Smarr formula for Lovelock black holes: A Lagrangian approach

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The mass formula for black holes can be formally expressed in terms of a Noether charge surface integral plus a suitable volume integral, for any gravitational theory. The integrals can be constructed as an application of Wald's formalism. We apply this formalism to compute the mass and the Smarr formula for static Lovelock black holes. Finally, we propose a new prescription for Wald's entropy in the case of Lovelock black holes, which takes into account topological contributions to the entropy functional.

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I. INTRODUCTION

The Smarr formula (SF) expresses the mass of a black hole in terms of its geometrical and dynamical parameters (angular momentum, electromagnetic potential, area, etc.), and it was first derived in the context of general relativity. For vacuum GR solutions the SF has a geometrical interpretation: it is equivalent to the Komar integral, a boundary surface integral of the covariant derivative of a Killing vector field. The question naturally raises if it is possible to find a similar geometrical interpretation of the SF for arbitrary theories of gravitation, i.e., a generalization of Komar's construction to other theories than GR.

Progress in this direction has been made in recent years: in particular, Kastor *et al.* [1] have shown that, for the particular class of Lovelock theories, it is indeed possible to construct a surface integral generalizing the Komar one.

If one weakens the requirement that Komar integral be a pure surface integral, and allows for volume integral contributions, then a strong result holds: any diffeomorphism invariant theory of gravity admits a geometrical identity—we call it the Smarr identity (SI)—which reduces to the Komar one in the GR case. In this paper, we show that the volume term reduces for Lovelock theories to a surface one, and the SI reproduces the Smarr formula found in [1]. We will discuss extension beyond Lovelock theory in the conclusion of the paper.

We proceed as follows: in Sec. II, we review Wald's derivation of BH entropy and illustrate the Smarr identity; in Sec. III, we compute the mass and the SF for static vacuum black hole solutions in Lovelock theories; in Sec. IV, we discuss the role of topological contributions to the Smarr formula; Sec. V contains an overview of the results with concluding remarks.

II. THE SMARR IDENTITY

In [2], the first law of black hole mechanics is derived for diffeomorfism invariant theories, by making use of the conserved Noether current associated with a special vector field. The BH entropy is then identified as a geometric functional of the Noether potential: this is the main result of Wald's construction, that we briefly review.

Some comments are in order. The derivation makes certain nontrivial assumptions on the spacetime geometry: in particular one starts with a stationary spacetime with an internal boundary, identified with the future event horizon of a single black hole. In GR, it is proved that the event horizon of a stationary BH is a Killing horizon generated by a Killing field ξ^a; although there is no generalization of the proof to higher curvature theories, all the known solutions are of this kind. Therefore, we restrict our attention to black holes whose event horizon is Killing, and we assume that its generators can be regularly extended in both directions. Hypersurface orthogonality ensures that ξ^a is tangent to nonaffinely parametrized geodesics, whose inaffinity κ is defined by

$$\xi^a \nabla_a \xi_b = \kappa \xi_b. \tag{1}$$

If $\kappa \neq 0$, one can show that (i) κ is constant over the horizon and (ii) the horizon contains a spacelike $(D-2)$-dimensional surface where ξ^a vanishes, called the "bifurcation surface" B. In the following, we will assume this to be always the case.

Apart from time translations, the spacetime will admit other possible spatial symmetries: we specialize to the case of rotational symmetries generated by a set of vector fields $\{\psi_i^a\}$ collectively denoted by $\vec{\psi}^a$. The Killing field ξ^a can then be expressed as

$$\xi^a = t^a + \vec{\Omega} \cdot \vec{\psi}^a, \tag{2}$$

where Ω_i is called the "angular velocity" of the horizon around the ith axis.

Given this preliminary setup, let us review Wald's derivation. Consider a collection of dynamical fields in D spacetime dimensions, collectively denoted by ϕ, including a metric tensor g_{ab} plus other possible matter fields, whose dynamics is determined by a Lagrangian D-form $\mathbb{L} = \mathcal{L}\epsilon$, with ϵ the spacetime volume element.

Under a generic variation $\delta\phi$ of the fields, the variation of $\mathbb{L}$ can be expressed as a sum of a bulk term plus a boundary one,

$$\delta\mathbb{L} = \mathbb{E}_\phi \delta\phi + d\Theta(\phi, \delta\phi), \tag{3}$$

where the $(D-1)$-form Θ is locally constructed out of ϕ and $\delta\phi$. From (3) we read that the equations of motion (EOM) are $\mathbb{E}_\phi \doteq 0$ for each ϕ.[1]

In particular, one can consider infinitesimal variations along a vector field ξ, $\delta\phi = \pounds_\xi\phi$. By diffeomorphism invariance, to any vector field ξ there corresponds a Noether current $(D-1)$-form

$$\mathbb{J}[\xi] = \Theta(\phi, \pounds_\xi\phi) - \xi\cdot\mathbb{L}, \tag{4}$$

which is conserved on shell:

$$d\mathbb{J}[\xi] = d\Theta(\phi, \pounds_\xi\phi) - d\xi\cdot\mathbb{L} = -\mathbb{E}_\phi\pounds_\xi\phi \doteq 0. \tag{5}$$

The conservation of $\mathbb{J}$ implies the existence of a $(D-2)$-form $\mathbb{Q}[\xi]$ [3]

$$\mathbb{J}[\xi] \doteq d\mathbb{Q}[\xi] \tag{6}$$

called the "Noether potential" associated to ξ.

