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Gauss' law and nonlinear plane waves for Yang-Mills theory

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We investigate nonlinear plane-wave solutions of the classical Minkowskian Yang-Mills (YM) equations of motion. By imposing a suitable ansatz which solves Gauss' law for the SU(3) theory, we derive solutions which consist of Jacobi elliptic functions depending on an enumerable set of elliptic modulus values. The solutions represent periodic anharmonic plane waves which possess arbitrary nonzero mass and are exact extrema of the nonlinear YM action. Among them, a unique harmonic plane wave with a nontrivial pattern in phase, spin, and color is identified. Similar solutions are present in the SU(4) case, while they are absent from the SU(2) theory.

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I. INTRODUCTION

Classical solutions of field theories remain of central interest in the understanding of the structure and dynamics of the Standard Model particle interactions. Regarding the pure Yang-Mills (YM) theory, major attention has been drawn to the extrema of the Euclidean action—the instantons—whose structure and properties are determined by topology (for a review see Ref. [1]). A lot of work has been produced on the relevance of the instanton configurations to the confining properties of the YM quantum vacuum as well as the quark dynamics coupled to the gauge bosons. On the other hand, in the Minkowskian $3+1$-dimensional spacetime, solitary waves, solitons, or traveling localized lumps with a finite energy are forbidden in the SU(N) gauge theory, as Coleman has shown generally [2] that finite-energy nonsingular gauge field configurations that do not radiate out to spatial infinity are not allowed to exist. Minkowskian Nonlinear plane waves (NLPWs) have been shown to exist for the SU(2) theory [3] and have the form of the Jacobian elliptic function $\mathrm{cn}(\omega t - \vec{k}\cdot\vec{x}; 1/2)$ with fixed elliptic parameter $1/2$ and relativistic dispersion relation $\omega^2 = \vec{k}^2 + m^2$ for an arbitrary mass parameter m. The self-interaction term is reflected in the anharmonic form of the cn-wave while scale invariance leaves free the value of mass m. Other solutions of the SU(2) Minkowskian equations of motion (EoM) are generated from the massless scalar ϕ^4 theory extrema via the Corrigan-Fairlie-'t Hooft-Wilczek ansatz [4,5] or generalized forms [6].

Regarding the SU(3) theory, a restrictive form of massless plane waves was initially shown to exist by Coleman [7], and in a more general form in Ref. [8], technically eliminating the interaction. A class of massive plane-wave solutions of the non-Abelian theory can be constructed from classical solutions of the massless ϕ^4 theory [9]. These are anharmonic waves of the $\mathrm{cn}(\omega t - \vec{k}\cdot\vec{x}; 1/2)$ type and are essentially waves constructed via the "multiple copies trick" [10]. Such solutions have also been shown to become relevant to the properties of the quantum theory in the strong coupling limit [9].

It is the purpose of this work to investigate if more general NLPW solutions exist for the SU(3) gauge theory EoM. For this reason, in Sec. II we present the detailed form of the EoM for the general SU(N) NLPW ansatz based on Lorentz symmetry and the scale invariance of the theory. In Sec. III we review the so-called "multiple copies trick" [10] which determines SU(N) solutions proportional to the $\mathrm{cn}(\omega t - \vec{k}\cdot\vec{x}; 1/2)$ field with appropriate non-Abelian constant factors and derive in particular the form of the constants for the SU(3) theory. In Sec. IV we propose a more general ansatz that solves the Gauss-law constraint equations for the SU(3) gauge theory. We arrive at a set of coupled cubic equations for complex fields which correspond to planar point-particle dynamics bounded by the r^4 potential. The gauge field color and polarization indices of the solution are mixed in a nontrivial scheme. The general solution is fixed by the angular momentum L of the particle in the sense that the elliptic parameter k^2 of the Jacobian elliptic functions is connected to L via a nonlinear equation. It is interesting that for the highest value of L allowed, a harmonic massive plane wave is shown to exist solving the SU(3) EoM even in the presence of the interaction terms. In Sec. V we show that plane waves other than the ones in Ref. [3] do not exist for the SU(2) theory. We finally outline the relevant ansatz for SU(4) and the embedding of the SU(3) solutions in it. The possible utility of such solutions is commented on in the final section.

II. NONLINEAR PLANE WAVES AND THE YANG-MILLS EQUATIONS OF MOTION

The Lagrangian density of the YM theory is defined via

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{\mu\nu a}, \qquad (1)$$

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where $\mathcal{F}^a_{\mu\nu}$ is the antisymmetric field tensor of the gauge field $\mathcal{A}^a_\mu$ [μ, $\nu = 0$, 1, 2, 3 are spacetime indices with a $(1,-1,-1,-1)$ metric assumed everywhere]:

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f_{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{2}$$

The structure constants f_{abc} define the non-Abelian SU(N) algebra via the commutators of the $N^2 - 1$ generators T^a in the fundamental $(N \times N)$ representation:

$$[T^a, T^b] = i f_{abc} T^c. \tag{3}$$

The corresponding classical EoMs for the gauge field are[1]

$$\partial_\mu \mathcal{F}^{\mu\nu a} + g f_{abc} \mathcal{A}^b_\mu \mathcal{F}^{\mu\nu c} = 0, \tag{4}$$

which in component form read explicitly

$$\begin{aligned}
&\Box \mathcal{A}^{\nu a} - \partial^\nu \partial_\mu \mathcal{A}^{\mu a} \\
&\quad + g f_{abc}[(\partial_\mu \mathcal{A}^{\mu b})\mathcal{A}^{\nu c} + 2\mathcal{A}^{\mu b}\partial_\mu \mathcal{A}^{\nu c} - \mathcal{A}^{\mu b}\partial^\nu \mathcal{A}^c_\mu)] \\
&\quad + g^2 f_{abc} f_{cde} \mathcal{A}^b_\mu \mathcal{A}^{\mu d} \mathcal{A}^{\nu e} \\
&\quad = 0.
\end{aligned} \tag{5}$$

