PHYSICAL REVIEW D **93,** 084030 (2016)

Stable coherent states

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(Received 11 February 2016; published 15 April 2016)

We analyze the stability under time evolution of complexifier coherent states (CCS) in one-dimensional mechanical systems. A system of coherent states is called stable if it evolves into another coherent state. It turns out that a system can only possess stable CCS if the classical evolution of the variable $\mathbf{z} = e^{-i\mathcal{L}_{\chi C}}\mathbf{q}$ for a given complexifier C depends only on $\mathbf{z}$ itself and not on its complex conjugate. This condition is very restrictive in general so that only a few systems exist that obey this condition. However, it is possible to access a wider class of models that in principle may allow for stable coherent states associated with certain regions in the phase space by introducing action-angle coordinates.

DOI: 10.1103/PhysRevD.93.084030

I. INTRODUCTION

Coherent states have proven to be a powerful tool in many areas of physics as well as mathematics. The name "coherent" goes back to Glauber [1,2] who rediscovered Schrödinger's states in the context of quantum optics. They are also used, for example, in geometric quantization [3], harmonic analysis, and representation theory [4–6]. This broad applicability entailed a vast number of generalizations, just to mention some [7–10].

In quantum gravity coherent states are employed to derive a semiclassical limit of the model in question. Especially in the absence of experimental data, this can provide important insights on quantization ambiguities and possible inconsistencies. In canonical loop quantum gravity (LQG) one uses, for example, so-called complexifier coherent states [11–15], going back to the pioneering work of Hall [8], in order to define such a limit. For constraint systems such as gravity one has to decide on which space, the kinematical or the physical Hilbert space, the states shall be defined. Which strategy is chosen depends, of course, on the problem in question but in many cases it is easier to build coherent states on the kinematical rather than the physical Hilbert space. This truly applies to LQG where physical states are only known formally and brings in a new aspect that has to be respected as the states designed on the kinematical Hilbert space should not lose their properties when solving the constraints.

From an heuristic point of view the implementation of a constraint $\hat{H}$ is related to a sort of time evolution generated by $\hat{H}$ since

$$\psi_{\text{phys}} = \delta(\hat{H})\psi = \int \mathrm{d}t e^{it\hat{H}}\psi \tag{1}$$

gives a formal solution. In fact, many strategies such as group averaging and rigging map procedures (see e.g. [16]) take this as a starting point. Evidently, this ansatz is also advocating itself in order to solve the Hamiltonian constraint of LQG and is the initial idea from which spin foam [17–19] models arose. So instead of asking "Is a coherent state maintaining its coherence when solving the constraints?" it is tempting to simplify matters and ask "Is the coherent state ψ_z stable under the evolution generated by $\hat{U}(t) \coloneqq e^{it\hat{H}/\hbar}$?" or likewise "Is $\hat{U}(t)\psi_z$ still coherent?" These questions are of interest in quantum mechanics as well because mostly one is interested not only in the semiclassical behavior at a certain time but in the dynamical evolution.

In this work, the necessary conditions for the existence of stable complexifier coherent states are investigated. It is found that in general it is very hard to construct a complexifier adapted to the dynamics of a given model. Nevertheless, the derived criteria are form invariant under canonical transformation which opens the possibility to excess a wider class of models, namely, those that show a quasiperiodic motion.

In the following section, the semiclassical properties of coherent states (Sec. II B) and the construction principle of complexifier coherent states (Sec. II C) are reviewed based on [16]. Thereafter, a stability criterion for finite dimensional models will be derived, and in Sec. III B we will discuss a simplified ansatz to find solutions to this condition. Section III C contains a proof that it is, in fact, not possible to use this simplified ansatz to determine systems other than the harmonic and radial oscillator that possess stable complexifier states. A generalized construction principle using so-called action-angle coordinates and the Hamilton-Jacobi approach is given in Sec. IV A, and some examples are analyzed in Sec. IV B. The paper closes with a short discussion of the results.

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II. COHERENT STATES

A. Preliminaries and conventions

If not stated otherwise, it will be assumed that the phase space $\mathcal{M}$ of a given system with a finite number of degrees of freedom $\mathfrak{f}$ is the cotangent bundle $\mathrm{T}^*\mathcal{C}$ of the configuration space $\mathcal{C}$. The Hamiltonian vector field χ_f of a continuous differentiable function f on $\mathcal{M}$ is the vector field that satisfies the condition $0 \equiv \mathcal{L}_{\chi_f}\Omega$ where Ω is the symplectic 2-form on $\mathcal{M}$ and $\mathcal{L}_{\chi_f}$ the Lie derivative along χ_f. The Poisson bracket corresponding to Ω is given by

$$\{f,g\} \coloneqq \Omega(\chi_f,\chi_g) = \chi_f[g],$$

and multiple Poisson brackets are defined through the recursion relation $\{f,g\}_{(n+1)} \coloneqq \{f,\{f,g\}_{(n)}\}$ with $\{f,g\}_{(0)} \coloneqq g$. The Liouville measure is the measure on $\mathcal{M}$ which is invariant under the action of the symplectic group that preserves Ω.

Throughout the rest of this paper, $\mathbf{x}$ will denote the $\mathfrak{f}$-tuple $(x_1,\ldots,x_\mathfrak{f})$, $\mathbf{x}\cdot\mathbf{y} = \sum_{j=1}^{\mathfrak{f}} x_j y_j$ the usual Euclidean scalar product, and $(\mathbf{p},\mathbf{q})$ a canonical conjugated pair; that is, they satisfy

$$\{p_j,q_k\} = \delta_{jk} \quad \text{and} \quad \{p_j,p_k\} = 0 = \{q_j,q_k\}.$$

Furthermore, $\mathcal{O}$ will denote a subalgebra of the Poisson algebra $\mathrm{C}^\infty(\mathcal{C})$ that separates the points of $\mathcal{M}$ and $\bar{z}$ the complex conjugate.

Under quantization we understand a map $(\mathcal{M},\{\cdot,\cdot\},\mathcal{O}) \to (\mathcal{H},\frac{1}{i\hbar}[\cdot,\cdot],\hat{\mathcal{O}})$ where $\mathcal{H}$ is a Hilbert space and $\hat{\mathcal{O}}$ is a subalgebra of the algebra of linear operators $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$ that is a representation of $\mathcal{O}$. If not said otherwise, the Hilbert space $\mathcal{H}$ is the space of square integrable functions $L^2(\bar{\mathcal{C}},\mathrm{d}\mu)$ on a suitable extension $\bar{\mathcal{C}}$ of the configuration space with measure $\mathrm{d}\mu$. The scalar product on $\mathcal{H}$ is usually given by

$$\langle f,g\rangle = \int \mathrm{d}\mu(x)\overline{f(x)}g(x).$$

B. Semiclassical and coherent states

In his lecture "Über die Spektraltheorie der Elemente" [20], given in 1920 at a meeting of the German Physical Society in Berlin, Bohr introduced the principle that the behavior of a quantum system should mimic the classical one for high energies. To formulate this statement in a more precise manner it is useful to introduce the notion of *semiclassical states*. These are elements $\psi_\mathfrak{m}$ in the Hilbert space $\mathcal{H}$ that are associated with points $\mathfrak{m}$ in $\mathcal{M}$. They are constructed in such a way that the expectation value of a quantum observable $\hat{O}$ in a given subalgebra $\hat{\mathcal{O}} \subset \mathcal{L}(\mathcal{H})$ that separates the points of $\mathcal{M}$ is close to the classical value $O(\mathfrak{m})$ of the corresponding phase space function O. Stated differently, the following properties have to hold for all generic points $\mathfrak{m} \in \mathcal{M}$ [i.e. points for which the denominators in (2), (3), and (4) are nonzero]:

(i) *Expectation value property*

$$\left|\frac{\langle\psi_\mathfrak{m},\hat{O}\psi_\mathfrak{m}\rangle}{O(\mathfrak{m})} - 1\right| \ll 1; \tag{2}$$

(ii) *Ehrenfest property*

$$\left|\frac{\langle\psi_\mathfrak{m},[\hat{O},\hat{O}']\psi_\mathfrak{m}\rangle}{i\hbar\{O,O'\}} - 1\right| \ll 1; \tag{3}$$

(iii) *Fluctuation property*

$$\left|\frac{\langle\psi_\mathfrak{m},\hat{O}^2\psi_\mathfrak{m}\rangle}{\langle\psi_\mathfrak{m},\hat{O}\psi_\mathfrak{m}\rangle^2} - 1\right| \ll 1. \tag{4}$$

For most systems it is not possible to design semiclassical states for *all* observables simultaneously but it highly depends on the chosen subalgebra. Good examples are the coherent states of the harmonic oscillator introduced by Schrödinger in 1926 [21] which have "good" semiclassical properties for the linear span of the annihilator $\hat{a}$, the creator $\hat{a}^\dagger$, and $\mathbb{1}$. In addition, these states have several other desirable properties which motivate the following definition.

Definition 1 *(Coherent states).* A system of states $\{\psi_\mathfrak{m}\}_{\mathfrak{m}\in\mathcal{M}} \subset \mathcal{H}$ is said to be coherent provided that in addition to (2), (3), and (4) the states also obey as follows:

(i) *Overcompleteness (Resolution of identity):*

$$\mathbb{1}_\mathcal{H} = \int_\mathcal{M} \mathrm{d}\nu(\mathfrak{m})|\psi_\mathfrak{m}\rangle\langle\psi_\mathfrak{m}| \tag{5}$$

for some measure ν on $\mathcal{M}$.

(ii) *Annihilation operator property:* There exist operators $\hat{z}$ such that $\hat{z}\psi_\mathfrak{m} = z(\mathfrak{m})\psi_\mathfrak{m}$.

(iii) *Minimal uncertainty:* For the self-adjoint operators $\hat{x} = (\hat{z}+\hat{z}^\dagger)/2$ and $\hat{y} = (\hat{z}-\hat{z}^\dagger)/(2i)$ the Heisenberg uncertainty relation is saturated, i.e.

$$\begin{aligned}\langle(\hat{x}-\langle\hat{x}^2\rangle_\mathfrak{m})^2\rangle_\mathfrak{m} &= \langle(\hat{y}-\langle\hat{y}^2\rangle_\mathfrak{m})^2\rangle_\mathfrak{m} \\ &= \frac{\hbar}{2}|\langle[\hat{x},\hat{y}]\rangle_\mathfrak{m}|,\end{aligned} \tag{6}$$

where $\langle\cdot\rangle_\mathfrak{m} \coloneqq \langle\psi_\mathfrak{m},\cdot\psi_\mathfrak{m}\rangle$.

(iv) *Peakedness property:* For any $\mathfrak{m} \in \mathcal{M}$, the overlap function

$$\mathfrak{m}' \mapsto |\langle\psi_\mathfrak{m},\psi_{\mathfrak{m}'}\rangle|^2 \tag{7}$$

is concentrated in a phase space cell of Liouville volume $\frac{1}{2}|\langle[\hat{p},\hat{q}]\rangle_\mathfrak{m}|$.

As stated in the Introduction, coherent states have a broad application in many areas of physics and mathematics, which entailed a vast number of generalizations, so that by now "coherent state" is not a clear-cut expression in the literature. In the subsequent section the generalization suggested in [11–15] is reviewed, which in most cases preserves the properties mentioned in Definition 1.

C. The complexifier method

The central point of semiclassical/coherent states is that they are *continuously labeled by points in the classical phase space* $\mathcal{M}$. In the case of the harmonic oscillator, this is achieved by constructing the eigenstates of the operator $\hat{a}$ corresponding to the *complex parametrization*

$$a := \sqrt{\frac{m\omega}{2}}\left(q - \frac{i}{m\omega}p\right). \tag{8}$$

It is, of course, just one possible choice out of many different parametrizations of $\mathcal{M}$, and therefore many other state systems are imaginable that are coherent in the sense of Definition 1. This is the starting point of the complexifier method introduced in [11–15] that enables one to directly relate a complex parametrization of $\mathcal{M}$ with a state system in $\mathcal{H}$.

