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# Degenerate spacetimes in first order gravity

Romesh K. Kaul[1,*] and Sandipan Sengupta[2,†]

[1]*The Institute of Mathematical Sciences, Taramani, Chennai 600113, India*
[2]*Indian Institute of Technology, Gandhinagar 382355, India*
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We present a systematic framework to obtain the most general solutions of the equations of motion in first order gravity theory with degenerate tetrads. There are many possible solutions. Generically, these exhibit nonvanishing torsion even in the absence of any matter coupling. These solutions are shown to contain a special set of eight configurations which are associated with the homogeneous model three-geometries of Thurston.

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## I. INTRODUCTION

The usual theory of gravity based on the Einstein-Hilbert action functional involves an invertible metric. Solutions of the vacuum equation of motion are, by construction, torsion-free. On the other hand, first order gravity based on the Hilbert-Palatini action accommodates invertible as well as noninvertible tetrad configurations. The phase containing degenerate tetrads can support solutions of the vacuum equations of motion with torsion. In the quantum theory in first order formalism, configurations with both invertible and noninvertible tetrads are to be integrated over in the functional integral.

Gravity theory with degenerate metrics has evoked interest for a long time [1–12]. These metrics are expected to be relevant to the discussion of topology change [2,5,6,12–14]. Such a topology change may have a quantum and even a classical origin [12].

In this article, we present, in the first order formalism, a detailed analysis of degenerate tetrads with one zero eigenvalue. An elaborate procedure to solve the equations of motion will be developed. In particular, a set of eight explicit solutions of the equations of motion of pure gravity will be presented. These are associated with eight independent homogeneous model three-geometries of Thurston [15–17]. These include, besides the three isotropic constant curvature three-geometries $E_3$, $S_3$ and $H_3$, others which are homogeneous but not isotropic. It is remarkable that all such degenerate solutions of four-dimensional gravity theory are not generically torsion-free.

Examples of degenerate tetrad configurations as solutions of equations of motion have appeared earlier in the interesting work of Tseytlin [5]. In particular, two explicit solutions reported in this reference correspond to two special cases, $S_3$ and $S_2 \times R$, as discussed in Sec. V.

The article is organized as follows. In Sec. II, we recall the Hilbert-Palatini action functional without any cosmological constant or matter fields and write down the consequent equations of motion. Section III outlines the standard analysis for invertible tetrads to demonstrate the well-known fact that such a theory is equivalent to the usual theory based on the Einstein-Hilbert action. The equations of motion are exactly the same as the vacuum Einstein field equations. Section IV contains an elaborate discussion of degenerate tetrads with one zero eigenvalues. Equations of motion are shown to exhibit many possible solutions. Eight explicit solutions corresponding to Thurston's homogeneous three-geometries are displayed in detail in Sec. V. Next, the nature of the underlying geometry of these degenerate solutions is argued to be represented by Sen Gupta geometry [18] in Sec. VI. Finally, some concluding remarks are presented in Sec. VII. The Appendix contains details of the calculations used earlier in Sec. IV.

## II. HILBERT-PALATINI ACTION

Euclidean gravity in the first order formulation is described in terms of tetrad fields $e_\mu^I$ and connection fields ${\omega_\mu}^{IJ}$ corresponding to the local Lorentz group SO(4). Both these sets of fields are treated as independent in the Hilbert-Palatini action functional:

$$S = \frac{1}{8\kappa^2} \int d^4x \; \epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e_\mu^I e_\nu^J {R_{\alpha\beta}}^{KL}(\omega), \qquad (1)$$

where the curvature ${R_{\mu\nu}}^{IJ}(\omega) = \partial_{[\mu} {\omega_{\nu]}}^{IJ} + {\omega_{[\mu}}^{IK} {\omega_{\nu]}}^{KJ}$ is the field strength of the gauge connection ${\omega_\mu}^{IJ}$ of the local SO(4) symmetry of Euclidean gravity. Here the Greek indices $\mu \equiv (a, \tau)$ are associated with the spatial coordinates $a \equiv (x, y, z)$ and Euclidean time coordinate $\tau$. Internal indices are $I \equiv (i, 4)$, $i = 1, 2, 3$. Completely antisymmetric epsilon symbols take constant values 0 and $\pm 1$ with $\epsilon^{xyz\tau} = +1$ and $\epsilon_{1234} = +1$. Internal indices are raised and lowered by the flat metric $\eta^{IJ} = \delta^{IJ} = \eta_{IJ}$.

Euler-Lagrange equations of motion are obtained by varying the action (1) with respect to ${\omega_\mu}^{IJ}$ and $e_\mu^I$ independently:

*kaul@imsc.res.in
†ssandipan@iitgn.ac.in

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$$\frac{\delta S}{\delta \omega_\beta{}^{IJ}}: \qquad \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e_\mu^K D_\nu(\omega)e_\alpha^L = 0, \qquad (2)$$

$$\frac{\delta S}{\delta e_\mu^I}: \qquad \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e_\nu^J R_{\alpha\beta}{}^{KL}(\omega) = 0. \qquad (3)$$

An equivalent way to display these equations of motion is

$$e_{[\mu}^{[I}D_\nu(\omega)e_{\alpha]}^{J]} = 0, \qquad (4)$$

$$e_{[\mu}^{[I}R_{\nu\alpha]}{}^{JK]}(\omega) = 0. \qquad (5)$$

We need to solve these equations for the tetrads and connections. Since the Hilbert-Palatini action functional (1) accommodates both invertible and noninvertible tetrads, we may consider these two cases separately.