The $\nu = 0$ set of the above equations constitute the Gauss-law constraint obeyed by the nondynamical fields A^a_0, while the $\nu = 1$, 2, 3 equations provide the evolution of the dynamical spatial components. We introduce generic plane-wave solutions, i.e., fields depending explicitly on the plane-wave phase

$$\xi = \omega t - \vec{k} \cdot \vec{x}, \tag{6}$$

with the momentum four-vector $\mathbb{k}^\mu = (\omega, \vec{k})$ satisfying the dispersion relation $\omega^2 = \vec{k}^2 + m^2$ for an arbitrary mass parameter m. In the physical system of units, the gauge field also has the dimension of mass so we scale the fields by completely eliminating at the same time the coupling g from the classical EoM:

$$\mathcal{A}^a_\mu = \frac{m}{g} A^a_\mu(\xi). \tag{7}$$

Spacetime derivatives are replaced by derivatives with respect to ξ (denoted by dots):

$$\partial_\mu A^a_\nu = \mathbb{k}_\mu \dot{A}^a_\nu(\xi), \tag{8}$$

and Eq. (5) becomes

$$\begin{aligned}
&m^2 \ddot{A}^{\nu a} - \mathbb{k}^\nu \mathbb{k}_\mu \ddot{A}^{\mu a} \\
&\quad + m f_{abc}[\mathbb{k}_\mu \dot{A}^{\mu b} A^{\nu c} + 2\mathbb{k}_\mu A^{\mu b} \dot{A}^{\nu c} - \mathbb{k}_\nu A^{\mu b} \dot{A}^c_\mu] \\
&\quad + m^2 f_{abc} f_{cde} A^b_\mu A^{\mu d} A^{\nu e} \\
&\quad = 0.
\end{aligned} \tag{9}$$

Now we make use of the Lorentz covariance of Eq. (9) by boosting the vector fields A^a_μ in the proper time frame, where $\mathbb{k}_\mu \to (m, \vec{0})$ and $\xi \to mt$, via a Lorentz transformation $\Lambda(\vec{\beta})$ with boost parameters

$$\vec{\beta} = \frac{\vec{k}}{\omega}, \qquad \gamma = \frac{\omega}{m}, \tag{10}$$

and the explicit spin-1 representation

$$\Lambda^\mu_\nu(\vec{\beta}) = \begin{pmatrix} \gamma & -\vec{\beta}\gamma \\ -\vec{\beta}\gamma & \delta_{ij} + \frac{\gamma^2}{\gamma+1}\beta_i\beta_j \end{pmatrix}. \tag{11}$$

The Gauss-law equation in the proper frame becomes[2]

$$f_{abc} A^b_j \dot{A}^c_j - f_{abc} f_{cde} A^b_j A^d_j A^{0e} = 0, \tag{12}$$

while the dynamical equations read (latin indices i, $j = 1$, 2, 3)

$$\begin{aligned}
&\ddot{A}^a_i + f_{abc}[\dot{A}^{0b} A^c_i + 2A^{0b} \dot{A}^c_i] \\
&\quad + f_{abc} f_{cde}[A^{0b} A^{0d} - A^b_j A^d_j] A^e_i = 0.
\end{aligned} \tag{13}$$

Next, we use the remaining t-dependent gauge freedom to fix $A^a_0 = 0$. The gauge group element $g(t)$ which solves the equation

$$g A^a_0 T^a g^{-1} - i \frac{dg}{dt} g^{-1} = 0 \tag{14}$$

is formally provided by the Polyakov line, $g(t) = \mathcal{P} e^{-i\int^t A^a_0 T^a}$, and the equations become

$$\begin{aligned}
f_{abc} A^b_j \dot{A}^c_j &= 0, \quad \text{(Gauss law)} \\
\ddot{A}^a_i - f_{abc} f_{cde} A^b_j A^d_j A^e_i &= 0.
\end{aligned} \tag{15}$$

Introducing also the matrices $\mathbf{A_i} = A^a_i T^a$ and the matrix-vector potential $\vec{\mathbf{A}} = A^a_i T^a \hat{e}_i$ ($\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ is an R^3 basis), the equations (15) are written as

$$[\vec{\mathbf{A}}, \dot{\vec{\mathbf{A}}}] = 0, \tag{16}$$

$$\ddot{\mathbf{A}}_i + \sum_{j \neq i} [\mathbf{A_j}, [\mathbf{A_j}, \mathbf{A_i}]] = 0. \tag{17}$$

[1] In covariant form $\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = 0$, with the covariant derivative defined as $\mathcal{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu$ and $-ig\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$.

[2] For convenience we use the same symbol for the rotated fields $\Lambda^\mu_\nu A^{\nu a} \to A^{\mu a}$.

The above set of equations maintain only the global SU(N) rotations, $\vec{\mathbf{A}} \to g\vec{\mathbf{A}}g^{-1}$. The chromoelectric and chromomagnetic fields for the above configurations are easily obtained:

$$E_i^a = \frac{m^2}{g}\dot{A}_i^a, \qquad B_i^a = \frac{m^2}{2g}\epsilon_{ijk}f_{abc}A_j^b A_k^c. \tag{18}$$

III. THE MULTIPLE COPIES TECHNIQUE

A standard trick which solves the Gauss-law constraint [for any SU(N)] is the so-called *multiple copies* technique [10]. This is the selection of copies of a color-independent field (Φ_x, Φ_y, Φ_z) as

$$(A_x^a, A_y^a, A_z^a) = (C_x^a\Phi_x, C_y^a\Phi_y, C_z^a\Phi_z) \tag{19}$$

for some constant vectors C_x^a, C_y^a, C_z^a. Due to the antisymmetric structure of f_{abc}, each of the three terms in Gauss' law

$$f_{abc}[C_x^b C_x^c \Phi_x \dot{\Phi}_x + C_y^b C_y^c \Phi_y \dot{\Phi}_y + C_z^b C_z^c \Phi_z \dot{\Phi}_z] = 0 \tag{20}$$

vanishes independently. Of course the dynamical equations have to be consistent for all color indices a and these impose restrictive algebraic constraints on the $3(N^2-1)$ constants C_i^a. Even in this case, the equations