$\mathbb{Q}$ enters in the definition of the conserved charges [4]: indeed the Hamiltonian variation, associated with the flow of ξ, over an initial value surface Σ with boundary $\partial\Sigma$, is given by [2]

$$\delta H[\xi] = \int_{\partial\Sigma} [\delta\mathbb{Q}[\xi] - \xi\cdot\Theta(\phi, \delta\phi)]; \tag{7}$$

it is then natural to identify the variations of the energy E and the angular momentum $\vec{J}$ at infinity as[2]

$$\delta E = \int_{S_\infty} [\delta\mathbb{Q}[t] - t\cdot\Theta(\phi, \delta\phi)], \tag{8}$$

$$\delta\vec{J} = -\int_{S_\infty} [\delta\mathbb{Q}[\vec{\psi}] - \vec{\psi}\cdot\Theta(\phi, \delta\phi)] = -\int_{S_\infty} \delta\mathbb{Q}[\vec{\psi}], \tag{9}$$

where S_∞ is the outer boundary of $\partial\Sigma$, and the last equality of (9) follows from the fact that $\vec{\psi}$ is tangential to S_∞. (Notice that, as usual, the angular charges are defined up to a conventional minus sign.). If there is a $D-2$ form $\mathbb{B}(\phi)$ such that $\int \xi\cdot\Theta(\phi, \delta\phi) = \delta\int \xi\cdot\mathbb{B}(\phi)$, one defines the conserved Hamiltonian charge as

$$H[\xi] = \int_{S_\infty} \mathbb{Q}[\xi] - \xi\cdot\mathbb{B}; \tag{10}$$

in particular the angular momentum is exactly the Noether charge at infinity, modulo a sign:

$$\vec{J} = -\int_{S_\infty} \mathbb{Q}[\vec{\psi}]. \tag{11}$$

If the field ξ is taken to be the Killing field (2) generating the horizon, then Eq. (7) implies the first law of black hole mechanics: let (i) ξ be a dynamical symmetry, meaning that $\pounds_\xi\phi \doteq 0$ for all the ϕ's, and (ii) $\delta\phi$ be a variation of the dynamical fields around the BH solution, such that $\delta\phi$ solves the linearized EOM; then $\delta H[\xi] \doteq 0$, from which it follows [2,5]

$$\delta E \doteq \frac{\kappa}{2\pi}\delta S + \vec{\Omega}\cdot\delta\vec{J}, \tag{12}$$

where S is $2\pi/\kappa$ times the integral of $\mathbb{Q}$ over the bifurcation surface:

$$S = \frac{2\pi}{\kappa}\int_B \mathbb{Q}[\xi], \tag{13}$$

and Eq. (12) is obtained by the vanishing of the integral (7) over an initial value surface with boundary $\partial\Sigma = S_\infty \cup B$, with B the bifurcation surface of the black hole. Since $\kappa/2\pi$ is the Hawking temperature, one interprets S as the thermodynamical entropy of the BH.[3]

Notice that, in order for Eq. (12) to hold, the variation $\delta\phi$ doesn't need to satisfy the same exact symmetries of the background solution ϕ: in particular, $\delta\phi$ can be nonstationary and not satisfying $\pounds_\xi\delta\phi = 0$.[4]

Finally, it is worth noting that for a general gravitational Lagrangian Eq. (13) can be expressed as [5]

$$S = -2\pi\int_B E_R^{abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\bar{\epsilon}, \qquad E_R^{abcd} = \frac{\delta\mathcal{L}}{\delta R_{abcd}}, \tag{14}$$

where $\bar{\epsilon}$ is the area element of B and $\hat{\epsilon}_{ab}$ is the binormal to B.

As shown in [6], the integral (13) needs not to be evaluated at the bifurcation surface, since it gives the correct entropy on any other cross section of the horizon. The proof makes use of the fact that, being ξ a dynamical symmetry, Eq. (4) becomes

[1]From now on, the dot indicates equalities holding on shell.

[2]δE contains also work term contributions from long range fields, such as gauge fields.

[3]Note, however, that this identification fails if the dynamical fields have divergent components at the bifurcation surface. This circumstance occurs, for example, in the case of gauge fields, but one can see that in this case the divergences at the horizon can be gauged out by an appropriate gauge fixing, thus recovering the correct expression for the entropy.