## III. INVERTIBLE TETRADS

For tetrads with $\det e_\mu^I \neq 0$, the inverse tetrad $e_I^\mu$ is given by $e_I^\mu e_\nu^I = \delta_\nu^\mu$, $e_I^\mu e_\mu^J = \delta_I^J$. Multiplying Eq. (4) by inverse tetrads, it is straightforward to check the following identities:

$$e_I^\mu e_{[\mu}^{[I}D_\nu(\omega)e_{\alpha]}^{J]} \equiv D_{[\nu}(\omega)e_{\alpha]}^J - e_\nu^J(e_I^\mu D_{[\alpha}(\omega)e_{\mu]}^I) + e_\alpha^J(e_I^\mu D_{[\nu}e_{\mu]}^I) = 0$$

$$\text{and} \quad e_I^\mu e_J^\nu e_{[\mu}^{[I}D_\nu(\omega)e_{\alpha]}^{J]} \equiv -4e_I^\mu D_{[\alpha}(\omega)e_{\mu]}^I = 0.$$

From these, it readily follows that, for invertible tetrads, 24 equations of motion in (4) are equivalent to the fact that the torsion is zero:

$$T_{\mu\nu}{}^I \equiv D_{[\mu}(\omega)e_{\nu]}^I = 0. \qquad (6)$$

As is well known, these 24 equations can in turn be solved for 24 connection fields showing that these are not independent but can be written in terms of the tetrad fields as

$$\omega_\mu^{IJ} = \omega_\mu^{IJ}(e) \equiv \frac{1}{2}[e_I^\nu \partial_{[\mu}e_{\nu]}^J - e_J^\nu \partial_{[\mu}e_{\nu]}^I - e_\mu^K e_I^\lambda e_J^\rho \partial_{[\lambda}e_{\rho]}^K]. \quad (7)$$

Another set of 16 equations of motion in (5), by multiplying with $e_I^\mu e_J^\nu$, yield the standard 16 equations of motion:

$$R_\alpha{}^K - \frac{1}{2}e_\alpha^K R = 0, \qquad (8)$$

where $R_\alpha{}^K \equiv e_I^\mu R_{\mu\alpha}{}^{IK}(\omega)$ and $R \equiv e_K^\alpha R_\alpha{}^K$. These equations are the same as Einstein field equations. This follows readily by realizing that the local Lorentz field strength, for invertible tetrads, is related to the Riemann curvature as

$$R_{\mu\nu}{}^{IJ}(\omega)e_\lambda^I e_J^\rho = R_{\mu\nu\lambda}{}^\rho(\Gamma). \qquad (9)$$

Thus, the first order formalism for invertible tetrads is exactly equivalent to the second order formalism based on the Einstein-Hilbert action functional.

It is important to notice that for invertible tetrads, solutions of the equations of motion would all be torsion-free.

As stated earlier, the Hilbert-Palatini action functional (1), and also the equations of motion (2) and (3) or (4) and (5), are well defined both for invertible tetrads ($\det e_\mu^I \neq 0$) and noninvertible tetrads ($\det e_\mu^I = 0$). Unlike the case above where $\det e_\mu^I \neq 0$, any solution of the equations of motion with degenerate tetrads can, in general, possess torsion. Degenerate tetrads can have one or more zero eigenvalues. We consider the case of tetrads with only one zero eigenvalue here.

## IV. DEGENERATE TETRADS WITH ONE ZERO EIGENVALUE

Through appropriate local $SO(4)$ rotations and general coordinate transformations, any degenerate tetrad $e_\mu^I$ with one zero eigenvalue can be cast as an invertible $3\times 3$ block of triads $e_a^i$ ($a = x, y, z$ and $i = 1, 2, 3$) with $e_\tau^I = e_a^4 = 0$ as follows:

$$e_\mu^I = \begin{pmatrix} e_a^i & 0 \\ 0 & 0 \end{pmatrix}. \qquad (10)$$

The four-dimensional metric is

$$g_{\mu\nu} = e_\mu^I e_\nu^I = \begin{pmatrix} g_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \qquad g_{ab} = e_a^i e_b^i.$$

We denote the determinant of the triad as $e$, $\det e_a^i \equiv e(\neq 0)$ and its inverse as $\hat{e}_i^a$; $\hat{e}_i^a e_b^i = \delta_b^a$, $\hat{e}_i^a e_a^j = \delta_i^j$. Note that the triad fields $e_a^i$ and the inverse $\hat{e}_i^a$ depend on all four spacetime coordinates $(x, y, z, \tau)$. The four-dimensional infinitesimal length element is $ds_{(4)}^2 = 0 + g_{ab}dx^a dx^b$.

Let us analyze the set of 24 equations in (4) for such degenerate tetrads. Unlike the case of invertible tetrads where these equations can be solved for all 24 components of the connection fields $\omega_\mu{}^{IJ}$ as in Eq. (7), here for the degenerate tetrads (10), Eq. (4) cannot be solved for all the components.

