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New soft theorems for the gravity dilaton and the Nambu-Goldstone dilaton at subsubleading order

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We study the soft behavior of two seemingly different particles that are both referred to as dilatons in the literature, namely the one that appears in theories of gravity and in string theory and the Nambu-Goldstone boson of spontaneously broken conformal invariance. Our primary result is the discovery of a soft theorem at subsubleading order for each dilaton, which in both cases contains the operator of special conformal transformations. Interesting similarities as well as differences between the dilaton soft theorems are discussed.

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I. INTRODUCTION

The word dilaton is currently used in particle physics for two different particles that, *a priori*, seem unrelated. One is the dilaton that appears in (super)gravity and string theories. This dilaton is a massless scalar particle accompanied by the graviton and the Kalb-Ramond antisymmetric tensor in the massless sector of the closed bosonic string and in the Neveu-Schwarz sector of the closed superstring. We refer to this particle as the "gravity dilaton." The other dilaton is a Nambu-Goldstone (NG) boson arising from the spontaneous breaking of scale and conformal invariance, which is frequently encountered in many phenomenological scenarios, for example in beyond-the-standard-model physics and inflationary cosmology. We refer to this as the "NG dilaton." Although they are different particles, they share the feature of satisfying soft theorems, according to which the scattering of a low-energy particle is entirely determined by symmetry properties. We demonstrate that both dilatons obey soft theorems through subsubleading order in the low-energy expansion, i.e., through $\mathcal{O}(q^1)$ in the soft-dilaton momentum q.

The leading and subleading soft behavior of the gravity dilaton has been known since the seventies [1,2]. Furthermore, recent work has shown the leading and subleading behavior to be a direct consequence of the same gauge invariance that reveals the soft theorems for the graviton [3]. In this article, we extend this analysis to subsubleading order, revealing the role of special conformal transformations. In the case of a NG dilaton, the leading soft behavior vanishes for massless external states, while the subleading soft behavior is known to be a consequence of the Ward identity of broken scale invariance [4]. (See also the recent work of Refs. [5–6].) In this article, we show that the subsubleading behavior is entirely fixed by the Ward identity of broken special conformal invariance.

The article is organized as follows. In Sec. II, we demonstrate that the low-energy behavior of the gravity dilaton through $\mathcal{O}(q^1)$ is entirely determined by gauge invariance when the other (hard) particles are all either (massive) scalars or all gravitons or dilatons. In Sec. III, we show that the Ward identities of broken scale and conformal invariance completely determine the soft behavior of the NG dilaton through $\mathcal{O}(q^1)$. Finally, in Sec. IV, we compare the two behaviors and provide some conclusions. We note that our derivation for the gravity dilaton soft theorem is valid at tree level. For the NG dilaton, the formal arguments in our derivation hold at tree level in classically conformal theories. We also expect similar behavior at the quantum level in theories without a conformal anomaly.

II. GRAVITY DILATON

We consider the low-energy behavior of the dilaton that appears in theories of (super)gravity and in string theories. The behavior in either case can be obtained using two different methods that, apart from possible string corrections, turn out to give the same result.

The first method that we employ is completely independent of string theory. It consists of determining the low-energy behavior of a tensor $M_{\mu\nu}(q;k_i)$ that, when saturated with the polarization $\epsilon_{\mu\nu}$ of the graviton or of the dilaton, describes the scattering amplitude of a graviton or a dilaton with momentum q and n other particles with momenta k_i. In this case, the soft behavior through $\mathcal{O}(q^1)$ of $M_{\mu\nu}(q;k_i)$ is fixed by imposing the following conditions,

$$q^\mu M_{\mu\nu}(q;k_i) = q^\nu M_{\mu\nu}(q;k_i) = 0, \tag{1}$$

dictated by gauge invariance. The procedure is the one discussed in Ref. [7] without restricting oneself to the assumption that the polarization of the soft particle is traceless as in the case of the graviton. Proceeding in this way, when the other particles are (massive) scalars, i.e. not dilatons, one gets the following soft behavior:

$$M^{\mu\nu}(q;k_i) = \kappa_D \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{k_i \cdot q} - i \frac{k_i^\mu q_\rho L_i^{\nu\rho}}{2k_i \cdot q} - i \frac{k_i^\nu q_\rho L_i^{\mu\rho}}{2k_i \cdot q} - \frac{1}{2} \frac{q_\rho L_i^{\mu\rho} q_\sigma L_i^{\nu\sigma}}{k_i \cdot q} + \frac{1}{2} \left(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma} - \frac{k_i^\mu q^\nu q^\sigma}{k_i \cdot q} \right) \frac{\partial}{\partial k_i^\sigma} \right] \times \mathcal{T}_n(k_1, \dots, k_n) + O(q^2), \tag{2}$$

where κ_D is related to the D-dimensional Newton's constant by $\kappa_D = \sqrt{8\pi G_N^{(D)}}$, $\mathcal{T}_n(k_1, \dots, k_n)$ is the scattering amplitude of n scalar particles and

$$L_i^{\mu\nu} = i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right). \tag{3}$$

When instead the other particles are gravitons and/or dilatons, one gets an extra polarization-dependent piece:

$$M^{\mu\nu}(q;k_i) = \kappa_D \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{k_i \cdot q} - i \frac{k_i^\mu q_\rho J_i^{\nu\rho}}{2k_i \cdot q} - i \frac{k_i^\nu q_\rho J_i^{\mu\rho}}{2k_i \cdot q} - \frac{1}{2} \frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q} + \frac{1}{2} \left(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma} - \frac{k_i^\mu q^\nu q^\sigma}{k_i \cdot q} \right) \frac{\partial}{\partial k_i^\sigma} - \frac{1}{2} \frac{q_\rho q_\sigma \eta_{\mu\nu} - q_\sigma q_\nu \eta_{\rho\mu} - q_\rho q_\mu \eta_{\sigma\nu}}{k_i \cdot q} \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}} \right] \times M_n(k_1, \dots, k_n) + \mathcal{O}(q^2), \tag{4}$$

where $M_n(k_1, \dots, k_n)$ is the n-point scattering amplitude involving gravitons and/or dilatons and

$$(J_i)_{\mu\nu} = (L_i)_{\mu\nu} + (S_i)_{\mu\nu},$$
$$(S_i)_{\mu\nu} = i \left(\epsilon_{i\mu} \frac{\partial}{\partial \epsilon_i^\nu} - \epsilon_{i\nu} \frac{\partial}{\partial \epsilon_i^\mu} \right). \tag{5}$$

It is easy to check that both tensors $M_{\mu\nu}(q;k_i)$ satisfy Eq. (1). By saturating the two previous tensors with the polarization of the dilaton, given by $\epsilon_d^{\mu\nu} = (\eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu)/\sqrt{D-2}$, where $q^2 = \bar{q}^2 = 0$ and $q \cdot \bar{q} = 1$, we get the soft behavior of a dilaton in an amplitude with n scalars:

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}} \left\{ -\sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q} \left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \right) + 2 - \sum_{i=1}^{n} \left[k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2} \left(2k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) \right] \right\} \mathcal{T}_n + \mathcal{O}(q^2), \tag{6}$$

where m_i is the mass of the ith scalar particle.

Similarly, by saturating Eq. (4) with the dilaton polarization, one gets the soft behavior of a dilaton in an amplitude with hard gravitons and/or dilatons:

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}} \left\{ 2 - \sum_{i=1}^{n} \left[k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2} \left(2k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) - i q^\rho S_{\mu\rho}^{(i)} \frac{\partial}{\partial k_{i\mu}} \right] + \sum_{i=1}^{n} \frac{q^\rho q^\sigma}{2k_i \cdot q} \left((S_{\rho\mu}^{(i)}) \eta^{\mu\nu} (S_{\nu\sigma}^{(i)}) + D \epsilon_{i\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \right) \right\} \times M_n + \mathcal{O}(q^2). \tag{7}$$

In all of our expressions, whenever a momentum derivative is acting on an amplitude, it is implicitly assumed that momentum conservation is applied to one of the external momenta in the amplitude.

The second method is to consider a string theory,[1] compute an amplitude with a dilaton and study its behavior when the momentum of the dilaton is soft. It turns out that, if the other particles are tachyons of the bosonic string, one gets exactly the behavior in Eq. (6) with $m_i^2 = -\frac{4}{\alpha'}$, as shown in Ref. [3]. On the other hand, if the other particles are also dilatons and/or gravitons, one finds the behavior given in Eq. (7) [1,3,16].