[4]In contrast, the first law derived in the next section requires exact Killing symmetries over all spacetime.

$$\mathbb{J}[\xi] + \xi \cdot \mathbb{L} \doteq 0, \tag{15}$$

provided $\Theta(\phi, \delta\phi)$ vanishes when $\delta\phi = 0$. Indeed, the authors of [5] suggest an algoritm giving a preferred "canonical" Θ_0, among all the possible Θ's, which is covariant, depends linearly on $\delta\phi$ and vanishes if $\delta\phi = 0$. However, the definition of theta suffers of the ambiguity associated to the freedom of adding a closed form $\Theta \to \Theta + d\alpha$ which, in principle, can spoil the above properties: we follow the authors of [5,6] and restrict only to those α's preserving the mentioned properties of Θ. Equation (15) is then ensured. Integration over Σ then gives

$$\oint_{\partial\Sigma} \mathbb{Q}[\xi] + \int_{\Sigma} \xi \cdot \mathbb{L} \doteq 0. \tag{16}$$

By linearity of $\mathbb{Q}$ with respect to ξ, using Eqs. (11) and (13), we obtain

$$\oint_{S_\infty} \mathbb{Q}[t] \doteq TS + \vec{\Omega} \cdot \vec{J} - \int_{\Sigma} \xi \cdot \mathbb{L} \quad \text{(Smarr identity)}, \tag{17}$$

where we used $\partial\Sigma = S_\infty \cup B$. This is the Smarr identity, corresponding to Eq. (29) in [2]: in the next section, we implement it to derive a generalized Smarr formula for Lovelock theories.

III. SMARR FORMULA FROM THE SMARR IDENTITY

In the very simple example of four-dimensional GR, the Smarr identity gives exactly the Komar integral

$$\oint_{\partial\Sigma} \mathbb{Q}[\xi] \doteq 0 \tag{18}$$

because the Einstein-Hilbert Lagrangian vanishes on shell. In the following, we show how the SI provides a generalized Smarr formula for the more general class of Lovelock theories. In III A, we review general features of Lovelock theories; in III B, we obtain a general expression for the mass of static spherically symmetric Lovelock black holes; finally, in III C, the desired Smarr formula is obtained.

A. Lovelock theories

Lovelock theories generalize Λ-GR theory and are the most general vacuum second order gravity theories in higher-dimensional spacetimes [7]. The peculiar structure of the Lagrangian makes them easier to deal with, if compared with more general higher curvature theories. The Lagrangian in D dimensions is

$$\mathbb{L} = \mathcal{L}\epsilon = \sum_{k=0}^{m} c_k \mathcal{L}^{(k)} \epsilon,$$
$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R^{c_1 d_1}_{ a_1 b_1} \ldots R^{c_k d_k}_{ a_k b_k} \tag{19}$$

for generic constants c_k. Since for $m > [\frac{D}{2}]$ the antisymmetrized delta symbol vanishes, m is restricted to be $m \leq [\frac{D}{2}]$; moreover, if $m = [\frac{D}{2}]$, the integral of $\mathcal{L}^{(m)}$ is a topological invariant proportional to the Euler characteristic in D dimensions, and therefore it doesn't contribute to the dynamics. The EOM are

$$\sum_{k=0}^{m} \mathcal{R}^{(k)r}_{s} - \frac{1}{2}\delta^r_s \mathcal{L} \doteq 0, \tag{20}$$

$$\mathcal{R}^{(k)r}_{s} = \frac{k c_k}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 s \ldots c_k d_k} R^{c_1 r}_{ a_1 b_1} \ldots R^{c_k d_k}_{ a_k b_k} \tag{21}$$

Following the procedure described in [5], the "canonical" Θ is

$$\Theta_0(\phi, \delta\phi) = \sum_{k=0}^{m} \frac{k c_k}{2^{k-1}} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} \nabla^{d_1} \delta g^{c_1}_{b_1} \ldots R^{c_k d_k}_{ a_k b_k} \epsilon_{a_1} \tag{22}$$

and the corresponding Noether charge is

$$\mathbb{Q}[\xi] = \sum_{k=0}^{m} \frac{k c_k}{2^{k-1}} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} \nabla_{[a_1} \xi^{d_1]} \ldots R^{c_k d_k}_{ a_k b_k} \epsilon_{b_1}{}^{c_1} \tag{23}$$

where the squared brackets indicate total antisymmetrization. Through Eq. (14), this gives the entropy of a Lovelock BH ([8], see also [9]):

$$S = \sum_{k=0}^{m} 4\pi k c_k \oint_B \underset{\leftarrow}{\mathcal{L}}^{(k-1)} \bar{\epsilon} \tag{24}$$

where the under-left arrow means that the object is evaluated with respect to the induced metric on B.

