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Dynamical spacetime symmetry

Benjamin C. Lovelady* and James T. Wheeler†

Department of Physics, Utah State University, 4415 Old Main Hill, Logan, Utah, 84322-4415, USA

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According to the Coleman-Mandula theorem, any gauge theory of gravity combined with an internal symmetry based on a Lie group must take the form of a direct product in order to be consistent with basic assumptions of quantum field theory. However, we show that an alternative gauging of a *simple* group can lead *dynamically* to a spacetime with compact internal symmetry. The biconformal gauging of the conformal symmetry of n-dimensional Euclidean space doubles the dimension to give a symplectic manifold. Examining one of the Lagrangian submanifolds in the flat case, we find that in addition to the expected $SO(n)$ connection and curvature, the solder form necessarily becomes Lorentzian. General coordinate invariance gives rise to an $SO(n-1,1)$ connection on the spacetime. The principal fiber bundle character of the original $SO(n)$ guarantees that the two symmetries enter as a direct product, in agreement with the Coleman-Mandula theorem.

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I. INTRODUCTION

The Coleman-Mandula theorem [1] and generalizations [2] show that, given certain assumptions likely true of a satisfactory quantum field theory, any unification of general relativity with internal symmetries based on Lie groups must take the form of a direct product of the Poincaré or conformal group with a compact internal symmetry group. By extending to a graded Lie algebra, supersymmetric theories escape this conclusion and give a nontrivial unification of gravity with the standard model interactions. In the present examination, we find an alternative to such unification, using a quotient of the conformal group of Euclidean space which doubles the original dimension to a symplectic manifold. We show that the solution of the field equations *dynamically* produces a Lorentzian metric on a Lagrangian submanifold. The class of orthonormal frame fields of this Lorentzian metric is invariant under $SO(n-1,1)$ and enters as a direct product with the $SO(n)$ symmetry of the original principal fiber bundle, satisfying the Coleman-Mandula theorem. Of course, while this approach satisfies the Coleman-Mandula theorem without supersymmetry, it does not preclude supersymmetric extension.

Our method uses the standard construction of a Cartan geometry [3], in which a principal fiber bundle is produced from the quotient of a Lie group by a Lie subgroup. The Lie subgroup provides the local symmetry over the quotient manifold. Group quotients allow group-theoretic insights into gravity models, and thus improve on the early work of Utiyama [4] and Kibble [5], who extended global Lorentz and Poincaré symmetries, respectively, to local symmetries. The group quotient method was first employed for gravity by Ne'eman and Regge [6,7] who used it as a systematic way to study general relativity and supergravity. Later, Ivanov and Niederle used the same method to develop a variety of gauge theories, including the biconformal gauging we use here [8,9], though the appropriate gravity field equations appear to arise from the curvature-linear action found by Wehner and Wheeler [10].

Starting from the generators of the conformal group of Euclidean n-space, we present the biconformal gauging (developed in [9–11] and presented concisely in [12]). Rather than continuing by introducing Cartan curvatures, we work directly with the homogeneous quotient manifold, using a known solution to the flat Maurer-Cartan equations. This solution is naturally expressed in terms of a pair of involute 1-forms which together span a 2n-dimensional symplectic manifold. The Killing metric induces a Lorentzian metric on one of the resulting Lagrangian submanifolds ([13], also see [12,14]), and we interpret this submanifold as spacetime.

In previous work studying these solutions [12,14], the $SO(4)$ connection was separated into an $SO(3,1)$ piece and additional fields. The additional fields, constructed from certain invariant scalar fields, were interpreted as contributions to dark matter and dark energy. The $SO(3,1)$ piece of their decomposition becomes compatible with the usual soldering of the base manifold to the fiber.

The novelty of our approach is to leave the $SO(4)$ symmetry intact and distinct from the Lorentzian symmetry that necessarily [13] develops for the metric. Because the gauging includes the $SO(n)$ symmetry of the original Euclidean space, the fiber symmetry includes its own $SO(n)$ connection. However, the solder form now fails to "solder", displaying instead a Lorentzian inner product and inducing the spacetime metric. General coordinate covariance induces local Lorentz symmetry on orthonormal frame fields, so the final model has both $SO(n)$ and

*benjamin.lovelady@aggiemail.usu.edu
†jim.wheeler@usu.edu

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$SO(n-1,1)$ symmetries and corresponding connections, in the direct product form required by the Coleman-Mandula theorem. We introduce adapted coordinates that enable us to clearly display the simultaneous presence of the two symmetries.

In a further new contribution to understanding these spaces, we include a discussion of the meaning of the full biconformal manifold. We show that the homogeneous solution permits identification of the full $2n$-dimensional space as the cotangent bundle of spacetime with orthogonal Lagrangian submanifolds, one of which may be made flat by conformal transformation.

While we deal with only the flat case here, we expect further examination to permit curvatures of both connections. For the $n=4$ case, the $SO(4)$ symmetry is naturally replaced by $SU(2)\times SU(2)$ and interpreted as a left-right symmetric electroweak model. Symmetry breaking of one of the $SU(2)$ groups should then give a grav-electroweak unification. Larger n could incorporate $SO(n)$ or $\mathrm{Spin}(n)$ GUT models.

II. BICONFORMAL GAUGING

A. Quotient manifold method

To develop the biconformal space, we use the quotient manifold method [3,6,7]. Starting with a Lie group $\mathcal{G}$, (in our case the conformal group, $\mathcal{C}$), we construct a principal fiber bundle by taking the quotient by a Lie subgroup, $\mathcal{H}$ [in our case the Euclidean Weyl group, $\mathcal{W}$, comprised of $SO(n)$ transformations and dilatations]. This subgroup becomes the local symmetry of the $2n$-dimensional quotient manifold.

Lie groups have natural connection 1-forms $\boldsymbol{\omega}^A$ dual to the group generators G_A, defined by the linear mapping $\boldsymbol{\omega}^A(G_B)=\delta^A_B$, with the coordinate bases being dual, $\langle\frac{\partial}{\partial x^\mu},\mathbf{d}x^\nu\rangle=\delta^\nu_\mu$. Rewriting the commutation relations of the generators

$$[G_A,G_B]=c_{AB}{}^{C}G_C, \tag{1}$$

where $c^A{}_{BC}$ are the group structure constants, in terms of these dual 1-forms, we arrive at the Maurer-Cartan structure equations,

$$\mathbf{d}\boldsymbol{\omega}^A=-\frac{1}{2}c^A{}_{BC}\boldsymbol{\omega}^B\wedge\boldsymbol{\omega}^C. \tag{2}$$

The integrability conditions of Eq. (2) follow from the Poincare lemma, $\mathbf{d}^2=0$, and exactly reproduce the Jacobi identity,

$$[G_A,[G_B,G_C]]+[G_B,[G_C,G_A]]+[G_C,[G_A,G_B]]=0$$

so that Eq. (2) is equivalent to the Lie algebra.

It is convenient to define $\mathbf{C}^A_B\equiv c^A{}_{CB}\boldsymbol{\omega}^C$ so the structure equation becomes $\mathbf{d}\boldsymbol{\omega}^A=\frac{1}{2}\boldsymbol{\omega}^C\wedge\mathbf{C}^A_C$.

The quotient of $\mathcal{G}$ by a Lie subgroup divides the Maurer-Cartan into horizontal and vertical parts. Thus, if we write the connection forms as $\tilde{\boldsymbol{\omega}}^A=(\tilde{\boldsymbol{\omega}}^a_{\mathcal{H}},\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}})$ with $\tilde{\boldsymbol{\omega}}^a_{\mathcal{H}}, a,b,c=1,\ldots K$ spanning $\mathcal{H}$ and $\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}}, m,n,p=1,\ldots N-K$ spanning the homogeneous quotient manifold, $\mathcal{M}^{N-K}=\mathcal{G}/\mathcal{H}$, then we have

$$\mathbf{C}^A_B=c^A{}_{pB}\boldsymbol{\omega}^p_{\mathcal{M}}+c^A{}_{bB}\boldsymbol{\omega}^b_{\mathcal{H}}$$

with the condition $c^m_{ab}=0$ guaranteeing the subgroup condition. The structure equations for $\tilde{\boldsymbol{\omega}}^a_{\mathcal{H}}$ and $\tilde{\boldsymbol{\omega}}^a_{\mathcal{M}}$ are

$$\mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\mathcal{H}}=-\frac{1}{2}\mathbf{C}^a_B\wedge\boldsymbol{\omega}^B$$
$$\mathbf{d}\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}}=-\frac{1}{2}\mathbf{C}^m_B\wedge\boldsymbol{\omega}^B.$$

Because $c^m_{ab}=0$, the second of these takes the special form

$$\begin{aligned}\mathbf{d}\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}}&=-\frac{1}{2}c^m{}_{np}\tilde{\boldsymbol{\omega}}^n_{\mathcal{M}}\wedge\tilde{\boldsymbol{\omega}}^p_{\mathcal{M}}-c^m{}_{nb}\tilde{\boldsymbol{\omega}}^n_{\mathcal{M}}\wedge\tilde{\boldsymbol{\omega}}^b_{\mathcal{H}}\\&=\left(-\frac{1}{2}c^m{}_{pn}\tilde{\boldsymbol{\omega}}^p_{\mathcal{M}}-c^m{}_{bn}\tilde{\boldsymbol{\omega}}^b_{\mathcal{H}}\right)\wedge\tilde{\boldsymbol{\omega}}^n_{\mathcal{M}}\end{aligned}$$

thereby placing $\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}}$ in involution. This involution means there exist submanifolds found by setting $\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}}=0$, and we recover the subgroup Lie structure equations of the fibers.