III. DILATON OF BROKEN CONFORMAL INVARIANCE

We consider a field theory whose action is invariant under some transformation, with corresponding Noether current j^μ, and we study the matrix element $T^* \langle 0 | j^\mu(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle$, where T^* denotes the T-product with the derivatives placed outside of the time-ordering symbol. For the sake of simplicity, here and in the following we restrict ourselves to scalar fields. Taking the derivative of this quantity with respect to the variable x and subsequently performing a Fourier transformation in x yields a form of the Ward identity that we use to derive low-energy theorems,

[1] String theory soft theorems have been recently studied in [8–15].

$$\int d^D x e^{-iq\cdot x}[-\partial_\mu T^* \langle 0|j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle + T^*\langle 0|\partial_\mu j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle] = -\sum_{i=1}^n e^{-iq\cdot x_i} T^*\langle 0|\phi(x_1)\ldots\delta\phi(x_i)\ldots\phi(x_n)|0\rangle, \quad (8)$$

where $\delta\phi$ is the infinitesimal transformation of the field ϕ under the generators of the symmetry. We consider the case where the Noether current corresponds to scale and special conformal transformations of a spontaneously broken conformal field theory in D space-time dimensions.[2]

Let us start by discussing the scale transformation. The Noether current and its divergence are equal to

$$j^\mu_{\mathcal{D}} = x_\nu T^{\mu\nu}; \qquad \partial_\mu j^\mu_{\mathcal{D}} = T_\mu{}^\mu, \quad (9)$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the theory. The action of the generator of scale transformation $\mathcal{D}$ on a scalar field is given by

$$\delta\phi(x) = [\mathcal{D}, \phi(x)] = i(d + x^\mu\partial_\mu)\phi(x), \quad (10)$$

where d is the scaling dimension of the scalar field $\phi(x)$.

Considering the left-hand side of Eq. (8), we may neglect the first term by only keeping terms up to $\mathcal{O}(q^0)$ in the soft expansion, assuming that $T^*\langle 0|j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle$ does not have a pole at $q = 0$. In a theory with spontaneously broken conformal symmetry, the second term contributes. Notably, in such a theory the NG dilaton $\xi(x)$, which is the massless fluctuation around the nonconformal vacuum, is by its equation of motion related to the trace of the energy-momentum tensor in the following way:

$$T_\mu{}^\mu(x) = -v\partial^2\xi(x), \quad (11)$$

where v is related to the vacuum expectation value of the dilaton field, denoted by $\langle\xi\rangle$. [In the specific theory considered in Ref. [5], we find that $v = \frac{D-2}{2}\langle\xi\rangle$, where Eq. (11) is a consequence of the classical equations of motion. See also Sec. II of Ref. [18] and Sec. III of Ref. [19].] It follows from Eq. (9) that

$$\partial_\mu j^\mu_{\mathcal{D}}(x) = v(-\partial^2)\xi(x). \quad (12)$$

To translate the correlation function identity in Eq. (8) to an identity among amplitudes, we apply the Lehmann-Symanzik-Zimmerman (LSZ) reduction. We define the LSZ operator,

$$[\mathrm{LSZ}] \equiv i^n \left(\prod_{j=1}^n \lim_{k_j^2\to -m_j^2} \int d^D x_j e^{-ik_j\cdot x_j}(-\partial_j^2 + m_j^2)\right),$$

where the limits $k_j^2 \to -m_j^2$ put the external states on shell, which has to be performed only at the end. Inserting Eq. (12) into Eq. (8) and applying the LSZ reduction we find the left-hand side of Eq. (8) to yield

$$[\mathrm{LSZ}] \int d^D x e^{-iq\cdot x} T^*\langle 0|\partial_\mu j^\mu_{\mathcal{D}}(x)\phi(x_1)\ldots\phi(x_n)|0\rangle = (-i)v(2\pi)^D\delta^{(D)}\left(\sum_{j=1}^n k_j + q\right)\mathcal{T}_{n+1}(q; k_1, \ldots, k_n), \quad (13)$$

where we have Fourier transformed and extracted the poles of the correlation function to identify the amplitude $\mathcal{T}_{n+1}$.[3] Notice that the operator $(-\partial^2)$ from the divergence of the current effectively amputates the dilaton propagator $\langle\xi\xi\rangle \sim 1/q^2$. It is implicitly assumed that this expression holds only up to terms of $\mathcal{O}(q^0)$, and that the k_i on-shell limits are taken after the soft expansion.