Let $C:\mathcal{M}\to\mathbb{R}$ be a positive definite function with the dimension of an action that is smooth with respect to the Liouville measure on $\mathcal{M}$, has a nowhere vanishing Hamiltonian vector field χ_C, and grows stronger than linearly in $\mathbf{p}$ for each fixed point $\mathbf{q}\in\mathcal{C}$. A function satisfying these properties is called a *complexifier* since

$$q_i \mapsto z_i := e^{-i\mathcal{L}_{\chi_C}} q_i \tag{9}$$

yields a complex coordinate system on $\mathcal{M}$ for a given parametrization $\{q_i | i = 1,\ldots,\mathfrak{f}\}$ of the configuration space $\mathcal{C}$. The smoothness of C and the fact that χ_C is nowhere vanishing guarantee that C induces a nondegenerate, smooth transformation. If $\mathcal{M} = \mathrm{T}^*\mathcal{C}$, then $e^{-i\mathcal{L}_{\chi_C}} q_i$ defines a symplectomorphism[1] that maps a point $\mathfrak{m}\in\mathcal{M}$ to a point $z(\mathfrak{m})$ in the complex extension $\mathcal{C}^{\mathbb{C}}$ of the configuration space. For example, the complexifier $C = \frac{p^2}{2}$ generates the symplectomorphism

$$\mathcal{M} = \mathbb{R}^2 \to \mathbb{C}, \qquad (q,p)\mapsto a = q - ip.$$

According to Bargmann and Segal [4,5], the associated coherent states ψ_a also give rise to a transformation from $L^2(\mathbb{R},\mathrm{d}\mu)$ onto the space $\mathfrak{H}^2(\mathbb{C},\mathrm{d}\nu)$ of square integrable, holomorphic functions on $\mathbb{C}$ through the integral transform

$$f(q)\mapsto [Bf](\alpha) = \int_{\mathbb{R}} \mathrm{d}\mu(q)\psi_\alpha(q)f(q). \tag{10}$$

If the measure ν is fixed by the resolution of the identity (5), then (10) is even unitary. Thus, the harmonic oscillator states play a similar role as plane waves for the Fourier transformation. In fact, for this specific example with $C = \frac{p^2}{2}$ one can show that up to a complex phase $\psi_a(x)$ is equal to

$$\exp\left(-\frac{1}{2\hbar}(x-a)^2\right) \propto \frac{1}{2\pi}\int \mathrm{d}k e^{-k^2/(2t)} e^{ik(x-a)} = [e^{-\hat{C}/\hbar}\delta_y(x)]_{y\to a}, \tag{11}$$

where $y\to a$ denotes analytic continuation.

Definition 2 *(Complexifier coherent states).* A coherent state associated with a complexifier $C\in \mathrm{C}^\infty(\mathcal{M})$ is an element of $\mathcal{H}$ of the form

$$\psi_{\mathfrak{m}}(q) = [e^{-C/\hbar}\delta_{q'}(q)]_{q'\to z(\mathfrak{m})}, \tag{12}$$

where $q'\to z(\mathfrak{m})$ denotes analytic continuation to $z(\mathfrak{m}) = [e^{-i\mathcal{L}_{\chi_C}} q](\mathfrak{m})$.

This also explains why it was required that C has the dimension of an action and has to satisfy a growth condition. Namely, if $\hat{C}/\hbar$ would not be dimensionless, then the exponential would not be well defined. Apart from that, $e^{-\hat{C}/\hbar}$ must decay fast enough to smooth out the divergence of δ. In Eq. (11) this is achieved by the Gaussian factor that is added through the action of $e^{-\hat{C}/\hbar}$.

By construction, the states (12) are eigenstates of the annihilators

$$\hat{z}_i = e^{-\hat{C}/\hbar}\hat{q}_i e^{\hat{C}/\hbar} \tag{13}$$

resulting from quantizing (9) and therefore automatically obey minimal uncertainty (6). Also condition (2) is clearly satisfied for the subalgebra spanned by $\mathbf{z}$ and $\bar{\mathbf{z}}$. All the other properties required in Definition 1 do not follow directly but are very plausible since the states are essentially a regularization of distributions. Therefore, it is reasonable to expect that they generate a resolution of identity and are well peaked on the phase space.

III. STABILITY OF COHERENT STATES IN QUANTUM MECHANICS

A. A stability criterion

Recall that the classical evolution is generated by the flow of the Hamiltonian vector field χ_H of the Hamiltonian H. For a not explicitly time dependent Hamiltonian this means that a phase space function f evolves as

[1]This is no longer true for $\mathcal{M}\neq \mathrm{T}^*\mathcal{C}$, but z_i still provide good local coordinates due to Darboux's theorem.

$f(t) = e^{(t-t_0)\mathcal{L}_{\chi_H}} f(t_0)$ and the quantum evolution of such a model is described by the operator $\hat{U}(t,t_0) = e^{\frac{1}{i\hbar}(t-t_0)\hat{H}}$.

Definition 3 *(Stable coherent states).* A coherent state $\psi_{z(t_0)}$ is stable under the evolution $\hat{U}(t,t_0)$ iff

$$\hat{U}(t,t_0)\psi_{z(t_0)} = e^{i\lambda(t)}\psi_{z(t)} \quad \forall\ t \in \mathbb{R}^+, \tag{14}$$

where $z(t)$ follows the classical motion of $z(t_0)$ on the phase space. The system of coherent states, $\{\psi_{z(\mathfrak{m})}|\mathfrak{m} \in \mathcal{M}\}$, is called stable iff all states are stable.

A simple example for a stable state system is the original coherent states ψ_a of the harmonic oscillator. By a short calculation using $\psi_a = e^{|a|^2/2}\sum_n \frac{a^n}{n!}|n\rangle$ and $\hat{H}_{ho}|n\rangle = \hbar\omega(n+1/2)|n\rangle$ one finds

$$\hat{U}_{ho}(t)\psi_a = e^{it\omega/2}\psi_{a(t)}.$$

Even more general, the set $\{\psi_a\}$ is stable under the evolution generated by a Hamiltonian[2] that satisfies $\frac{1}{i\hbar}[\hat{H},\hat{a}] = if(\hat{a},t)$ (see e.g. [24,25]). For complexifier coherent states one can prove a very similar criterion.

Theorem 1. Suppose the set of complexifier states $\mathcal{S}_{t_0} \coloneqq \{\psi_{z(\mathfrak{m}(t_0))}|\mathfrak{m}(t_0) \in \mathcal{M}\}$ is overcomplete and stable and the time evolution $\hat{U}(t,t_0)$ is unitary; then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_j(t_0) = if_j(\hat{z}_1,\ldots,\hat{z}_{\mathfrak{f}},t_0) \quad \forall\ j = 1,\ldots,\mathfrak{f}. \tag{15}$$

On the other hand, if Eq. (15) holds at t_0, then there exists an $\epsilon > 0$ such that $\mathcal{S}_t$ is stable with respect to $\hat{U}(t,t_0)$ for all $|t_0 - t| < \epsilon$.

Proof.—According to Definition 3 the system $\mathcal{S}_{t_0}$ is stable iff

$$\begin{aligned}
&\hat{z}_j(t_0)\hat{U}(t,t_0)|\psi_{z(t_0)}\rangle = z_j(t)\hat{U}(t,t_0)|\psi_{z(t_0)}\rangle \\
&\quad\Leftrightarrow \hat{z}_j(t)|\psi_{z(t_0)}\rangle = z_j(t)|\psi_{z(t_0)}\rangle \\
&\Rightarrow \left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_j(t_0)\right]|\psi_{z(t_0)}\rangle = \left[\frac{\mathrm{d}}{\mathrm{d}t}z_j(t_0)\right]|\psi_{z(t_0)}\rangle.
\end{aligned}$$

Here, $\hat{z}_j(t) \coloneqq \hat{U}^\dagger(t,t_0)\hat{z}_j(t_0)\hat{U}(t,t_0)$ is the time-dependent operator in the Heisenberg picture. Since $\mathcal{S}_{t_0}$ is overcomplete, the last equation can hold only if $\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_j(t_0)$ is a function of the annihilators $\hat{z}_j(t_0)$, that is, if $\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_j(t_0) = if_j(\hat{z}_1,\ldots,\hat{z}_{\mathfrak{f}},t_0)$ where i is just introduced for convenience.On the other hand, if Eq. (15) holds, then in some region $|t_0 - t| < \epsilon$ the Taylor expansion of $\hat{z}(t)$ exists and reads

$$\begin{aligned}
\hat{z}_j(t) = \sum_n \frac{t^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\hat{z}_j(t_0) &= \hat{z}_j(t_0) \\
&+ i(t-t_0)f_j(\hat{z}_1(t_0),\ldots,\hat{z}_{\mathfrak{f}}(t_0),t_0) + \cdots.
\end{aligned}$$

Therefore $\hat{z}_j(t)|\psi_{z(t_0)}\rangle = z_j(t)|\psi_{z(t_0)}\rangle$ in that region. □

This implies the following two conditions that are necessary for coherent states to be stable.

Corollary 1. Stable coherent states can exist only if it is possible to find a complexifier C such that $\mathbf{z} = e^{-i\mathcal{L}_{\chi_C}}\mathbf{q}$ obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = if_j(z_1,\ldots,z_{\mathfrak{f}},t_0) \quad \forall\ j = 1,\ldots,\mathfrak{f}. \tag{16}$$

Corollary 2. If neither the complexifier C nor the Hamiltonian H are explicitly time dependent, then $e^{i\mathcal{L}_{\chi_C}}H$ must satisfy

$$e^{i\mathcal{L}_{\chi_C}}H = i\sum_{j=1}^{\mathfrak{f}} p^j f_j(q_1,\ldots,q_{\mathfrak{f}}) + g(q_1,\ldots,q_{\mathfrak{f}}) \tag{17}$$

for some functions f_j and g on $\mathcal{C}$ in order that the complexifier coherent states are stable.

Proof.—Since $\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = \{H,z_j\}$ and since $e^{-i\mathcal{L}_{\chi_C}}$ is a symplectomorphism, it follows from Corollary 1 that

$$\begin{aligned}
0 &= \{H,z_j\} - if_j(z_1(t),\ldots,z_{\mathfrak{f}}(t)) \\
&= e^{-i\mathcal{L}_{\chi_C}}[\{e^{i\mathcal{L}_{\chi_C}}H,q_j\} - if_j(q_1,\ldots,q_{\mathfrak{f}})].
\end{aligned}$$

Yet, $\{e^{i\mathcal{L}_{\chi_C}}H,q_j\}$ is equal to $f_j(q_1,\ldots,q_{\mathfrak{f}})$ if and only if Eq. (17) holds. □

To summarize: The existence of stable states is closely tied to the classical behavior of the model. The subtlety hereby is that conditions (16) and (17) are local. Even though it might be possible to find a complexifier satisfying these conditions in a neighborhood of a point $(p_0,q_0) \in \mathcal{M}$, it does not ensure that the resulting parametrization $\mathbf{z} = e^{-i\mathcal{L}_{\chi_C}}\mathbf{q}$ is sensible. For example, $\mathbf{z}$ might not be defined everywhere on the phase space or it might be multivalued. Yet, for the system $\mathcal{S}$ to be overcomplete it is important that $\mathbf{z}$ provides a "good" parametrization.

Lemma 1. If the system S is overcomplete, then $\{z_j\}$ is a proper parametrization of $\mathcal{M}$ in the sense that almost every point $\mathfrak{m} \in \mathcal{M}$ is uniquely determined by a $\mathfrak{f}$-tuple $(z_1,\ldots,z_{\mathfrak{f}})$. Here, "almost" means up to a set of measure zero with respect to the measure (5).

Proof.—If this were not the case, then there would exist a subset U in $\mathcal{M}$ with a parametrization $\mathbf{w}$ such that states $\psi_{\mathrm{w}(\mathfrak{m})}$ associated with $\mathfrak{m} \in U$ cannot be expanded in terms of states in $\mathcal{S}$. But then

[2]If H is required to be self-adjoint, then f must be linear in $\hat{a}$ and H must be of the form $H = \omega(t)a^\dagger a + f(t)a^\dagger + f^*(t)a + \beta(t)$. Note there are even more general Hamiltonians under which certain proper subsets of coherent states are stable (see [22,23]).

$$\int_{\mathcal{M}-U} \mathrm{d}\nu(\mathfrak{m})|\psi_{z(\mathfrak{m})}\rangle\langle\psi_{z(\mathfrak{m})}| \neq \mathbb{1}_{\mathcal{H}},$$

which contradicts the assumptions. □

Note this does not exclude parametrizations that break down at single points nor does it exclude parametrizations that differ in region A from this in region B if the resolution can be modified accordingly, that is, if $\int_{\mathcal{M}}$ splits into $\int_A + \int_B$. Especially the last point is of interest since in many models the dynamics can be fundamentally different in separated regions, and therefore it might be necessary to consider different complexifiers for those regions. It is even possible that in certain areas of $\mathcal{M}$ one cannot find stable states while for others there exist some. For example, inside a finite potential well the motion is periodic which suggests that there exist stable states while outside the well the motion is unbounded. These issues will play a significant role in Sec. IV. For now the main aim is to find solutions to (17).