$$\begin{aligned} &C_x^a\ddot{\Phi}_x - f_{abc}f_{cde}C_x^e[C_y^b C_y^d \Phi_y^2 + C_z^b C_z^d \Phi_z^2]\Phi_x = 0,\\ &C_y^a\ddot{\Phi}_y - f_{abc}f_{cde}C_y^e[C_x^b C_x^d \Phi_x^2 + C_z^b C_z^d \Phi_z^2]\Phi_y = 0,\\ &C_z^a\ddot{\Phi}_z - f_{abc}f_{cde}C_z^e[C_x^b C_x^d \Phi_x^2 + C_y^b C_y^d \Phi_y^2]\Phi_z = 0 \end{aligned} \tag{21}$$

will in general present chaotic behaviour [11]. Integrability is expected only for the diagonal case $\Phi_x = \Phi_y = \Phi_z = \Phi$, in which case the compatibility of the system

$$\begin{aligned} &C_x^a\ddot{\Phi} - f_{abc}f_{cde}C_x^e[C_y^b C_y^d + C_z^b C_z^d]\Phi^3 = 0,\\ &C_y^a\ddot{\Phi} - f_{abc}f_{cde}C_y^e[C_x^b C_x^d + C_z^b C_z^d]\Phi^3 = 0,\\ &C_z^a\ddot{\Phi} - f_{abc}f_{cde}C_z^e[C_x^b C_x^d + C_y^b C_y^d]\Phi^3 = 0 \end{aligned} \tag{22}$$

still allows a large space of constants C_i^a that can be traced numerically. We investigated in particular the SU(3) group and based on insight from Sec. IV we confirmed that the (nonunique) structure

$$\begin{aligned} C_x^a &= (\cos\phi_1, \sin\phi_1, 0, 0, 0, 0, 0, 0),\\ C_y^a &= (0, 0, 0, \cos\phi_2, \sin\phi_2, 0, 0, 0),\\ C_z^a &= (0, 0, 0, 0, 0, \cos\phi_3, \sin\phi_3, 0) \end{aligned} \tag{23}$$

with arbitrary constant angles ϕ_1, ϕ_2, ϕ_3 satisfies Eq. (22) and leads to a single equation for Φ,

$$\ddot{\Phi} + \frac{1}{4}\Phi^3 = 0, \tag{24}$$

which is solved by [12]

$$\Phi(\xi) = \mathrm{sn}\left[\frac{1}{2}\xi; -1\right] = \mathrm{cn}\left[\frac{1}{2}\xi; \frac{1}{2}\right]. \tag{25}$$

For the SU(2) theory, the choice $C_x^1 = C_y^2 = C_z^3 = 1$ (with all others zero) leads to the original solution presented in Ref. [3] with $A_x^1 = A_y^2 = A_z^3 = \Phi$, and Φ is the solution of

$$\ddot{\Phi} + 2\Phi^3 = 0. \tag{26}$$

IV. SU(3)

We present here a more general way to solve the Gauss-law constraint [Eq. (16)] for the SU(3) theory. The idea is to arrange the matrix-vector $\vec{\mathbf{A}}$ in such a way that it becomes orthogonal to the matrix-vector multiplication with itself. This can be achieved by "staggering" the color fields of the fundamental 3×3 matrix along the three orthogonal vectors of the R^3 basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ in the following way:

$$\vec{\mathbf{A}} = \begin{pmatrix} D_3\hat{e}_3 & \Psi_1^*\hat{e}_1 & \Psi_2\hat{e}_2 \\ \Psi_1\hat{e}_1 & D_2\hat{e}_2 & \Psi_3^*\hat{e}_3 \\ \Psi_2^*\hat{e}_2 & \Psi_3\hat{e}_3 & D_1\hat{e}_1 \end{pmatrix} - \frac{1}{3}(D_1\hat{e}_1 + D_2\hat{e}_2 + D_3\hat{e}_3)\mathbf{I}_3, \tag{27}$$

where D_1, D_2, D_3 are real functions and Ψ_1, Ψ_2, Ψ_3 are complex functions of ξ. Essentially, each complex SU(3) root pair—which corresponds to a doublet of nondiagonal color fields—is aligned along one of the three spatial directions. The explicit connection with the octet fields is given below (with all other components zero):

$$\begin{aligned} &A_x^1 + iA_x^2 = 2\Psi_1,\\ &A_y^4 - iA_y^5 = 2\Psi_2,\\ &A_z^6 + iA_z^7 = 2\Psi_3,\\ &A_x^3 = 0, \qquad A_y^3 = -D_2, \qquad A_z^3 = D_3,\\ &A_x^8 = -\frac{2D_1}{\sqrt{3}}, \qquad A_y^8 = \frac{D_2}{\sqrt{3}}, \qquad A_z^8 = \frac{D_3}{\sqrt{3}}. \end{aligned} \tag{28}$$

A direct check of Gauss' law is straightforward, since the trace piece [second term in Eq. (27)] drops out of the commutator. By construction each column (row) of the first term in Eq. (27) is orthogonal to any other column (row) and thus Gauss' law is written as

$$G^aT^a = \vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{A}}} - \dot{\vec{\mathbf{A}}}\cdot\vec{\mathbf{A}} = 2i\begin{pmatrix} L_1 - L_2 & 0 & 0 \\ 0 & L_3 - L_1 & 0 \\ 0 & 0 & L_2 - L_3 \end{pmatrix}, \tag{29}$$

where we introduced the real quantities

$$L_1 = \frac{i}{2}(\Psi_1\dot{\Psi}_1^* - \Psi_1^*\dot{\Psi}_1),$$
$$L_2 = \frac{i}{2}(\Psi_2\dot{\Psi}_2^* - \Psi_2^*\dot{\Psi}_2),$$
$$L_3 = \frac{i}{2}(\Psi_3\dot{\Psi}_3^* - \Psi_3^*\dot{\Psi}_3). \tag{30}$$

The implementation of Gauss' law requires

$$L_1 = L_2 = L_3 = L, \tag{31}$$

with the classes of solutions distinguished from now on by the $L \neq 0$ and $L = 0$ cases.