Next, performing the same operations on the right-hand side of Eq. (8) gives

$$[\mathrm{LSZ}]\left(-\sum_{i=1}^n e^{-iq\cdot x_i} T^*\langle 0|\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)|0\rangle\right) = -\sum_{i=1}^n \left[\lim_{k_i^2\to -m_i^2}(k_i^2 + m_i^2) i\left(d - D - (k_i+q)^\mu \frac{\partial}{\partial k_i^\mu}\right) \times \frac{(2\pi)^D\delta^{(D)}(\sum_{j=1}^n k_j + q)}{(k_i+q)^2 + m_i^2}\mathcal{T}_n(k_1, \ldots, k_i + q, \ldots, k_n)\right], \quad (14)$$

where all states $j \neq i$ have already been amputated and put on shell. The next step is to commute the differential operator past the ith propagator and the δ-function, using the identity

$$\sum_{i=1}^n k_i^\mu \frac{\partial}{\partial k_{i\nu}}\left[\delta^{(D)}\left(\sum_{j=1}^n k_j\right)\mathcal{T}_n(k_1, \ldots, k_n)\right] = \delta^{(D)}\left(\sum_{j=1}^n k_j\right)\left(-\eta^{\mu\nu} + \sum_{i=1}^n k_i^\mu \frac{\partial}{\partial k_{i\nu}}\right)\mathcal{T}_n \times \left(k_1, \ldots, -\sum_{j=1}^{n-1} k_j\right). \quad (15)$$

To ensure a simple form, notice the need to apply momentum conservation on one of the momenta of the n-point amplitude before differentiating. As we remarked in the previous section, it is necessary to enforce this condition whenever a derivative is acting on the amplitude.

[2] For a review on the conformal Ward identities, see Coleman's beautiful book [17].

[3] For more details, see for instance Ref. [20], whose conventions we follow up to an immaterial factor i.

We denote this procedure for brevity by $\bar{k}_n = -\sum_{j=1}^{n-1} k_j$ (see also Ref. [21] for a more general discussion). Expanding $\mathcal{T}_n$ in the soft momentum q and following through with this procedure, we find Eq. (14), up to terms of $\mathcal{O}(q^1)$, to be equal to (the δ-function is kept unexpanded because it appears in the same form on the left-hand side)

$$-i(2\pi)^D\delta^{(D)}\left(\sum_{j=1}^n k_j + q\right)\left\{D - nd - \sum_{i=1}^n k_i^\mu \frac{\partial}{\partial k_i^\mu}\right.$$
$$\left. - \sum_{i=1}^n \lim_{k_i^2\to -m_i^2} \frac{2m_i^2(k_i^2+m_i^2)}{[(k_i+q)^2+m_i^2]^2}\left(1 + q^\mu \frac{\partial}{\partial k_i^\mu}\right)\right\}$$
$$\times \mathcal{T}_n(k_1,\ldots,\bar{k}_n), \tag{16}$$

where we have used $d = (D-2)/2$, and neglected terms of $\mathcal{O}(q^1)$. The second line clearly has a singularity problem, and it would be incorrect to expand the denominator in the soft momentum q, since the on-shell limit is then divergent. The soft momentum here must instead be understood as a physical regulator that leaves the on-shell limit finite, which physically means that we should identify $k_i \simeq k_i + q$, leaving us with

$$-i(2\pi)^D\delta^{(D)}\left(\sum_{j=1}^n k_j + q\right)\left\{D - nd - \sum_{i=1}^n k_i^\mu \frac{\partial}{\partial k_i^\mu}\right.$$
$$\left. - \sum_{i=1}^n \frac{m_i^2}{k_i\cdot q}\left(1 + q^\mu \frac{\partial}{\partial k_i^\mu}\right)\right\}\mathcal{T}_n(k_1,\ldots,\bar{k}_n). \tag{17}$$