B. First solutions

For a one-dimensional not-explicitly time dependent system Eq. (17) reduces to

$$\sum_{n=0}^{\infty} \frac{i^n}{n!}\{C,H\}_{(n)} = ipf(q) + g(q), \tag{18}$$

where $\{C,H\}_{(n)} = \{C,\{C,H\}_{(n-1)}\}$ and f and g are arbitrary smooth functions. To find a first solution of this equation consider the following ansatz:

$$H = \frac{p^2}{2m} + V(q), \qquad C = \frac{p^2}{2m\omega} + U(q), \tag{19}$$

$$\text{and} \quad \{C,H\}_{(2N+1)} = 0 \quad \text{for some } N \geq 1. \tag{20}$$

Here, m is the mass, V and U are twice continuously differentiable real valued functions, and $\omega > 0$ is a free parameter. Note that the even (odd) Poisson brackets in (19) can only contain summands of even (odd) powers of the momentum. Therefore, f and g in (19) have to be real and satisfy

$$\sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!}\{C,H\}_{(2n+1)} = pf(q), \tag{21}$$

$$\sum_{n=0}^{N} \frac{(-1)^n}{(2n)!}\{C,H\}_{(2n)} = g(q). \tag{22}$$

A model with $\mathfrak{f}$ degrees of freedom can allow at most $\mathfrak{f}$ independent commuting Hamiltonian vector fields (first integrals of motion), which is why for $\mathfrak{f} = 1$ condition (20) is equivalent to

$$\{C,H\}_{(2N)} = \sum_{j=0}^{N} \beta_j C^j. \tag{23}$$

The restriction on the degree of the polynomial is due to the fact that $\{C,H\}_{(2N)}$ can be at most a polynomial of degree $2N$ in p. For $N = 1$ Eqs. (21), (22), and (23) are replaced by

$$\{C,H\} = pf(q), \tag{24}$$

$$H - \frac{1}{2}\{C,H\}_{(2)} = g(q), \tag{25}$$

$$\text{and} \quad \{C,H\}_{(2)} = \beta_1 C + \beta_0. \tag{26}$$

For this choice the first two conditions in (19) are equivalent to

$$(\omega^{-1}V' - U')/m = f \quad \text{and} \quad \frac{p^2}{2m\omega}f' - fU' = 2(H-g),$$

where $'$ denotes the derivative with respect to q. Thus f' must be equal to ω and β_1 must be equal to 2ω. Inserting this into (26) yields

$$fU' = -2\omega[U + \beta_2],$$

which is solved by $f = \omega q + \alpha$ and $U = \frac{\lambda}{\omega}(q + \alpha/\omega)^{-2} + \beta_2$ for some real constants λ and α. Since α just defines a shift in the configuration variable, it can be set to zero without loss of generality. Concluding,

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2 + \lambda q^{-2}, \qquad C = \frac{p^2}{2m\omega} + \frac{\lambda}{\omega}q^{-2} \tag{27}$$

solves (19) and (20) for $N = 1$. If λ is zero, this reduces to the usual Hamiltonian and complexifier of the harmonic oscillator. The term λ/q^2 can be interpreted as the angular momentum contribution of a two-dimensional oscillator with constant radius which justifies the name *radial oscillator*. The complexification $z := e^{-i\mathcal{L}_{\chi_C}}q$ can be computed by solving the "equation of motion"

$$\frac{\mathrm{d}}{\mathrm{d}s}q = \{C,q\} \tag{28}$$

for $q(s) := e^{s\mathcal{L}_{\chi_C}}q$ and then extending it analytically $(s \to -i)$. Instead of integrating Eq. (28) one can integrate

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2 + \frac{\lambda}{m\omega^2}q^{-2}\right] = 0 \Rightarrow \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2 + \frac{\lambda}{m\omega^2}q^{-2} = \frac{C}{m\omega}$$

since $\frac{\mathrm{d}}{\mathrm{d}s}C = 0$ and $\{C, q\} = p/(m\omega)$. This finally gives

$$z = q(s = -i) = \left[\left(q - i\frac{p}{m\omega}\right)^2 - 2\frac{\lambda}{m\omega^2}q^{-2}\right]^{1/2}.$$

Despite the fact that z depends on a square root, one can derive a well-defined quantum operator whose eigenfunctions $\psi_z = [e^{-\hat{C}/\hbar}\delta_y]_{y\to z}$ can be expressed in terms of modified Bessel functions of the second kind. Furthermore, the obtained states share all desired properties; that is, they form an overcomplete set, minimize the uncertainty, have small fluctuations, and are, of course, stable under the dynamics. More details on the quantization of this system can be found in Appendix A.

C. A no-go theorem

After the method applied in the preceding section has been proven so successful, it is worth testing whether more solutions to (18) can be found by generalizing the ansatz (19). Given two real valued, strictly positive, continuously differentiable functions α and β on the configuration space let

$$H = \alpha(q)\frac{p^2}{2} + V(q) \quad \text{and} \quad C = \frac{1}{2}[\beta(q)p]^2 + U(q)$$
$$\text{and suppose } \{C, H\}_{(2N+1)} = 0 \quad \text{for some } N > 1. \tag{29}$$

Since β should be nonzero in order that the complexifier is well defined, there exists a canonical transformation sending

$$p \to p\beta(q) \quad \text{and} \quad q \to \int_0^q \mathrm{d}x[\beta(x)]^{-1}$$

so that the first line in (29) may be exchanged through

$$H = \alpha(q)p^2 + V(q) \quad \text{and} \quad C = \frac{p^2}{2} + U(q) \tag{30}$$

without loss of generality. To compute the multiple Poisson brackets of (30) it turns out to be handy to introduce the operators $\hat{X} = p\frac{\partial}{\partial q}$ and $\hat{P} = -U'\frac{\partial}{\partial p}$ which yields

$$\{C, H\}_{(n)} = \hat{X}^n H + \sum_{\nu=0}^{n-1}\hat{X}^{n-1-\nu}\hat{P}\hat{X}^\nu H + \sum_{\mu+\nu=0}^{n-2}\hat{X}^{n-1-\nu-\mu}\hat{P}\hat{X}^\nu\hat{P}\hat{X}^\mu H + \cdots. \tag{31}$$

The general strategy in the subsequent analysis is to reorder the terms by powers in the momentum. The degree in p of the terms $\hat{X}^{\nu_1}\hat{P}\hat{X}^{\nu_2}\cdots$ only depends on the number of operators $\sharp\hat{O}$ that are applied, i.e.

$$\deg_p \hat{X}^{\nu_1}\hat{P}\hat{X}^{\nu_2}\cdots V = \sharp\hat{X} - \sharp\hat{P}$$

and

$$\deg_p \hat{X}^{\nu_1}\hat{P}\hat{X}^{\nu_2}\cdots\alpha p^2 = \sharp\hat{X} - \sharp\hat{P} + 2,$$

where $\deg_p$ denotes the degree in p. Note that the total number of operators, $\sharp\hat{X} + \sharp\hat{P}$, corresponds to the grade n of the bracket $\{C, H\}_{(n)}$. For this reason the bracket contains only either even or odd powers in p depending on whether n is even or odd, respectively. This can be made more explicit by replacing (31) through

$$\{C, H\}_{(n)} = \sum_{\substack{\mu\in\mathbb{N}\\ 0\le\mu\le(n+2)/2}} f^n_{n+2-2\mu}p^{n+2-2\mu}. \tag{32}$$

The coefficients $f^n_{n+2-2\mu}$ are derived from (31); that is,

$$f^n_{n+2} \coloneqq \alpha^{(n)},$$
$$f^n_n \coloneqq -\sum_\nu^{n-1}(\nu+2)\left(\frac{\partial}{\partial q}\right)^{n-1-\nu}U'\alpha^{(\nu)} + V^{(n)}, \ldots, \tag{33}$$

where $g^{(n)}$ is the nth derivative of a function g with respect to q and the upper index of f^n_m refers to the total number of operators while the lower one indicates the power in p. Instead of trying to directly compute these coefficients it is much more useful to consider the following recursion relation:

$$f^n_m = \frac{\partial}{\partial q}f^{n-1}_{m-1} - U'(2m+1)f^{n-1}_{m+1}, \quad m \le n+2, \tag{34}$$

with $f^0_0 \equiv V$ and $f^0_2 \equiv \alpha$.

Furthermore, because of $\frac{1}{2}\deg_p\{C, H\}_{(2N)} \le N+1$ and $\{C, H\}_{(2N+1)} = 0$ one finds that

$$\{C, H\}_{(2N)} = \sum_{m=0}^{N+1} a_m C^m \tag{35}$$

for some constants a_m. Apart from that, all summands in (18) whose degree in p is exceeding one must cancel which implies that

$$\sum_{n=m-1}^{N}\frac{(-)^n}{(2n)!}f^{2n}_{2m} = 0 \quad \text{and} \quad \sum_{n=m-1}^{N-1}\frac{(-)^n}{(2n+1)!}f^{2n+1}_{2m+1} = 0 \tag{36}$$

for all $m > 1$. Note there can be only one nontrivial term of power $2N+2$ and one of $2N+1$ which is why f^{2N}_{2N+2} and f^{2N-1}_{2N+1} have to vanish. This in turn implies $\deg_C\{C, H\}_{(2N)} \le N$ and $\deg_q\alpha \le 2N-2$ since f^{2N-1}_{2N+1} is

proportional to $\alpha^{(2n-1)}$. The next nontrivial contributions are those of degree $2N$ and $2N-1$. Because of Eq. (36) the coefficients f^{2N}_{2N} and f^{2N-1}_{2N-1} must satisfy

$$f^{2N}_{2N} = (2N)(2N-1)f^{2N-2}_{2N} \quad \text{and}$$
$$f^{2N-1}_{2N-1} = (2N-1)(2N-2)f^{2N-3}_{2N-1}. \tag{37}$$

Yet, taking the derivative of the equation on the right hand side and inserting equality (34) yields

$$\frac{\partial}{\partial q} f^{2N-1}_{2N-1} = (2N-1)(2N-2)\frac{\partial}{\partial q} f^{2N-3}_{2N-1} \Leftrightarrow f^{2N}_{2N}$$
$$= (2N-1)(2N-2)f^{2N-2}_{2N},$$

which obviously contradicts the first equation in (37). Thus, f^m_{2N} must vanish for all m. With the next term one can proceed in the same manner. Combining (36), (34), and $f^{2N+1}_{2N-1} = 0$ gives

$$0 = \frac{\partial}{\partial q}\sum_{n=N-2}^{N}\frac{(-)^n}{(2n)!}f^{2n}_{2N-2} = \sum_{n=N-2}^{N-1}\frac{(-)^n}{(2n)!}f^{2n+1}_{2N-1}.$$

This in turn contradicts the second equation in (37) and therefore implies $f^m_{2N-1} = 0$ for all m which also reduces the degree of the polynomials α and $\{H, C\}_{2N}$ and proves the following lemma.

Lemma 2. For $N > 1$ there exists an integer M with $2 \le M < 2N-1$ such that $f^n_m = 0$ for all $m > M$ and all n.

The above reasoning can be repeated to show also that summands of a lower degree in the momentum have to vanish. But note the number of conditions needed to derive a contradiction is increasing when the degree in p is decreasing since there are more and more nontrivial coefficients that contribute. Nevertheless, one can obtain a new condition for the coefficients f^{2n+1}_* from

$$0 = \sum_{n=m-2}^{N}\frac{(-)^n}{(2n)!}f^{2n}_{2m} \quad \forall\ m > 0 \tag{38}$$

by taking the derivative and using (34). More specifically, this yields

$$0 = \sum_{n=m-2}^{N-1}\frac{(-)^n}{(2n)!}[f^{2n+1}_{2m+1} - (2m+2)U' f^{2n}_{2m+2}]$$
$$= \sum_{n=m-2}^{N-1}\frac{(-)^n}{(2n)!}f^{2n+1}_{2m+1} \quad \forall\ m > 0. \tag{39}$$

From this one can deduce another condition,

$$\sum_{n=m-2}^{N-1}\frac{(-)^n}{(2n)!}f^{2n+2}_{2m+2} \quad \forall\ \mu > 0, \tag{40}$$

by first taking the derivative and then applying (34) and (39) and so forth. Repeating this procedure also for the odd terms generates $2m-1$ independent conditions for the terms of power $2m$ and $2m+1$. On the other hand, the number of nontrivial terms $\sharp f^*_m$ in a tower of constant power m increases as $\sharp f^*_{2m} = N - m + 1$ and $\sharp f^*_{2m+1} = N - m + 2$. Since the procedure breaks up as soon as $\sharp f^*_m$ is greater than the number of available conditions, namely if $\sharp f^*_{2m/2m+1} \ge 2m-1$, another trick is needed to eliminate more coefficients.

Lemma 3. Let $C = \frac{p^2}{2} + U(q)$, $F = \sum_{n=0}^m F_n C^n$, and let $G = G(p,q)$ be a phase space function that is polynomial in p.

(a) If $\deg_p G = 2m+1$, then $\{C, G\} = F(C)$ has a solution iff $U = c_1 q + c_0$ for some constants $c_1, c_0 \in \mathbb{R}$.
(b) If $\deg_p G = 2m-1$, then $\{C, G\} = F(C)$ has a solution iff $U = c_1(q+c_2)^{-2} + c_0$ for some constants $c_2, c_1, c_0 \in \mathbb{R}$.