As shown in the Appendix, Eq. (4) can be solved to yield the following constraints for triads $e_a^i$ and connection fields $\omega_\mu{}^{IJ}$:

$$D_\tau(\omega)e_a^k = 0 \quad \text{where } \omega_\tau{}^{ij} = \bar{\omega}_\tau{}^{ij}(e) \equiv \hat{e}_i^a\partial_\tau e_a^j = e_a^i \partial_\tau \hat{e}_j^a, \qquad (11)$$

$$\omega_\tau{}^{4k} = 0; \qquad \omega_a{}^{4k} \equiv M_a^k = M^{kl}e_a^l \quad \text{with} \quad M^{kl} = M^{lk} \qquad (12)$$

$$\text{and} \quad \omega_a{}^{ij} = \bar{\omega}_a{}^{ij}(e) + \kappa_a{}^{ij}; \quad \kappa_a{}^{ij} \equiv \epsilon^{ijk}N_a^k = \epsilon^{ijk}N^{kl}e_a^l \quad \text{with} \quad N^{kl} = N^{lk}$$

$$\bar{\omega}_a{}^{ij}(e) \equiv \frac{1}{2}[\hat{e}_i^b \partial_{[a}e_{b]}^j - \hat{e}_j^b \partial_{[a}e_{b]}^i - e_a^l \hat{e}_i^b \hat{e}_j^c \partial_{[b}e_{c]}^l]. \qquad (13)$$

Here $\bar{\omega}_a{}^{ij}(e)$ and $\kappa_a{}^{ij}$ are the torsion-free Levi-Civita connection and contortion fields, respectively.

These equations state that the triads $e^i_a$ are covariantly conserved with respect to $\tau$, and the connection components $\omega_\tau{}^{4k}$ are fixed to zero. Of the nine independent fields $\omega_a{}^{4k}$, the three represented by the antisymmetric part of the matrix $M^{ij}$ are zero, the other six represented by the symmetric matrix $M^{ij}(=M^{ji})$ are not determined at all. Similarly, of the nine components of the contortion fields $\kappa_a{}^{ij}$, the six represented by the symmetric matrix $N^{ij}$ are left undetermined. Thus, for degenerate tetrads (10), Eq. (4) fixes only 12 independent fields in $\omega_\mu{}^{IJ}$ and leaves the other 12, as encoded by two symmetric matrices $M^{ij}$ and $N^{ij}$, undetermined. Some of these will be further fixed by other equations of motion (5) as discussed below.

Notice that Eq. (11) implies that the three-metric is $\tau$ independent: $\partial_\tau g_{ab} \equiv D_\tau(\omega)(e^i_a e^i_b) = 0$. Therefore, $\tau$ dependence of triads $e^i_a$ is only a pure gauge artifact and can be rotated away by an $SO(3)$ transformation. That is, for an appropriate orthogonal matrix $O^{ij}$, it is always possible to write

$$e^i_a = O^{ij} e'^j_a, \quad \bar{\omega}_\mu{}^{ij} = O^{il} O^{jk} \bar{\omega}'_\mu{}^{lk} + O^{il}\partial_\mu O^{jl}$$
$$\text{such that } \partial_\tau e'^i_a = 0, \quad \partial_\tau \bar{\omega}'_a{}^{ij}(e') = 0 \quad \text{and} \quad \bar{\omega}'_\tau{}^{ij} = 0. \tag{14}$$

As shown in the Appendix, the 16 tetrad equations of motion in (5), for degenerate tetrads (10), are equivalent to the following four sets of 3, 9, 3 and 1 equations, respectively:

$$\hat{e}^a_i R_{\tau a}{}^{ij}(\omega) = 0, \tag{15}$$

$$R_{\tau a}{}^{4k}(\omega) = D_\tau(\omega) M^k_a = 0, \tag{16}$$

$$\hat{e}^a_k R_{ab}{}^{4k}(\omega) = (e^l_b \hat{e}^a_i - \delta^a_b \delta^l_i) D_a(\bar{\omega}) M^{il} = 0, \tag{17}$$

$$\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) + (M^{ij}M^{ji} - M^{ii}M^{jj}) + (N^{ij}N^{ji} - N^{ii}N^{jj}) = 0, \tag{18}$$

where $D_a(\bar{\omega})M^{il} \equiv \partial_a M^{il} + \bar{\omega}_a{}^{ij}(e)M^{jl} + \bar{\omega}_a{}^{lj}(e)M^{ij}$ and $\bar{R}_{ab}{}^{ij}(\bar{\omega})$ is the curvature for the torsion-free Levi-Civita spin connection $\bar{\omega}_a{}^{ij}(e)$ of (13):

$$\bar{R}_{ab}{}^{ij}(\bar{\omega}) \equiv \partial_{[a}\bar{\omega}_{b]}{}^{ij} + \bar{\omega}_{[a}{}^{il}\bar{\omega}_{b]}{}^{lj}.$$