The final step in the construction of a Cartan geometry is to allow horizontal curvature 2-forms, $\Sigma^A=(\Sigma^a,\Omega^m)$. This changes the connection 1-forms, $(\tilde{\boldsymbol{\omega}}^a_{\mathcal{H}},\tilde{\boldsymbol{\omega}}^m_{\mathcal{M}})$ to a new connection, $(\boldsymbol{\omega}^a_{\mathcal{H}},\boldsymbol{\omega}^m_{\mathcal{M}})$, and we have the Cartan equations, $\mathbf{d}\boldsymbol{\omega}^A=-\frac{1}{2}c^A{}_{BC}\boldsymbol{\omega}^B\wedge\boldsymbol{\omega}^C+\Sigma^A$, or

$$\mathbf{d}\boldsymbol{\omega}^a_{\mathcal{H}}=-\frac{1}{2}\mathbf{C}^a_B\wedge\boldsymbol{\omega}^B+\Sigma^a$$
$$\mathbf{d}\boldsymbol{\omega}^m_{\mathcal{M}}=-\frac{1}{2}\mathbf{C}^m_B\wedge\boldsymbol{\omega}^B+\Omega^m.$$

Separating out the subgroup components,

$$\begin{aligned}\mathbf{d}\boldsymbol{\omega}^a_{\mathcal{H}}&=-\frac{1}{2}c^a{}_{bc}\boldsymbol{\omega}^b_{\mathcal{H}}\wedge\boldsymbol{\omega}^c_{\mathcal{H}}-\frac{1}{2}c^a{}_{mp}\boldsymbol{\omega}^m_{\mathcal{M}}\wedge\boldsymbol{\omega}^p_{\mathcal{M}}\\&\quad-c^a{}_{mb}\boldsymbol{\omega}^m_{\mathcal{M}}\wedge\boldsymbol{\omega}^b_{\mathcal{H}}+\Sigma^a\\\mathbf{d}\boldsymbol{\omega}^m_{\mathcal{M}}&=-\frac{1}{2}c^m{}_{np}\boldsymbol{\omega}^n_{\mathcal{M}}\wedge\boldsymbol{\omega}^p_{\mathcal{M}}-c^m{}_{nb}\boldsymbol{\omega}^n_{\mathcal{M}}\wedge\boldsymbol{\omega}^b_{\mathcal{H}}+\Omega^m.\end{aligned} \tag{3}$$

The curvature forms are tensorial under $\mathcal{H}$, and because they are horizontal, describe curvature of $\mathcal{M}$ only. Consistency requires the integrability of these equations, which is no longer guaranteed by the Jacobi identity. Integrability of $\mathbf{d}\boldsymbol{\omega}^A=-\frac{1}{2}c^A{}_{BC}\boldsymbol{\omega}^B\wedge\boldsymbol{\omega}^C+\Sigma^A$ requires

$$
\begin{aligned}
0 &\equiv \mathbf{d}^2\boldsymbol{\omega}^A \\
&= -\frac{1}{2}c^A_{\ BC}\mathbf{d}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C + \frac{1}{2}c^A_{\ BC}\boldsymbol{\omega}^B \wedge \mathbf{d}\boldsymbol{\omega}^C + \mathbf{d}\Sigma^A \\
&= -\frac{1}{2}c^A_{\ BC}\left(-\frac{1}{2}c^B_{\ \mathrm{DE}}\boldsymbol{\omega}^D \wedge \boldsymbol{\omega}^E + \Sigma^B\right) \wedge \boldsymbol{\omega}^C + \frac{1}{2}c^A_{\ BC}\boldsymbol{\omega}^B \wedge \left(-\frac{1}{2}c^C_{\ \mathrm{DE}}\boldsymbol{\omega}^D \wedge \boldsymbol{\omega}^E + \Sigma^C\right) + \mathbf{d}\Sigma^A \\
&= \frac{1}{2}c^A_{\ B[C}c^B_{\ DE]}\boldsymbol{\omega}^C \wedge \boldsymbol{\omega}^D \wedge \boldsymbol{\omega}^E + \mathbf{d}\Sigma^A - \frac{1}{2}c^A_{\ BC}\Sigma^B \wedge \boldsymbol{\omega}^C + \frac{1}{2}c^A_{\ BC}\boldsymbol{\omega}^B \wedge \Sigma^C \\
&= \mathbf{d}\Sigma^A + c^A_{\ BC}\boldsymbol{\omega}^B \wedge \Sigma^C \\
&\equiv \mathbf{D}\Sigma^A
\end{aligned}
$$

where we use the Jacobi identity, $c^A_{\ B[C}c^B_{\ DE]} \equiv 0$, and define the covariant exterior derivative. The result holds for both Σ^a and Ω^a, leaving us with

$$
\begin{aligned}
\mathbf{D}\Sigma^a &= 0 \\
\mathbf{D}\Omega^a &= 0.
\end{aligned}
$$

These are the Bianchi identities in gravitational models.

A locally $\mathcal{H}$-invariant physical theory is now found by writing any scalar Lagrange density built from the tensors available from the construction of this principal bundle, Ω^a, Σ^m, along with tensors from the original linear representation. The nonvertical basis forms $\boldsymbol{\omega}^m_{\mathcal{M}}$ also become tensors because the curvature breaks the corresponding symmetries.

B. Conformal structure equations and the Cartan geometry

Applying the quotient method to the conformal group of a compactified space with flat metric, η_{ab}, of signature (p, q), we express the generators as

$$
\begin{aligned}
M^a_{\ b} &= \frac{1}{2}(x^a\partial_b - \eta^{ac}\eta_{bd}x^d\partial_c) \\
&\equiv \Delta^{ac}_{db}x^d\partial_c \\
P_a &= \partial_a \\
K^a &= \frac{1}{2}(x^2\eta^{ab}\partial_b - 2x^a x^c\partial_c) \\
D &= x^c\partial_c
\end{aligned}
$$

where the A index on the generators G_A includes all possible antisymmetric pairs for the group, i.e., $A \in \{\binom{a}{b}, \binom{\cdot}{a}, \binom{a}{\cdot}, \binom{\cdot}{\cdot}\}$ and $\Delta^{ac}_{db} \equiv \frac{1}{2}(\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd})$ is the antisymmetric projection operator on $\binom{1}{1}$ tensors. The operators $M^a_{\ b}$, P_a, K^a, D generate $SO(p, q)$ transformations, translations, special conformal transformations, and dilatations, respectively.