The dynamical EoMs (24 in total) are derived according to Eq. (27). We separate them into three different groups. Group 1:

$$\ddot{\Psi}_1 + \Psi_1[|\Psi_2|^2 + |\Psi_3|^2 + D_2^2 + D_3^2] = 0,$$
$$\ddot{\Psi}_2 + \Psi_2[|\Psi_1|^2 + |\Psi_3|^2 + D_1^2 + D_3^2] = 0,$$
$$\ddot{\Psi}_3 + \Psi_3[|\Psi_1|^2 + |\Psi_2|^2 + D_1^2 + D_2^2] = 0. \tag{32}$$

Group 2:

$$\ddot{D}_1 + 6D_1|\Psi_2|^2 = 0, \qquad \ddot{D}_1 + 6D_1|\Psi_3|^2 = 0,$$
$$\ddot{D}_2 + 6D_2|\Psi_1|^2 = 0, \qquad \ddot{D}_2 + 6D_2|\Psi_3|^2 = 0,$$
$$\ddot{D}_3 + 6D_3|\Psi_1|^2 = 0, \qquad \ddot{D}_3 + 6D_3|\Psi_2|^2 = 0. \tag{33}$$

Group 3:

$$D_1\Psi_1\Psi_2 = 0, \qquad D_1\Psi_1\Psi_3 = 0,$$
$$D_2\Psi_1\Psi_2 = 0, \qquad D_2\Psi_2\Psi_3 = 0,$$
$$D_3\Psi_1\Psi_3 = 0, \qquad D_3\Psi_2\Psi_3 = 0. \tag{34}$$

A. $L \neq 0$ solutions

Admitting $L \neq 0$ requires $\Psi_1 \neq 0$, $\Psi_2 \neq 0$, and $\Psi_3 \neq 0$. From the "Group 3" equations we are led to

$$D_1 = D_2 = D_3 = 0, \tag{35}$$

and the coupled system of equations

$$\ddot{\Psi}_1 + \Psi_1[|\Psi_2|^2 + |\Psi_3|^2] = 0,$$
$$\ddot{\Psi}_2 + \Psi_2[|\Psi_1|^2 + |\Psi_3|^2] = 0,$$
$$\ddot{\Psi}_3 + \Psi_3[|\Psi_1|^2 + |\Psi_2|^2] = 0. \tag{36}$$

The above system has the interpretation of three coupled planar point dynamics on the Ψ_1, Ψ_2, and Ψ_3 planes, respectively (which are the 1–2, 4–5, and 6–7 planes in the adjoint representation). One may use equivalently a polar coordinate description

$$\Psi_1 = r_1 e^{i\theta_1}, \qquad \Psi_2 = r_2 e^{i\theta_2}, \qquad \Psi_3 = r_3 e^{i\theta_3}, \tag{37}$$

where (r_i, θ_i) are functions of the phase ξ. The dynamics on each plane is invariant under independent global U(1) rotations,

$$\Psi_1 \to \Psi_1 e^{i\omega_1}, \qquad \Psi_2 \to \Psi_2 e^{i\omega_2}, \qquad \Psi_3 \to \Psi_3 e^{i\omega_3}, \tag{38}$$

or equivalently under independent SO(2) rotations on the (r_i, θ_i) planes. From this we identify L_1, L_2, L_3 in Eq. (29) as the *conserved angular momenta* of the three coupled rotating particles,

$$L_1 = A_x^1\dot{A}_x^2 - A_x^2\dot{A}_x^1 = r_1^2\dot{\theta}_1,$$
$$L_2 = A_y^5\dot{A}_y^4 - A_y^4\dot{A}_y^5 = r_2^2\dot{\theta}_2,$$
$$L_3 = A_z^6\dot{A}_z^7 - A_z^7\dot{A}_z^6 = r_3^2\dot{\theta}_3. \tag{39}$$

Imposing Gauss' law $L_1 = L_2 = L_3$ relates the complex functions Ψ_1, Ψ_2, Ψ_3 via linear transformations. These are well known, e.g.,

$$\Psi_2 = a\Psi_1 + b\Psi_1^*, \qquad aa^* - bb^* = 1, \tag{40}$$

where a, b are arbitrary complex constants. Equivalently, in the Cartesian form they act as $SL(2,R)$ transformations,[3] e.g.,

$$\Psi_2 = \begin{pmatrix} A_y^4 \\ -A_y^5 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\cdot\begin{pmatrix} A_x^1 \\ A_x^2 \end{pmatrix} = \Pi\Psi_1, \tag{41}$$

where $\det(\Pi) = \alpha\delta - \beta\gamma = 1$.

The linear relations in conjunction with the dynamical equations (36) further constrain the amplitudes:

$$|\Psi_1|^2 = |\Psi_2|^2 = |\Psi_3|^2 = r^2. \tag{42}$$

Up to constant angles, this also leads to $\theta_1 = \theta_2 = \theta_3 = \theta$, and thus we conclude that the $L \neq 0$ class of solutions is

[3] The explicit relation is $a = \frac{1}{2}(\alpha + \delta - i(\beta - \gamma))$, $b = \frac{1}{2}(\alpha - \delta + i(\beta + \gamma))$.

described by a single planar rotor bound by a central potential $V(r) = r^4/2$ and satisfying the EoM

$$\ddot{r} - \frac{L^2}{r^3} + 2r^3 = 0. \tag{43}$$

The system is integrable via the energy constant

$$E = \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} + \frac{1}{2}r^4. \tag{44}$$

Defining the rescaled function u via

$$r^2 = \sqrt{\frac{8E}{3}}\, u\left(\left(\frac{8E}{3}\right)^{1/4}\xi\right), \tag{45}$$

it satisfies

$$\dot{u}^2 = 3u - 4u^3 - \lambda, \qquad \lambda = 4L^2. \tag{46}$$

We scaled the constant $(8E/3)^{1/4} = 1$ in Eq. (45) by absorbing it into the mass parameter m. Equation (46) is solved by a *Weierstrass elliptic function*, $\mathcal{P}$, which is a doubly periodic function on the complex plane [12]. We look for real, positive, bounded solutions to describe periodic closed orbits on the (r, θ) plane and the suitable solution is expressed in terms of the Jacobi elliptic functions $\mathrm{sn}(\xi; k^2)$, $\mathrm{cn}(\xi; k^2)$, $\mathrm{dn}(\xi; k^2)$ of elliptic modulus k (or elliptic parameter k^2),

$$u(\xi) = e_1 + (e_2 - e_1)\frac{1}{\mathrm{dn}^2(\alpha\xi; k^2)}, \qquad k = \sqrt{\frac{e_3 - e_2}{e_3 - e_1}}. \tag{47}$$