Equation (15) is identically valid for all configurations which satisfy Eqs. (11)–(13). To show this, note that $R_{\tau a}{}^{ij}(\omega) = \bar{R}_{\tau a}{}^{ij}(\bar{\omega}) + D_\tau(\bar{\omega})\kappa_a{}^{ij}$ where $\bar{R}_{\tau a}{}^{ij}(\bar{\omega}) \equiv \partial_{[\tau}\bar{\omega}_{a]}{}^{ij} + \bar{\omega}_{[\tau}{}^{il}\bar{\omega}_{a]}{}^{lj}$. We can write $\bar{R}_{\tau a}{}^{ij}(\bar{\omega}) = O^{il}O^{jk}\bar{R}'_{\tau a}{}^{lk}(\bar{\omega}')$ where the gauge rotated primed quantities are as defined in Eq. (14). Now, since $\bar{\omega}'_\tau{}^{ij}(e') = 0$ and $\partial_\tau \bar{\omega}'_a{}^{ij}(e') = 0$ for the primed connections of (14), the curvature $\bar{R}'_{\tau a}{}^{ij}(\bar{\omega}') \equiv \partial_{[\tau}\bar{\omega}'_{a]}{}^{ij} + \bar{\omega}'_{[\tau}{}^{il}\bar{\omega}'_{a]}{}^{lj} \equiv 0$ and hence $\bar{R}_{\tau a}{}^{ij}(\bar{\omega}) = 0$. This thus implies $R_{\tau a}{}^{ij}(\omega) = D_\tau(\bar{\omega})\kappa_a{}^{ij}$. Contracting with $\hat{e}^a_i$, we note that $\hat{e}^a_i R_{\tau a}{}^{ij}(\omega) = \hat{e}^a_i D_\tau(\bar{\omega})\kappa_a{}^{ij} = D_\tau(\bar{\omega})(\hat{e}^a_i \kappa_a{}^{ij}) = 0$ because $\hat{e}^a_i \kappa_a{}^{ij} = 0$ for $\kappa_a{}^{ij} = \epsilon^{ijk} N^{kl} e^l_a$ where $N^{kl} = N^{lk}$.

Next, using Eq. (11), we note that the constraints (16) and (17) are solved by the choice

$$M^i_a = \lambda e^i_a \Rightarrow M^{ij} \equiv M^i_a \hat{e}^a_j = \lambda \delta^{ij} \tag{19}$$

where $\lambda$ is a spacetime constant. This further implies that

$$M^{ij}M^{ji} - M^{ii}M^{jj} = -6\lambda^2. \tag{20}$$

Using this, the last constraint (18) can then be recast as

$$\zeta = 6\lambda^2 - \hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) \tag{21}$$

where

$$\zeta \equiv N^{ij}N^{ji} - N^{ii}N^{jj} = 2(\eta_1^2 + \eta_2^2 + \eta_3^2 - \alpha\beta - \beta\gamma - \gamma\alpha) \tag{22}$$

for the symmetric matrix

$$N^{ij} = \begin{pmatrix} \alpha & \eta_3 & \eta_2 \\ \eta_3 & \beta & \eta_1 \\ \eta_2 & \eta_1 & \gamma \end{pmatrix}. \tag{23}$$

We conclude this section by noting that the action (1) for any configuration with degenerate tetrads (10) satisfying the equations of motion is zero:

$$S = \frac{1}{8\kappa^2}\int d^4x \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL} e^I_\mu e^J_\nu R_{\alpha\beta}{}^{KL}(\omega)$$
$$= \frac{1}{2\kappa^2}\int d^4x \epsilon^{abc}\epsilon_{ijk} e^i_a e^j_b R_{c\tau}{}^{k4}(\omega) = 0, \tag{24}$$

where we have used the constraint (16) in the last step.

## V. EXPLICIT SOLUTIONS WITH DEGENERATE TETRADS

To obtain explicit solutions of the equations of motion (11)–(13) and (15)–(17), all we need to do is prescribe a set of triads $e^i_a$ and associated torsion-free Levi-Civita spin connections $\bar{\omega}_a{}^{ij}(e)$ and evaluate the spatial (three-) curvature scalar $\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega})$ to fix the combination $\zeta$ of Eq. (21). There are many possible solutions. A set of solutions for homogeneous three-geometries described by the triads can be put in eight classes as given by Thurston's model three-geometries [15]. We now display all eight solutions.

### A. $E_3$ geometry

This flat solution is the simplest where, for affine coordinates $x^a \equiv (x, y, z)$, the infinitesimal (squared) length element is $ds^2_{(4)} = dx^2 + dy^2 + dz^2$. The triads here are simply $e^1_x = e^2_y = e^3_z = 1$, and all others are zero. The corresponding spin connection $\bar{\omega}_a{}^{ij}(e) = 0$, and so the three-curvature is $\bar{R}_{ab}{}^{ij}(\bar{\omega}) = 0$. The contortion components as given by the symmetric matrix $N^{ij}$ are constrained as

$$\zeta \equiv 2(\eta_1^2 + \eta_2^2 + \eta_3^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = 6\lambda^2. \quad (25)$$

### B. $S_3$ geometry

The metric in terms of the angular coordinates $x^a = (\theta, \phi, \chi)$ for this spherical three-geometry is

$$ds^2_{(4)} = l^2[d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\chi^2)].$$

The only nonzero components of the triad are

$$e^1_\theta = l, \qquad e^2_\phi = l\sin\theta, \qquad e^3_\chi = l\sin\theta\sin\phi.$$

