_\mu + c_3 u_\mu) u_\nu. \end{aligned} \tag{B3}$$

Applying the Ward identity $p^\mu \Pi_{\mu\nu} = 0$ and $p^\nu \Pi_{\mu\nu} = 0$, this directly leads to $a_1 = a_2 = a_3 = b_1 = c_1 = 0$. Hence, we are left with the expression

$$\begin{aligned} \Pi_{\mu\nu} &= (b_2 \tilde{p}_\mu + b_3 u_\mu) \tilde{p}_\nu + (c_2 \tilde{p}_\mu + c_3 u_\mu) u_\nu \\ &= b_2 \tilde{p}_\mu \tilde{p}_\nu + d_1 (u_\mu \tilde{p}_\nu + u_\nu \tilde{p}_\mu) \\ &\quad + d_2 (\tilde{p}_\mu u_\nu - \tilde{p}_\nu u_\mu) + c_3 u_\mu u_\nu, \end{aligned} \tag{B4}$$

in which, by convenience, we have rewritten the terms $b_3 u_\mu \tilde{p}_\nu + c_2 \tilde{p}_\mu u_\nu$ as symmetric and antisymmetric parts. Furthermore, it is easy to show that the antisymmetric part can be revised as

$$\tilde{p}_\mu u_\nu - \tilde{p}_\nu u_\mu = \epsilon_{\mu\nu\lambda} p^\lambda \tilde{p}^2. \tag{B5}$$

Using this result and also the completeness relation (B2), we can write the photon self-energy in a clear and appropriated form; see (B8). Now, at this step, we construct the general form for the $1PI$ function $\Gamma_{\mu\nu}$ for the NC Maxwell-Chern-Simons theory using the defined basis

$$\Gamma^{\mu\nu} = \Gamma^{\mu\nu}_{\text{tree-level}} + \Pi^{\mu\nu}_{\text{loop-level}}, \tag{B6}$$

where the $1PI$ two-point function and the polarization tensor are, respectively, given by

$$\Gamma^{\mu\nu}_{\text{tree-level}} = -p^2 \eta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu + i m \epsilon^{\mu\nu\lambda} p_\lambda \tag{B7}$$

$$\begin{aligned} \Pi^{\mu\nu}_{\text{loop-level}} = {} & \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \Pi^\star_{\rm e} + \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2} \tilde{\Pi}^\star_{\rm e} + i \epsilon^{\mu\nu\lambda} p_\lambda \Pi^{\rm A}_{\rm o} \\ & + (\tilde{p}^\mu u^\nu + \tilde{p}^\nu u^\mu) \Pi^{\rm S}_{\rm o}, \end{aligned} \tag{B8}$$

with which the tensor structure is in agreement [43]. We can obtain the complete propagator expression by means of the standard functional relation $\Gamma^{\lambda\mu} \mathcal{D}_{\mu\nu} = i \delta^\lambda_\nu$. After some laborious calculation, we find that the complete propagator has the following general expression,

$$\begin{aligned} i \mathcal{D}_{\mu\nu} = {} & \frac{p^2 - \Pi^\star_{\rm e} - \tilde{\Pi}^\star_{\rm e}}{\mathcal{R}} \eta_{\mu\nu} + \left(\frac{-p^2 + \Pi^\star_{\rm e} + \tilde{\Pi}^\star_{\rm e}}{\mathcal{R}} + \frac{\xi}{p^2} \right) \frac{p_\mu p_\nu}{p^2} \\ & + \frac{\tilde{\Pi}^\star_{\rm e}}{\mathcal{R}} \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \frac{\Pi^{\rm S}_{\rm o}}{\mathcal{R}} (\tilde{p}_\mu u_\nu + u_\mu \tilde{p}_\nu) + \frac{m + \Pi^{\rm A}_{\rm o}}{\mathcal{R}} i \varepsilon_{\mu\nu\lambda} p^\lambda, \end{aligned} \tag{B9}$$

where $\mathcal{R} = (p^2 - \Pi^\star_{\rm e})(p^2 - \Pi^\star_{\rm e} - \tilde{\Pi}^\star_{\rm e}) + p^2 [(\tilde{p}^2 \Pi^{\rm S}_{\rm o})^2 - (m + \Pi^{\rm A}_{\rm o})^2]$.