The commutators of the generators then give the Lie algebra,

$$
\begin{aligned}
[M^a_{\ b}, M^c_{\ d}] &= \frac{1}{2}[\delta^c_b\delta^a_e\delta^f_d + \eta_{bd}\delta^c_e\eta^{fa} + \eta^{ac}\eta_{ed}\delta^f_b \\
&\quad + \delta^a_d\eta_{be}\eta^{fc}]M^e_{\ f} \\
[M^a_{\ b}, P_c] &= \Delta^{ad}_{cb}P_d \\
[M^a_{\ b}, K^c] &= -\Delta^{ac}_{db}K^d \\
[P_a, K^b] &= 2\Delta^{bd}_{ca}M^c_{\ d} - \delta^b_a D \\
[D, P_a] &= -\delta^c_a P_c \\
[D, K^a] &= \delta^a_c K^c.
\end{aligned}
$$

Then we introduce basis 1-forms dual to the generators

$$
\begin{aligned}
\langle M^a_{\ b}, \tilde{\boldsymbol{\omega}}^c_{\ d}\rangle &= 2\Delta^{ac}_{db} \\
\langle P_a, \tilde{\boldsymbol{\omega}}^b\rangle &= \delta^b_a \\
\langle K^a, \tilde{\boldsymbol{\omega}}_b\rangle &= \delta^a_b \\
\langle D, \tilde{\boldsymbol{\omega}}\rangle &= 1.
\end{aligned}
$$

Notice that index position corresponds to conformal weight, so $\tilde{\boldsymbol{\omega}}^a$ and $\tilde{\boldsymbol{\omega}}_a$ are distinct fields. These lead directly to the Maurer-Cartan structure equations

$$
\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\ b} &= \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + 2\Delta^{ad}_{cb}\tilde{\boldsymbol{\omega}}_d \wedge \tilde{\boldsymbol{\omega}}^c \\
\mathbf{d}\tilde{\boldsymbol{\omega}}^a &= \tilde{\boldsymbol{\omega}}^c \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + \tilde{\boldsymbol{\omega}} \wedge \tilde{\boldsymbol{\omega}}^a \\
\mathbf{d}\tilde{\boldsymbol{\omega}}_a &= \tilde{\boldsymbol{\omega}}^c_{\ a} \wedge \tilde{\boldsymbol{\omega}}_c + \tilde{\boldsymbol{\omega}}_a \wedge \tilde{\boldsymbol{\omega}} \\
\mathbf{d}\tilde{\boldsymbol{\omega}} &= \tilde{\boldsymbol{\omega}}^c \wedge \tilde{\boldsymbol{\omega}}_c.
\end{aligned} \tag{4}
$$

With the quotient $\mathcal{C}/\mathcal{W}$, these equations describe a $2n$-dimensional homogeneous manifold spanned by $\tilde{\boldsymbol{\omega}}^c$ and $\tilde{\boldsymbol{\omega}}_c$. Notice that $\mathbf{d}\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}}^c \wedge \tilde{\boldsymbol{\omega}}_c$ is a symplectic form since it is manifestly both closed and nondegenerate.

To complete the construction of a Cartan geometry, these connection forms are now generalized, $(\tilde{\boldsymbol{\omega}}^a_{\ b}, \tilde{\boldsymbol{\omega}}^a, \tilde{\boldsymbol{\omega}}_a, \tilde{\boldsymbol{\omega}}) \to (\boldsymbol{\omega}^a_{\ b}, \boldsymbol{\omega}^a, \boldsymbol{\omega}_a, \boldsymbol{\omega})$, to permit horizontal curvature 2-forms,

$$\mathbf{d}\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + 2\Delta^{ad}_{cb}\boldsymbol{\omega}_d \wedge \boldsymbol{\omega}^c + \Omega^a{}_b$$
$$\mathbf{d}\boldsymbol{\omega}^a = \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\omega} \wedge \boldsymbol{\omega}^a + \mathbf{T}^a$$
$$\mathbf{d}\boldsymbol{\omega}_a = \boldsymbol{\omega}^c{}_a \wedge \boldsymbol{\omega}_c + \boldsymbol{\omega}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a$$
$$\mathbf{d}\boldsymbol{\omega} = \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}_c + \Omega. \tag{5}$$

The final step in producing a gravity model is to write the most general action functional linear in the curvatures [10],

$$S = \int (\alpha\Omega^a{}_b + \beta\delta^a_b\Omega + \gamma\boldsymbol{\omega}^a \wedge \boldsymbol{\omega}_b)\varepsilon_{ac\ldots d}{}^{be\ldots f}\boldsymbol{\omega}^d \wedge \ldots \wedge \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}_e \wedge \ldots \wedge \boldsymbol{\omega}_f.$$

We will be concerned only with the solution to the homogeneous geometry, Eqs. (4), but the complete $\mathbf{T}^a = 0$ class of solutions has been shown to reduce to locally scale-invariant general relativity on a Lagrangian submanifold [15].

For the rest of our discussion, we return to the homogeneous manifold described by Eqs. (4). An extended derivation, similar to that in [10,15] but straightforward to verify by direct substitution, gives the general solution to these structure equations up to gauge and coordinate choices. The solution takes the form first given in [11], but a coordinate choice removes the function $\alpha_a(x)$ present there. Here we use the symmetry between $\tilde{\boldsymbol{\omega}}^a$ and $\tilde{\boldsymbol{\omega}}_a$ to recast that general solution as

$$\tilde{\boldsymbol{\omega}}^a{}_b = -2\Delta^{ac}_{db}x^d\mathbf{d}y_c \tag{6}$$

$$\tilde{\boldsymbol{\omega}}^a = \mathbf{d}x^a - \left(x^a x^b - \frac{1}{2}\eta^{ab}x^2\right)\mathbf{d}y_b \tag{7}$$

$$\tilde{\boldsymbol{\omega}}_a = \mathbf{d}y_a \tag{8}$$

$$\tilde{\boldsymbol{\omega}} = x^c\mathbf{d}y_c \tag{9}$$

where $x^2 \equiv \eta_{ab}x^a x^b$. For convenience, we define

$$b^{ab} \equiv 2x^a x^b - \eta^{ab}x^2 \tag{10}$$

which simplifies Eq. (8) to $\tilde{\boldsymbol{\omega}}^a = \mathbf{d}x^a - \frac{1}{2}b^{ab}\mathbf{d}y_b$.

Our Euclidean starting point gives us a specific model within which we can explicitly verify and more clearly understand the signature theorem of [13]. Note that when $\eta_{ab} = \delta_{ab}$ is Euclidean, b^{ab} has Lorentzian signature

$$b^{ab} = -|x^2|\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

but other initial signatures (p, q) for η_{ab} do not give consistent signature for b_{ab}. Specifically, for non-Euclidean η_{ab}, if x^a is timelike $(x^2 < 0)$, a transformation to $x^a = +\sqrt{|x^2|}(1, 0, \ldots, 0)$ gives signature $(p', q') = (p+1, q-1)$ for b_{ab}, while if x^a is spacelike $x^2 > 0$, boosting and rotating to $x^a = +\sqrt{|x^2|}(0, 1, 0, 0)$ at any given point gives b_{ab} signature $(p'', q'') = (q+1, p-1)$. Equating $(p', q') = (p'', q'')$ requires $p = q$. Therefore, b_{ab} only has a consistent signature for all x^a if η_{ab} is Euclidean or if $p = q$ and in the Euclidean case b^{ab} is necessarily Lorentzian. This is in agreement with the signature theorem of [13]. For the remainder of this investigation, we will take $\eta_{ab} = \delta_{ab} = \mathrm{diag}(1, \ldots, 1)$ so that $b^{ab} = 2x^a x^b - \delta^{ab}x^2$ is Lorentzian for all x^a.

We show in the next section that b_{ab} is the restriction of the Killing metric to certain Lagrangian submanifolds. The argument above then shows how the uniqueness of the signature theorem occurs. If the initial signature differs from Euclidean or $p = q$, the restriction of the Killing metric to the Lagrangian submanifolds is degenerate.

C. The Killing form

The Killing metric of the biconformal manifold is the restriction of the conformal Killing form $K_{AB} = tr(G_A G_B)$ (equal to $c^C{}_{AD}c^D{}_{BC}$ in the adjoint representation) to the quotient manifold. The nondegeneracy of K_{AB} allows us to define the inner product

$$\langle\boldsymbol{\omega}^A, \boldsymbol{\omega}^B\rangle \equiv K^{AB} \tag{11}$$

in either the curved $(\boldsymbol{\omega}^A)$ or the homogeneous $(\tilde{\boldsymbol{\omega}}^A)$ case. Restricting K^{AB} to its form on the quotient manifold, we have

$$\langle\boldsymbol{\omega}^a, \boldsymbol{\omega}^b\rangle = 0$$
$$\langle\boldsymbol{\omega}^a, \boldsymbol{\omega}_b\rangle = \langle\boldsymbol{\omega}_b, \boldsymbol{\omega}^a\rangle = \delta^a_b$$
$$\langle\boldsymbol{\omega}_a, \boldsymbol{\omega}_b\rangle = 0 \tag{12}$$

which is readily seen to be nondegenerate. This means that, unlike the Poincaré case, the group Killing form provides a metric on the quotient manifold.

It is useful to express this inner product in terms of the x^a and y_a coordinates as well. Substituting the solution of Eqs. (6)–(9) into the Killing metric, Eq. (12), we find the inner product of the coordinate basis forms to be

$$\langle\mathbf{d}x^a, \mathbf{d}x^b\rangle = b^{ab}$$
$$\langle\mathbf{d}x^a, \mathbf{d}y_b\rangle = \langle\mathbf{d}y_b, \mathbf{d}x^a\rangle = \delta^a_b$$
$$\langle\mathbf{d}y_a, \mathbf{d}y_b\rangle = 0.$$

In the next section, we show that there is a Lagrangian submanifold of flat $SO(n)$ biconformal space with natural Lorentzian signature and both $SO(n)$ and spacetime connections.