Associated torsion-free spin connections for this set of triads are

$$\bar{\omega}_\phi{}^{12} = -\cos\theta, \quad \bar{\omega}_\chi{}^{23} = -\cos\phi, \quad \bar{\omega}_\chi{}^{31} = \cos\theta\sin\phi,$$

and all others are zero. This is a constant curvature three-geometry with the curvature components given by $\bar{R}_{ab}{}^{ij}(\bar{\omega}) = \frac{1}{l^2} e^i_{[a} e^j_{b]}$ so that the spatial curvature scalar is $\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) = \frac{6}{l^2}$. The contortion components are given by

$$N^1_a = l(\alpha, \eta_3\sin\theta, \eta_2\sin\theta\sin\phi),$$
$$N^2_a = l(\eta_3, \beta\sin\theta, \eta_1\sin\theta\sin\phi),$$
$$N^3_a = l(\eta_2, \eta_1\sin\theta, \gamma\sin\theta\sin\phi),$$

where the six fields $(\alpha, \beta, \gamma, \eta_1, \eta_2, \eta_3)$ are as in (23). The final constraint (21) takes the form

$$\zeta = 6\lambda^2 - \frac{6}{l^2}. \quad (26)$$

For the special choice, $N^i_a = l\mu e^i_a$, this $S_3$ configuration is exactly a gauge rotated version of the first of the two solutions obtained by Tseytlin [5].

### C. $H_3$ geometry

The metric for this hyperbolic three-geometry is

$$ds^2_{(4)} = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2), \qquad z > 0.$$

Only nonzero components of the triad are $e^1_x = e^2_y = e^3_z = \frac{l}{z}$, and those of the torsion-free connection are $\bar{\omega}_x{}^{31} = \frac{1}{z} = -\bar{\omega}_y{}^{23}$. This is again a constant curvature three-geometry with the curvature components as $\bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{1}{l^2} e^i_{[a} e^j_{b]}$ so that the spatial curvature scalar becomes $\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{6}{l^2}$. The contortion is given by

$$N^1_a = \frac{l}{z}(\alpha, \eta_3, \eta_2), \quad N^2_a = \frac{l}{z}(\eta_3, \beta, \eta_1), \quad N^3_a = \frac{l}{z}(\eta_2, \eta_1, \gamma),$$

and the constraint (21) becomes

$$\zeta = \frac{6}{l^2} + 6\lambda^2. \quad (27)$$

### D. $R \times S_2$ geometry

The metric here is

$$ds^2_{(4)} = dx^2 + l^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Nontrivial triad components are $e^1_x = 1$, $e^2_\theta = l$, $e^3_\phi = l\sin\theta$, and the only nonzero component of the associated spin connection is $\bar{\omega}_\phi{}^{23} = -\cos\theta$. There is only one nonvanishing curvature component $\bar{R}_{\theta\phi}{}^{23}(\bar{\omega}) = \sin\theta$, so the spatial three-curvature scalar is $\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) = \frac{2}{l^2}$. The contortion components are given by

$$N^1_a = (\alpha, l\eta_3, l\eta_2\sin\theta), \qquad N^2_a = (\eta_3, l\beta, l\eta_1\sin\theta),$$
$$N^3_a = (\eta_2, l\eta_1, l\gamma\sin\theta),$$

and the master constraint (21) is

$$\zeta = 6\lambda^2 - \frac{2}{l^2}. \quad (28)$$

This solution is a gauge rotated version of the second solution obtained earlier by Tseytlin [5].

### E. $R \times H^2$ geometry

The infinitesimal arc length square is

$$ds^2_{(4)} = dx^2 + \frac{l^2}{z^2}(dy^2 + dz^2), \qquad z > 0.$$

Nonzero components of the triad and the corresponding torsion-free connection are $e^1_x = 1$, $e^2_y = e^3_z = \frac{l}{z}$ and $\bar{\omega}_y{}^{23} = -\frac{1}{z}$. The curvature has only one nonzero component, $\bar{R}_{yz}{}^{23}(\bar{\omega}) = -\frac{1}{z^2}$, leading to the spatial curvature scalar $\hat{e}^a_i \hat{e}^b_j \bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{2}{l^2}$. The contortion is given by

$$N^1_a = \left(\alpha, \frac{l}{z}\eta_3, \frac{l}{z}\eta_2\right), \qquad N^2_a = \left(\eta_3, \frac{l}{z}\beta, \frac{l}{z}\eta_1\right),$$
$$N^3_a = \left(\eta_2, \frac{l}{z}\eta_1, \frac{l}{z}\gamma\right).$$

Finally we have the constraint

$$\zeta = \frac{2}{l^2} + 6\lambda^2. \tag{29}$$

### F. Sol-geometry

Here the metric is

$$ds^2_{(4)} = e^{\frac{2z}{l}}dx^2 + e^{-\frac{2z}{l}}dy^2 + dz^2,$$

with nonzero components of the triads and spin-connection fields as

$$e^1_x = e^{\frac{z}{l}}, \qquad e^2_y = e^{-\frac{z}{l}}, \qquad e^3_z = 1,$$
$$\bar{\omega}_y{}^{23} = -\frac{e^{-\frac{z}{l}}}{l}, \qquad \bar{\omega}_x{}^{31} = -\frac{e^{\frac{z}{l}}}{l}.$$

Nonvanishing curvature components are

$$\bar{R}_{xy}{}^{12}(\bar{\omega}) = \frac{1}{l^2}, \quad \bar{R}_{yz}{}^{23}(\bar{\omega}) = -\frac{e^{-\frac{z}{l}}}{l^2}, \quad \bar{R}_{zx}{}^{31}(\bar{\omega}) = -\frac{e^{\frac{z}{l}}}{l^2},$$

so $\hat{e}^a_i\hat{e}^b_j\bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{2}{l^2}$. The contortion fields are

$$N^1_a = (\alpha e^{\frac{z}{l}}, \eta_3 e^{-\frac{z}{l}}, \eta_2), \qquad N^2_a = (\eta_3 e^{\frac{z}{l}}, \beta e^{-\frac{z}{l}}, \eta_1),$$
$$N^3_a = (\eta_2 e^{\frac{z}{l}}, \eta_1 e^{-\frac{z}{l}}, \gamma).$$