The Smarr identity (17) reads

$$\oint_{S_\infty} \mathbb{Q}[t] \doteq TS + \vec{\Omega} \cdot \vec{J} - W. \tag{25}$$

Observe that the work term

$$W = \sum_{k=0}^{m} c_k \int_{\Sigma} \mathcal{L}^{(k)} \xi \cdot \epsilon \tag{26}$$

contains powers of the Riemann tensor up to degree m; one can, however, use the EOM to lower the degree by one, thus reducing W to an expression easier to work with: it is sufficient to trace (20) and solve for $\mathcal{L}^{(m)}$, the resulting $\mathcal{L}$ being

$$\mathcal{L} \doteq \sum_{k=0}^{m-1} \left(\frac{2k-2m}{D-2m}\right) c_k \mathcal{L}^{(k)}. \tag{27}$$

Plugging this expression into W, we get the equivalent form,

$$W \doteq \sum_{k=0}^{m-1} \left(\frac{2k-2m}{D-2m}\right) c_k \int_\Sigma \mathcal{L}^{(k)} \xi \cdot \epsilon, \tag{28}$$

for the work term. For example, the $\Lambda - GR$ Lagrangian in D dimensions gives the Smarr identity

$$\oint_{\partial\Sigma} \nabla^a \xi^b \epsilon_{ab} + \frac{4\Lambda}{(D-2)} \int_\Sigma \xi^a \epsilon_a \doteq 0 \tag{29}$$

in agreement with the results of [1,10].

So far, we have been general. The main difficulties of Eq. (25) are that (i) the integral of $\mathbb{Q}[\xi]$ is not yet expressed in terms of the mass M of the BH, and (ii) the work term W is a volume integral and, therefore, it requires the knowledge of the solution over the entire spacetime. These difficulties can be addressed under the additional hypothesis of staticity. As a preliminary, we derive a general expression for the mass of a static black hole in Lovelock theories.

B. Mass of a static Lovelock black hole

Consider a black hole solution. One is tempted to define the total mass as the value of the Hamiltonian $H[t]$ at spatial infinity. However in general the Hamiltonian at infinity receives divergent contributions from the maximally symmetric background. To regularize these divergences, we define the total mass as $H[t] - H_0[t]$, where $H_0[t]$ refers to the background metric. Thus we can use the expression (7) for δH:

$$M = \delta H[t] = \int_{S_\infty} [\delta\mathbb{Q}[t] - t \cdot \Theta(\phi, \delta\phi)]. \tag{30}$$

We need to identify the asymptotic form of the line element: if we assume staticity, then the metric at infinity approaches a maximally symmetric background, i.e., Minkowski or (anti–) de Sitter (AdS). It is known [11] that static spherically symmetric BH solutions of Lovelock theory are all of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^{D-2} d\Omega_{D-2}^2. \tag{31}$$

For definiteness, we specify to the AdS case and keep $f(r)$ to scale as

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{\mu}{r^{D-3}} + o(r^{-(D-3)}). \tag{32}$$

The Minkowski case is recovered in the limit $l \to \infty$.

Let us compute the two terms in (30) separately. For a metric of the form (31) the integral of $\mathbb{Q}[t]$ simplifies drastically: from Eq. (23), one gets

$$\oint_{S_\infty} \mathbb{Q}[t] = \lim_{r\to\infty} \sum_{k=0}^{m} \left[\frac{k c_k \gamma_k (1-f)^{k-1} f'}{r^{2k-2}}\right] r^{D-2} \Omega_{D-2}, \tag{33}$$

where we defined $\gamma_k = (D-2)!/(D-2k)!$. Therefore,

$$\oint_{S_\infty} \delta\mathbb{Q}[t] = \lim_{r\to\infty} \sum_{k=0}^{m} \frac{k\gamma_k c_k}{r^{2k-D}} \frac{d}{dr}[(1-f)^{k-1}\delta f]\Omega_{D-2}. \tag{34}$$

This is a variation around the maximally symmetric background, so we have to take

$$f(r) = 1 + \frac{r^2}{l^2} \qquad \delta f(r) = -\frac{\mu}{r^{D-3}} \tag{35}$$

which yields

$$\begin{aligned}\oint_{S_\infty} \delta\mathbb{Q}[t] &= \sum_{k=0}^{m} (-1)^{k+1} \frac{k\gamma_k c_k (D-2k-1)}{l^{2k-2}} \mu\Omega_{D-2} \\ &= (\sigma - \gamma)\mu\Omega_{D-2},\end{aligned} \tag{36}$$

where, for later convenience, we defined

$$\gamma = \sum_{k=0}^{m} (-1)^{k+1} \frac{k c_k (D-2)!}{l^{2k-2}(D-2k)!}, \tag{37}$$

$$\sigma = \sum_{k=0}^{m} (-1)^{k+1} \frac{k c_k (D-2)!}{l^{2k-2}(D-2k-1)!}. \tag{38}$$