Proof.—Proof of (a): Recall that the complexifier is the only constant of motion with respect to the flow generated by C itself so that the homogeneous solutions to the first order partial differential equations (PDE) $\{C, G\} = F(C)$ are functions of C. Therefore, the most general solution G with $\deg_p G = 2m+1$ is of the form

$$G = F(C)g(p,q) + g_h(C).$$

Here, g_h is any polynomial of C whose degree is not exceeding $\deg_C F$ and $g = g(p,q)$ is a linear function in p for which

$$1 \overset{!}{=} \{C, g\} = p\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}U'.$$

Since U and $\frac{\partial g}{\partial p}$ are independent of p, this can be solved only if g and U' are constant in q.

Proof of (b): Without loss of generality one can assume that $F_0 = 0$ because the Poisson bracket does not change when C is shifted by a constant. Suppose $g = ph_1(q) + h_0(q)$; then

$$G = \frac{F(C)}{C}g(p,q) + g_h(C)$$

is a generic solution that has the right degree in p iff

$$C \overset{!}{=} \{C, g\} = p(ph_1' + h_0') - h_1 U'.$$

This implies that h_0 is constant, $h_1 = \frac{1}{2}(x + c_2)$, and $2U = -(x+c_2)U'$ which is solved by $U = c_1(q+c_2)^{-2} + c_0$. □

Remember that Lemma 2 guarantees the existence of a nonzero integer $m_0 < N$, for which $\deg_C\{C, H\}_{(2N)} = m_0$ and $\deg_p\{C, H\}_{(2N-1)} = 2m_0 \pm 1$, and that the potential U must be either linear in q (for $2m_0 + 1$) or proportional to

q^{-2} (for $2m_0 - 1$) according to Lemma 3. The degree in q of the functions f^{μ}_{ν} can be directly determined from Eq. (33), that is,

$$\deg_q f^n_{n+2} = \deg_q \alpha - n$$

and

$$\deg_q f^n_{n+2-2m} = \max\left[\deg_q \alpha - n + m + m\deg_q U', \deg_q V - n + m - 1 + (m-1)\deg_q U'\right].$$

Suppose the coefficients $f^*_{2m_0+1}$ are not zero; then they have to be constant as $f^*_{2m_0+2} = 0$. In this case, Lemma 3 states that U' also has to be constant, which is why

$$0 \overset{!}{=} \deg_q f^{2m_0-1}_{2m_0+1} = \deg_q \alpha - (2m_0 - 1)$$

and

$$0 \overset{!}{=} \deg_q f^{2m_0+1}_{2m_0+1} = \max\left[\deg_q \alpha - (2m_0+1) + 1, \deg_q V - (2m_0+1) + 1 - 1\right].$$

This forces $\deg_q \alpha = 2m_0 - 1$ and $\deg_q V = 2m_0 + 1$. Yet,

$$0 \overset{!}{=} \deg_q f^{2m_0+3}_{2m_0+1} = \max\left[2m_0 - 1 - (2m_0+3) + 2, 2m_0 - (2m_0+3) + 1\right] < 0$$

leads to an inconsistency, and thus $f^*_{2m_0+1}$ has to vanish, which means that $f^*_{2m_0}$ must be constant. By using Lemma 3(b) one can deduce that the potential U is proportional to q^{-2} and consequently $\deg_q \alpha$ must be equal to $2m_0 - 2$ and $\deg_q V$ must equal $2m_0$. Again

$$0 \overset{!}{=} \deg_q f^{2m_0+2}_{2m_0} < 0$$

leads to a contradiction so that all $f^*_{2m_0}$ have to vanish. By repeating this argument one can finally show that f^*_m must vanish for all $m > 2$. Hence, f^*_2 is constant and $f^{n-1}_1 = -f^n_2 x + c^{n-1}_1$ for some $c^{n-1}_1 \in \mathbb{R}$ which can be achieved only if α is constant as well and U is proportional to x^{-2}. Obviously, the brackets $\{H, C\}_{(2n)}$ and $\{H, C\}_{(2m)}$, n, $m \neq 0$, can only differ by an overall constant so that already $\{H, C\}_{(3)} = 0$. This proves the following theorem.

Theorem 2. Suppose the Hamiltonian and the complexifier are quadratic in the momentum p; then the only system that solves $\{C, H\}_{(2N+1)} = 0$ for some $N > 0$ and (18) is the radial oscillator (27) and canonical conjugates thereof. Furthermore, N equals 1.

Note all the equations used to prove this theorem are directly related to Poisson brackets and for this reason only hold up to canonical transformations. This point will be exploited heavily in the next section to derive a more general construction principle for a stable coherent state system.

IV. ADAPTED COMPLEXIFIERS FOR INTEGRABLE SYSTEMS

The preceding investigations have revealed that it is in general very hard to construct a complexifier adapted to the dynamics of a given model. Nevertheless, the derived criteria are form invariant under canonical transformation which opens the possibility to access a wider class of models than those examined above. A common feature of the harmonic and the radial oscillators is their periodic motion, which is why integrable systems that show a quasiperiodic motion seem to be especially promising candidates. This idea is to be elucidated in more detail in the subsequent section. In the first part a generalized construction principle for adapted complexifiers will be worked out using so-called action-angle coordinates and the Hamilton-Jacobi approach. This will then be tested on several examples.

A. Generalized construction principle

Throughout this section it will be assumed that the mechanical models in question are not explicitly time dependent and possess only a finite number $\mathfrak{f}$ of degrees of freedom.

A *first integral of motion*[3] is a C^1 function f on the phase space $\mathcal{M}$ that Poisson commutes with the Hamiltonian H, i.e. $\{H, f\} = 0$ on the entire phase space. Two functions $g, f \in C^1(\mathcal{M})$ are said to be in *involution* if $\{g, f\} = 0$ and (functionally) independent on a subset $U \subset \mathcal{M}$ if the one-forms $\mathrm{d}f$ and $\mathrm{d}g$ are linearly independent on U. If f and g are independent, then in particular $\mathrm{d}f \wedge \mathrm{d}g$ is nonzero on U.

Definition 4 *(Integrable system).* A system with $\mathfrak{f}$ degrees of freedom is integrable if there exist $\mathfrak{f}$ first integrals H_j, $j = 1, \ldots, \mathfrak{f}$, in involution that are independent of a dense subset of $\mathcal{M}$.

The name is motivated by the fact that such systems are integrable by quadratures; that is, they are solvable by a finite number of algebraic operations. This insight goes back to Liouville and was later enlarged by Arnold by the following (see e.g. [26]): If the system is integrable, then the level sets

[3]The term "first integral" often refers to a global property while "constant of motion" is used in a more local context. Yet, the nomenclature is far from being unique; here the conventions of [26] will be used.

$$M_{\mathbf{h}} = \{\mathfrak{m} \in \mathcal{M} | H_j(\mathfrak{m}) = h_j\} \tag{41}$$

are smooth submanifolds that are invariant under the phase flow generated by the Hamiltonian H. If $M_{\mathbf{h}}$ is compact and connected, then it is diffeomorphic to the torus $T^{\mathfrak{f}}$ and the motion is conditionally periodic. This means that $M_{\mathbf{h}}$ has coordinates Θ_j which parametrize the circles S^1 in $T^{\mathfrak{f}}$ and which evolve as

$$\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{h}). \tag{42}$$

For this statement to hold it actually suffices that the first integrals are independent on $M_{\mathbf{h}}$. Furthermore, this angle coordinates can be used to parametrize the phase space (in the neighborhood of the invariant torus). Its conjugated momenta I_j can be found by a canonical transformation that, by definition, leaves the symplectic structure invariant. So

$$\sum_{j=1}^{\mathfrak{f}} \mathrm{d}p_j \wedge \mathrm{d}q_j = \sum_{j=1}^{\mathfrak{f}} \mathrm{d}I_j \wedge \mathrm{d}\Theta_j.$$

The parameters $(\mathbf{I}, \boldsymbol{\Theta})$ are called action-angle coordinates in the literature and are widely used in classical perturbation theory (see e.g. [26,27] for more details). Here, their simple time dependence is of interest, which is given by

$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{I}). \tag{43}$$

This immediately shows that the complex parametrization,

$$\mathrm{w}_j \coloneqq \sqrt{I_j} e^{i\theta_j} \quad \text{and} \quad \overline{\mathrm{w}}_j \coloneqq \sqrt{I_j} e^{-i\theta_j}, \tag{44}$$

is "stable" in the sense that it obeys

$$\frac{\mathrm{d}\mathrm{w}_j}{\mathrm{d}t} = \{H, \mathrm{w}_j\} = i\omega_j(\mathbf{I})\mathrm{w}_j. \tag{45}$$

In addition, the pair $(\overline{\mathbf{w}}, \mathbf{w})$ is canonical conjugated as

$$\{\overline{\mathrm{w}}_j, \mathrm{w}_k\} = i\delta_{jk} \quad \text{and} \quad \{\mathrm{w}_j, \mathrm{w}_k\} = \{\overline{\mathrm{w}}_j, \overline{\mathrm{w}}_k\} = 0$$

and the polar decomposition of w_j reads

$$\mathrm{w}_j = \frac{1}{\sqrt{2}}(Q_j - iP_j),$$

where

$$Q_j = \sqrt{2I_j}\cos\theta_j \quad \text{and} \quad P_j = -\sqrt{2I_j}\sin\theta_j.$$

Since $(\mathbf{P}, \mathbf{Q})$ are also canonical conjugated, w_j can be written as a complexifier coordinate,

$$\mathrm{w}_j = \frac{1}{\sqrt{2}} e^{-i\mathcal{L}_{\chi^C}} Q_j,$$

with $C = \frac{1}{2}\mathbf{P}\cdot\mathbf{P}$. But note the parametrization given by (44) does not satisfy the criteria of theorem 1 since the frequencies[4] ω_j still depend on $I_j = \mathrm{w}_j\overline{\mathrm{w}}_j$. If the system is nondegenerate, which is the case if $\det\frac{\partial\omega_j}{\partial I_k} \neq 0$, then the invariant tori are uniquely defined and independent of the initial choice of the coordinates $(\boldsymbol{\Theta}, \mathbf{I})$. This implies that, no matter which action-angle coordinates are selected, the frequencies will always depend on $\mathbf{I}$. In other words (44) can only give rise to stable states if the system is degenerate. More specifically, if $H = \sum_j \omega_j I_j$ for constant ω_j, then the associated coherent states are stable because $\frac{\partial\omega_j}{\partial I_k} = 0$ for all j, k.

Before bothering about degeneracy one has to solve the more practical problem of how to determine action variables in the first place. A very useful tool for that is the Hamilton-Jacobi formalism: Given a Hamiltonian H as a function of $\mathbf{p}$ and $\mathbf{q}$, the goal is to find a function $S(q_1, \ldots, q_{\mathfrak{f}}, h_1, \ldots, h_{\mathfrak{f}})$ with $\det\frac{\partial^2 S}{\partial q_j \partial H_k} \neq 0$ and $\frac{\partial S}{\partial q_j} = p_j$ such that

$$H\left(q_1, \ldots, q_{\mathfrak{f}}, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_{\mathfrak{f}}}\right) = K(h_1, \ldots, h_{\mathfrak{f}}). \tag{46}$$

Note S generates a canonical transformation since

$$\mathrm{d}S = \sum_j (p_j \mathrm{d}q_j + \Theta_j \mathrm{d}h_j) \Rightarrow \sum_j \mathrm{d}p_j \wedge \mathrm{d}q_j = \mathrm{d}h_j \wedge \mathrm{d}\Theta_j \tag{47}$$

with $\Theta_j \coloneqq \frac{\partial S}{\partial h_j}$. Suppose one can find such a solution S to the first order PDE (46); then, by (47), one has also found $\mathfrak{f}$ independent constants of motion[5] h_j in involution. If $\det\frac{\partial^2 S}{\partial q_j \partial H_k} \neq 0$ as required, then $h_j = h_j(\mathbf{p}, \mathbf{q})$ can be extracted out of the equation $\frac{\partial S}{\partial q_j} = p_j$ due to the inverse function theorem. Apart from that the dynamics of Θ_j also simplifies, i.e.

$$\frac{\mathrm{d}\Theta_j}{\mathrm{d}t} = \{H, \Theta_j\} = \frac{\partial K}{\partial h_j}.$$

By setting $K(h_1, \ldots, h_{\mathfrak{f}}) = E = h_1$ the dynamics becomes especially simple, instead of $\mathfrak{f}$ constants of motion one now has $2\mathfrak{f} - 1$ constants. This seems rather odd, notably if it would hold globally, as it effectively reduces the system to a one-dimensional free particle. Astonishingly, the

[4] In general, $\frac{\partial\omega_j}{\partial I_k} = \frac{\partial^2 H}{\partial I_k \partial I_j} = \frac{\partial\omega_k}{\partial I_j}$.