With these, the constraint (21) becomes

$$\zeta = \frac{2}{l^2} + 6\lambda^2. \tag{30}$$

### G. Nil-geometry

This geometry is characterized by the metric

$$ds^2_{(4)} = dx^2 + dy^2 + \left(dz - \frac{x}{l}dy\right)^2$$

with nonzero triad components as $e^1_x = 1$; $e^2_y = 1$, $e^3_y = -\frac{x}{l}$; $e^3_z = 1$ and the nontrivial components of the inverse as $\hat{e}^x_1 = 1$; $\hat{e}^y_2 = 1$; $\hat{e}^z_2 = \frac{x}{l}$, $\hat{e}^z_3 = 1$. Nonvanishing components of the torsion-free spin connection are $\bar{\omega}_y{}^{12} = -\frac{x}{2l^2}$, $\bar{\omega}_z{}^{12} = -\bar{\omega}_x{}^{23} = -\bar{\omega}_y{}^{31} = \frac{1}{2l}$. These lead to $\bar{R}_{xy}{}^{12}(\bar{\omega}) = -\frac{3}{4l^2}$, $\bar{R}_{yz}{}^{23}(\bar{\omega}) = \frac{1}{4l^2} = \bar{R}_{zx}{}^{31}(\bar{\omega})$, $\bar{R}_{xy}{}^{31}(\bar{\omega}) = \frac{x}{4l^3}$ as the only nonzero curvature components. Thus, the curvature scalar is $\hat{e}^a_i\hat{e}^b_j\bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{1}{2l^2}$. The contortion fields are

$$N^1_a = \left(\alpha, \eta_3 - \frac{x}{l}\eta_2, \eta_2\right), \qquad N^2_a = \left(\eta_3, \beta - \frac{x}{l}\eta_1, \eta_1\right),$$
$$N^3_a = \left(\eta_2, \eta_1 - \frac{x}{l}\gamma, \gamma\right),$$

and the constraint (21) reads

$$\zeta = \frac{1}{2l^2} + 6\lambda^2. \tag{31}$$

### H. $\widetilde{SL_2R}$-geometry

The metric is given by [17]

$$ds^2_{(4)} = dr^2 + l^2[c^2s^2d\theta^2 + (d\phi + s^2d\theta)^2]$$

where $c \equiv \cosh(\frac{r}{l})$ and $s \equiv \sinh(\frac{r}{l})$. The nonvanishing components of the triad, inverse triad and torsion-free spin connection are

$$e^1_r = 1; \quad e^2_\theta = lsc, \quad e^3_\theta = ls^2; \quad e^3_\phi = l;$$
$$\hat{e}^r_1 = 1; \quad \hat{e}^\theta_2 = \frac{1}{lsc}; \quad \hat{e}^\phi_2 = -\frac{s}{lc}, \quad \hat{e}^\phi_3 = \frac{1}{l};$$
$$\bar{\omega}_\theta{}^{12} = -(c^2 + 2s^2), \quad \bar{\omega}_\phi{}^{12} = -1, \quad \bar{\omega}_r{}^{23} = \frac{1}{l}, \quad \bar{\omega}_\theta{}^{31} = cs.$$

These imply that only the following curvature components are nonvanishing:

$$\bar{R}_{r\theta}{}^{12}(\bar{\omega}) = -\frac{7cs}{l}, \quad \bar{R}_{\theta\phi}{}^{23}(\bar{\omega}) = cs, \quad \bar{R}_{r\theta}{}^{31}(\bar{\omega}) = -\frac{s^2}{l},$$
$$\bar{R}_{\phi r}{}^{31}(\bar{\omega}) = \frac{1}{l},$$

so the curvature scalar is $\hat{e}^a_i\hat{e}^b_j\bar{R}_{ab}{}^{ij}(\bar{\omega}) = -\frac{10}{l^2}$. The contortion components are

$$N^1_a = (\alpha, lsc\eta_3 + ls^2\eta_2, l\eta_2), \quad N^2_a = (\eta_3, lsc\beta + ls^2\eta_1, l\eta_1),$$
$$N^3_a = (\eta_2, lsc\eta_1 + ls^2\gamma, l\gamma).$$

The final constraint (21) now is

$$\zeta = \frac{10}{l^2} + 6\lambda^2. \tag{32}$$

With this we have completed the discussion of various explicit solutions associated with Thurston's eight model three-geometries. All these solutions generically contain torsion as reflected by the symmetric matrix $N^{ij}$ where the contortion is parametrized as $\kappa_a{}^{ij} = \epsilon^{ijk}N^{kl}e^l_a$. Six component fields of symmetric $N^{ij}$ depend on all four spacetime coordinates $(x, y, z, \tau)$. These are independent except for one constraint, so the combination $\zeta = (N^{ij}N^{ji} - N^{ii}N^{jj})$ has fixed values as dictated by the condition (21) for various solutions. For all eight solutions above, $\zeta$ as given by Eqs. (25)–(32) is a spacetime constant in each case.