III. AN $SO(n)$ FIELD ON SPACETIME

We now come to our principal results: the simultaneous presence on a Lagrangian submanifold of an $SO(n)$ connection and its Yang-Mills field strength, together with Lorentzian metric, connection and curvature. The result is achieved by using a different involution from that of [12,14].

From the Maurer-Cartan equations, Eqs. (4), we see that both basis forms, $\tilde{\boldsymbol{\omega}}^a$ and separately, $\tilde{\boldsymbol{\omega}}_a$, are in involution. This implies the existence of complementary submanifolds, and the vanishing of the symplectic form when one or the other basis form vanishes shows that the submanifolds are Lagrangian. In this section, we use a general solution to the flat structure equations to develop the Killing metric on both the $\tilde{\boldsymbol{\omega}}_a = 0$ submanifold and the $\tilde{\boldsymbol{\omega}}^a = 0$ submanifold and explore properties of the solution, showing the presence of an $SO(n)$ Yang-Mills field on spacetime from the elements of the solution.

A. The structure equations on the spacetime submanifold

Consider the involution of the basis forms, $\tilde{\boldsymbol{\omega}}^a$. This means that there exist coordinates, found in the next section, such that $\tilde{\boldsymbol{\omega}}^a = \alpha^a{}_b \mathbf{d}u^b$, and holding u^a constant sets $\tilde{\boldsymbol{\omega}}^a = 0$ and selects a submanifold spanned by the remaining basis forms. Setting $\tilde{\boldsymbol{\omega}}^a = 0$, Eq. (7) shows that $\mathbf{d}y_a = 2b_{ab}\mathbf{d}x^b$ and the structure equations reduce to

$$\begin{aligned}
\tilde{\boldsymbol{\omega}}^a{}_b &= -\frac{4}{x^2}\Delta^{ac}_{db} x_c \mathbf{d}x^d \\
\tilde{\boldsymbol{\omega}}_a &= 2b_{ab}\mathbf{d}x^b \\
\tilde{\boldsymbol{\omega}} &= \mathbf{d}(\ln x^2).
\end{aligned}$$

These describe what we will call the spacetime submanifold; it is a Lagrangian submanifold since the symplectic form $\mathbf{d}\tilde{\boldsymbol{\omega}} = \mathbf{d}^2(\ln x^2) \equiv 0$ vanishes.

B. Symmetries

The inner products of the restricted basis forms $\tilde{\boldsymbol{\omega}}_a$ are now

$$\begin{aligned}
\langle \tilde{\boldsymbol{\omega}}_a, \tilde{\boldsymbol{\omega}}_b \rangle &= \langle 2b_{ac}\mathbf{d}x^c, 2b_{bd}\mathbf{d}x^d \rangle \\
&= 4b_{ac}b_{bd}\langle \mathbf{d}x^c, \mathbf{d}x^d \rangle \\
&= 4b_{ac}b_{bd}b^{cd} \\
&= 4b_{ab}
\end{aligned}$$

so the basis and coordinate differentials specify a Lorentzian inner product.

Unlike previous treatments, this metric symmetry is different from the symmetry of the connection, which is constructed from $SO(n)$ invariants including the antisymmetric projection Δ^{ac}_{db}. Since

$$\begin{aligned}
\Delta^{ac}_{db}\tilde{\boldsymbol{\omega}}^d_c &= -\frac{4}{x^2}\Delta^{ac}_{db}\Delta^{de}_{fc}\delta_{eg}x^g\mathbf{d}x^f \\
&= -\frac{4}{x^2}\Delta^{ae}_{fb}\delta_{eg}x^g\mathbf{d}x^f \\
&= \tilde{\boldsymbol{\omega}}^a{}_b
\end{aligned}$$

the infinitesimal change in the Euclidean metric δ_{ab} produced by $\tilde{\boldsymbol{\omega}}^a{}_b$ vanishes:

$$\delta_{ac}\tilde{\boldsymbol{\omega}}^c{}_b + \delta_{bc}\tilde{\boldsymbol{\omega}}^c{}_a = 0.$$

Therefore, $\tilde{\boldsymbol{\omega}}^a{}_b$ is a generator of $SO(n)$.

Evidently, though the biconformal bundle as a whole has $SO(n)$ fiber symmetry, and this symmetry together with dilatations preserve the structure equations, the symmetry of the metric restricted to this submanifold is Lorentzian. We develop this further as follows.

Let the $SO(n)$ gauge be fixed, and consider coordinate transformations. The vielbein, $\tilde{\boldsymbol{\omega}}_a$, gives rise to a Lorentzian metric:

$$\langle \tilde{\boldsymbol{\omega}}_a, \tilde{\boldsymbol{\omega}}_b \rangle = 4b_{ab} \tag{13}$$

as do the coordinates

$$\langle \mathbf{d}x^\alpha, \mathbf{d}x^\beta \rangle = b^{\alpha\beta}.$$

Given a metric manifold we may construct the frame bundle, $\mathbf{B}(\pi, G, \mathcal{M})$. There then exists the subbundle of orthonormal frames [16], $\mathbf{O}(\pi, SO(n-1,1), \mathcal{M})$, a principal fiber bundle with Lorentz symmetry group. Therefore, implicit in the set of general coordinate transformations, we have all Lorentz transformations of the corresponding orthonormal frame fields. In this way, we generate a representation of $SO(n-1,1)$ which is clearly independent of the $SO(n)$ fiber symmetry. The bundle structure guarantees that these symmetries are independent, satisfying the Coleman-Mandula theorem. We regard the combined bundle as having symmetry $SO(n) \times SO(n-1,1) \times SO(1,1)$.

We define the Christoffel connection, $\Gamma^\alpha{}_{\mu\nu}$, and Riemann curvature, $R^\alpha{}_{\beta\mu\nu}$, in the usual way, from the metric $b_{\alpha\beta}$ and without reference to the $SO(n)$ bundle symmetry. The orthogonal bundle symmetry has its own connection, $\tilde{\boldsymbol{\omega}}^a{}_b$, and corresponding Yang-Mills field,

$$\mathbf{F}^a{}_b = \mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b - \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c.$$

This development within the solution of a Lorentzian metric is in accordance with the signature theorem proved in [13] and developed further in [12,14].

We now compute the spacetime curvature and the Yang-Mills field strength of this model.

C. Spacetime curvature of the Lagrangian submanifold

The inverse spacetime metric is

$$\langle \mathbf{d}x^\alpha, \mathbf{d}x^\beta \rangle = b^{\alpha\beta} = 2x^\alpha x^\beta - \delta^{\alpha\beta} x^2.$$

In these coordinates, $\delta^{\alpha\beta} = \mathrm{diag}(1, \ldots, 1)$. Inverting,

$$g_{\alpha\beta} = \frac{1}{(x^2)^2}(2x_\alpha x_\beta - \delta_{\alpha\beta} x^2).$$

The Christoffel connection is readily found to be,

$$\Gamma^\alpha_{\ \beta\mu} = \frac{1}{x^2}(x^\alpha \delta_{\beta\mu} - \delta^\alpha_\beta x_\mu - \delta^\alpha_\mu x_\beta)$$

and the curvature is

$$\begin{aligned} R^\alpha_{\ \beta\mu\nu} &= \Gamma^\alpha_{\ \beta\nu,\mu} - \Gamma^\alpha_{\ \beta\mu,\nu} - \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \beta\mu} + \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \beta\nu} \\ &= \frac{1}{x^2}\left(\left(\delta^\alpha_\mu - \frac{1}{x^2}x^\alpha x_\mu\right)\left(\delta_{\beta\nu} - \frac{1}{x^2}x_\beta x_\nu\right) \right. \\ &\quad \left. - \left(\delta^\alpha_\nu - \frac{1}{x^2}x^\alpha x_\nu\right)\left(\delta_{\beta\mu} - \frac{1}{x^2}x_\beta x_\mu\right)\right). \end{aligned}$$

Defining the projection operator $P^\alpha_\mu \equiv (\delta^\alpha_\mu - \frac{1}{x^2}x^\alpha x_\mu)$, orthogonal to x^α, we may write this as

$$R^\alpha_{\ \beta\mu\nu} = \frac{1}{x^2}(P^\alpha_{\ \mu}P_{\beta\nu} - P^\alpha_{\ \nu}P_{\beta\mu}).$$

D. The Yang-Mills field strength

Finally, we compute the Yang-Mills field strength,

$$\mathbf{F}^a_{\ b} = \mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\ b} - \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c}$$

where the $SO(n)$ connection is given by

$$\tilde{\boldsymbol{\omega}}^a_{\ b} = -\frac{4}{x^2}\Delta^{ac}_{db} x_c \mathbf{d}x^d.$$

We may find this directly from the $SO(n)$ structure equation,

$$\mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\ b} = \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + 2\Delta^{ad}_{cb}\tilde{\boldsymbol{\omega}}_d \wedge \tilde{\boldsymbol{\omega}}^c$$

which shows immediately that

$$\mathbf{F}^a_{\ b} = -2\Delta^{ad}_{cb}\tilde{\boldsymbol{\omega}}_d \wedge \tilde{\boldsymbol{\omega}}^c|_{\tilde{\boldsymbol{\omega}}^c = 0} = 0.$$

The Yang-Mills field strength therefore vanishes on the spacetime Lagrangian submanifold.