[5] We here use the term "constants of motion" to indicate that they are in general not globally defined.

Hamilton-Jacobi equation (46) with $K = E$ always has a *local* solution[6] given by the classical action functional (see e.g. [26–28]). This demonstrates that in the above construction the global properties are essential. On the other hand, one can also fix $K(h_1, \ldots, h_\mathfrak{f}) = \sum_j h_j$ locally which proves the following lemma.

Lemma 4.—For each generic point $\mathfrak{m} \in \mathcal{M}$ exists a neighborhood $U_\mathfrak{m}$ on which the functions

$$\mathrm{w}_j \coloneqq \sqrt{h_j} e^{i\omega_j \Theta_j} \quad \text{and} \quad \overline{\mathrm{w}}_j \coloneqq \sqrt{h_j} e^{-i\omega_j \Theta_j}, \quad j = 1, \ldots, \mathfrak{f},$$

are well defined, are functionally independent, are canonical conjugated ($\{\overline{\mathrm{w}}_j, \mathrm{w}_k\} = i\omega_j \delta_{jk}$), and obey

$$H = \sum_j \mathrm{w}_j \overline{\mathrm{w}}_j \quad \text{and} \quad \{H, \mathrm{w}_j\} = i\omega_j \mathrm{w}_j.$$

Similarly one can introduce coordinates $Q_j \coloneqq \sqrt{\frac{h_j}{2\omega_j}} \cos \omega_j \Theta_j$ and $P_j \coloneqq -\sqrt{\frac{h_j}{2\omega_j}} \sin \omega_j \Theta_j$ which are well defined, functionally independent, and canonical conjugated on $U_\mathfrak{m}$. In terms of these coordinates w_j can be rephrased as

$$\mathrm{w}_j = \frac{1}{\sqrt{2\omega_j}} e^{-i\mathcal{L}_{\chi_C}} Q_j \quad \text{with} \quad C = \frac{1}{2}\mathbf{P}^2.$$

As already mentioned the w_j's will in general not define good coordinates on $\mathcal{M}$ but can only be defined locally. Below it will be demonstrated on the simplest example of a free particle what happens if the global properties are ignored (see Sec. IV B 2). In contrast to that the complex parametrization defined through proper action-angle coordinates is at least well defined in the neighborhood of the whole torus.[7]

In order to better understand what are the necessary criteria for the existence of well-defined parametrizations w_j it is a good idea to investigate further the relation between the w variables and those obtained by the complexifier method. The question is whether $z_j = e^{i\mathcal{L}_{\chi_C}} q_j$ can be obtained by Lemma 4 given that $\mathbf{z}$ define good coordinates, i.e. they are everywhere defined and functionally independent, and given that

$$\{H, z_j\} = i f_j(\mathbf{z}). \tag{48}$$

[6]That means that a solution exists in a neighborhood of any generic point $\mathfrak{m} \in \mathcal{M}$ on which H is not extremal.

[7]In fact, this parametrization only breaks down at separatrices where $M_\mathbf{h}$ even ceases to be a manifold. Such separatrices divide the phase space into several regions on which the level sets may have different properties. An example for that is the mathematical pendulum where the phase space divides into an oscillatory and a rotational branch (see e.g. [26]).

In general, z_j will *not* be equal to w_j since w_j is canonically conjugated to its complex conjugate $\bar{\mathrm{w}}_j$, but

$$\{\bar{z}_j, z_k\} = \{e^{i\mathcal{L}_{\chi_C}} q_j, e^{-i\mathcal{L}_{\chi_C}} q_k\} = e^{i\mathcal{L}_{\chi_C}} \{q_j, e^{-2i\mathcal{L}_{\chi_C}} q_k\}$$

is generically not even constant (see Sec. III B for an example). The momenta conjugated to z_j and $\bar{z}_j$ are $\Pi_j = e^{-i\mathcal{L}_{\chi_C}}(p_j + u_j(q_j))$ and $\Pi_j = e^{i\mathcal{L}_{\chi_C}}(ip_j + u_j(q_j))$ where u_j is any function that should depend only on q_j to ensure that the momenta are Poisson commuting. Note that the u_j are not arbitrary but, because of the stability criterion (17), must be such that

$$H = \sum_j (i f_j(\mathbf{z}) \Pi_j + g_j(\mathbf{z})) = \sum_j (-i\bar{f}_j(\bar{\mathbf{z}}) \bar{\Pi}_j + \bar{g}(\bar{\mathbf{z}})) \tag{49}$$

with $g_j(\mathbf{q}) = \frac{1}{\mathfrak{f}} g(\mathbf{q}) - u_j(q_j)$. In general the functions f_j, u_j, and g_j can depend on complex parameters, which is why $\overline{f_j(\mathbf{z})} = \bar{f}_j(\bar{\mathbf{z}})$.

Even though the z and w-parametrizations are generically different, they are still closely related. To see this let us first derive a set of $\mathfrak{f}$ constants of motion of (49). A function H_k is a constant of motion of (49) iff

$$\{e^{i\mathcal{L}_{\chi_C}} H, e^{i\mathcal{L}_{\chi_C}} H_k\} = \left\{ \sum_j f_j(\mathbf{q}) p^j + g(\mathbf{q}), F_k \right\} \overset{!}{=} 0,$$

where $F_k \coloneqq e^{i\mathcal{L}_{\chi_C}} H_k$. Additionally, the maps F_k must be in involution which suggests the ansatz $F_1 \coloneqq e^{i\mathcal{L}_{\chi_C}} H$ and $F_j = F_j(\mathbf{q})$ for $j = 2, \ldots, \mathfrak{f}$. This yields a system of $\mathfrak{f} - 1$ linear, first order PDE

$$0 \overset{!}{=} \sum_j f_j(\mathbf{q}) \frac{\partial F_k}{\partial q_j} \tag{50}$$

that can be solved locally by the method of characteristics (see Appendix B). A local solution of such an equation depends on a set of $\mathfrak{f}$ arbitrary initial functions so that it should in general not be problematic to find $\mathfrak{f} - 1$ functionally independent ones at least for a sufficiently small neighborhood. Yet in practice, it can be very hard to actually determine them, and it might be easier to use the Hamilton-Jacobi formalism. In the following, let

$$F(\mathbf{p}, \mathbf{q}) \coloneqq e^{i\mathcal{L}_{\chi_C}} H = \sum_j (i f_j(\mathbf{q}) p^j + g_j(\mathbf{q})) \tag{51}$$

and

$$\bar{F}(\mathbf{p}, \mathbf{q}) \coloneqq e^{-i\mathcal{L}_{\chi_C}} H = \sum_j (-i\bar{f}_j(\mathbf{q}) p^j + \bar{g}_j(\mathbf{q})). \tag{52}$$

As before, we want to find maps $S(q_1,\ldots,q_{\mathfrak{f}},F_1,\ldots,F_{\mathfrak{f}})$ and $\tilde{S}(q_1,\ldots,q_{\mathfrak{f}},\bar{F}_1,\ldots,\bar{F}_{\mathfrak{f}})$ that satisfy $\frac{\partial S}{\partial q_j}=\frac{\partial \tilde{S}}{\partial q_j}=p_j$,

$$F\left(q_1,\ldots,q_{\mathfrak{f}},\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_{\mathfrak{f}}}\right)=\sum_j F_j \quad \text{and}$$
$$\bar{F}\left(q_1,\ldots,q_{\mathfrak{f}},\frac{\partial \tilde{S}}{\partial q_1},\ldots,\frac{\partial \tilde{S}}{\partial q_{\mathfrak{f}}}\right)=\sum_j \bar{F}_j. \tag{53}$$

If $\det\frac{\partial^2 S}{\partial q_j\partial F_k}\neq 0$ and $\det\frac{\partial^2 \tilde{S}}{\partial q_j\partial \bar{F}_k}\neq 0$, then S and $\tilde{S}$ are generators of canonical transformations that map $(\mathbf{p},\mathbf{q})$ on the conjugated pairs $(\mathbf{F},\Phi)$ and $(\bar{\mathbf{F}},\bar{\boldsymbol{\phi}})$, respectively, where $\frac{\partial S}{\partial F_j}\coloneqq\Phi_j$ and $\frac{\partial \tilde{S}}{\partial \bar{F}_j}\coloneqq\bar{\phi}_j$. In these variables the reality condition (49) can be replaced by

$$\begin{aligned} e^{-i\mathcal{L}_{\chi_C}}F_j &= if_j(\mathbf{z})\Pi_j+g_j(\mathbf{z})=-i\bar{f}_j(\bar{\mathbf{z}})\Pi_j+\bar{g}_j(\bar{\mathbf{z}}) \\ &= e^{i\mathcal{L}_{\chi_C}}\bar{F}_j\coloneqq h_j. \end{aligned} \tag{54}$$

Together with the angles

$$\Theta_j\coloneqq\frac{1}{2}(e^{-i\mathcal{L}_{\chi_C}}\Phi_j+e^{i\mathcal{L}_{\chi_C}}\bar{\phi}_j) \tag{55}$$

the action variables h_j build a new canonical pair $(\mathbf{h},\boldsymbol{\Theta})$ because

$$\{h_j,\Theta_k\}=\frac{1}{2}[e^{-i\mathcal{L}_{\chi_C}}\{F_j,\Phi_k\}+e^{i\mathcal{L}_{\chi_C}}\{\bar{F}_j,\bar{\phi}_k\}]=\delta_{jk},$$

which can be used to construct a w-parametrization as in Lemma 4. Since Φ_j and $\bar{\phi}_j$ only depend on $\mathbf{q}$ and $\mathbf{F}$ and $\bar{\mathbf{F}}$, the new angle coordinates are of the form

$$\Theta_j=\frac{1}{2}[\Phi_j(\mathbf{z},\mathbf{h})+\bar{\phi}_j(\bar{\mathbf{z}},\mathbf{h})]. \tag{56}$$

It still remains to check whether $\mathrm{w}_j=\sqrt{h_j}e^{i\omega_j\Theta_j}$ are good coordinates that are everywhere defined and independent. For this it is unavoidable to solve the Hamilton-Jacobi equations (53). This is particularly easy if the Hamilton-Jacobi equation is completely separable, that is, if F is such that $F=\sum_j F_j(q_j,\frac{\partial S_1}{\partial q_j})$. In this case, $S=\sum_j S_j(q_j,F_j)$ and $\tilde{S}=\sum_j\tilde{S}_j(q_j,\bar{F}_j)$ are solutions of (53) with

$$\begin{aligned} S_j(q,F) &= c_j(F)+\int_{q_0}^{q}\mathrm{d}x\,\frac{F_j-g_j(x)}{if_j(x)}, \\ \tilde{S}_j(q,\bar{F}) &= \tilde{c}_j(\bar{F}_j)-\int_{q_0}^{q}\mathrm{d}x\,\frac{\bar{F}_j-\bar{g}_j(x)}{i\bar{f}_j(x)} \end{aligned} \tag{57}$$

Here, c_j and $\tilde{c}_j$ are arbitrary functions which will be set to zero in the following. To simplify the notation also the label j will be left away. From ansatz (57) one can deduce that

$$\begin{aligned} \Phi(q) &= -i\int_{q_0}^{q}\mathrm{d}x\,\frac{1}{f(x)} \quad \text{and} \\ \bar{\phi}(q) &= \bar{\Phi}(q)=i\int_{q_0}^{q}\mathrm{d}x\,\frac{1}{\bar{f}(x)}, \end{aligned} \tag{58}$$

which yields

$$\Theta=-\frac{i}{2}\left[\int_{z_0}^{z}\mathrm{d}x\,\frac{1}{f(x)}-\int_{\bar{z}_0}^{\bar{z}}\mathrm{d}x\,\frac{1}{f(x)}\right].$$

Thus,

$$\mathrm{d}\Theta=\frac{-i}{2}(f(z)^{-1}\mathrm{d}z-\overline{f(z)}^{-1}\mathrm{d}\bar{z}) \tag{59}$$

is well defined for all points $\mathfrak{m}\in\mathcal{M}$ on which $f(\mathfrak{m})=f(q)\neq 0$. Moreover, since the Hamiltonian is real, it will in most cases be a function of $z\bar{z}$ so that

$$\begin{aligned} \mathrm{dw}\wedge\mathrm{d}\bar{\mathrm{w}} &= \frac{i\omega}{2}\mathrm{d}\Theta\wedge\mathrm{d}h \\ &= \frac{\omega}{4}\left(f(z)^{-1}\frac{\partial h}{\partial\bar{z}}+\overline{f(z)}^{-1}\frac{\partial h}{\partial z}\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \end{aligned} \tag{60}$$

is most likely nonvanishing and $\mathrm{w}=\sqrt{h}e^{i\omega\Theta}$ will provide a good parametrization if z does.