To emphasize, unlike the case of invertible tetrads where torsion enters into the theory through matter couplings such as fermions, here in the phase with degenerate tetrads torsion is exhibited by the solutions even in the case of pure gravity without any torsion-inducing matter fields.

Our discussion of degenerate tetrads above has been set up in Euclidean gravity. As is obvious, it holds equally well for Lorentzian signature where the zero eigenvalue of the tetrad is in the time direction. Also, the analysis has a straightforward generalization even when the zero eigenvalue is in a spatial direction where the nontrivial three-geometry would now be Lorentzian and corresponding changes for three of the six torsional components will appear.

## VI. SEN GUPTA GEOMETRY

The degenerate tetrad solutions we have discussed here do not represent the usual geometry as seen in the Einsteinian gravity. To understand the nature of these solutions, let us go to the flat spacetime limit.

We use Lorentzian signature in the discussion that follows. In the flat limit, the square of the infinitesimal length element is given by

$$ds^2_{(4)} = -c^2 dt^2 + dx^2 + dy^2 + dz^2.$$

The degenerate tetrad with one zero eigenvalue as considered here corresponds to the limit where the metric component $g_{tt} \equiv -c^2 \to 0$ in this flat spacetime case.

Under a change of frame, the length element stays unaltered:

$$\begin{aligned} ds^2_{(4)} &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2. \end{aligned}$$

There are two ways of writing transformations which leave $ds^2_{(4)}$ invariant. First is the standard Lorentz transformation:

$$dt' = \frac{dt - \frac{v}{c^2} dx}{\sqrt{1-\frac{v^2}{c^2}}}, \quad dx' = \frac{dx - v dt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad dy' = dy, \quad dz' = dz \tag{33}$$

where we have introduced the boost transformation in the $t - x$ plane. Here the parameter $v$, bounded from above as $v^2 < c^2$, is the relative velocity between the frames. In other words, $v = \frac{dx}{dt}$ (for $\Delta x' = 0$) is the velocity of a fixed point in the primed frame in the spacetime of the unprimed frame. As pointed out by Sen Gupta [18], there is another transformation which leaves the length element $ds^2_{(4)}$ invariant:

$$dt' = \frac{dt - \frac{dx}{w}}{\sqrt{1-\frac{c^2}{w^2}}}, \quad dx' = \frac{dx - \frac{c^2}{w} dt}{\sqrt{1-\frac{c^2}{w^2}}}, \quad dy' = dy, \quad dz' = dz. \tag{34}$$

Here the parameter $w$ is bounded from below as $w^2 > c^2$. Despite its dimensions, $w$ is not a relative frame velocity. Since $w = \frac{dx}{dt}$ for $\Delta t' = 0$, it rather represents the rate of change of an event that occurs at a fixed time in the primed system as measured in the unprimed system. The two transformations (33) and (34) are dual to each other. They go to each other under the changes $v \to \frac{c^2}{w}$ and $w \to \frac{c^2}{v}$.

The nonrelativistic limit of the Lorentz transformation is obtained by taking the $c \to \infty$ limit in (33) to yield the standard Galilean transformation:

$$dt' = dt, \quad dx' = dx - v dt, \quad dy' = dy, \quad dz' = dz. \tag{35}$$

On the other hand, it is the transformation (34) that is appropriate for studying the limit $c \to 0$. In this limit, as was pointed out by Sen Gupta, transformation (34) leads to the following dual transformation:

$$dt' = dt - \frac{dx}{w}, \quad dx' = dx, \quad dy' = dy, \quad dz' = dz. \tag{36}$$

This transformation [18,19], though analogous to the Galilean transformation (35), yet is different, with the roles of space and time interchanged. We may refer to the spacetime with transformation properties (36) as Sen Gupta spacetime.

The phase of degenerate tetrads in the first order formalism discussed in this article describes the curved spacetime generalizations of the Sen Gupta spacetime. This is in contrast to the phase with invertible tetrads which corresponds to the usual Einstein curved spacetime.

## VII. CONCLUDING REMARKS

The phase containing invertible tetrads in the first order gravity based on the Hilbert-Palatini action is exactly the same as the usual Einstein geometry described by the second order formalism based on the Einstein-Hilbert action. However, in the first order formulation there is another phase containing noninvertible tetrads. Thus, even classically the two formalisms are not equivalent.

Here we have studied in detail possible degenerate tetrad solutions with one zero eigenvalue in first order gravity. Many such solutions are possible. A special class of solutions obtained are associated with Thurston's eight homogeneous three-geometries. All these solutions generically possess torsion without the presence of any matter fields such as fermions.

While the solutions with invertible tetrads correspond to the usual Einstein geometry, the degenerate ones with one zero eigenvalue are curved spacetime generalizations of Sen Gupta (flat) spacetime geometry.

In the quantum theory of gravity we need to integrate over all possible configurations, including those with degenerate tetrads, in the functional integral as prescribed by the Feynman path integral formulation. Such noninvertible configurations can play an important role in the quantum theory.

Although our analysis has been presented in the framework of Euclidean gravity, it is also valid for Lorentzian gravity where the zero eigenvalue of the tetrad is in the time direction. In particular, the eight explicit solutions displayed in Sec. V are valid for this case as well. The analysis with the null eigenvalue in a spatial direction is also a mere simple generalization of the analysis elucidated here.