We easily check this directly. Substituting the submanifold form, $-\frac{4}{x^2}\Delta^{ac}_{db}x_c\mathbf{d}x^d$ leads after some algebra to $\mathbf{F}^a_{\ b} = 0$ (see Appendix A). Naturally, this will change when curved biconformal spaces are considered.

We end the section by finding a set of coordinates adapted to the involution of $\tilde{\boldsymbol{\omega}}^a$.

E. Adapted coordinates for the involution of the translational gauge fields

The involution of $\boldsymbol{\omega}^a$ means that we may write $\boldsymbol{\omega}^a = \alpha^\alpha_{\ \beta}\mathbf{d}u^\beta$ for n coordinates u^α. To find $\alpha^\alpha_{\ \beta}$ and u^β, we need to solve

$$\alpha^\alpha_{\ \beta}\mathbf{d}u^\beta = \mathbf{d}x^\alpha - \frac{1}{2}b^{\alpha\beta}\mathbf{d}y_\beta. \tag{14}$$

With the metric $b_{\alpha\beta}$ given by

$$b_{\alpha\beta} = \frac{1}{(x^2)^2}(2x_\alpha x_\beta - \delta_{\alpha\beta}x^2)$$

where we define $x_\alpha \equiv \delta_{\alpha\beta}x^\beta$, we rewrite Eq. (14) as $b_{\alpha\mu}\alpha^\mu_{\ \beta}\mathbf{d}u^\beta = b_{\alpha\mu}\mathbf{d}x^\mu - \frac{1}{2}\mathbf{d}y_\alpha$ and note that $b_{\alpha\mu}\mathbf{d}x^\mu = -\delta_{\alpha\mu}\mathbf{d}(\frac{x^\mu}{x^2})$. Therefore,

$$2b_{\alpha\mu}\alpha^\mu_{\ \beta}\mathbf{d}u^\beta = -\delta_{\alpha\mu}\mathbf{d}\left(\frac{2x^\mu}{x^2} + \delta^{\mu\beta}y_\beta\right),$$

and we may identify

$$u^\alpha \equiv \frac{2x^\alpha}{x^2} + \delta^{\alpha\beta}y_\beta$$

and require

$$2b_{\alpha\mu}\alpha^\mu_{\ \beta} = \delta_{\alpha\beta}.$$

Inverting the metric on this expression now shows that

$$\alpha^\mu_{\ \beta} = x^\alpha x_\beta - \frac{1}{2}\delta^\alpha_\beta x^2.$$

In terms of the adapted coordinates u^α, we may solve for and replace x^α,

$$x^\alpha(u^\alpha, y_\beta) = \frac{2(u^\alpha - \delta^{\alpha\beta}y_\beta)}{u^2 - 2u^\alpha y_\alpha + y^2}$$

so that the full structure equations take the form

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^a_{\ b} &= -2\Delta^{ac}_{db}\delta^d_\mu\delta^\nu_c x^\mu(u^\alpha, y_\beta)\mathbf{d}y_\nu \\ \tilde{\boldsymbol{\omega}}^a &= \delta^a_\beta \mathbf{d}u^\beta \\ \tilde{\boldsymbol{\omega}}_a &= \delta^\beta_a \mathbf{d}y_\beta \\ \tilde{\boldsymbol{\omega}} &= x^\mu(u^\alpha, y_\beta)\mathbf{d}y_\mu, \end{aligned} \tag{15}$$

and we note that the second involution, for $\tilde{\boldsymbol{\omega}}_a$, is simultaneously expressed in adapted coordinates y_α. It is straightforward to check both directly and in terms of the (x^α, y_β) inner products that $\langle \mathbf{d}u^\alpha, \mathbf{d}u^\beta \rangle = 0$.

IV. PHASE SPACE

The full biconformal space is a symplectic manifold, but unlike our usual basis for phase spaces, the inner product of the $(\tilde{\boldsymbol{\omega}}^a, \tilde{\boldsymbol{\omega}}_a)$ basis is off diagonal, Eq. (12). To make a full connection between the background biconformal manifold and a one-particle phase space, we introduce a further change of basis.

A. An orthogonal basis of Lagrangian submanifolds

Having both metric and symplectic form, we may find a subspace orthogonal to the spacetime manifold. To find it, we introduce new coordinates z_a such that $\langle \mathbf{d}z_a, \mathbf{d}x^b \rangle = 0$. Making the ansatz $\mathbf{d}z_a = \xi_a{}^b \mathbf{d}y_b + \Xi_{ab}\mathbf{d}x^b$, we solve for $\xi_a{}^b$ and Ξ_{ab}

$$\begin{aligned} 0 &= \langle \mathbf{d}z_a, \mathbf{d}x^c \rangle \\ &= \xi_a{}^b \delta_b^c + \Xi_{ab} b^{bc}. \end{aligned}$$

This is satisfied if $\xi_a^c = \delta_a^c$ and $\Xi_{ab} = -b_{ab}$, giving

$$\mathbf{d}z_a = \mathbf{d}y_a - b_{ab}\mathbf{d}x^b$$

for the differential of the coordinates.

These subspaces are better aligned with our usual notion of a single particle phase space. In addition to the inner product of the spacetime coordinate differentials $\langle \mathbf{d}x^a, \mathbf{d}x^b \rangle = b^{ab}$, and orthogonality of the spacetime and momentum directions, we have a metric on the momentum space as well,

$$\begin{aligned} \langle \mathbf{d}z_a, \mathbf{d}z_b \rangle &= \langle \mathbf{d}y_a - b_{ac}\mathbf{d}x^c, \mathbf{d}y_b - b_{bd}\mathbf{d}x^d \rangle \\ &= -b_{ab}. \end{aligned}$$

In terms of the (x^α, z_β) coordinates, the solution for the connection is

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^a{}_b &= -2\Delta^{ac}_{db} x^d (\mathbf{d}z_c + b_{ce}\mathbf{d}x^e) \\ \tilde{\boldsymbol{\omega}}^a &= \frac{1}{2}(\mathbf{d}x^a - b^{ab}\mathbf{d}z_b) \\ \tilde{\boldsymbol{\omega}}_a &= \mathbf{d}z_a + b_{ac}\mathbf{d}x^c \\ \tilde{\boldsymbol{\omega}} &= x^a \mathbf{d}z_a + \frac{1}{x^2} x_a \mathbf{d}x^a. \end{aligned}$$

However, in order for (x^α, z_β) coordinates to fully match our expectations for canonically conjugate variables, spacetime and momentum space must be Lagrangian submanifolds. Therefore the differentials $\mathbf{d}x^\alpha$ and $\mathbf{d}z_\alpha$ need to be integrable. In general, this is not the case. If we substitute $\mathbf{d}x^a \to \boldsymbol{\kappa}^a$ and $\mathbf{d}z_a \to \boldsymbol{\lambda}^a$ in the structure equations and solve for $\mathbf{d}\boldsymbol{\kappa}^a$ and $\mathbf{d}\boldsymbol{\lambda}^a$, they are not involute. For $\mathbf{d}\boldsymbol{\kappa}^a$, for example, we find

$$\begin{aligned} 2\mathbf{d}\boldsymbol{\kappa}^a &= \boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}}^a{}_c + \boldsymbol{\kappa}^a \wedge \tilde{\boldsymbol{\omega}} - b^{ac}\mathbf{d}b_{cb}\boldsymbol{\kappa}^b + \tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\kappa}^a \\ &\quad + b^{ae}\tilde{\boldsymbol{\omega}}^c{}_e \wedge b_{cb}\boldsymbol{\kappa}^b - b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c - \tilde{\boldsymbol{\omega}} \wedge b^{ab}\boldsymbol{\lambda}_b \\ &\quad + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b + b^{ab}\tilde{\boldsymbol{\omega}}^c{}_b \wedge \boldsymbol{\lambda}_c + b^{ab}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}. \end{aligned}$$

Therefore, in order for $\mathbf{d}x^a$ and $\mathbf{d}z_a$ to span a pair of Lagrangian submanifolds, we require the $\boldsymbol{\lambda}_a \wedge \boldsymbol{\lambda}_b$ terms to vanish. Equivalently, we must have

$$\begin{aligned} \Lambda^a \equiv& -b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c - \tilde{\boldsymbol{\omega}} \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b + b^{ab}\tilde{\boldsymbol{\omega}}^c{}_b \wedge \boldsymbol{\lambda}_c \\ &+ b^{ab}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}} \end{aligned}$$

linear in $\boldsymbol{\kappa}^a$, with a similar condition $\Sigma^a \sim \boldsymbol{\lambda}^a$ arising for the involution of $\mathbf{d}\boldsymbol{\lambda}_b$.