As we will see, this regulation leads to physically sensible results when one considers the decomposition of $\mathcal{T}_{n+1}$ as

$$\mathcal{T}_{n+1} = \left[\sum_{i=1}^n \frac{\mathbf{S}_i^{(-1)}(q)}{k_i\cdot q} + \mathbf{S}^{(0)} + q^\mu \mathbf{S}_\mu^{(1)}\right]$$
$$\times \mathcal{T}_n(k_1,\ldots,k_n) + \mathcal{O}(q^2), \tag{18}$$

where the $\mathbf{S}^{(0)}$ and $\mathbf{S}_\mu^{(1)}$ are operators dependent only on the momenta k_j, while $\mathbf{S}^{(-1)}$ may depend on q as well. Then, from equating Eqs. (13) and (17), we find

$$v\mathbf{S}_i^{(-1)} = -m_i^2\left(1 + q^\mu \frac{\partial}{\partial k_i^\mu}\right) + \mathcal{O}(q^2), \tag{19a}$$

$$v\mathbf{S}^{(0)} = D - nd - \sum_{i=1}^n k_i^\mu \frac{\partial}{\partial k_i^\mu}. \tag{19b}$$

The previous expressions agree with Eq. (4) of Ref. [4] for the four-dimensional massless case. One can see that the Ward identity for the scale transformation completely determines the $\mathcal{O}(q^{-1})$ and $\mathcal{O}(q^0)$ contributions of the amplitude with a soft dilaton in terms of the amplitude without the dilaton.

We now show that the Ward identity of the special conformal transformation [22] determines also the $\mathcal{O}(q^1)$ contribution of the amplitude with a soft dilaton. In this case, the Noether current and its divergence are given by

$$j^\mu{}_{(\lambda)} = T^{\mu\nu}(2x_\nu x_\lambda - \eta_{\nu\lambda}x^2),$$
$$\partial_\mu j^\mu{}_{(\lambda)} = 2x_\lambda T^\mu{}_\mu = 2vx_\lambda(-\partial^2)\xi(x), \tag{20}$$

while the action of a special conformal transformation on a scalar field is equal to

$$\delta_{(\lambda)}\phi(x) = [\mathcal{K}_\lambda, \phi(x)]$$
$$= i((2x_\lambda x_\nu - \eta_{\lambda\nu}x^2)\partial^\nu + 2dx_\lambda)\phi(x). \tag{21}$$

(Note that, in general, the special conformal transformation has an extra term when acting on fields with spin.)

Analyzing the right-hand side of Eq. (8), mirroring the procedure for scale transformations utilized above, we find an expression analogous to Eq. (17),

$$[\mathrm{LSZ}]\left(-\sum_{i=1}^n \mathrm{e}^{-iq\cdot x_i}T^*\langle 0|\phi(x_1)\cdots\delta_{(\lambda)}\phi(x_i)\cdots\phi(x_n)|0\rangle\right)$$
$$= (2\pi)^D\delta^{(D)}\left(\sum_{j=1}^n k_j + q\right)\times 2\sum_{i=1}^n\left\{\right.$$
$$\times \frac{m_i^2}{k_i\cdot q}\left[\frac{k_{i\lambda}}{k_i\cdot q} - \frac{\partial}{\partial k_i^\lambda}\right]\left(1 + q^\mu\frac{\partial}{\partial k_i^\mu} + \frac{1}{2}q^\mu q^\nu \frac{\partial^2}{\partial k^\mu \partial k^\nu}\right)$$
$$-\left[k_i^\mu\left(\frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - \frac{1}{2}\eta_{\mu\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right) + d\frac{\partial}{\partial k_i^\lambda}\right]$$
$$\left. + \mathcal{O}(q)\right\}\mathcal{T}_n(k_1,\ldots,\bar{k}_n), \tag{22}$$

where again we use $d = (D-2)/2$. This calculation is very similar to that of scale transformations except that the action of the differential operators on the δ-function does not introduce extra terms, like D in the case of scale transformations. Similarly, analyzing the left-hand side of Eq. (8), we find an expression analogous to Eq. (13), but now with a derivative of q acting on the amplitude,

$$[\mathrm{LSZ}]\int d^Dx\mathrm{e}^{-iq\cdot x}T^*\langle 0|\partial_\mu j^\mu_{(\lambda)}(x)\phi(x_1)\ldots\phi(x_n)|0\rangle$$
$$= 2v(2\pi)^D\delta^{(D)}\left(\sum_{j=1}^n k_j + q\right)$$
$$\times \frac{\partial}{\partial q^\lambda}\mathcal{T}_{n+1}\left(q; k_1,\ldots,-q-\sum_{j=1}^{n-1}k_j\right). \tag{23}$$