The parameter $\alpha = \sqrt{e_3 - e_1}$ and e_1, e_2, e_3 are the three real roots of $3u - 4u^3 - \lambda = 0$. Three real roots exist only for $0 \le \lambda \le 1$ and are conveniently expressed via an angle ϕ which satisfies $\lambda = \cos\phi$,

$$\begin{aligned} e_1 &= -\cos(\phi/3), \\ e_2 &= \frac{1}{2}\cos(\phi/3) - \frac{\sqrt{3}}{2}\sin(\phi/3), \\ e_3 &= \frac{1}{2}\cos(\phi/3) + \frac{\sqrt{3}}{2}\sin(\phi/3). \end{aligned} \tag{48}$$

The solution (47) oscillates between e_2 and e_3 with a period equal to $T = 2K(k)/\alpha$ [$K(k)$ is the complete integral of the first kind of elliptic modulus k]. The angular field θ is determined from

$$\theta = \frac{\sqrt{\lambda}}{2}\int \frac{d\xi}{u} \tag{49}$$

and using properties of the Jacobi elliptic functions [12] is written as

$$\theta(\xi) = \frac{\sqrt{\lambda}}{2e_1}\left[\xi - \frac{e_2 - e_1}{\alpha e_2}\Pi\left(\frac{e_1 k^2}{e_2}; \mathrm{am}(\alpha\xi; k^2); k^2\right)\right]. \tag{50}$$

$\Pi(n; x; k^2)$ denotes the incomplete elliptic integral of the third kind with modulus k and characteristic n, while $\mathrm{am}(x; k^2)$ is the Jacobi amplitude function, which satisfies $\sin(\mathrm{am}(x; k^2)) = \mathrm{sn}(x; k^2)$. A periodic solution for the gauge fields is equivalent to a *closed orbit* for the rotor (43) on the (r, θ) plane. Thus a "quantization" condition on the parameter λ of the solution is enforced from the periodicity of θ for integers N, N' such that

$$\theta(NT) = 2\pi N'. \tag{51}$$

This, in turn, leads to the highly nonlinear relation

$$N\frac{\sqrt{\lambda}}{e_1\alpha}\left[K(k) - \frac{e_2 - e_1}{e_2}\bar{\Pi}\left(\frac{e_1 k^2}{e_2}; k^2\right)\right] = 2\pi N'. \tag{52}$$

$\bar{\Pi}(n; k^2) = \Pi(n; \pi/2; k^2)$ here denotes the complete elliptic integral of the third kind. Therefore we look for all values of the angle ϕ in $(0, \pi/2)$ such that the function

$$\mathcal{Q}(\phi) = \frac{\sqrt{\lambda}}{2\pi e_1\alpha}\left[K(k) - \frac{e_2 - e_1}{e_2}\bar{\Pi}\left(\frac{e_1 k^2}{e_2}; k^2\right)\right] \tag{53}$$

takes the value N'/N, i.e., on the set of rationals. From known properties and a Taylor analysis of $\mathcal{Q}(\phi)$ for ϕ near $\pi/2$, we deduce that $\mathcal{Q}(\phi)$ increases monotonically in the $(1/\sqrt{6}, 1/2)$ interval (see Fig. 1). Effectively, Eq. (52) becomes a "quantization" condition on the angular momentum L of the rotor since $\lambda = 4L^2$. Numerical solutions of Eq. (52) are easily obtained by selecting pairs of integers (N', N) such that $1/\sqrt{6} < N'/N < 1/2$. The lowest pairs of integers satisfying Eq. (52) are shown in Table I (see Figs. 3–5). For a given (large) N, one expects based on the

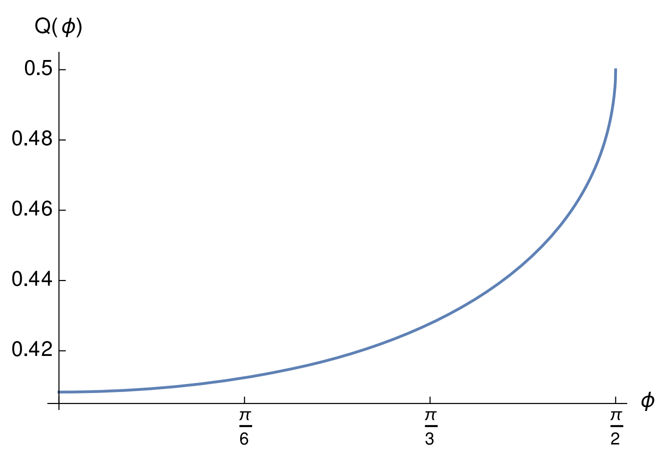

FIG. 1. The function $\mathcal{Q}(\phi)$, Eq. (53), as a function of ϕ. Each rational value of $\mathcal{Q}$ provides a solution of the SU(3) EoM.

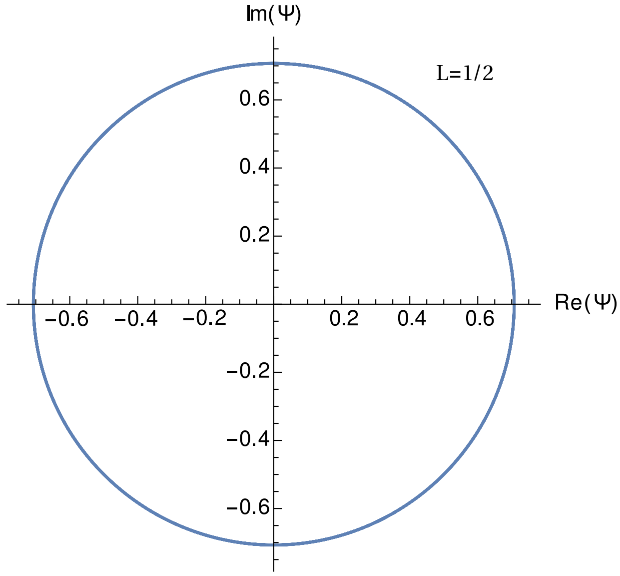

FIG. 2. The harmonic plane-wave solution for $L = 1/2$ corresponds to the circular orbit on the plane.

density of primes that the number of solutions is roughly $\sim 0.1N/\log N$.