In order to conclude this discussion, we shall now determine the coefficients appearing in the expression (B8) for the $1PI$ form factors $\Pi^\star_{\rm e}$, $\tilde{\Pi}^\star_{\rm e}$, $\Pi^{\rm A}_{\rm o}$ and $\Pi^{\rm S}_{\rm o}$. These are found from the following identities:

$$\Pi^\star_{\rm e} = \eta_{\mu\nu} \Pi^{\mu\nu} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \Pi^{\mu\nu}, \tag{B10}$$

$$\tilde{\Pi}^\star_{\rm e} = -\eta_{\mu\nu} \Pi^{\mu\nu} + 2 \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \Pi^{\mu\nu}, \tag{B11}$$

$$\Pi^{\rm A}_{\rm o} = \frac{i}{2p^2} \epsilon_{\mu\nu\alpha} p^\alpha \Pi^{\mu\nu}, \tag{B12}$$

$$\Pi^{\rm S}_{\rm o} = -\frac{1}{2 \tilde{p}^4 p^2} (u_\mu \tilde{p}_\nu + u_\nu \tilde{p}_\mu) \Pi^{\mu\nu}. \tag{B13}$$

APPENDIX C: NONPLANAR INTEGRAL

Throughout the paper, we have made use of some known results involving momentum integration. We shall recall some of these results, in particular those involving a nonplanar factor. The simplest integration reads

$$\begin{aligned} & \int \frac{d^\omega q}{(2\pi)^\omega} \frac{1}{(q^2 - s^2)^a} e^{ik\wedge q} \\ & \quad = \frac{2i(-)^a}{(4\pi)^{\frac{\omega}{2}}} \frac{1}{\Gamma(a)} \frac{1}{(s^2)^{a-\frac{\omega}{2}}} \left(\frac{|\tilde{k}| s}{2} \right)^{a - \frac{\omega}{2}} K_{a-\frac{\omega}{2}}(|\tilde{k}| s). \end{aligned} \tag{C1}$$

Next, we have the integration

$$\int \frac{d^\omega q}{(2\pi)^\omega} \frac{q^\mu q^\nu}{(q^2 - s^2)^a} e^{ik\wedge q} = \eta^{\mu\nu} F_a + \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2} G_a, \tag{C2}$$

where we have introduced the following quantities,

$$\{F_a, G_a\} = \frac{i(-)^{a-1}}{(4\pi)^{\frac{\omega}{2}}}\frac{1}{\Gamma(a)}\frac{1}{(s^2)^{a-1-\frac{\omega}{2}}}\{f_a, g_a\}, \quad \text{(C3)}$$

with

$$f_a = \left(\frac{s|\tilde{k}|}{2}\right)^{a-1-\frac{\omega}{2}} K_{a-1-\frac{\omega}{2}}(|\tilde{k}|s), \quad \text{(C4)}$$

$$g_a = (2a-2-\omega)\left(\frac{s|\tilde{k}|}{2}\right)^{a-1-\frac{\omega}{2}} K_{a-1-\frac{\omega}{2}}(|\tilde{k}|s) - 2\left(\frac{s|\tilde{k}|}{2}\right)^{a-\frac{\omega}{2}} K_{a-\frac{\omega}{2}}(|\tilde{k}|s). \quad \text{(C5)}$$