Equating Eqs. (23) and (22) and contracting with q^λ, we find

$$
\begin{aligned}
& v q^\lambda \frac{\partial}{\partial q^\lambda} \mathcal{T}_{n+1}\left(q; k_1, \ldots, -q - \sum_{j=1}^{n-1} k_j\right) \\
&= \sum_{i=1}^{n} \left\{ \frac{m_i^2}{k_i \cdot q} \left(1 - \frac{1}{2} q^\mu q^\lambda \frac{\partial^2}{\partial k^\mu \partial k^\lambda}\right) \right. \\
&\quad \left. -q^\lambda \left[k_i^\mu \left(\frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - \frac{1}{2} \eta_{\mu\lambda} \frac{\partial^2}{\partial k_{i\nu} \partial k_i^\nu} \right) + d \frac{\partial}{\partial k_i^\lambda} \right] \right\} \\
&\quad \times \mathcal{T}_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q^2). \qquad (24)
\end{aligned}
$$

With Eqs. (19) in hand, we may use Eq. (18) to replace $\mathcal{T}_{n+1}$. The $\mathcal{O}(q^{-1})$ terms then exactly cancel on both sides of the above equation, leading to an equation that uniquely determines the $\mathcal{O}(q^1)$ terms of $\mathcal{T}_{n+1}$, i.e.

$$
v \mathbf{S}_i^{(-1)}|_{\mathcal{O}(q^2)} = -m_i^2 \left(\frac{1}{2} q^\mu q^\lambda \frac{\partial^2}{\partial k^\mu \partial k^\lambda} \right), \qquad (25a)
$$

$$
v \mathbf{S}_\lambda^{(1)} = - \sum_{i=1}^{n} \left[k_i^\mu \left(\frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - \frac{1}{2} \eta_{\mu\lambda} \frac{\partial^2}{\partial k_{i\nu} \partial k_i^\nu} \right) + d \frac{\partial}{\partial k_i^\lambda} \right]. \qquad (25b)
$$

In conclusion, the Ward identities of the scale and conformal transformations determine completely the low-energy behavior, through the $\mathcal{O}(q^1)$, of an amplitude with a soft dilaton in terms of the amplitude without the dilaton. Inserting the results from Eqs. (19) and (24) into Eq. (18), we have in total

$$
\begin{aligned}
& v \mathcal{T}_{n+1}(q; k_1, \ldots, k_n) \\
&= \left\{ - \sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q} \left(1 + q^\mu \frac{\partial}{\partial k_i^\mu} + \frac{1}{2} q^\mu q^\nu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\nu} \right) \right. \\
&\quad + D - nd - \sum_{i=1}^{n} k_i^\mu \frac{\partial}{\partial k_i^\mu} \\
&\quad \left. -q^\lambda \sum_{i=1}^{n} \left[\frac{1}{2} \left(2 k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - k_{i\lambda} \frac{\partial^2}{\partial k_{i\nu} \partial k_i^\nu} \right) + d \frac{\partial}{\partial k_i^\lambda} \right] \right\} \\
&\quad \times \mathcal{T}_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q^2). \qquad (26)
\end{aligned}
$$

Notice that the terms proportional to m_i^2 can be considered as the expansion of $\mathcal{T}_n(k_1, \ldots, k_i + q, \ldots, k_n)$ in q. Indeed, this is exactly what we expect from the structure of tree-level amplitudes; however, here it comes out as a consequence of the Ward identities. We have checked the above expression against three-, four-, five- and six-point amplitudes in a simple four-dimensional two-scalar model, and against 3-, 4-, and 5-point amplitudes in the generalized D-dimensional model, discussed in Ref. [5]. The term of order $\mathcal{O}(q^0)$ agrees with the one proposed in Ref. [6].[4]

[4]We thank Congkao Wen for communicating to us that the dilaton actions constructed for proving the a-theorem [23–25] satisfy the complete soft behavior in Eq. (26).