The elliptic modulus k as determined from Eq. (47) takes an enumerable, infinite set of values in the interval $0 \leq k^2 \leq 1/2$. A particularly interesting solution is represented by the circular orbit, $\lambda = 1$ $(\phi = 0)$, which has the maximal angular momentum $L = 1/2$ (see Fig. 2). At this point $k = 0$ and the Jacobi elliptic functions become harmonic $[\mathrm{sn}(\xi;0) = \sin\xi, \mathrm{cn}(\xi;0) = \cos\xi, \mathrm{dn}(\xi;0) = 1]$. Since $r = 1/\sqrt{2}$ and $\theta = \xi$, a massive, harmonic wave

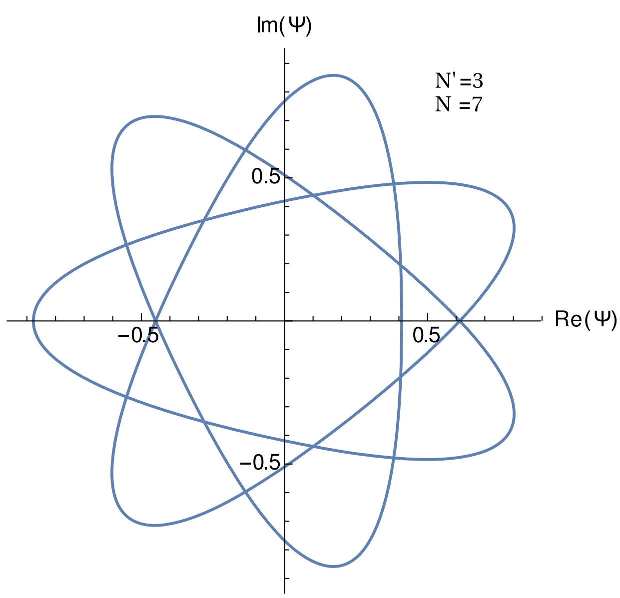

FIG. 3. The nonlinear plane-wave solution, Eq. (57), with $N' = 3$, $N = 7$.

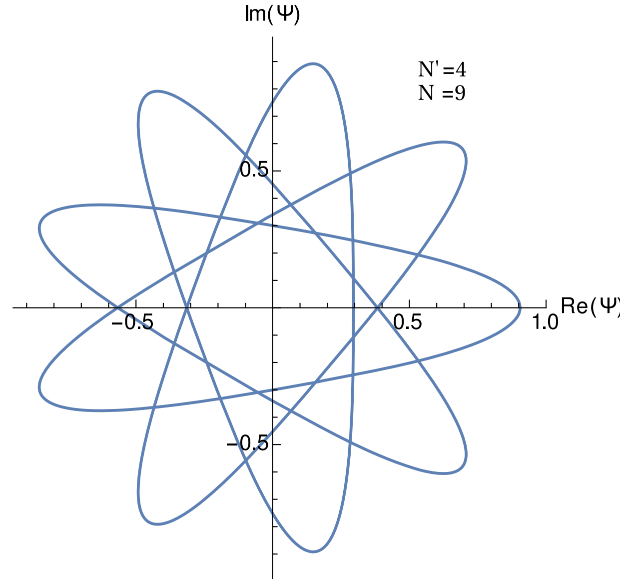

FIG. 4. The nonlinear plane-wave solution, Eq. (57), with $N' = 4$, $N = 9$.

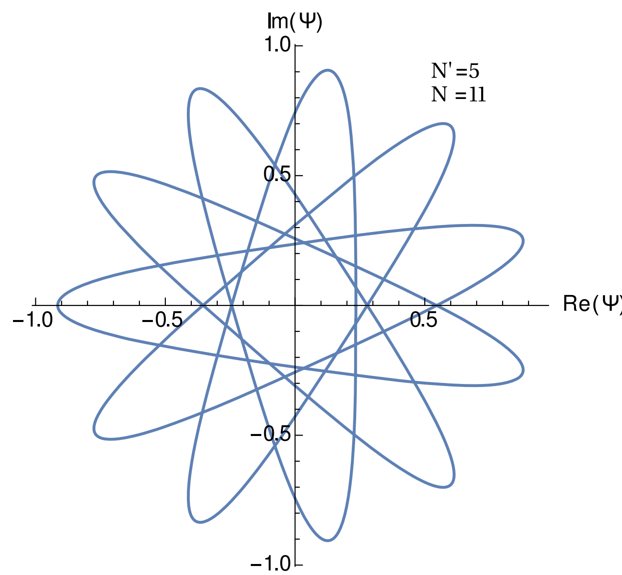

FIG. 5. The nonlinear plane-wave solution, Eq. (57), with $N' = 5$, $N = 11$.

solution of the interacting YM EoM is given in the proper frame by[4]

$$\vec{\mathbf{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-imt-i\phi_1}\hat{e}_1 & e^{imt+i\phi_2}\hat{e}_2 \\ e^{imt+i\phi_1}\hat{e}_1 & 0 & e^{-imt-i\phi_3}\hat{e}_3 \\ e^{-imt-i\phi_2}\hat{e}_2 & e^{imt+i\phi_3}\hat{e}_3 & 0 \end{pmatrix}. \tag{54}$$

[4]We include also arbitrary phase shifts ϕ_1, ϕ_2, ϕ_3 on the complex pairs.

TABLE I. Solutions of Eq. (52) obtained numerically.