Finally, we have

$$\int \frac{d^\omega q}{(2\pi)^\omega}\frac{q^\mu q^\nu q^\lambda q^\beta}{(q^2-s^2)^a} e^{ik\wedge q} = (\eta^{\mu\nu}\eta^{\lambda\beta} + \eta^{\nu\lambda}\eta^{\mu\beta} + \eta^{\nu\beta}\eta^{\lambda\mu})H_a + \frac{\tilde{k}^\lambda\tilde{k}^\beta\tilde{k}^\nu\tilde{k}^\mu}{\tilde{k}^4} J_a + \left(\eta^{\lambda\beta}\frac{\tilde{k}^\nu\tilde{k}^\mu}{\tilde{k}^2} + \eta^{\nu\lambda}\frac{\tilde{k}^\beta\tilde{k}^\mu}{\tilde{k}^2} + \eta^{\nu\beta}\frac{\tilde{k}^\lambda\tilde{k}^\mu}{\tilde{k}^2} + \eta^{\lambda\mu}\frac{\tilde{k}^\beta\tilde{k}^\nu}{\tilde{k}^2} + \eta^{\beta\mu}\frac{\tilde{k}^\lambda\tilde{k}^\nu}{\tilde{k}^2} + \eta^{\nu\mu}\frac{\tilde{k}^\lambda\tilde{k}^\beta}{\tilde{k}^2}\right) I_a, \quad \text{(C6)}$$

where the quantities H_a, I_a and J_a are defined as the following,

$$\{H_a, I_a, J_a\} = -\frac{i(-)^a}{(4\pi)^{\frac{\omega}{2}}}\frac{1}{\Gamma(a)}\frac{1}{(s^2)^{a-2-\frac{\omega}{2}}}\frac{1}{s^2\tilde{k}^2}\{h_a, i_a, j_a\}, \quad \text{(C7)}$$

with

$$h_a = (2a-2-\omega)\left(\frac{s|\tilde{k}|}{2}\right)^{a-1-\frac{\omega}{2}} K_{a-1-\frac{\omega}{2}}(|\tilde{k}|s) - 2\left(\frac{s|\tilde{k}|}{2}\right)^{a-\frac{\omega}{2}} K_{a-\frac{\omega}{2}}(|\tilde{k}|s), \quad \text{(C8)}$$

$$i_a = [(2a-2-\omega)(2a-4-\omega) + s^2\tilde{k}^2]\left(\frac{s|\tilde{k}|}{2}\right)^{a-1-\frac{\omega}{2}} \times K_{a-1-\frac{\omega}{2}}(|\tilde{k}|s) - 2(2a-4-\omega)\left(\frac{s|\tilde{k}|}{2}\right)^{a-\frac{\omega}{2}} K_{a-\frac{\omega}{2}}(|\tilde{k}|s), \quad \text{(C9)}$$

$$j_a = (2a-4-\omega)[(2a-2-\omega)(2a-6-\omega) + 2s^2\tilde{k}^2]\left(\frac{s|\tilde{k}|}{2}\right)^{a-1-\frac{\omega}{2}} K_{a-1-\frac{\omega}{2}}(|\tilde{k}|s) - 2(2(2a-4-\omega) + s^2\tilde{k}^2)\left(\frac{s|\tilde{k}|}{2}\right)^{a-\frac{\omega}{2}} K_{a-\frac{\omega}{2}}(|\tilde{k}|s). \quad \text{(C10)}$$

APPENDIX D: ONE-LOOP FORM FACTORS

In this section, we write down explicitly some lengthy expressions from the planar and nonplanar parts from the self-energy form factors, discussed in Sec. IV. First, for the planar contribution of the transverse part (Π_e), Eq. (4.6), we immediately find

$$(\Pi_\mathrm{e})_\mathrm{p}(p^2) = -\frac{e^2}{16\pi}\left\{-\frac{1}{p^2}\int d\Phi \frac{1}{\sqrt{\Delta_1^2}}(6p^2 + 16m^2 - 4\Delta_1^2) - \frac{1}{2p^2}\int d\Upsilon \frac{1}{(\Delta_2^2)^{\frac{3}{2}}}(16(y+z)^2p^4 - 10(y+z)p^4 + p^4 + (26(y+z)-12)m^2p^2 - 16m^4 - 2[(7-2(y+z))p^2 + 8m^2]\Delta_2^2) + \frac{1}{4}\int d\Xi \frac{1}{(\Delta_3^2)^{\frac{5}{2}}}(3p^2[4(z+w)(1-2(z+w))m^2 + (z+w)^2(3-4(z+w))p^2] + [14(z+w)m^2 + (4(z+w) + 4(z+w)^2 - 1)p^2]\Delta_3^2 - 4\Delta_3^4)\right\}, \quad \text{(D1)}$$