As we show in detail in Appendix B, these conditions *do* hold for the homogeneous solution above—substituting the form of the connection into Λ^a and Σ^a so in the model considered (x^α, z_β) do characterize Lagrangian submanifolds.

Restricting to the constant z_α submanifold, the solution is

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^a{}_b &= -2\Delta^{ac}_{db} x^d b_{ce}\mathbf{d}x^e \\ \tilde{\boldsymbol{\omega}}^a &= \frac{1}{2}\mathbf{d}x^a \\ \tilde{\boldsymbol{\omega}} &= \frac{1}{x^2} x_a \mathbf{d}x^a = \frac{1}{2}\mathbf{d}\ln x^2 \end{aligned}$$

with metric $4b_{ab}$ and $SO(4)$ connection and $\tilde{\boldsymbol{\omega}}_a = 2b_{ac}\tilde{\boldsymbol{\omega}}^c$, as before. If instead, we hold x^α constant, then the solution is

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^a{}_b &= -2\Delta^{ac}_{db} x_0^d \mathbf{d}z_c \\ \tilde{\boldsymbol{\omega}}_a &= \mathbf{d}z_a \\ \tilde{\boldsymbol{\omega}} &= x_0^a \mathbf{d}z_a \end{aligned}$$

with $\tilde{\boldsymbol{\omega}}^a = -\frac{1}{2}b^{ab}\mathbf{d}z_b$, *constant* metric $-b^{ab}(x_0^c)$ and constant $SO(4)$ connection.

While the constancy of the $SO(4)$ connection does not imply vanishing of the Yang-Mills field, it may be gauged to zero by a conformal transformation. The constancy of the metric, $SO(4)$ connection, and Weyl vector show the momentum space to be a Lagrangian submanifold with vanishing curvature and vanishing Yang-Mills field. It is therefore consistent to identify the entire biconformal manifold with the cotangent bundle.

It is unclear which of these properties survive in curved biconformal spaces. Torsion-free biconformal spaces are known to be fully determined by the spacetime solder form and Weyl vector with $\boldsymbol{\omega}_a$ spanning the cotangent spaces, but the solutions apply to different submanifolds than we consider in this section. Still, it is conceivable that the cotangent interpretation is always possible.

It is amusing to speculate on the meaning of the momentum space if it is found to be curved in some solutions. If so, it might provide a novel approach to canonical field theory by allowing canonically conjugate fields to coexist on what is essentially a particle phase space. Expressing a relativistic field theory on a particle phase space has had only measured success. Born [17] suggested introducing a curved momentum space to complement gravitating spacetime, but with no clear indication of what would determine its curvature. In the nonrelativistic case, the Wigner distribution extends the wave function to a distribution on phase space, but it is not obvious how this generalizes to the relativistic case. Typically, phase space for field theory employs a natural symplectic structure on field space, but what occurs here seems to exist midway between the particle and field cases. The symplectic base manifold allows fields to acquire a momentum component automatically, and the field equations determine the structure of the entire phase space, apparently restricting these fields so that the only independent degrees of freedom are those from the spacetime Lagrangian submanifold.

V. CONCLUSION

The Coleman-Mandula theorem shows that unifying gauge theories that include gravity and are based on Lie groups require a direct product between the internal and gravitational symmetries. This seemed to be an unnatural starting point for a unified theory and led to an increased emphasis on supersymmetric theories. These avoid being direct products by allowing graded Lie groups, which in turn give improved quantum convergence and useful restrictions on possible models.

In the present work, we show that it is possible to write a unified theory as a gauge theory of a simple Lie group which dynamically enforces the Coleman-Mandula theorem. Although the starting point is the conformal group $SO(n-1,1)$ of Euclidean n-space, the general solution of the Maurer-Cartan structure equations shows that the connection retains its original $SO(n)$ symmetry but the Killing metric (which is nondegenerate in these models) restricts to a Lorentzian signature on certain Lagrangian submanifolds.

We outlined the quotient manifold method and applied it to the biconformal gauging of the conformal group. By starting from the generators, we constructed the Maurer-Cartan structure equations for the conformal group. The known solution to these equations was introduced, though we choose to use different coordinates better adapted to the Lagrangian submanifolds.

Our principal contribution was to identify a complementary pair of Lagrangian submanifolds on which the connection (of the principal fiber bundle) is orthogonal $[SO(n)]$ while the restriction of the Killing metric is Lorentzian. The Lorentzian metric allows calculation of its Christoffel connection and curvature, while the $SO(n)$ connection gives rise to a Yang-Mills field (trivial, since we only consider the flat case). The Riemannian curvature of the submanifolds was found to be constructible from projection operators orthogonal to the time direction. Notice that the time direction emerges dynamically in accordance with the signature theorem of [13] and further developed in [12].

By the definition of a fiber bundle, the $SO(n)$ fibers are in a direct product with the $SO(n-1,1)$ symmetry of the base manifold. This direct product relation naturally satisfies the Coleman-Mandula theorem.

We showed that the full biconformal space may be given the structure of a cotangent bundle to the curved spacetime.

Preliminary work with curved biconformal spaces suggests that these results extend to those cases as well, though the computations pose some interesting challenges. Since these constructions depend only on the Lie algebra they will work in the same way with spinor representations. There is no obstruction to considering supersymmetric generalizations [18], which among other benefits provide a principled way of introducing spinor fields into bosonic field theories.

APPENDIX A: VANISHING OF THE SUBMANIFOLD YANG-MILLS FIELD

The $SO(n)$ Yang-Mills field strength is

$$\mathbf{F}^a{}_b = -\Delta^{ad}_{cb} b^{cf}(\mathbf{d}z_d \wedge \mathbf{d}z_f + 2b_{de}\mathbf{d}x^e \wedge \mathbf{d}z_f),$$

which vanishes on the $z_a = \text{constant}$ Lagrangian submanifold. To verify this explicitly, we substitute the submanifold form of the connection, $\tilde{\boldsymbol{\omega}}^a{}_b|_{z_a=z_a^0} = -\frac{4}{x^2}\Delta^{ac}_{db}x_c\mathbf{d}x^d$ into $\mathbf{F}^a{}_b = \mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b - \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c$:

$$
\begin{aligned}
\mathbf{F}^a{}_b &= \mathbf{d}\left(-\frac{4}{x^2}\Delta^{ac}_{db}x_c\mathbf{d}x^d\right) - \frac{4}{x^2}\Delta^{ce}_{db}x_e\mathbf{d}x^d \wedge \frac{4}{x^2}\Delta^{af}_{gc}x_f\mathbf{d}x^g \\
&= \frac{8}{(x^2)^2}\Delta^{ac}_{db}x_e x_c\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{x^2}\Delta^{ac}_{db}\mathbf{d}x_c \wedge \mathbf{d}x^d - \frac{16}{(x^2)^2}\Delta^{ce}_{db}\Delta^{af}_{gc}x_e x_f\mathbf{d}x^d \wedge \mathbf{d}x^g \\
&= \frac{8}{(x^2)^2}\Delta^{ac}_{db}x_e x_c\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{x^2}\Delta^{ac}_{db}\delta_{ce}\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}(\delta^c_d\delta^e_b - \delta^{ce}\delta_{bd})(\delta^a_g\delta^f_c - \delta^{af}\delta_{gc})x_e x_f\mathbf{d}x^d \wedge \mathbf{d}x^g \\
&= \frac{8}{(x^2)^2}\Delta^{ac}_{db}\left(x_e x_c - \frac{1}{2}x^2\delta_{ce}\right)\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}(\delta^a_e x_b x_d + \delta_{bd}x_e x^a - x^2\delta_{bd}\delta^a_e)\mathbf{d}x^d \wedge \mathbf{d}x^e \\
&= \frac{8}{(x^2)^2}\Delta^{ac}_{db}\left(x_e x_c - \frac{1}{2}x^2\delta_{ce}\right)\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}\left(\left(x_b x_d - \frac{1}{2}x^2\delta_{bd}\right)\delta^a_e + \left(x^a x_e - \frac{1}{2}x^2\delta^a_e\right)\delta_{bd}\right)\mathbf{d}x^d \wedge \mathbf{d}x^e \\
&= \frac{8}{(x^2)^2}\Delta^{ac}_{db}\left(x_e x_c - \frac{1}{2}x^2\delta_{ce}\right)\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}\left(x_c x_d - \frac{1}{2}x^2\delta_{cd}\right)(\delta^c_b\delta^a_e - \delta^{ac}\delta_{be})\mathbf{d}x^d \wedge \mathbf{d}x^e \\
&= \frac{4}{(x^2)^2}\Delta^{ac}_{db}(2x_e x_c - x^2\delta_{ce})\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}\Delta^{ac}_{eb}(2x_c x_d - x^2\delta_{cd})\mathbf{d}x^d \wedge \mathbf{d}x^e \\
&= \frac{4}{(x^2)^2}\Delta^{ac}_{db}(2x_e x_c - x^2\delta_{ce})\mathbf{d}x^e \wedge \mathbf{d}x^d - \frac{4}{(x^2)^2}\Delta^{ac}_{db}(2x_c x_e - x^2\delta_{cd})\mathbf{d}x^e \wedge \mathbf{d}x^d \\
&= 0.
\end{aligned}
$$