N'	N	ϕ (deg)	k^2	L
3	7	61.0208	0.352594	0.348027
4	9	74.9299	0.423877	0.254951
5	11	80.5348	0.452263	0.202761
5	12	41.9829	0.251578	0.431087
6	13	83.4493	0.466980	0.168880

The other limiting solution is obtained for $\phi = \pi/2$ ($\lambda = 0$, $L = 0$) where the elliptic modulus takes the value $k = 1/\sqrt{2}$. In this limit the solution degenerates to a straight line on the plane ($\theta = \text{const}$). From known properties it can be shown that

$$r(\xi) = \left(\frac{\sqrt{3}}{2}\right)^{1/2} \text{cn}(3^{1/4}\xi + K(1/\sqrt{2}); 1/2). \tag{55}$$

In a general frame, the solution is obtained by a Lorentz boost [Eqs. (10)–(11)]. The R^3 basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is boosted to the three orthonormal spacelike[5] polarization vectors $\varepsilon_\mu^{(\sigma)}$ given for $\sigma = 1, 2, 3$ and $\mu = 0, 1, 2, 3$,

$$\varepsilon_\mu^{(\sigma)} = \begin{pmatrix} -k_1/m & -k_2/m & -k_3/m \\ 1 + \frac{k_1^2}{m(\omega+m)} & \frac{k_1 k_2}{m(\omega+m)} & \frac{k_1 k_3}{m(\omega+m)} \\ \frac{k_1 k_2}{m(\omega+m)} & 1 + \frac{k_2^2}{m(\omega+m)} & \frac{k_2 k_3}{m(\omega+m)} \\ \frac{k_1 k_3}{m(\omega+m)} & \frac{k_2 k_3}{m(\omega+m)} & 1 + \frac{k_3^2}{m(\omega+m)} \end{pmatrix}, \tag{56}$$

and the color fields are written as

$$A_\mu^1 + iA_\mu^2 = 2\varepsilon_\mu^{(1)} r(\xi) e^{i\theta(\xi)+i\phi_1},$$
$$A_\mu^4 - iA_\mu^5 = 2\varepsilon_\mu^{(2)} r(\xi) e^{i\theta(\xi)+i\phi_2},$$
$$A_\mu^6 + iA_\mu^7 = 2\varepsilon_\mu^{(3)} r(\xi) e^{i\theta(\xi)+i\phi_3} \tag{57}$$

for any selected L such that Eq. (52) is satisfied. The harmonic plane-wave solution is recovered for $r = 1/\sqrt{2}$ and $\theta = \omega t - \vec{k} \cdot \vec{x}$. Note that the solution (57) satisfies the Lorentz gauge condition $\partial_\mu A^{\mu a} = 0$ *in any frame* and this is a direct consequence of the ansatz (27) chosen in the proper frame. It may also be called "diagonal" in the sense that the complex SU(3) algebra roots are aligned with the three gluon polarization states. Global SU(3) transformations on the solution (57) are allowed since they do not spoil the Lorentz gauge condition. They rotate the solutions

$$\mathbf{A}_\mu \to g\mathbf{A}_\mu g^{-1} \tag{58}$$

[where g is a constant SU(3) matrix in the fundamental] and thus generate additional color fields in the Cartan subalgebra (T^3, T^8). The general field component A_μ^a is a linear superposition of the "diagonal" solution (57) with weights equal to the adjoint matrix elements $R^{ab}(g)$.

B. $L = 0$ solutions

Solutions satisfying $L = 0$ are also possible. In that case the angles θ_1, θ_2, θ_3 are necessarily constants.

(i) The case $\Psi_1 = \Psi_2 = \Psi_3 = 0$ does not lead to interesting (periodic) solutions.

(ii) The case $\Psi_1 \neq 0$, $\Psi_2 = \Psi_3 = 0$, from the "Group 2" equations, in order to avoid unbounded solutions for D_2 and D_3 also leads to $D_1 = D_2 = D_3 = 0$, and finally forbids any bounded solution.

(iii) The case $\Psi_1 \neq 0$, $\Psi_2 \neq 0$, $\Psi_3 = 0$, from the "Group 3" equations, leads to $D_1 = D_2 = 0$. This allows the following set of coupled equations for the remaining fields:

$$\ddot{r}_1 + r_1[r_2^2 + D_3^2] = 0,$$
$$\ddot{r}_2 + r_2[r_1^2 + D_3^2] = 0,$$
$$\ddot{D}_3 + 6D_3 r_1^2 = 0,$$
$$\ddot{D}_3 + 6D_3 r_2^2 = 0. \tag{59}$$

In general this is a chaotic system but the following integrable cases are included:

$$r_1 = r_2 = \text{cn}\left(\xi; \frac{1}{2}\right), \qquad D_3 = 0,$$
$$r_1 = r_2 = \frac{D_3}{\sqrt{5}} = \text{cn}\left(\sqrt{6}\xi; \frac{1}{2}\right). \tag{60}$$

(iv) The case $\Psi_1 \neq 0$, $\Psi_2 \neq 0$, $\Psi_3 \neq 0$ leads to the coupled system

$$\ddot{r}_1 + r_1[r_2^2 + r_3^2] = 0,$$
$$\ddot{r}_2 + r_2[r_1^2 + r_3^2] = 0,$$
$$\ddot{r}_3 + r_3[r_1^2 + r_2^2] = 0, \tag{61}$$

which in general presents chaotic behavior. Integrable cases are

$$r_1 = r_2 = r_3 = \text{cn}\left(\sqrt{2}\xi; \frac{1}{2}\right),$$
$$r_1 = r_2 = \text{cn}\left(\xi; \frac{1}{2}\right), \qquad r_3 = 0, \tag{62}$$

and the similar permutations.

[5]Note that $\varepsilon_\mu^{(\sigma)} \varepsilon^{\mu(\sigma')} = -\delta^{\sigma\sigma'}$ and $\varepsilon_\mu^{(\sigma)} \mathbb{k}^\mu = 0$ hold.

From the above analysis we conclude that the $L = 0$ solutions are less interesting and are always of the $\mathrm{cn}(\xi; 1/2)$ type which also solves the SU(2) theory.