while for the nonplanar contribution of (4.6), we obtain

$$(\Pi_\mathrm{e})_{\mathrm{n-p}}(p^2) = \frac{e^2}{16\pi}\left\{-\frac{1}{p^2}\int d\Phi \frac{e^{-\Delta_1|\tilde{p}|}}{\sqrt{\Delta_1^2}}[(6p^2+16m^2) - 4\Delta_1^2] - \frac{1}{2p^2}\int d\Upsilon \frac{e^{-\Delta_2|\tilde{p}|}}{(\Delta_2^2)^{\frac{3}{2}}}(-[16p^2 + ((4(y+z)+2)p^2 - 16m^2)[-1+|\tilde{p}|\Delta_2]]\Delta_2^2\right.$$

$$+ [16(y+z)^2 p^4 - 10(y+z)p^4 + (26(y+z) - 12)m^2 p^2 + p^4 - 16m^4](1 + \Delta_2|\tilde{p}|))$$
$$+ \frac{1}{4}\int d\Xi \frac{e^{-\Delta_3|\tilde{p}|}}{(\Delta_3^2)^{\frac{5}{2}}}(-[3p^2 - 8m^2 - 12(z+w)p^2](1 + \Delta_3|\tilde{p}|)\Delta_3^2 + 4\Delta_3^4(\Delta_3|\tilde{p}| - 1)$$
$$+ [4(z+w)(1 - 2(z+w))m^2 + (z+w)^2(3 - 4(z+w))p^2](3 + 3\Delta_3|\tilde{p}| + \Delta_3^2\tilde{p}^2)p^2$$
$$- ((8 - 14(z+w))m^2 - 4(z+w)^2 p^2 - 2p^2 + 8(z+w)p^2)[1 + \Delta_3|\tilde{p}| - \tilde{p}^2\Delta_3^2]\Delta_3^2)\Big\}. \tag{D2}$$

Moreover, similar expressions follow for the NC transverse part $(\tilde{\Pi}_{\mathrm{e}})$, Eq. (4.9). Without any complication, the planar contribution results in

$$(\tilde{\Pi}_{\mathrm{e}})_{\mathrm{p}}(p^2) = -\frac{e^2}{16\pi}\Big\{\frac{1}{p^2}\int d\Phi \frac{1}{\sqrt{\Delta_1^2}}((16m^2 - 4p^2) - 2\Delta_1^2) - \frac{1}{2p^2}\int d\Upsilon \frac{1}{(\Delta_2^2)^{\frac{3}{2}}}(m^2(-30(y+z)p^2 + 7p^2 + 16m^2)$$
$$- 3p^4 + 10(y+z)p^4 - 2(2p^2 + 4(y+z)p^2 - 16m^2)\Delta_2^2)$$
$$+ \frac{1}{4}\int d\Xi \frac{1}{(\Delta_3^2)^{\frac{5}{2}}}(3p^2[-4m^2(z+w) + 7m^2(z+w)^2 + (z+w)^2p^2 - 4(z+w)^3p^2]$$
$$- [8(z+w)^2p^2 + 5p^2 - 28(z+w)p^2 - m^2[9 - 28(z+w)]]\Delta_3^2 + 8\Delta_3^4)\Big\}, \tag{D3}$$