IV. COMPARISON AND CONCLUSIONS

In this paper, we have extracted the tree-level soft behavior of two, *a priori*, different objects that are both referred to as dilatons in the literature. We have shown that in both cases the symmetry properties determine the soft behavior through the $\mathcal{O}(q^1)$ in the dilaton momentum. In the case of the gravity dilaton, the symmetry is the same gauge invariance that determines the soft behavior of the graviton, while the soft behavior of the NG dilaton is determined by the Ward identities of scale and special conformal transformations. The soft behavior of the gravity dilaton is given in Eq. (6) in an amplitude with scalar particles and in Eq. (7) in an amplitude with other massless particles, while that of a NG dilaton is given in Eq. (26). In both cases, we get a term of $\mathcal{O}(q^{-1})$, which is proportional to the squared mass of the other particles. This follows from the fact that, in both cases, there is a three-point amplitude involving a dilaton and two identical particles, which is proportional to their squared mass.

Furthermore, for both dilatons we have a term of $\mathcal{O}(q^0)$ and a term of $\mathcal{O}(q^1)$ that are fixed by the symmetry properties and which contain terms connected to the conformal operators $\mathcal{D}$ and $\mathcal{K}_\mu$. The generators of space-time scale transformations, $\hat{\mathcal{D}}$, and special conformal space-time transformation $\hat{\mathcal{K}}_\mu$, are equal to

$$
\hat{\mathcal{D}} = x_\mu \hat{\mathcal{P}}^\mu; \qquad \hat{\mathcal{K}}_\mu = (2 x_\mu x_\lambda - x^2 \eta_{\mu\lambda}) \hat{\mathcal{P}}^\lambda. \qquad (27)
$$

where $\hat{\mathcal{P}}^\mu$ is the generator of space-time translations. Going to momentum space they become

$$
\begin{aligned}
\hat{\mathcal{D}} &= -i k_\mu \frac{\partial}{\partial k_\mu}; \\
\hat{\mathcal{K}}_\mu &= - \left(2 k^\nu \frac{\partial^2}{\partial k^\nu \partial k^\mu} - k_\mu \frac{\partial^2}{\partial k^\nu \partial k_\nu} \right), \qquad (28)
\end{aligned}
$$

which are precisely the operators that appear in both soft behaviors. Apart from these similarities, there seems to be some difference in the soft behavior as well; looking at the term of $\mathcal{O}(q^0)$ in Eqs. (6) and (26), we find that in the first case the kinematically invariant part equals 2, while in the second case it is equal to $D - nd$. The term $D - nd = D - n\frac{D-2}{2}$ represents the fact that $\mathcal{T}_n$ has exactly this mass dimension. In fact, for dimensional reasons, the amplitude $\mathcal{T}_n$ has the following form:

$$
\mathcal{T}_n(m; k_i) = m^{D - n\frac{D-2}{2}} g^{n-2} G_n(k_i/m), \qquad (29)
$$

where m is the mass scale of the theory, which is typically given by $(gv)^{\frac{2}{D-2}}$, where v is related to the vacuum expectation value of the dilaton field, $\langle \xi \rangle$, and g is a typical dimensionless coupling constant of the theory. It follows immediately from Eq. (29) that the term of $\mathcal{O}(q^0)$ in Eq. (26) can be rewritten as

$$\frac{1}{v}\left(D - n\frac{D-2}{2} - \sum_{i=1}^{n} k_i^{\mu}\frac{\partial}{\partial k_i^{\mu}}\right)\mathcal{T}_n = \frac{m}{v}\frac{\partial}{\partial m}\mathcal{T}_n = \left(\frac{D-2}{2}\right)\frac{\partial}{\partial v}\mathcal{T}_n \sim \frac{\partial}{\partial\langle\xi\rangle}\mathcal{T}_n, \tag{30}$$

where $\sim$ means up to a numerical constant.

The term of $\mathcal{O}(q^0)$ in Eq. (6) instead seems to have another meaning. In string theory M_n has, of course, the same physical dimension as $\mathcal{T}_n$ and the following form:

$$M_n = \frac{4\pi}{\alpha'}\left(\frac{\kappa_D}{\pi}\right)^{n-2} F_n(\sqrt{\alpha'}k_i) = C_n m_s^{D-n\frac{D-2}{2}} g_s^{n-2} F_n(k_i/m_s), \tag{31}$$

where α' is the inverse string tension, and in the second line we rewrote the expression into a form similar to Eq. (29), with C_n being a numerical constant, $m_s \equiv 1/\sqrt{\alpha'}$, g_s the string coupling constant and

$$\kappa_D = \frac{1}{2^{\frac{D-10}{4}}}\frac{g_s}{\sqrt{2}}(2\pi)^{\frac{D-3}{2}}(\sqrt{\alpha'})^{\frac{D-2}{2}}. \tag{32}$$