In the same way, we compute the second piece of the lhs of (30):

$$\begin{aligned}-\oint_{S_\infty} t\cdot\Theta &= \lim_{r\to\infty} \sum_{k=0}^{m} \frac{k\gamma_k c_k (1-f)^{k-1} f}{r^{2k-2}} 2r^a \nabla_{[b}\delta g^b_{a]} r^{D-2}\Omega_{D-2} \\ &= \lim_{r\to\infty} \sum_{k=0}^{m} \frac{k\gamma_k c_k (1-f)^{k-1}}{r^{2k-2}} \left(-\frac{d\delta f}{dr} - \frac{(D-2)\delta f}{r}\right) \\ &\quad \times r^{D-2}\Omega_{D-2} \\ &= \sum_{k=0}^{m} (-1)^{k+1} \frac{k c_k \gamma_k}{l^{2k-2}} \mu\Omega_{D-2} = \gamma\mu\Omega_{D-2}.\end{aligned} \tag{39}$$

Putting the two pieces together, we get

$$M = \sigma\mu\Omega_{D-2}. \tag{40}$$

Notice that σ, and thus M, doesn't receive contributions from the topological term of the Lagrangian. Expression (40) for the mass of a static asymptotically AdS Lovelock BH had been already obtained in [12] by means of an Hamiltonian analysis: in particular, the mass is there defined as the ADM (Regge-Teitelboim) Hamiltonian evaluated at spatial infinity. Our Lagrangian derivation agrees, and shows that $H[t]$ is exactly the ADM energy.

C. Smarr formula for Lovelock black holes

The expression (40) for the mass allows to rewrite the Smarr identity (25) as a Smarr formula, namely, as an identity expressing the mass in terms of geometric and dynamical parameters. It is sufficient to plug the asymptotic form of f, Eq. (32), into Eq. (33). The result is

$$\oint_{S_\infty} \mathbb{Q}[t] = \lim_{r\to\infty} \sum_{k=1}^{m} \frac{(-1)^{k+1} k c_k \gamma_k}{l^{2k-2}} \times \left[\frac{2r^{D-1}}{l^2} + (D-2k-1)\mu\right]\Omega_{D-2}. \tag{41}$$

The first term in parentheses is divergent: this divergence can be regularized as we did for the BH mass, i.e., by subtracting the same integral evaluated with respect to the background AdS metric. This subtraction cancels the divergence exactly, and one has

$$\oint_{S_\infty - \mathrm{AdS}} \mathbb{Q}[t] = \left(1 - \frac{\gamma}{\sigma}\right) M. \tag{42}$$

Thus, by adopting this regularization prescription, the Smarr identity (25) becomes

$$\left(1 - \frac{\gamma}{\sigma}\right) M \doteq TS - \hat{W}, \tag{43}$$

where $\hat{W}$ is now the regularized work term

$$\hat{W} = \sum_{k=0}^{m} c_k \int_{\Sigma - \mathrm{AdS}} \mathcal{L}^{(k)} \xi \cdot \epsilon \tag{44}$$

and $\vec{J} = 0$ because of staticity.

Now we have to deal with the fact that the work term is a volume integral. As we anticipated, this constitutes a difficulty because it forces to know the solution on a whole hypersurface; by contrast, a surface integral would allow to specify only the asymptotic behaviors of the solution. However, in the case of static solutions under consideration, W becomes a surface integral over $\partial\Sigma$: this follows from the fact that static solutions of Lovelock theories are all of the form (31). Substitution into $\mathcal{L}^{(k)}$ yields

$$\mathcal{L}^{(k)} = \frac{\gamma_k}{r^{D-2}} \frac{d^2}{dr^2}[(1-f)^k r^{D-2k}] \tag{45}$$

and the regularized work term becomes

$$\hat{W} = \sum_{k=0}^{m} c_k \oint_{\partial\Sigma - \mathrm{AdS}} W^{(k)} d\Omega_{D-2}, \tag{46}$$

$$W^{(k)} = \gamma_k \frac{d}{dr}[(1-f)^k r^{D-2k}], \tag{47}$$

which, as anticipated, is a surface integral. Therefore, in Lovelock theories the generalized Smarr formula (43) holds for static Lovelock black holes, where $\hat{W}$ is now a surface integral. It is interesting and insightful to compare (43) with the expression obtained in [12,13]: the authors there start from an Hamiltonian analysis and derive an extended first law with dynamical Lovelock couplings; integration of such a differential law produces the Smarr formula. The two formulas can, of course, be shown to be equivalent; see Appendix A.

In addition, notice that the expansion (31)–(32) of the metric at infinity still holds in the case of rotating asymptotically flat black holes: therefore, by taking the limit of Eq. (42) for $l \to \infty$, we obtain the Smarr formula,

$$\frac{(D-3)}{(D-2)} M \doteq TS - \vec{\Omega}\cdot\vec{J} - W, \tag{48}$$

for such BHs, where no regularization for W is needed in the asymptotically flat case; now, however, W is not generically expressible as a surface integral.