APPENDIX B: INVOLUTION OF THE ORTHOGONAL SUBSPACES

We find the involution conditions for the $(\mathbf{d}x^a, \mathbf{d}u_b)$ basis and show that they are satisfied by the homogeneous solution.

The $(\mathbf{d}x^a, \mathbf{d}u_b)$ basis is related to the original basis by

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^a &= \frac{1}{2}(\mathbf{d}x^a - b^{ab}\mathbf{d}u_b) \\
\tilde{\boldsymbol{\omega}}_a &= \mathbf{d}u_a + b_{ab}\mathbf{d}x^b.
\end{aligned}
$$

If we write this instead in the form of an alternative basis,

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^a &= \frac{1}{2}(\boldsymbol{\kappa}^a - b^{ab}\boldsymbol{\lambda}_b) \\
\tilde{\boldsymbol{\omega}}_a &= \boldsymbol{\lambda}_a + b_{ab}\boldsymbol{\kappa}^b
\end{aligned}
$$

and substitute into the original structure equations,

$$
\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^a &= \tilde{\boldsymbol{\omega}}^c \wedge \tilde{\boldsymbol{\omega}}^a{}_c + \tilde{\boldsymbol{\omega}} \wedge \tilde{\boldsymbol{\omega}}^a \\
\mathbf{d}\tilde{\boldsymbol{\omega}}_a &= \tilde{\boldsymbol{\omega}}^c{}_a \wedge \tilde{\boldsymbol{\omega}}_c + \tilde{\boldsymbol{\omega}}_a \wedge \tilde{\boldsymbol{\omega}}
\end{aligned}
$$

we find the structure equations for $\boldsymbol{\kappa}^a$ and $\boldsymbol{\lambda}^a$,

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\kappa}^a &= \frac{1}{2}(\boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}}^a{}_c + \boldsymbol{\kappa}^a \wedge \tilde{\boldsymbol{\omega}} - b^{ac}\mathbf{d}b_{cb}\boldsymbol{\kappa}^b + \tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\kappa}^a + b^{ae}\tilde{\boldsymbol{\omega}}^c{}_e \wedge b_{cb}\boldsymbol{\kappa}^b) \\
&\quad + \frac{1}{2}(-b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c - \tilde{\boldsymbol{\omega}} \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b + b^{ab}\tilde{\boldsymbol{\omega}}^c{}_b \wedge \boldsymbol{\lambda}_c + b^{ab}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}) \\
\mathbf{d}\boldsymbol{\lambda}_a &= \frac{1}{2}(b_{ad}b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^d{}_c + \tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\lambda}_a - b_{ac}\mathbf{d}b^{cb}\boldsymbol{\lambda}_b + \tilde{\boldsymbol{\omega}}^c{}_a \wedge \boldsymbol{\lambda}_c + \boldsymbol{\lambda}_a \wedge \tilde{\boldsymbol{\omega}}) \\
&\quad + \frac{1}{2}(\tilde{\boldsymbol{\omega}}^c{}_a \wedge b_{cb}\boldsymbol{\kappa}^b + b_{ac}\boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}} - \mathbf{d}b_{ab}\boldsymbol{\kappa}^b - b_{ab}\boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}}^b{}_c - b_{ab}\tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\kappa}^b).
\end{aligned}
$$

Therefore, involution of the new basis requires

$$
\begin{aligned}
\Lambda^a \equiv {}& -b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c - \tilde{\boldsymbol{\omega}} \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b + b^{ab}\tilde{\boldsymbol{\omega}}^c{}_b \wedge \boldsymbol{\lambda}_c \\
&+ b^{ab}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}
\end{aligned}
$$

to be proportional to $\boldsymbol{\kappa}^a$ and

$$
\begin{aligned}
\Sigma^a \equiv {}& b^{ae}\tilde{\boldsymbol{\omega}}^c{}_e \wedge b_{cb}\boldsymbol{\kappa}^b + \boldsymbol{\kappa}^a \wedge \tilde{\boldsymbol{\omega}} - b^{ac}\mathbf{d}b_{cb}\boldsymbol{\kappa}^b - \boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\
&- \tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\kappa}^a
\end{aligned}
$$

to be proportional to $\boldsymbol{\lambda}^a$.

The involution does not generically hold, but it only needs to hold for the solution. Expressing the solution in terms of $(\boldsymbol{\kappa}^a, \boldsymbol{\lambda}_b)$,

$$\tilde{\boldsymbol{\omega}}^a{}_b = -2\Delta^{ac}_{db}\left(x^d\mathbf{d}u_c - b_{cf}x^d\mathbf{d}x^f + \frac{2}{x^2}x_c\mathbf{d}x^d\right)$$
$$= -2\Delta^{ac}_{db}\left(x^d\boldsymbol{\lambda}_c - b_{cf}x^d\boldsymbol{\kappa}^f + \frac{2}{x^2}x_c\boldsymbol{\kappa}^d\right)$$
$$\tilde{\boldsymbol{\omega}}^a = \frac{1}{2}(\boldsymbol{\kappa}^a - b^{ab}\boldsymbol{\lambda}_b)$$
$$\tilde{\boldsymbol{\omega}}_a = \boldsymbol{\lambda}_a + b_{ab}\boldsymbol{\kappa}^b$$
$$\tilde{\boldsymbol{\omega}} = x^a(\boldsymbol{\lambda}_a - b_{ab}\boldsymbol{\kappa}^b) + \mathbf{d}(\ln x^2)$$

substitution into Λ^a gives

$$\Lambda^a \equiv b^{ab}\tilde{\boldsymbol{\omega}}^c{}_b \wedge \boldsymbol{\lambda}_c - b^{cb}\boldsymbol{\lambda}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c - 2\tilde{\boldsymbol{\omega}} \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b$$
$$= b^{ab}\left(-2\Delta^{ce}_{db}\left(x^d\boldsymbol{\lambda}_e - b_{ef}x^d\boldsymbol{\kappa}^f + \frac{2}{x^2}x_e\boldsymbol{\kappa}^d\right)\right) \wedge \boldsymbol{\lambda}_c$$
$$- b^{cb}\boldsymbol{\lambda}_b \wedge \left(-2\Delta^{ae}_{dc}\left(x^d\boldsymbol{\lambda}_e - b_{ef}x^d\boldsymbol{\kappa}^f + \frac{2}{x^2}x_e\boldsymbol{\kappa}^d\right)\right)$$
$$- 2(x^e(\boldsymbol{\lambda}_e - b_{ef}\boldsymbol{\kappa}^f) + \mathbf{d}(\ln x^2)) \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b.$$

Dropping the irrelevant $\boldsymbol{\kappa}^a$ terms (since only $\boldsymbol{\lambda}_a \wedge \boldsymbol{\lambda}_b$ terms violate the involution),

$$\Lambda^a \cong b^{ab}(-2\Delta^{ce}_{db}x^d\boldsymbol{\lambda}_e) \wedge \boldsymbol{\lambda}_c + b^{cb}\boldsymbol{\lambda}_b \wedge 2\Delta^{ae}_{dc}(x^d\boldsymbol{\lambda}_e)$$
$$- 2(x^e\boldsymbol{\lambda}_e + \mathbf{d}(\ln x^2)) \wedge b^{ab}\boldsymbol{\lambda}_b + \mathbf{d}b^{ab}\boldsymbol{\lambda}_b.$$