and the nonplanar contribution reads

$$(\tilde{\Pi}_{\mathrm{e}})_{\mathrm{n-p}}(p^2) = \frac{e^2}{16\pi}\Big\{\frac{1}{p^2}\int d\Phi \frac{e^{-\Delta_1|\tilde{p}|}}{\sqrt{\Delta_1^2}}\left[(-4p^2 + 16m^2) - 4\frac{1}{|\tilde{p}|}\Delta_1 - 6\Delta_1^2\right]$$
$$- \frac{1}{2p^2}\int d\Upsilon \frac{e^{-\Delta_2|\tilde{p}|}}{(\Delta_2^2)^{\frac{3}{2}}}(-(4p^2 + 8(y+z)p^2 - 32m^2)[1 - \Delta_2|\tilde{p}|]\Delta_2^2$$
$$+ [m^2(-30(y+z)p^2 + 7p^2 + 16m^2) - 3p^4 + 10(y+z)p^4](1 + \Delta_2|\tilde{p}|))$$
$$+ \frac{1}{4}\int d\Xi \frac{e^{-\Delta_3|\tilde{p}|}}{(\Delta_3^2)^{\frac{5}{2}}}(-(7m^2 + p^2 - 12(z+w)p^2)(1 + \Delta_3|\tilde{p}|)\Delta_3^2 + 8\Delta_3^4(1 - \Delta_3|\tilde{p}|)$$
$$+ p^2[-4m^2(z+w) + 7m^2(z+w)^2 + (z+w)^2p^2 - 4(z+w)^3p^2](3 + 3\Delta_3|\tilde{p}| + \Delta_3^2\tilde{p}^2)$$
$$- [8(z+w)^2p^2 + 4p^2 - 16(z+w)p^2 - 2m^2[8 - 14(z+w)]](1 + \Delta_3|\tilde{p}| - \tilde{p}^2\Delta_3^2)\Delta_3^2)\Big\}. \tag{D4}$$

Finally, for the *CP* odd part $(\Pi_{\mathrm{o}}^{\mathrm{A}})$, Eq. (4.11), we find the planar part of the expression,

$$(\Pi_{\mathrm{o}}^{\mathrm{A}})_{\mathrm{p}}(p^2) = -\frac{me^2}{8\pi}\Big\{\int d\Phi \frac{5}{\sqrt{\Delta_1^2}} - \frac{1}{2}\int d\Upsilon \frac{1}{(\Delta_2^2)^{\frac{3}{2}}}(((-6 + 15x - 5x^2)p^2 + 7m^2) + 5\Delta_2^2)$$
$$- \frac{1}{4}\int d\Xi \frac{1}{(\Delta_3^2)^{\frac{5}{2}}}(3(z+w)^2\{2m^2 - 5(z+w)p^2 + 4p^2\}p^2 - (2m^2 + 4p^2 - 15(z+w)p^2)\Delta_3^2)\Big\}, \tag{D5}$$

whereas, for the nonplanar part, we obtain the following contribution:

$$(\Pi_{\rm o}^{\rm A})_{\rm n-p}(p^2)=\frac{me^2}{8\pi}\left\{5\int d\Phi\frac{e^{-\Delta_1|\tilde{p}|}}{\sqrt{\Delta_1^2}}+\frac{1}{2}\int d\Upsilon\frac{e^{-\Delta_2|\tilde{p}|}}{(\Delta_2^2)^{\frac{3}{2}}}(((6-15x+5x^2)p^2-7m^2)(1+\Delta_2|\tilde{p}|)-5\Delta_2^2)\right.$$
$$-\frac{1}{4}\int d\Xi\frac{e^{-\Delta_3|\tilde{p}|}}{(\Delta_3^2)^{\frac{5}{2}}}((z+w)^2\{2m^2+(4-5z-5w)p^2\}(3+3\Delta_3|\tilde{p}|+\Delta_3^2\tilde{p}^2)p^2$$
$$\left.-(2m^2+4p^2-15(z+w)p^2)(1+\Delta_3|\tilde{p}|)\Delta_3^2)\right\}. \tag{D6}$$