The last thing that remains to check is whether the level sets (41) are compact, that is, whether $(\boldsymbol{\Theta},\mathbf{h})$ are proper action variables. Note that if Θ is not a proper angle, then w is multivalued and therefore the parametrization will break down on the points where $\Theta(\mathfrak{m}')=\Theta(\mathfrak{m})+n\pi$. Concluding, if (59) and (60) are well-defined nowhere vanishing differential forms, then $\boldsymbol{\Theta}$ is most likely a parametrization of an invariant torus (for a counterexample see Sec. IV B 2).

Even though the above considerations do not prove that integrability and compactness of the level sets are necessary conditions for stable coherent states, it nevertheless uncovers a close relation between the existence of (global) action-angle coordinates and those states. In particular, one can only hope to find a good global z-parametrization if the system is integrable, has compact level sets, and has a degenerate dynamics ($\det\frac{\partial^2 H}{\partial I_j\partial I_k}=0$).

B. Examples

To illustrate the construction principle described in Sec. IV A and to emphasize its advantages and drawbacks, some examples will be discussed.

1. Radial oscillator

Apart from the harmonic oscillator the radial oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 + \lambda q^{-2}$$

and phase space $\mathcal{M} = \{(p,q) \in \mathbb{R}^2 | q \neq 0\}$ was the only one-dimensional model with stable coherent states found in Sec. III B. To obtain stable states one has to consider the complexifier

$$C = \frac{1}{\omega}\left(\frac{p^2}{2} + \lambda q^{-2}\right)$$

for which

$$e^{i\mathcal{L}_{\chi_C}} H = i\omega pq + g(q) \coloneqq F$$

and

$$z \coloneqq e^{-i\mathcal{L}_{\chi_C}} q = \left[\left(q - \frac{i}{\omega} p\right)^2 - 2\frac{\lambda}{\omega^2} q^{-2}\right]^{\frac{1}{2}}.$$

Starting with that, it is possible to construct a w parametrization displaying all the properties mentioned in Lemma 4 by applying the Hamilton-Jacobi method with generating function

$$S = -i \int_{q_0}^{q} \mathrm{d}x \frac{1}{\omega x}(F - g(x)).$$

One easily verifies $\frac{\partial^2 S}{\partial q \partial F} = -\frac{i}{\omega q} \neq 0$ for all $\mathfrak{m} \in \mathcal{M}$, $\frac{\partial S}{\partial q} = p$, and $F(q, \frac{\partial S}{\partial q}) = F$. The angles, which are canonically conjugated to F and $\bar{F}$, are

$$\Phi = \frac{\partial S}{\partial F} = -\frac{i}{\omega} \ln x \quad \text{and} \quad \bar{\Phi} = \frac{\partial \bar{S}}{\partial \bar{F}} = \frac{i}{\omega} \ln x, \tag{61}$$

and the angle conjugated to h is

$$\Theta \coloneqq -\frac{i}{2\omega}(\ln z - \ln \bar{z}).$$

This shows that the w-parametrization is given by

$$\mathrm{w} \coloneqq \sqrt{h} e^{i\omega\phi} = \sqrt{\frac{hz}{\bar{z}}} = \sqrt{\frac{h}{\bar{z}z}} z = \sqrt{\frac{\omega h/2}{\sqrt{h^2/\omega^2 - 2\lambda}}} z,$$

where the last equality follows from

$$z\bar{z} = \left[\left(q^2 - \frac{p^2}{\omega^2}\right)^2 + 4\frac{\lambda}{\omega^2}\left(\frac{\lambda}{\omega^2} q^{-4} + \frac{p^2 q^2}{\omega^2} - 1\right)\right]^{\frac{1}{2}} = \frac{2}{\omega}\left[\frac{h^2}{\omega^2} - 2\lambda\right]^{\frac{1}{2}}.$$

Of course, the functional dependence of w and $\bar{\mathrm{w}}$ on the original parametrization (p,q) is much more complicated than for z and $\bar{z}$. However, their algebraic properties are nicer. By construction $\{\bar{\mathrm{w}}, \mathrm{w}\} = i\omega$, while for z and $\bar{z}$ holds

$$\{\bar{z}, z\} = \left\{\sqrt{\frac{|z|^2}{h}}\bar{\mathrm{w}}, \sqrt{\frac{|z|^2}{h}}\mathrm{w}\right\} = \frac{4i}{\omega^3 |z|^2} H.$$

Since the Hamiltonian is strictly positive on the phase space also $\frac{h^2}{\omega^2} - 2\lambda$ must be greater than zero. Because of that and since z and $\bar{z}$ are well defined and independent, w and $\bar{\mathrm{w}}$ are everywhere defined and independent. Besides that, the level sets are compact.

The complex parametrization w can as well be brought into the complexifier form defining

$$Q \coloneqq \sqrt{\frac{h/4}{\sqrt{h^2/\omega^2 - 2\lambda}}} \mathrm{Re}(z) \quad \text{and}$$

$$P \coloneqq -\sqrt{\frac{h/4}{\sqrt{h^2/\omega^2 - 2\lambda}}} \mathrm{Im}(z)$$

and $C = P^2/2$. Then $w = \frac{1}{\sqrt{2\omega}} e^{-i\mathcal{L}_{\chi_C}} Q = \frac{1}{\sqrt{2\omega}}(Q - iP)$.

Instead of starting with the z-parametrization it is equally well allowed to construct directly the action-angle coordinates of the model. With the generating function

$$\tilde{S}(q,h) = \int_{q_0}^{q} \mathrm{d}x \sqrt{2(h - V(x))}$$

the Hamilton-Jacobi approach leads to $H(q, \frac{\partial \tilde{S}}{\partial q}) = h$, $\frac{\partial \tilde{S}}{\partial q} = p$, and

$$\tilde{\Phi} \coloneqq \frac{\partial \tilde{S}}{\partial h} = \int_{q_0}^{q} \frac{\mathrm{d}x}{\sqrt{2(h - V(x))}} = \frac{1}{2\omega} \arcsin \frac{y}{\Omega}, \tag{62}$$

where $\Omega^2 = h^2/\omega^2 - 2\lambda$, $y = \omega q^2 - h/\omega$, and q_0 is fixed such that $y_0 = 1$ for simplicity. Since $\arcsin x = -i \ln(ix + \sqrt{1 - x^2})$, the w-parametrization, which one obtains from this ansatz, is equal to the above up to a complex phase depending on the value of q_0. Here, $\tilde{\mathrm{w}} = e^{i\pi/4}\mathrm{w}$. Yet, the integration of (62) is more involved than that of (61).

2. Free particle

The free particle with Hamiltonian $H = \frac{p^2}{2}$ is a good example for what can go wrong if the global properties are

ignored. Locally the Hamilton-Jacobi equation, $H(q, \frac{\partial S}{\partial q}) = h$, is solved by the generating function

$$S(q,h) = \sqrt{2h}q$$

with conjugated "angle"

$$\Theta := \frac{\partial S}{\partial h} = \frac{q}{\sqrt{2h}} = \frac{q}{p},$$

which is well defined only for $p \neq 0$. This yields

$$\begin{aligned} \mathrm{w} &:= \sqrt{h}e^{i\omega\Theta} = p(\cos q/p + i\sin \omega p/q) \\ &= \frac{1}{\sqrt{2\omega}} e^{-i\mathcal{L}_{\chi_C}} Q, \end{aligned}$$

where $Q = \frac{1}{\sqrt{2\omega}} p\cos\frac{q}{p}$, $P = -\frac{1}{\sqrt{2\omega}} p\sin\frac{q}{p}$, and $C = P^2/2$. Locally, the differentials dw and $\mathrm{d}\bar{\mathrm{w}}$ are well defined and

$$\mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}h \wedge \mathrm{d}\Theta = \mathrm{d}P \wedge \mathrm{d}Q = \frac{1}{i\omega}\mathrm{dw} \wedge \mathrm{d}\bar{\mathrm{w}}$$

is nondegenerate. But it is not possible to extend that to all of $\mathcal{M}$ since, first, the whole parametrization is ill-defined for $p = 0$ and, second, the coordinates $(\mathrm{w}, \bar{\mathrm{w}}, P, Q)$ are multivalued; that is, $\mathrm{w}(q,p) = \mathrm{w}(q',p)$ for $\omega\frac{q}{p} = 2\pi n + \omega\frac{q'}{p}$ and $n \in \mathbb{N}$. Even more severe, the motion of w seems to be periodic,

$$\begin{aligned} \mathrm{w}(t) &= \mathrm{w}(p_0, q(t)) \\ &= p_0[\cos(\omega t + q_0/p_0) + i\sin(\omega t + q_0/p_0)]. \end{aligned}$$

All these problems arise from Θ not being a proper angle since the level sets $h = \text{const}$ are not compact. On the other hand, for $\omega \ll q/p$ the parametrization is not too bad since up to second order

$$P \approx -p + \mathcal{O}(\omega^2) \quad \text{and} \quad Q = -\omega q + \mathcal{O}(\omega^3).$$

Another interesting property of w is that on the level sets, i.e. for constant momenta, w are the generators of the $*$-algebra of quasiperiodic functions.[8] The spectrum of this algebra constitutes the so-called Bohr compactification that lies at the heart of loop quantum cosmology (LQC, see e.g. [29–32]).

3. Anharmonic oscillators

For a generic one-dimensional Hamiltonian of the form $H := \frac{p^2}{2} + V(q)$ one can always solve the Hamilton-Jacobi equation locally by choosing

$$S = \int_{q_0}^{q} \mathrm{d}x\sqrt{2(h - V(x))}. \tag{63}$$

To conclude, we will discuss two examples of an anharmonic oscillator that are widely studied in the literature: the Pöschl-Teller potential [33]

$$V_{P.T.}(q) = -\frac{\lambda(\lambda+1)}{2}(\cosh(q))^{-2},$$

which is an effective potential to describe vibrations in diatomic molecules and for which the Schrödinger equation is solvable, and the quartic potential

$$V_4(q) = \frac{\omega^2}{2}q^2 + \frac{\lambda^2}{2}q^4,$$

which is a standard example in perturbation theory.

For large q the Pöschel-Teller potential is exponentially suppressed since for large q $\cosh(q) \approx e^{|q|}/2$, and hence the level sets are compact. Moreover, a local w-parametrization is conceivable. Ansatz (63) yields

$$\begin{aligned} \Theta := \frac{\partial S}{\partial h} &= \int_0^q \frac{\mathrm{d}x\cosh x}{\sqrt{2h\cosh^2 x + \lambda(\lambda+1)}} \\ &= (2h)^{-1/2}\int_0^q \frac{\mathrm{d}(\sinh x)}{\sqrt{\sinh^2 x + \Omega^2}} \\ &= (2h)^{-1/2}\sinh^{-1}\left(\frac{\sinh q}{\Omega}\right), \end{aligned}$$

where $\Omega^2 = \frac{\lambda(\lambda+1)}{2h} + 1$, and

$$\mathrm{w} = \sqrt{h}e^{i\omega\Theta} = \frac{\sqrt{h}}{\Omega}\left[\sinh q + \sqrt{\Omega^2 + (\sinh q)^2}\right]^{\frac{i}{\sqrt{2h}}}.$$

In the last equation the identity $\sinh^{-1} q = \ln[q + \sqrt{1+q^2}]$ was used. The corresponding P, Q variables are even more complicated so that the quantization via the complexifier method can cause severe problems.

While the motion for the quartic potential is also bounded, the angle Θ is even harder to determine. In fact,

$$\Theta := \int_{q_0}^{q} \frac{\mathrm{d}x}{\sqrt{2h - \omega^2 x^2 - \lambda^2 x^4}}$$

is an incomplete elliptic integral of the first kind[9] that *is not* solvable in terms of elementary functions. By replacing $x = -b\cos\phi$ and

[8] These are functions f of the form $f(x) = \sum_{j\in\mathcal{I}} f_j e^{ik_j x}$ for some finite label set $\mathcal{I}$ and $k_j \in \mathbb{R}$. Here, $k_j = \omega_j/p_0$.

[9] For more information see e.g. [34,35].

$$2h - \omega^2 x^2 - \lambda^2 x^4 = \lambda^2 (x^2 + a^2)(b^2 - x^2),$$

where $a^2 - b^2 = (\omega/\lambda)^2$ and $b^2\omega^2 + b^4\lambda^2 = 2h$, this integral can be brought into the so-called Legendre canonical form

$$\Theta = \frac{1}{\lambda}\int_{q_0}^{q} \frac{\mathrm{d}x}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{k}{\lambda b} F(\phi, k) + c. \quad (64)$$

Here, $k^2 = \frac{b^2}{b^2 + a^2}$, c is a constant, and

$$F(\phi, k) = \int_0^{\sin\phi} \frac{\mathrm{d}\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}} = \int_0^{\phi} \frac{\mathrm{d}\psi}{\sqrt{1 - k^2 \sin^2\psi}}.$$

A literal quantization of the corresponding complexifier seems hopeless. On the other hand, the quartic anharmonic oscillator can be treated, classically as well as quantum mechanically, by a perturbation theory. Especially the classical perturbation theory makes heavy use of action-angle coordinates of the unperturbed harmonic oscillator. It therefore seems much more promising to study the w-parametrizations in this context; however, this would go beyond the scope of this paper.