Also, since b_{ab} depends only on x^a, the derivatives $\mathbf{d}b_{ab}$ depend only on $\boldsymbol{\kappa}^a$ and may be dropped,

$$\Lambda^a \cong (b^{bc}2\Delta^{ae}_{db}x^d + b^{ab}2\Delta^{ce}_{db}x^d + 2x^e b^{ac})\boldsymbol{\lambda}_c \wedge \boldsymbol{\lambda}_e$$
$$= (b^{bc}(\delta^a_d\delta^e_b - \delta^{ae}\delta_{bd})x^d + b^{ab}(\delta^c_d\delta^e_b - \delta^{ce}\delta_{bd})x^d$$
$$+ 2x^e b^{ac})\boldsymbol{\lambda}_c \wedge \boldsymbol{\lambda}_e$$
$$= (b^{ac}x^e - x_b b^{bc}\delta^{ae})\boldsymbol{\lambda}_c \wedge \boldsymbol{\lambda}_e$$
$$= (2x^a x^c x^e - x^2(\delta^{ac}x^e + \delta^{ae}x^c))\boldsymbol{\lambda}_c \wedge \boldsymbol{\lambda}_e$$
$$= 0.$$

Therefore, $\boldsymbol{\kappa}^a$ is in involution for this solution.

For the involution of $\boldsymbol{\lambda}^a$ we require Σ^a to be linear in $\boldsymbol{\lambda}^a$. In fact, it vanishes identically:

$$\Sigma^a \equiv b^{ae}\tilde{\boldsymbol{\omega}}^c{}_e \wedge b_{cb}\boldsymbol{\kappa}^b + \boldsymbol{\kappa}^a \wedge \tilde{\boldsymbol{\omega}} - b^{ac}\mathbf{d}b_{cb}\boldsymbol{\kappa}^b - \boldsymbol{\kappa}^c \wedge \tilde{\boldsymbol{\omega}}^a{}_c - \tilde{\boldsymbol{\omega}} \wedge \boldsymbol{\kappa}^a$$
$$= b^{ae}\left(-2\Delta^{cg}_{de}\left(-b_{gf}x^d\boldsymbol{\kappa}^f + \frac{2}{x^2}x_g\boldsymbol{\kappa}^d\right)\right) \wedge b_{cb}\boldsymbol{\kappa}^b - \boldsymbol{\kappa}^c \wedge \left(-2\Delta^{ab}_{dc}\left(-b_{bf}x^d\boldsymbol{\kappa}^f + \frac{2}{x^2}x_b\boldsymbol{\kappa}^d\right)\right)$$
$$+ 2\boldsymbol{\kappa}^a \wedge (x^a(-b_{ab}\boldsymbol{\kappa}^b) + \mathbf{d}(\ln x^2)) - b^{ac}\mathbf{d}b_{cb}\boldsymbol{\kappa}^b$$
$$= \frac{1}{(x^2)^2}(x^2\delta^a_f x_b - x^a(4x_f x_b - 2x_b x_f - 2x_b x_f + x^2\delta_{bf}) + 4x^a x_b x_f - 2x^2\delta^a_f x_b)\boldsymbol{\kappa}^f \wedge \boldsymbol{\kappa}^b$$
$$+ \frac{1}{(x^2)^2}(2x^a x_c x_f - \delta^a_f x_c x^2)\boldsymbol{\kappa}^c \wedge \boldsymbol{\kappa}^f + \frac{2}{x^2}x_c\boldsymbol{\kappa}^c \wedge \boldsymbol{\kappa}^a - 2\frac{1}{(x^2)^2}x_b x^2\boldsymbol{\kappa}^a \wedge \boldsymbol{\kappa}^b + \frac{2}{x^2}2x_c\boldsymbol{\kappa}^a \wedge \mathbf{d}x^c$$
$$+ \frac{1}{(x^2)^2}(2x^2x_b\delta^a_c + 2x^a 2x_b x_c - 2x^a\delta_{bc}x^2 - 4x_b x_c x^a + 2\delta^a_b x_c x^2)\mathbf{d}x^c\boldsymbol{\kappa}^b$$
$$= -\frac{1}{x^2}\delta^a_f x_c\boldsymbol{\kappa}^f \wedge \boldsymbol{\kappa}^c + \frac{1}{x^2}\delta^a_f x_c\boldsymbol{\kappa}^f \wedge \boldsymbol{\kappa}^c + \frac{2}{x^2}x_c\boldsymbol{\kappa}^c \wedge \boldsymbol{\kappa}^a + \frac{2}{x^2}2x_c\boldsymbol{\kappa}^a \wedge \mathbf{d}x^c - 2\frac{1}{(x^2)^2}x_b x^2\boldsymbol{\kappa}^a \wedge \boldsymbol{\kappa}^b$$
$$+ \frac{1}{(x^2)^2}(2x^2x_b\delta^a_c + 2x^a 2x_b x_c - 2x^a\delta_{bc}x^2 - 4x_b x_c x^a + 2\delta^a_b x_c x^2)\mathbf{d}x^c\boldsymbol{\kappa}^b$$
$$= \frac{1}{x^2}(2\delta^a_c x_b + 2\delta^a_b x_c)\boldsymbol{\kappa}^c\boldsymbol{\kappa}^b$$
$$= 0$$

where we use $\mathbf{d}x^b = \boldsymbol{\kappa}^b$ in the last step.

The description of the spacetime and momentum submanifolds follow from the solution,

$$\tilde{\boldsymbol{\omega}}^a{}_b = -2\Delta^{ac}_{db}\left(x^d\mathbf{d}u_c - b_{cf}x^d\mathbf{d}x^f + \frac{2}{x^2}x_c\mathbf{d}x^d\right)$$
$$\tilde{\boldsymbol{\omega}}^a = \frac{1}{2}(\mathbf{d}x^a - b^{ab}\mathbf{d}u_b)$$
$$\tilde{\boldsymbol{\omega}}_a = \mathbf{d}u_a + b_{ab}\mathbf{d}x^b$$
$$\tilde{\boldsymbol{\omega}} = x^a(\mathbf{d}u_a - b_{ab}\mathbf{d}x^b) + \mathbf{d}(\ln x^2).$$

When we hold u_a constant

$$\tilde{\boldsymbol{\omega}}^a{}_b = 2\Delta^{ac}_{db}\left(b_{cf}x^d\mathbf{d}x^f - \frac{2}{x^2}x_c\mathbf{d}x^d\right)$$
$$\tilde{\boldsymbol{\omega}}^a = \frac{1}{2}\mathbf{d}x^a$$
$$\tilde{\boldsymbol{\omega}} = \frac{1}{2}\mathbf{d}(\ln x^2)$$

with $\tilde{\boldsymbol{\omega}}_a = b_{ab}\mathbf{d}x^b$. The submanifold is clearly Lagrangian, the connection $SO(4)$, and inner product of the basis forms is Lorentzian,

$$\langle \tilde{\boldsymbol{\omega}}^a, \tilde{\boldsymbol{\omega}}^b \rangle = \frac{1}{4} \langle \mathbf{d}x^a, \mathbf{d}x^b \rangle = \frac{1}{4} b^{ab}.$$

On the momentum submanifold, with $x^a = x_0^a$ constant,

$$\tilde{\boldsymbol{\omega}}^a{}_b = -2\Delta^{ac}_{db} x_0^d \mathbf{d}u_c$$
$$\tilde{\boldsymbol{\omega}}_a = \mathbf{d}u_a$$
$$\tilde{\boldsymbol{\omega}} = x_0^a \mathbf{d}u_a$$

where $\tilde{\boldsymbol{\omega}}^a = -\frac{1}{2} b^{ab}(x_0^\alpha) \mathbf{d}u_b$. This is again Lagrangian with inner product

$$\langle \tilde{\boldsymbol{\omega}}_a, \tilde{\boldsymbol{\omega}}_b \rangle = \langle \mathbf{d}u_a, \mathbf{d}u_b \rangle = b_{ab}(x_0^\alpha).$$

Since the metric is now constant, the momentum submanifold is flat. The Yang-Mills field strength has the form

$$\mathbf{F}^a{}_b = \mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b - \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c = -4\Delta^{ec}_{db}\Delta^{af}_{ge} x_0^d x_0^g \mathbf{d}u_c \wedge \mathbf{d}u_f = -\Delta^{ac}_{bd} b_0^{df} \mathbf{d}u_c \wedge \mathbf{d}u_f,$$

which is of the form of a pure rescaling and can therefore, be made to vanish by a conformal transformation, $\phi = -x_0^\alpha u_\alpha$.

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