[1] R. Jackiw, *Topics in Planar Physics*, Fields and particles (Springer, Berlin, 1990), p. 117; republished in Nucl. Phys. B, Proc. Suppl. **18**, 107 (1990).
[2] H. Nastase, *Introduction to the ADS/CFT Correspondence* (Cambridge University Press, Cambridge, England, 2015).
[3] M. Tinkham, *Introduction to Superconductivity*, 2nd ed. (Dover, New York, 2004).
[4] M. A. H. Vozmediano, M. I. Katsnelson, and F. Guinea, Gauge fields in graphene, Phys. Rep. **496**, 109 (2010).
[5] A. A. Burkov, M. D. Hook, and L. Balents, Topological nodal semimetals, Phys. Rev. B **84**, 235126 (2011).
[6] V. A. Miransky and I. A. Shovkovy, Quantum field theory in a magnetic field: From quantum chromodynamics to graphene and Dirac semimetals, Phys. Rep. **576**, 1 (2015).
[7] A. G. Grushin, Consequences of a condensed matter realization of Lorentz violating QED in Weyl semi-metals, Phys. Rev. D **86**, 045001 (2012).
[8] D. Mattingly, Modern tests of Lorentz invariance, Living Rev. Relativity **8**, 5 (2005).
[9] G. Amelino-Camelia, Quantum-spacetime phenomenology, Living Rev. Relativity **16**, 5 (2013).
[10] S. Hossenfelder, Minimal length scale scenarios for quantum gravity, Living Rev. Relativity **16**, 2 (2013).
[11] M. R. Douglas and N. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. **378**, 207 (2003).
[12] I. Hinchliffe, N. Kersting, and Y. L. Ma, Review of the phenomenology of noncommutative geometry, Int. J. Mod. Phys. A **19**, 179 (2004).
[13] P. Nicolini, Noncommutative black holes, the final appeal to quantum gravity: A review, Int. J. Mod. Phys. A **24**, 1229 (2009).
[14] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Space-time quantization induced by classical gravity, Phys. Lett. B **331**, 39 (1994).
[15] A. Connes, M. R. Douglas, and A. S. Schwarz, Noncommutative geometry and matrix theory: Compactification on tori, J. High Energy Phys. 02 (1998) 003.
[16] C. S. Chu and P. M. Ho, Noncommutative open string and D-brane, Nucl. Phys. **B550**, 151 (1999).
[17] Y. K. E. Cheung and M. Krogh, Noncommutative geometry from D0-branes in a background B field, Nucl. Phys. **B528**, 185 (1998).
[18] N. Seiberg and E. Witten, String theory and noncommutative geometry, J. High Energy Phys. 09 (1999) 032.
[19] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, Dirac quantization of open strings and noncommutativity in branes, Nucl. Phys. **B576**, 578 (2000).
[20] L. Alvarez-Gaume and S. R. Wadia, Gauge theory on a quantum phase space, Phys. Lett. B **501**, 319 (2001).
[21] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, Noncommutative perturbative dynamics, J. High Energy Phys. 02 (2000) 020.
[22] H. Grosse and R. Wulkenhaar, Renormalization of ϕ^4 theory on noncommutative R^2 in the matrix base, J. High Energy Phys. 12 (2003) 019.
[23] H. Grosse and R. Wulkenhaar, Renormalization of Φ^4 theory on noncommutative R^4 to all orders, Lett. Math. Phys. **71**, 13 (2005).
[24] L. Susskind, The quantum Hall fluid and noncommutative Chern-Simons theory (to be published).
[25] S. Hellerman and M. Van Raamsdonk, Quantum Hall physics equals noncommutative field theory, J. High Energy Phys. 10 (2001) 039.
[26] D. Xiao, J. Shi, and Q. Niu, Berry Phase Correction to Electron Density of States in Solids, Phys. Rev. Lett. **95**, 137204 (2005).
[27] C. Bastos, O. Bertolami, N. C. Dias, and J. N. Prata, Noncommutative graphene, Int. J. Mod. Phys. A **28**, 1350064 (2013).
[28] B. G. Sidharth, Graphene and high energy physics, Int. J. Mod. Phys. E **23**, 1450025 (2014).
[29] G. H. Chen and Y. S. Wu, One loop shift in noncommutative Chern-Simons coupling, Nucl. Phys. **B593**, 562 (2001).
[30] A. A. Bichl, J. M. Grimstrup, V. Putz, and M. Schweda, Perturbative Chern-Simons theory on noncommutative R^3, J. High Energy Phys. 07 (2000) 046.
[31] N. E. Grandi and G. A. Silva, Chern-Simons action in noncommutative space, Phys. Lett. B **507**, 345 (2001).
[32] A. K. Das and M. M. Sheikh-Jabbari, Absence of higher order corrections to noncommutative Chern-Simons coupling, J. High Energy Phys. 06 (2001) 028.
[33] M. M. Sheikh-Jabbari, A note on noncommutative Chern-Simons theories, Phys. Lett. B **510**, 247 (2001).
[34] D. N. Blaschke, S. Hohenegger, and M. Schweda, Divergences in non-commutative gauge theories with the Slavnov term, J. High Energy Phys. 11 (2005) 041.