In the field theory limit (gravity and supergravity), M_n behaves as follows:

$$\lim_{\alpha'\to 0} M_n \sim \left(\frac{\kappa_D}{\pi}\right)^{n-2}\lim_{\alpha'\to 0}\frac{4\pi F_n(\sqrt{\alpha'}k_i)}{\alpha'}, \tag{33}$$

where the limit is finite and, for dimensional reasons, has to provide a homogenous function in the momenta of the particles of degree 2. This means that, in this limit, the action of the full term of $\mathcal{O}(q^0)$ in Eqs. (6)–(7) gives zero. In other words, in the field theory limit, the amplitude with one soft dilaton is vanishing and this is consistent with the fact that an amplitude with an odd number of dilatons in (super)gravity vanishes. However, in the full string theory this is no longer true. Equation (31) implies that the $\mathcal{O}(q^0)$ term in Eqs. (6)–(7) can be written as

$$\kappa_D\left(2 - \sum_{i=1}^{n} k_i^{\mu}\frac{\partial}{\partial k_i^{\mu}}\right)M_n = \kappa_D\left(\frac{D-2}{2}g_s\frac{\partial}{\partial g_s} - \sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}}\right)M_n = \left(\frac{D-2}{2}\right)\kappa_D\frac{d}{d\phi_0}M_n, \tag{34}$$

where we used the relation between g_s and the vacuum expectation value of the string dilaton, ϕ_0, i.e. $g_s \equiv e^{\phi_0}$. [Equation (34) is valid also when the amplitude M_n contains massless open strings.] The operator that appears in the second line leaves κ_D invariant, which can be explicitly checked (see also [26,27]). This implies that in string theory one does not have two fundamental constants $g_s \equiv e^{\phi_0}$ and α' that can be fixed independently from each other; the physical amplitudes depend on α' and on κ_D where a change of ϕ_0 can be reabsorbed in a rescaling of α'. Thus, the last step in Eq. (34) means that we should differentiate with respect to ϕ_0 keeping κ_D fixed. To compare with the field theory dilaton, we should canonically normalize $\phi_0 = \sqrt{2}\kappa_D\phi_{\text{c.n.}}$, and since κ_D is kept fixed we simply get up to a numerical constant,

$$\kappa_D\left(2 - \sum_{i=1}^{n} k_i^{\mu}\frac{\partial}{\partial k_i^{\mu}}\right)M_n \sim \frac{d}{d\phi_{\text{c.n.}}}M_n. \tag{35}$$

Thus in both cases we find that, up to numerical constants, the $\mathcal{O}(q^0)$ term of the soft-dilaton amplitude is simply given by the derivative of the lower-point amplitude with respect to the vacuum expectation value.

Before we leave the term of $\mathcal{O}(q^0)$, let us conclude with a more intuitive argument for the kinematically invariant terms of $\mathcal{O}(q^0)$. In the case of a NG dilaton, all dimensional factors are rescaled by a scale transformation, while in string theory one rescales the factor $\frac{1}{\alpha'}$ in the front of Eq. (31) without rescaling κ_D. That is the reason why in one case one gets $D - n\frac{D-2}{2}$, while in the other case one gets 2.

Finally, the terms of $\mathcal{O}(q^1)$ are equivalent up to a single piece; in the case of the NG dilaton, Eq. (26), there is a term with a single derivative, which is not present in the case of the gravity dilaton. As mentioned, however, they both contain the operator related to special conformal space-time translations.

It would be interesting to extend our considerations to the loop diagrams of both dilatons. In string theory, the dilaton stays massless to any order of perturbation theory, while in field theory, the dilaton, in general, gets a mass because conformal invariance is explicitly broken in the quantum theory. (For a perturbatively controllable example, see for instance Refs. [28,29].) There are, however, theories such as $\mathcal{N} = 4$ super Yang-Mills on the Coulomb branch that are not plagued by a conformal anomaly. In these theories, it would be especially compelling to investigate the extent of agreement with our soft theorem at the quantum level.

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