V. DISCUSSION

It was found that a necessary criterion for the existence of stable coherent states is that the classical evolution of the variable $\mathbf{z} = e^{-i\mathcal{L}_{\chi_C}}\mathbf{q}$ depends only on $\mathbf{z}$ itself and not on its complex conjugate. However, it is not possible to determine other models than the harmonic and radial oscillators from ansatz (29) that allow for such parametrizations. This issue was circumvented by invoking the Hamilton-Jacobi formalism. In doing so, one can, indeed, construct local complex coordinates for almost any system that display the desired properties and are related to the complexifier formalism. In general, these cannot, however, be extended globally to what was explained exemplarily for the free particle. Another issue arises from the fact that the resulting complexifiers and parametrizations are in general not analytic functions of the "old" variables (p, q) (see Sec. IV B) that potentially cause severe problems when quantizing.

Let us speculate a bit more about the last point: One essential ingredient for quantization is the choice of a polarization. Very roughly, the polarization says how to split the classical phase space $\mathcal{M}$ of dimension $2\mathfrak{f}$ into an $\mathfrak{f}$-dimensional submanifold $\mathcal{P}$ whose elements are then represented as multiplication operators on an appropriate Hilbert space. Of course, for $\mathcal{M} = \mathrm{T}^*\mathcal{C}$ the natural choice of $\mathcal{P}$ is the configuration space $\mathcal{C}$, which corresponds to the usual Schrödinger quantization. In the case of the harmonic oscillator we already mentioned implicitly another possible polarization, namely, that defined by the complex parameter $a = q - ip$. This leads to the Hilbert space $\mathfrak{H}^2(\mathbb{C}, \mathrm{d}\nu)$ of a holomorphic square integrable function on which the annihilator $\hat{a}$ acts by multiplication. The resulting quantum theory is unitary equivalent to the Schrödinger quantization, and the transformation between the two representations is given by the Segal-Bargann transform (10). This close relation between coherent states and complex polarizations is much more generic and often used in geometric quantization.

As was mentioned in Sec. II C, the complexifier approach follows exactly the above line of thoughts: It intends to provide a link between an arbitrarily chosen complex Kähler structure and a quantization on the corresponding polarizations. This relation was examined in great detail in [36–39], focusing especially on unitary equivalence of the resulting quantum theories. An interesting result of [37] is that for compact Lie groups G with phase space T^*G only complexifiers of the form $\hat{C} = \frac{\Delta}{2}$ give rise to a quantization that is unitary equivalent to the Schrödinger one with Hilbert space $L^2(G, \mathrm{d}x)$. Here, Δ is the Laplacian of the group and $\mathrm{d}x$ is the Haar measure. But formally Δ can be identified with the square $\hat{p}^2$ of the original momentum operators while the complexifiers obtained for the various examples in Sec. IV B show a much more complicated functional dependence on p. Thus, one should not expect that a quantum model obtained by directly quantizing the systems in the new w-parametrization is equivalent to the Schrödinger representation.

Apart from the complicated structure of the complexifier it is also questionable whether the above strategy can be applied if the w-parametrization is ill-defined. All of these are interesting questions which deserve a further investigation but go beyond the scope of this paper.

Remember, the original motivation for investigating the stability of coherent states originated from the heuristic implementation of the Hamiltonian constraint in LQG by the formal expression (1). Yet, the formalism developed in Sec. IV can blow up already for "simple" one-dimensional models. The technical obstacles encountered in the finite dimensional examples of this paper are of course even more challenging in quantum field theories (QFT) such as LQG. On the other hand, the formalism developed in this paper in principle directly applies to symmetry reduced models of LQG such as LQC. In fact, there are two observations which support the assumption that the above formalism is conceivable in LQC: The first hint is the appearance of the Bohr compactification, which plays a crucial role in the quantization of LQC, in the stable state system of the free particle. The second hint is more vague; namely, the stable parametrizations z, $\bar{z}$ of the radial oscillator form an $\mathfrak{sl}(2, \mathbb{R})$ algebra together with the Hamiltonian H. But this algebra also shows up in the modified dynamics of deparametrized, homogeneous, and isotropic models.

ACKNOWLEDGMENTS

T. T. thanks José Mourão, Joao Nunes, and William Kirwin for several fruitful discussions. This project was supported in part by funds from the Emerging Field Initiative of the Friedrich-Alexander-University Erlangen Nürnberg to the Emerging Field Project "Quantum Geometry." A. Z. acknowledges financial support by the Grant No. 2012/05/E/ST2/03308 of Narodowe Centrum Nauki (NCN).

APPENDIX A: STABLE COMPLEXIFIER STATES FOR THE RADIAL OSCILLATOR

1. Classical and quantum properties of the radial oscillator

Note that the potential of the radial oscillator,

$$V(q) = \frac{m\omega^2}{2}q^2 + \lambda q^{-2},$$

with mass m, frequency ω, and coupling constant λ diverges in the limit $q \to 0$. Therefore, the classical motion is confined to the positive (or negative) axes. The equation of motion can be solved, for example, by integrating the law of conversation of energy via a separation of variables. This yields

$$q(t) = \pm\left[\frac{E}{m\omega^2} + \frac{\sqrt{\gamma}}{\sqrt{m}\omega}\sin\left(2\sqrt{m}\omega t + \phi\right)\right]^{\frac{1}{2}},$$

where $\gamma = \frac{E^2}{\sqrt{m}\omega}^2 - 2\lambda$ and the phase ϕ is determined by the initial condition.

From the form of the potential, one expects the quantized system to be discrete. In the limit $\lambda \to 0$, the solutions should approach the solutions of the harmonic oscillator. Because of the barrier at $q = 0$, only the odd solutions, i.e. those solutions which have a knot at zero, are allowed. For "big" λ the solutions on the positive and negative axes should decouple. This can be checked explicitly by replacing the peak at $q = 0$ through a box potential of width $d = 2\sqrt{\frac{\lambda m}{\omega^2}}$ and height V_0. The transmission coefficient T for such a system with energy E is then given by

$$T = \left(1 + \frac{V_0^2 \sinh^2(\sqrt{2(V_0 - E)}d)}{4E(V_0 - E)}\right)^{-1},$$

which vanishes for $V_0 \to \infty$. Therefore, also the quantum system can be restricted to the positive axis. The natural choice for a Hilbert space is the space $L^2(\mathbb{R}^+, \mathrm{d}q)$ of square integrable functions with respect to the Lebesgue measure $\mathrm{d}q$ where $\hat{q}$ and $\hat{p}$ are represented as multiplication and derivative operators, respectively.

The corresponding eigenvalue equation,

$$\hat{H}\psi(x) = E\psi(x) \Rightarrow \psi''(x) - \left(y^2 + \frac{\alpha^2 - 1}{4} + 2\epsilon\right)\psi(x) = 0$$

with $x := \sqrt{\frac{m\omega}{\hbar}}$, $\epsilon := \frac{E}{\omega\hbar}$, and $\alpha^2 := 8m\lambda + 1$, is a modified Whittaker equation (see [34], p. 505) whose solutions are given by a hypergeometric function of the first kind M; that is,

$$\psi(x) = Ne^{-\frac{x^2}{2}}x^{\frac{1+\alpha}{2}}M\left[\frac{1}{4}(2 + \alpha - 2\epsilon), \frac{2+\alpha}{2}, x^2\right],$$

where N is a normalization constant. To ensure that ψ is an element of the Hilbert space, the following conditions must be satisfied:

(i) $\lim_{x\to 0}\psi(x) < \infty$ requires $\alpha \geq -1$. Otherwise ψ cannot be square integrable.

(ii) In order that the Hamiltonian is well defined, i.e. $\|\hat{H}\psi\| < \infty$, ψ needs to be at least once continuously differentiable for $x \to 0$ (and twice everywhere else). This requires $\alpha^2 \geq 1 \Leftrightarrow \lambda \geq 0$.

(iii) ψ is square integrable iff $\frac{1}{4}(2+\alpha-2\epsilon) = -n, n \in \mathbb{N}$. Thus the energy spectrum is discrete and equidistant

$$E_n = \omega\hbar\left(2n + 1 + \frac{\alpha}{2}\right). \tag{A1}$$

In the limit $\lambda \to 0$ the eigenvalues approach the odd energy levels of the harmonic oscillator as expected, and M reduces to a Hermite polynomial ([34], p. 505).

For these specific values the hypergeometric function M becomes proportional to a generalized Laguerre polynomial $\mathrm{L}_n^{\frac{\alpha}{2}}(x)$, $n \in \mathbb{N}$, which builds a complete orthogonal system in $L^2(\mathbb{R}^+, y^{\frac{\alpha}{2}}e^{-y}\mathrm{d}y)$. Thus, the normalized eigenstates are given by

$$\psi_n(x) = \sqrt{\frac{2n!}{\Gamma(\frac{\alpha}{2} + 1 + n)}}e^{-\frac{x^2}{2}}x^{\frac{\alpha+1}{2}}\mathrm{L}_n^{\alpha/2}(x^2). \tag{A2}$$

Remark: Since the eigenvalue equation for the radial oscillator is a second order partial differential equation there exist of course two independent solutions. Here the second linear independent solution is

$$\tilde{\psi}(x) = Ne^{-\frac{x^2}{2}}x^{\frac{1-\alpha}{2}}M\left[\frac{1}{4}(2 - \alpha - 2\epsilon), \frac{2-\alpha}{2}, x^2\right]. \tag{A3}$$

These functions have to obey analogous normalization conditions

$$\alpha \leq 1, \qquad E_n = \omega\hbar\left(2n + 1 - \frac{\alpha}{2}\right).$$

In order that the Hamiltonian is well defined on these states also requires $\alpha^2 \geq 1$. Therefore λ has to be greater than zero. On the other hand, α is equal to $\pm\sqrt{1 + 8m\lambda}$. Thus either $\alpha \geq 1$ or else $\alpha \leq -1$. For $\alpha > 0$ we get the solutions above, and for $\alpha < 0$ we obtain (A3).

2. Coherent states

Recall that the complex parametrization $z = e^{-\imath\mathcal{L}_{\chi_C}} q$ associated with the complexifier

$$C = \frac{1}{m\omega}\left(\frac{1}{2}p^2 + m\lambda q^{-2}\right) \tag{A4}$$

is given by

$$z = \sqrt{\frac{2}{m\omega}}\sqrt{a^2 - m\lambda q^{-2}}, \tag{A5}$$

where $a = \frac{m\omega}{2}q - i\frac{p}{2}$. The Poisson relations for these variables are

$$\{z, \bar{z}\} = -\imath\frac{4}{\omega^3 m^2}\frac{H}{z\bar{z}},$$
$$\{H, z\} = \imath\omega z, \quad \text{and} \quad \{H, \bar{z}\} = -\imath\omega\bar{z}.$$

For the operators of the rescaled maps

$$L_3 := \frac{\omega}{2}H, \qquad L_- := \frac{m\omega^2}{4\sqrt{2}}z^2, \quad \text{and} \quad L_+ := \frac{m\omega^2}{4\sqrt{2}}\bar{z}^2,$$

one recovers the commutation relations of $\mathfrak{su}(1,1)$,

$$[\hat{L}_3, \hat{L}_\pm] = \pm\hat{L}_\pm \quad \text{and} \quad [\hat{L}_-, \hat{L^+}] = \hat{L}_3. \tag{A6}$$

Therefore, it is expected that the complexifier coherent states coincide with the coherent states defined by Barut and Giradello (see [40]).