[35] A. Matusis, L. Susskind, and N. Toumbas, The IR/UV connection in the noncommutative gauge theories, J. High Energy Phys. 12 (2000) 002.

[36] N. Seiberg, L. Susskind, and N. Toumbas, Space-time noncommutativity and causality, J. High Energy Phys. 06 (2000) 044.

[37] R. D. Pisarski and S. Rao, Topologically massive chromodynamics in the perturbative regime, Phys. Rev. D **32**, 2081 (1985).

[38] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406 (1988); **281**, 409 (2000).

[39] A. N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions, Phys. Rev. D **29**, 2366 (1984).

[40] R. Banerjee, T. Shreecharan, and S. Ghosh, Three dimensional noncommutative bosonization, Phys. Lett. B **662**, 231 (2008).

[41] O. F. Dayi, Noncommutative Maxwell-Chern-Simons theory, duality and a new noncommutative Chern-Simons theory in $d = 3$, Phys. Lett. B **560**, 239 (2003).

[42] E. Harikumar and V. O. Rivelles, Noncommutative Maxwell-Chern-Simons theory in three dimensions and its dual, Phys. Lett. B **625**, 156 (2005).

[43] N. Caporaso and S. Pasquetti, Noncommutative (Softly Broken) Supersymmetric Yang-Mills-Chern-Simons (to be published).

[44] A. F. Ferrari, M. Gomes, J. R. Nascimento, A. Y. Petrov, A. J. da Silva, and E. O. Silva, On the finiteness of the noncommutative supersymmetric Maxwell-Chern-Simons theory, Phys. Rev. D **77**, 025002 (2008).

[45] S. Deser and R. Jackiw, Higher derivative Chern-Simons extensions, Phys. Lett. B **451**, 73 (1999).

[46] D. S. Kaparulin, I. Y. Karataeva, and S. L. Lyakhovich, Higher derivative extensions of 3*d* Chern-Simons models: Conservation laws and stability, Eur. Phys. J. C **75**, 552 (2015).

[47] R. Bufalo and M. Ghasemkhani, Higher derivative Chern-Simons extension in the noncommutative QED_3, Phys. Rev. D **91**, 125013 (2015).

[48] H. O. Girotti, M. Gomes, V. O. Rivelles, and A. J. da Silva, A consistent noncommutative field theory: The Wess-Zumino model, Nucl. Phys. **B587**, 299 (2000).

[49] M. M. Sheikh-Jabbari, C, P, and T Invariance of Noncommutative Gauge Theories, Phys. Rev. Lett. **84**, 5265 (2000).

[50] M. M. Sheikh-Jabbari, Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus, J. High Energy Phys. 06 (1999) 015.

[51] R. Jackiw and S. Templeton, How superrenormalizable interactions cure their infrared divergences, Phys. Rev. D **23**, 2291 (1981).

[52] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic Press, New York, 2007).

[53] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED, Phys. Rev. Lett. **86**, 2716 (2001).

[54] M. Ghasemkhani and R. Bufalo, Noncommutative Maxwell-Chern-Simons theory (II): One-loop beta-function and renormalizability (to be published).

[55] A. Martín-Ruiz, M. Cambiaso, and L. F. Urrutia, Green's function approach to Chern-Simons extended electrodynamics: An effective theory describing topological insulators, Phys. Rev. D **92**, 125015 (2015).