IV. TOPOLOGICAL WORK TERM

As we observed, if $m = \frac{D}{2}$ the last term $\mathcal{L}^{(m)}$ of the sum (19) is topological, and it does not contribute to the EOM; nonetheless, the Smarr formula (43) receives contributions from it. This is evident already in the simple training case of the Einstein-Gauss-Bonnet theory of gravity in four dimensions: the Lagrangian of the EGB theory is

$$\mathcal{L} = \frac{1}{16\pi G}(\mathcal{L}^{(1)} + \alpha\mathcal{L}^{(2)}), \tag{49}$$

$$\mathcal{L}^{(1)} = R, \tag{50}$$

$$\mathcal{L}^{(2)} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \tag{51}$$

The second term $\mathcal{L}^{(2)}$ is topological in four dimensions, and therefore the BH solutions are the same as in vacuum GR; since they are Ricci flat, the Smarr formula becomes

$$\frac{M}{2} \doteq TS - \Omega J - W, \tag{52}$$

$$W = \alpha \int_{\Sigma} K\xi \cdot \epsilon \tag{53}$$

where K is the Kretschmann invariant $R_{abcd}R^{abcd}$.

On the other hand, the Smarr formula in vacuum GR is known to be

$$\frac{M}{2} \doteq T\frac{A}{4G} - \Omega J. \tag{54}$$

Now, Wald's entropy S in (52) is not simply the Bekenstein entropy, but it receives a topological contribution S^{top} from the Gauss-Bonnet part of the Lagrangian,

$$S = \frac{A}{4G} + \frac{\alpha}{2G}\oint_B \underset{\sim}{\mathcal{L}}^{(1)}\bar{\epsilon} = \frac{A}{4G} + \frac{2\pi\alpha}{G}\chi, \tag{55}$$

where χ is the Euler characteristic of the bifurcation surface. For a single BH $\chi = 2$, and therefore by consistency the work term (53) must be equal to

$$W = \frac{2\alpha\kappa}{G}. \tag{56}$$

This is indeed the case (for example, for the Schwarzschild solution the Kretschmann scalar is $K = 48G^2M^2/r^6$ and, using $\kappa = 1/2r_H$, Eq. (56) follows).

By generalizing the above argument, we can conclude that, if $m = \frac{D}{2}$, the Smarr formula always contains suitable topological terms, performing the task of compensating the topological correction to the entropy.

In the case of spherically symmetric solutions, it is very easy to verify explicitly how the compensation arises (see Appendix B): indeed, it turns out that the topological counterterms sum up to give the temperature $T = f'(r_H)/4\pi$, times a surface integral at the bifurcation surface, which reproduces exactly S^{top}. Thus, the compensation occurs between terms having the very same geometrical nature.

This fact suggests that S^{top} and its counter terms are not genuine physical contributions, respectively, to the entropy and to the work terms, but they are rather an artifact of Wald's formalism.

Indeed, the topological correction to the Bekenstein entropy in four dimensions has been addressed by several authors [6,14,15], arguing that it can lead to possible violations of the generalized second law[5]: this again suggests that the physical entropy should be identified with the Bekenstein one, rather than the Wald's one. After all, it would be quite strange that a physical quantity like the entropy be affected by terms in the Lagrangian not contributing to the dynamics.

[5]See, however, [16] in which the authors argue that such a violation does not occur if the Gauss-Bonnet term is viewed as an effective field theory contribution.

How does this reconcile with Eq. (55)? One could simply remove by hand the topological term from Wald's entropy, as suggested in [15]. However, having interpreted S^{top} as an artifact of the formalism, we wonder if there is a natural window inside the formalism itself: the answer is in the affirmative. One can make use of a further ambiguity in the definition of $\mathbb{Q}[\xi]$, in addition to those listed in [5,6]: as noted in [9], it is possible to rescale the Noether form by a term proportional to the volume element $\underset{\sim}{\epsilon}$ of S^2,

$$\mathbb{Q}[\xi] \to \mathbb{Q}[\xi] + \mathrm{const} \cdot \underset{\sim}{\epsilon}, \tag{57}$$

where $\underset{\sim}{\epsilon}$ is defined as

$$\underset{\sim}{\epsilon} = \frac{1}{2}\sin\theta d\theta \wedge d\phi, \tag{58}$$

without affecting the validity of Wald's construction, because $d\underset{\sim}{\epsilon} = 0$, and $\oint_{B\cup S_\infty} \underset{\sim}{\epsilon} = 0$.

Therefore, in four-dimensional EGB, we can redefine

$$\mathbb{Q}[\xi] \to \mathbb{Q}[\xi] - \frac{\kappa\alpha}{4\pi G}\chi\underset{\sim}{\epsilon}, \tag{59}$$

so that the modified Noether potential gives the correct physical entropy, i.e., the Bekenstein one. The procedure can be straightforwardly generalized to higher dimensions. Observe that $\underset{\sim}{\epsilon}$ is well defined also in the rotating case, and our prescription is thus completely general.