The easiest way to find the associated coherent states

$$\psi_z(q) = e^{-\frac{\hat{C}}{\hbar}}\delta_z(q) \tag{A7}$$

is to express the convolution δ in terms of the eigenfunctions of $\hat{C}$ that are given in terms of Bessel functions of the first kind $\mathrm{J}_\beta(x)$ (see e.g. [34], pp. 358–364). That is, the eigenfunction to the eigenvalue $\hbar\omega c^2, c > 0$ is

$$\Phi_c(x) = \sqrt{cx}\mathrm{J}_{\alpha/2}(cx),$$

where $x = \sqrt{\frac{m\omega}{\hbar}}$. These functions are not square integrable on $\mathbb{R}^+$ as expected but define a convolution,

$$\delta_y(x) = \int_0^\infty \mathrm{d}c\Phi_c(x)\Phi_c(y). \tag{A8}$$

Thus, one finds

$$\psi_z(x) = \left[\int_0^\infty \mathrm{d}c\, c e^{-\frac{c^2}{2\hbar}}\sqrt{xy}\mathrm{J}_{\alpha/2}(cx)\mathrm{J}_{\alpha/2}(cy)\right]_{y\to z}$$
$$= \exp\left(-\frac{x^2 + z^2}{2}\right)\mathrm{I}_{\alpha/2}(xz). \tag{A9}$$

Here, $\mathrm{I}_\beta(x) = \imath^{-\beta}\mathrm{J}_\beta(\imath x)$ is the modified Bessel function of the first kind. These states can also be expressed in terms of the eigenfunctions of the Hamiltonian by utilizing the relation of Bessel functions and Laguerre polynomials (see [34], p. 734). This yields

$$\psi_z(x) = e^{-\frac{z^2}{4}}\sum_{n=0}^{\infty}(-1)^n\frac{(z/\sqrt{2})^{2n+\frac{\alpha+1}{2}}}{\sqrt{n!\Gamma(n + \alpha/2 + 1)}}\Psi_n(x). \tag{A10}$$

For $z \to z^2/\sqrt{2}$ and $x \mapsto x^2/2$ one recovers the coherent states defined by Barut and Giradello [40] up to a constant factor as expected. However, the states (A10) are not normalized with respect to the real measure $\mathrm{d}x$. Instead, one finds

$$\|\psi_z\|_{L^2(\mathbb{R}^+)} = \frac{|z|}{2}e^{-\frac{z^2+\bar{z}^2}{4}}\mathrm{I}_{\alpha/2}\left(\frac{|z|^2}{2}\right). \tag{A11}$$

Properties: The states (A10) are overcomplete with respect to the measure

$$\mathrm{d}\mu(z) = \left[\mathrm{I}_0\left(\frac{r^2}{2}\right)\right]^{-1}\frac{r^2}{4\pi}\mathrm{K}_{\alpha/2}\left(\frac{r^2}{2}\right)\mathrm{d}r\mathrm{d}\theta \tag{A12}$$

expressed in polar coordinates $z = re^{\imath\theta} r \geq 0, \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Here, $\mathrm{K}_\beta(x)$ is a modified Bessel function of the second kind. The unusual integration domain of ϕ originates from the fact that the analytic continuation $q \to z$ for the radial oscillator is defined only on the upper complex plane. Moreover, the states are truly stable under the dynamics, i.e.,

$$e^{\imath\frac{\omega}{\hbar}t\hat{H}}\psi_z(x) = e^{\imath\omega/2}\psi_{z(t)}(x).$$

Since (A10) are the eigenvectors of the annihilation operator $\hat{z}$, they minimize the uncertainty of

$$\hat{a} := \frac{\hat{z}^2 + (\hat{z}^2)^\dagger}{2} = -\frac{2}{\omega}\hat{C} + \hat{x}^2 \tag{A13}$$

and of

$$\hat{b} := \frac{\hat{z}^2 - (\hat{z}^2)^\dagger}{2} = -\frac{1}{\omega}(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}). \tag{A14}$$

APPENDIX B: PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Partial differential equations emerge in all kinds of physical problems and are widely studied in mathematics. A detailed treatise would go beyond this work. It is rather intended to give a rough overview of properties and solution techniques for quasilinear, first order PDE mentioned in the main text. For the interested reader we recommend the very detailed book by Hilbert and Courant [28] or the more elementary book by Cohen [41].

A first order PDE is called quasilinear if it is of the form

$$\sum_{j=1}^{n} a_j(\mathbf{x}, u)\frac{\partial u}{\partial x_j} = b(\mathbf{x}, u), \tag{B1}$$

where a_j and b are continuous differentiable functions of $\mathbf{x} \coloneqq (x_1, \ldots, x_n)$ and u. The equation is linear if a_j and b do not depend on u. A solution $u(\mathbf{x})$ to (B1) defines n-dimensional surfaces, called *integral surfaces*, whose tangential vectors $v_i = \frac{\partial u}{\partial x_i}$ satisfy (B1) at every point with coordinates $(\mathbf{x}, u)$.

To solve (B1) it is sufficient to determine a family of $(n-1)$-parametric *characteristic curves* $\mathbf{x}(s, t_1, \ldots, t_{n-1})$, $u(s, t_1, \ldots, t_{n-1})$ that obey

$$\frac{\mathrm{d}x_j}{\mathrm{d}s} = a_j(\mathbf{x}, u) \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = b(\mathbf{x})$$

with initial values $\phi_j(t_1, \ldots, t_{n-1}) = x_j(0, t_1, \ldots, t_{n-1})$ and $u(0, t_1, \ldots, t_{n-1}) = \chi_k(t_1, \ldots, t_{n-1})$. For a given set of initial data on the $(n-1)$-dimensional initial manifold C there exists a unique solution of (B1) iff the functional determinant

$$\Delta \coloneqq \det \begin{vmatrix} a_1 & \cdots & a_n \\ \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix}$$

does not vanish. If Δ is zero, then solutions of the initial value problem can exist only if C is a characteristic manifold; that means that C is generated by a family of $(n-2)$-parametric characteristic curves itself (see [28] for details). In this case, there exist infinitely many solutions.

A generic first order PDE can always be written in the form $H(\mathbf{x}, \mathbf{p}, u) = 0$ where $p_j = \frac{\partial u}{\partial x_j}$. Suppose H is twice continuously differentiable; then solutions can be found by a method similar to the above. Namely, one replaces the PDE by the system of ordinary differential equations (ODE)

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \sum_{j=1}^{n}\frac{\partial H}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}s} + \sum_{j=1}^{n}\frac{\partial H}{\partial p_j}\frac{\mathrm{d}p_j}{\mathrm{d}s} + \frac{\partial H}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s}$$

together with characteristic equations

$$\frac{\mathrm{d}x_j}{\mathrm{d}s} = \frac{\partial H}{\partial p_j}, \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = \sum_{j=1}^{n} p_j\frac{\partial H}{\partial p_j},$$
$$\frac{\mathrm{d}p_j}{\mathrm{d}s} = -\left(\frac{\partial H}{\partial x_j} + \frac{\partial H}{\partial u}p_j\right).$$

Note that all these methods are local implying that the existence of solutions holds only in an appropriate neighborhood of a point where the initial functions and the coefficients in (B1) are well behaved.

The general strategy behind the above is to replace the PDE by a system of ODEs which are considered easier to integrate. Yet, often, it is exactly the other way around that the PDE is easier to solve, for example by the method of separation of variables, than the system of ODEs. The idea of turning a system of ODEs into a PDE goes back to Hamilton and Jacobi and will be explained in a bit more detail in Appendix C in the context of classical mechanics.

APPENDIX C: HAMILTON-JACOBI METHOD AND CANONICAL TRANSFORMATIONS

This short summary of the Hamilton-Jacobi approach is mainly based on [26], and more information can be found in any good mechanics book, e.g. in [27]. To keep this small discourse as simple as possible, the discussion is restricted to time-independent, one-dimensional systems. Most of the formulas can be immediately generalized to models with more degrees of freedom. To also include time-dependent systems a bit more work would be required.

Definition 5 (Canonical transformation). A map $g\colon \mathcal{M} \to \mathcal{M}$ is a canonical transformation iff

$$g^*\Omega = \Omega,$$

where g^* denotes the pullback and Ω the symplectic form on $\mathcal{M}$.

By applying the Stokes theorem one can easily show that this condition is equivalent to

$$\oint_\gamma p\mathrm{d}q - P\mathrm{d}Q = 0 \tag{C1}$$

for any closed curve γ in $\mathcal{M}$. It is also well known that a canonical transformation leaves the equations of motion form invariant, which can be proven by examining the transformation of the one-form $\Omega_1 \coloneqq p\mathrm{d}q - H\mathrm{d}t$ on the extended phase space $\mathcal{M} \times \mathbb{R}^+$. To see why, remember that any alternating two-form in odd dimensions has at least one

null direction; i.e. it has at least one eigenvector to the eigenvalue zero. Here, this null direction is given by the integral curve $(p(t), q(t))$ for which

$$0 = \mathrm{d}\Omega_1(p(t), q(t)) = \mathrm{d}p \wedge \mathrm{d}q - \frac{\partial H}{\partial p}\mathrm{d}p \wedge \mathrm{d}t - \frac{\partial H}{\partial q}\mathrm{d}q \wedge \mathrm{d}t = \left(\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{\partial H}{\partial p}\mathrm{d}p \wedge \mathrm{d}t\right) + \left(\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial H}{\partial q}\right)\mathrm{d}q \wedge \mathrm{d}t.$$

This obviously implies the canonical equations of motion, and consequently Ω_1 captures the full dynamics.

Theorem 3 [Ref. [26]] Suppose $g: \mathcal{M} \to \mathcal{M}$ is a canonical transformation that maps (p, q) to (P, Q); then there exist functions $K(P, Q)$ and $S(p, q)$ so that

$$p\mathrm{d}q - H\mathrm{d}t = P\mathrm{d}Q - K\mathrm{d}t + \mathrm{d}S, \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{\partial K}{\partial Q}, \quad \text{and} \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial K}{\partial P}.$$

Proof.—Since condition (C1) has to hold for all closed curves γ, the one-form $p\mathrm{d}q - P\mathrm{d}Q$ is exact, which means that there exists a potential S with $\mathrm{d}S = p\mathrm{d}q - P\mathrm{d}Q$. Now set $K(P(p, q), Q(p, q)) = H(p, q)$. This proves the first part of the theorem. The second part follows directly from $\mathrm{d}^2 S = 0$. □

The function S is called a *generating function* of the canonical transformation g. For one-dimensional models there exist two types of such functions[10]:

(A) Suppose $\det\frac{\partial(Q,q)}{\partial(p,q)} \neq 0$; then the momentum can be written as a function of Q and q by the inverse function theorem. Inserting $p = p(Q, q)$ in S leads to $S(p, q) = S_1(Q, q)$. By comparison of $\mathrm{d}S$ and $\mathrm{d}S_1$ one finds

$$\frac{\partial S_1(Q, q)}{\partial q} = p \quad \text{and} \quad \frac{\partial S_1(Q, q)}{\partial Q} = -P.$$

(B) If $\det\frac{\partial(P,q)}{\partial(p,q)} \neq 0$, then $p = p(P, q)$. The corresponding generating function $S_2(P, q)$ is obtained via a Legendre transformation, that is, $S_2(P, q) = PQ + S(p, q)$. A comparison of the differentials yields

$$\frac{\partial S_2(P, q)}{\partial q} = p \quad \text{and} \quad \frac{\partial S_2(P, q)}{\partial P} = Q.$$

A given function S_1/S_2 generates a canonical transformation iff $\frac{\partial^2 S_1}{\partial q \partial Q} \neq 0 / \frac{\partial^2 S_2}{\partial q \partial P} \neq 0$. This nondegeneracy condition is needed to ensure that Q/P can be extracted as functions of p and q. In more dimensions it must be replaced by $\det\frac{\partial^2 S_1}{\partial(q,Q)} \neq 0$ or $\det\frac{\partial^2 S_2}{\partial(q,P)} \neq 0$, respectively.

The simplest example of a generating function is $S_2(P, q) = Pq$ that gives rise to the identity transformation. Another application is the Hamilton-Jacobi method. The main idea, hereby, is to transform the system such that the dynamics is especially simple. This is, of course, always the case if some coordinates are cyclic. Thus, one tries to find functions S_1 or S_2 so that

$$H\left(\frac{\partial S}{\partial q}, q\right) = K(P, t).$$

Then P is obviously constant and $Q(t) = \int_0^t \frac{\partial K}{\partial P}$ (for specific examples see Sec. IV B).

This is closely related to the action-angle coordinates that can be introduced for models with compact level sets $M_h = \{(p, q) | H(p, q) = h\}$. Here, a function $S_2(I, q)$ generating the transformation $(p, q) \mapsto (I, \phi)$ is constructed that obeys

$$\frac{\partial S_2(I, q)}{\partial q} = p, \qquad \frac{\partial S_2(I, q)}{\partial I} = \phi, \quad \text{and} \quad H\left(\frac{\partial S}{q}, q\right) = h(I).$$

For models with 1 degree of freedom, M_h being compact is equivalent with M_h being a closed curve in $\mathcal{M}$ that should be parametrized by ϕ. This leads to the additional requirements

$$I = I(h) \quad \text{and} \quad \oint_{M_h} \mathrm{d}\phi = 2\pi.$$

As shown in [26], the function

$$S_2(I, q) = \int_{\gamma_I(q_0,q)} p\mathrm{d}q,$$

where $\gamma_I(q_0, q)$ is a curve in $M_{h(I)}$ joining q_0 and q, meets all these requirements.

[10]For more than 1 degree of freedom there exists a third type depending on a mixture of Q and P variables.

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