V. OTHER GAUGE GROUPS

A. SU(2)

For SU(2) we solve Gauss' law via the following ansatz:

$$\vec{\mathbf{A}} = \frac{1}{2}\begin{pmatrix} D\hat{e}_3 & \Psi_1^*\hat{e}_1 + \Psi_2\hat{e}_2 \\ \Psi_1\hat{e}_1 + \Psi_2^*\hat{e}_2 & -D\hat{e}_3 \end{pmatrix}, \quad (63)$$

where D is a real function and Ψ_1, Ψ_2 are complex functions of ξ. Gauss' law is written as

$$G^a T^a = \vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{A}}} - \dot{\vec{\mathbf{A}}}\cdot\vec{\mathbf{A}} = \frac{i}{2}\begin{pmatrix} L_1 - L_2 & 0 \\ 0 & L_2 - L_1 \end{pmatrix}, \quad (64)$$

where we introduced

$$L_j = \frac{i}{2}(\Psi_j\dot{\Psi}_j^* - \Psi_j^*\dot{\Psi}_j), \quad j = 1, 2. \quad (65)$$

The implementation of Gauss' law requires $L_1 = L_2 = L$. The EoMs for the ansatz (64) are

$$\begin{aligned} \ddot{D} + D[|\Psi_1|^2 + |\Psi_2|^2] &= 0, \\ \ddot{\Psi}_1 + \Psi_1 D^2 + \frac{1}{2}\Psi_2^*[\Psi_1\Psi_2 - \Psi_1^*\Psi_2^*] &= 0, \\ \ddot{\Psi}_2 + \Psi_2 D^2 + \frac{1}{2}\Psi_1^*[\Psi_1\Psi_2 - \Psi_1^*\Psi_2^*] &= 0. \end{aligned} \quad (66)$$

The above equations do not admit harmonic plane-wave solutions. Assuming the ansatz $\Psi_1 = c_1 e^{i\omega_1\xi}$, $\Psi_2 = c_2 e^{i\omega_2\xi}$ with constants c_1, c_2, ω_1, ω_2 and enforcing Gauss' law $L_1 = L_2$, it is easily shown that such solutions are not allowed. The only integrable case of Eq. (66) appears for $\Psi_1 = i\Psi_2 = D$, which is none other than the original solution in Ref. [3].

B. SU(4)

For the case of SU(4) gauge theory we propose the following ansatz for the gauge potential which [similar to Eq. (27)] has by construction rows (columns) that are orthogonal to each other:

$$\vec{\mathbf{A}} = \begin{pmatrix} 0 & \Psi_1\hat{e}_1 & \Psi_2\hat{e}_2 & \Psi_3\hat{e}_3 \\ \Psi_1^*\hat{e}_1 & 0 & \Psi_6\hat{e}_3 & \Psi_5\hat{e}_2 \\ \Psi_2^*\hat{e}_2 & \Psi_6^*\hat{e}_3 & 0 & \Psi_4\hat{e}_1 \\ \Psi_3^*\hat{e}_3 & \Psi_5^*\hat{e}_2 & \Psi_4^*\hat{e}_1 & 0 \end{pmatrix}. \quad (67)$$

Gauss' law for the above ansatz is written as

$$\begin{aligned} G^a T^a &= \vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{A}}} - \dot{\vec{\mathbf{A}}}\cdot\vec{\mathbf{A}} \\ &= -2i\,\mathrm{diag}(L_1 + L_2 + L_3, -L_1 + L_5 + L_6, -L_2 \\ &\quad + L_4 - L_6, -L_3 - L_4 - L_5), \end{aligned} \quad (68)$$

where we introduced

$$L_j = \frac{i}{2}(\Psi_j\dot{\Psi}_j^* - \Psi_j^*\dot{\Psi}_j), \quad j = 1, 2, \ldots, 6. \quad (69)$$

The implementation of Gauss' law requires

$$\begin{aligned} L_1 + L_2 + L_3 = 0, &\qquad -L_1 + L_5 + L_6 = 0, \\ -L_2 + L_4 - L_6 = 0, &\qquad -L_3 - L_4 - L_5 = 0. \end{aligned} \quad (70)$$

The EoMs contain cubic terms of the fields Ψ_i and a general solution goes beyond the scope of this work. Here, we simply note that SU(3) solutions can be readily immersed in SU(4) in four different ways:

$$\begin{aligned} \Psi_1 = \Psi_2 = \Psi_3 = 0, &\qquad \Psi_1 = \Psi_5 = \Psi_6 = 0, \\ \Psi_2 = \Psi_4 = \Psi_6 = 0, &\qquad \Psi_3 = \Psi_4 = \Psi_5 = 0. \end{aligned} \quad (71)$$

For each of the choices, the remaining three fields are identified with the SU(3) solution, Eq. (57). Thus, plane waves (and particularly harmonic waves) can also propagate in the SU(4) theory, but are non-Abelian to the SU(3) subspaces.

VI. CONCLUSIONS AND OUTLOOK

We have presented plane-wave solutions for the SU(3) Yang-Mills EoM in $3 + 1$ dimensions. The solutions represent massive nonlinear plane waves, for arbitrary values of the mass parameter (reflecting the scale invariance of the action) and obeying the relativistic dispersion relation. The solution is designed by aligning the non-diagonal color fields with the polarization states of the massive vector boson. Via a global SU(3) transformation all the octet fields participate in the solution. The functional form is described by the dynamics of a planar particle bounded with the r^4 potential. The periodic particle orbits on the plane are characterized from the value of the angular momentum L which is bounded (in scaled units) between zero and $1/2$. The value of the angular momentum is related to the elliptic modulus k of the Jacobi elliptic functions which describe the solution. Each rational number in the $(1/\sqrt{6}, 1/2)$ interval gives a solution, and at the edge of the interval, with $L = 1/2$, a harmonic massive plane-wave solution of the *interacting* YM theory is recovered. Compared to the SU(2) theory which possesses only the $k^2 = 1/2$ plane wave, the SU(3) theory presents a far richer spectrum of solutions, with k^2 obtaining an infinite, enumerable set of values densely covering the interval $0 \le k^2 \le 1/2$.

The coupling to fermions is straightforward for static quark matter in the Cartan (T^3, T^8) subalgebra. Gauss' law, Eq. (29), admits quark densities on the rhs if angular momenta values $L_1 \neq L_2 \neq L_3$ are used.

Plane-wave solutions, and in particular the harmonic ones, may be of use in quantization schemes or particular perturbative treatments of the quantum theory since they automatically incorporate a gluon mass. The impact of such configurations on the properties of the quantum theory is worth exploring. In addition, classical Minkowskian solutions may also be useful in the study of gluon radiation, or the thermodynamical properties near the phase transition where semiclassical configurations become relevant.

Finally, we note that higher-rank gauge groups admit a similar treatment. For the SU(4) theory, Gauss' law can be solved by following the same strategy. In the SU(3) subgroups the plane-wave solutions remain valid, while the issue of existence of more generic SU(4) solutions is left for future investigations.

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