V. DISCUSSION

In this work, we investigated a procedure to compute the Smarr formula for black holes in diffeoinvariant theories of gravity. The method makes use of Eq. (17), which is obtained integrating and expanding Eq. (15).

To the extent of our knowledge, the above equations have been considered before, but not in connection with the Smarr formula: in particular, Eq. (15) was used in [6] to show that Wald's entropy formula can be evaluated not only over the bifurcation surface, but over any spatial cross section of the horizon.

We applied our procedure to the case of Lovelock black holes, thus deriving the Smarr Fomulas (43) for static black holes, and (48) for rotating asymptotically flat black holes. In particular, static BHs show the preferable feature that the work term W is a surface integral, which follows from the simple form (31) that the line element assumes in the static BH solutions of Lovelock gravity. The derivation cannot be straightforwardly extended to the rotating case, because there is no general form of the line element. It would be interesting to investigate under which restrictions the relative extension can be done.

Moreover, one might wonder if the method presented here can be further extended beyond Lovelock theories. Indeed, one can realize that this is possible for scalar-tensor and $f(R)$ theories, by simply noticing that theorems guarantee that in these theories black hole solutions are coincident with those of GR under rather mild assumptions [17,18]. A general proof of applicability of our method for other extended theories of gravity is presently lacking, and we hope to further investigate this issue in future work.

In the final part of the paper, we examined the behavior of topological terms in the Lovelock Lagrangian; we argued that the corresponding topological terms in the Smarr formula, including the contribution S^{top} to the entropy, can be viewed as unphysical artifacts of the formalism; motivated by this, we proposed a modified prescription for the Noether charge, which incorporates topological effects and reconciles the results with the physical quantities.

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APPENDIX A: EQUIVALENCE BETWEEN THE LAGRANGIAN AND THE HAMILTONIAN SMARR FORMULAS

In this appendix, we show the equivalence between the Smarr formula (43) for static Lovelock black holes and the one obtained in Refs. [12,13] by means of an Hamiltonian analysis.

Let us define the following quantities for later convenience:

$$\sigma_n = \sum_{k=0}^{m} \frac{(-1)^{k+n} k!}{(k-n)! l^{2(k-1)}} \frac{(D-2)! b_k}{(D-2k-1)!} \quad \text{(A1)}$$

$$\gamma_n = \sum_{k=0}^{m} \frac{(-1)^{k+n} k!}{(k-n)! l^{2(k-1)}} \frac{(D-2)! b_k}{(D-2k)!}, \quad \text{(A2)}$$

where the b_k are related to the Lovelock couplings c_k by $b_k = 16\pi G c_k$.

The strategy of [12,13] consists of first deriving an extended first law for dynamical perturbations around static Lovelock black hole solutions, in which also the Lovelock coupling are allowed to vary:

$$\delta M = T\delta S - \frac{1}{16\pi G} \sum_{k=0}^{m} \delta b_k \Psi^{(k)}. \quad \text{(A3)}$$

Then, regarding the mass as an homogeneous function of S and the b_k, one can use Euler's theorem to extract the Smarr formula:

$$\begin{aligned}(D-3)M &= (D-2)TS - \frac{1}{8\pi G} \sum_{k=0}^{m} b_k (k-1) \Psi^{(k)} \\ &= (D-2)TS - \Psi. \end{aligned} \quad \text{(A4)}$$

The second term on the rhs can be split into three contributions:

$$\Psi = \Theta + B + TS', \quad \text{(A5)}$$

where B and S' are given by

$$B = 2\frac{\sigma_2}{\sigma_1} M \quad \text{(A6)}$$

$$S' = \frac{\Omega_{D-2}}{2G} \sum_{k=0}^{m} \frac{k(k-1)}{r_H^{2k-D}} \frac{(D-2)! b_k}{(D-2k)!}, \quad \text{(A7)}$$

while Θ is defined as

$$\Theta = \frac{1}{8\pi G} \sum_{k=0}^{m} (k-1) b_k \oint_{\partial\Sigma-\mathrm{AdS}} \Theta^{(k)} d\Omega_{D-2}, \quad \text{(A8)}$$

$$\Theta^{(k)} = \frac{(D-2)!}{(D-2k-1)!} (1-f)^k r^{D-2k-1}. \quad \text{(A9)}$$

We are going to manipulate Eq. (43) and show its equivalence to Eq. (A4). First, tracing (20) and solving for $\mathcal{L}^{(1)}$, we obtain the following on-shell equivalent expression for the Lovelock Lagrangian,

$$\mathcal{L} \doteq \frac{1}{8\pi G} \sum_{k=0}^{m} b_k \frac{(k-1)}{(D-2)} \mathcal{L}^{(k)}, \quad \text{(A10)}$$

which is the analogous of Eq. (27) for $m = 1$. The work term $\hat{W}$ is, thus, equivalent to

$$\hat{W} \doteq \frac{1}{8\pi G} \sum_{k=0}^{m} \frac{(k-1)}{(D-2)} b_k \oint_{\partial\Sigma-\mathrm{AdS}} W^{(k)} d\Omega_{D-2}. \quad \text{(A11)}$$

With the help of (47), one can easily show that

$$\begin{aligned}(D-2)\hat{W} \doteq{}& \Theta + TS' \\ & - \frac{\Omega_{D-2}}{8\pi G} \sum_{k=0}^{m} (-1)^{k+1} \frac{k(k-1)}{l^{2(k-1)}} \frac{(D-2)! b_k}{(D-2k)!} \\ & \times (D-2k-1)\mu. \end{aligned} \quad \text{(A12)}$$

Moreover,

$$\left(1-\frac{\gamma}{\sigma}\right)M = \left(1-\frac{\gamma_1}{\sigma_1}\right)M$$
$$= \frac{(D-3)}{(D-2)}M - \frac{\Omega_{D-2}}{16\pi G}\left(\frac{\sigma_1}{(D-2)} - \gamma_1\right)\mu$$
$$= \frac{(D-3)}{(D-2)}M - \frac{\Omega_{D-2}}{8\pi G(D-2)}$$
$$\times \sum_{k=0}^{m}(-1)^{k+1}\frac{k(k-1)}{l^{2(k-1)}}\frac{(D-2)!b_k}{(D-2k)!}\mu. \tag{A13}$$

Combining (A12) with (A13), we get

$$(D-2)\left(1-\frac{\gamma}{\sigma}\right)M + (D-2)\hat{W}$$
$$\doteq (D-3)M + \Theta + TS' + \frac{\Omega_{D-2}}{8\pi G}\sigma_2\mu$$
$$= (D-3)M + \Theta + TS' + 2\frac{\sigma_2}{\sigma_1}M$$
$$= (D-3)M + \Psi. \tag{A14}$$

Inserting (A14) into (43), the desired Eq. (A4) follows.

APPENDIX B: COMPENSATION OF THE TOPOLOGICAL TERMS IN THE SMARR FORMULA

Consider the Lovelock Lagrangian in even-dimension D and with $m \equiv \frac{D}{2}$, such that $\mathcal{L}^{(m)}$ is topological and doesn't contribute to the EOM; nevertheless, the Smarr formula (43) contains three different topological contributions: the first is the topological entropy component

$$S^{\text{top}} \coloneqq c_m\frac{dS}{dc_m} = 4\pi m c_m \oint_B \underset{\leftarrow}{\mathcal{L}}^{(m-1)}\bar{\epsilon}. \tag{B1}$$

Given that

$$\underset{\leftarrow}{R}{}^{cd}_{ab} = \frac{\bar{\delta}^{bd}_{ac}}{r_H^2}, \tag{B2}$$

where $\bar{\delta}^{bd}_{ac}$ is the antisymmetrized delta on the $(D-2)$-dimensional bifurcation surface, S^{top} becomes

$$S^{\text{top}} \equiv \frac{2\pi D!\, c_{D/2}\,\Omega_{D-2}}{(D-1)}. \tag{B3}$$

The other two contributions, as anticipated before in Sec. IV, compensate exactly TS^{top}. Let us show how the compensation occurs. The second contribution is the topological part of $\hat{W}$,

$$\hat{W}^{\text{top}} \coloneqq c_m\frac{d\hat{W}}{dc_m} = c_{D/2}\gamma_{D/2}\oint_{\partial\Sigma-\text{AdS}}\frac{d}{dr}(1-f)^{D/2}d\Omega_{D-2}$$
$$= \frac{D!c_{D/2}}{2(D-1)}\oint_B (1-f)^{\frac{D}{2}-1}f'(r)d\Omega_{D-2}$$
$$- \frac{D!c_{D/2}}{2(D-1)}\oint_{S_\infty-\text{AdS}}(1-f)^{\frac{D}{2}-1}f'(r)d\Omega_{D-2}. \tag{B4}$$

Finally, the last contribution comes from the lhs of (43):

$$M^{\text{top}} \coloneqq c_m\frac{d}{dc_m}\left(\frac{\gamma}{\sigma}M\right) = \frac{(-1)^{\frac{D}{2}-1}D!\,c_{D/2}\,\mu\,\Omega_{D-2}}{2(D-1)l^{D-2}}. \tag{B5}$$

Using (32), a direct calculation shows that the second term in (B4) cancels exactly (B5). Therefore, TS^{top} is ultimately compensated by the first term on the rhs of (B4): this consists of a surface integral over the bifurcation surface B; moreover, using $f(r_H)=0$ and $T=f'(r_H)/4\pi$, it is immediate to see that it factorizes precisely as T times S^{top}.

This shows that the topological terms in the Smarr formula compensate with the same geometrical structure.

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