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## APPENDIX

Here we present the details of the derivations of Eqs. (11)–(13) and (15)–(18).

For the degenerate tetrads (10), we may break the 24 equations in (4) into two sets of 18 and 6 equations, respectively, as

$$e^{[I}_{[\tau}D_a(\omega)e^{J]}_{b]} = 0, \tag{A1}$$

$$e^{[I}_{[a}D_b(\omega)e^{J]}_{c]} = 0. \tag{A2}$$

It is straightforward to see that Eq. (A1) can be recast as 18 equivalent equations:

$$D_\tau(\omega)(e^{[I}_a e^{J]}_b) = 0. \tag{A3}$$

Taking $I = i$ and $J = 4$, these result in nine equations: $e^i_{[a}e^k_{b]}\omega^{4k}_\tau = 0$. These in turn imply vanishing of ${\omega_\tau}^{4i}$ as claimed in (12). Again, for $I = i$, $J = j$, Eq. (A3) leads to the nine equations $D_\tau(\omega)(e^{[i}_a e^{j]}_b) = 0$. These are equivalent to nine equations $D_\tau(\omega)e^i_a = 0$ as claimed in (11). These further imply $D_\tau(\omega)\hat{e}^a_i = 0$ and $\partial_\tau e = 0$ where $e = \det e^i_a$. These equations can be solved for the connection components ${\omega_\tau}^{ij}$ as

$${\omega_\tau}^{ij} = {\bar{\omega}_\tau}^{ij}(e) \equiv \hat{e}^a_i\partial_\tau e^j_a = e^i_a\partial_\tau\hat{e}^a_j = -\hat{e}^a_j\partial_\tau e^i_a = -e^j_a\partial_\tau\hat{e}^a_i. \tag{A4}$$

Next, we take $I = i$, $J = 4$ in Eq. (A2) and multiply by $\hat{e}^a_i$ to show that $D_{[b}(\omega)e^4_{c]} \equiv {\omega_{[b}}^{4k}e^k_{c]} = 0$ which in turn implies that ${\omega_b}^{4k} \equiv M^k_b = M^{kl}e^l_b$ is such that the $3\times3$ matrix $M^{kl}$ is symmetric. Furthermore, for $I = i$, $J = j$ in (A2), multiplying by $\hat{e}^a_i\hat{e}^b_j$, it can readily be shown to lead to three conditions:

$$\hat{e}^a_i D_{[c}(\omega)e^i_{a]} = 0. \tag{A5}$$

Now let us split the connection fields as ${\omega_a}^{ij} = {\bar{\omega}_a}^{ij}(e) + {\kappa_a}^{ij}$ where ${\kappa_a}^{ij} \equiv \epsilon^{ijk}N^k_a \equiv \epsilon^{ijk}N^{kl}e^l_a$ are the contortion fields and ${\bar{\omega}_a}^{ij}(e)$, as given in Eq. (13), are the torsion-free Levi-Civita spin connections for the triads $e^i_a$:

$$D_{[a}(\bar{\omega})e^i_{b]} = 0. \tag{A6}$$

With this, Eq. (A5) can be shown to imply that $\hat{e}^a_i{\kappa_a}^{ij} = 0$. This further leads to the fact that the $3\times3$ contortion matrix $N^{ij}$ does not have any antisymmetric part, that is, $N^{ij} = N^{ji}$.

Next, we split the 16 equations in (5) into two sets of 12 and 4 equations, respectively, as

$$e^{[I}_{[\tau}{R_{ab]}}^{JK]}(\omega) = 0, \tag{A7}$$

$$e^{[I}_{[a}{R_{bc]}}^{JK]}(\omega) = 0. \tag{A8}$$

In Eq. (A7) we take $I = 4$, $J = j$, $K = k$ and multiply by inverse triads to note that $\hat{e}^a_j e^{[4}_{[\tau}{R_{ab]}}^{jk]}(\omega) = 4[{R_{b\tau}}^{k4}(\omega) + e^k_b(\hat{e}^a_j{R_{\tau a}}^{4j}(\omega))] = 0$ and $\hat{e}^b_k\hat{e}^a_j e^{[4}_{[\tau}{R_{ab]}}^{jk]}(\omega) = 16\hat{e}^a_k{R_{a\tau}}^{4k}(\omega) = 0$. This leads to the 9 conditions ${R_{a\tau}}^{4k}(\omega) = 0$. Further using the fact that ${\omega_\tau}^{4k} = 0$ as argued above, we note that ${R_{a\tau}}^{4k}(\omega) \equiv -(\partial_\tau{\omega_a}^{4k} + {\omega_\tau}^{kl}{\omega_a}^{4l}) \equiv -(\partial_\tau M^k_a + {\omega_\tau}^{kl}M^l_a) \equiv -D_\tau(\omega)M^k_a$. Thus we have the nine constraints:

$${R_{a\tau}}^{4k}(\omega) = -D_\tau(\omega)M^k_a = 0. \tag{A9}$$

Again in Eq. (A7), we take $I = i$, $J = j$, $K = k$ and use the fact that $\hat{e}^a_i\hat{e}^b_j e^{[i}_{[\tau}{R_{ab]}}^{jk]}(\omega) = 8\hat{e}^a_i{R_{\tau a}}^{ki}(\omega)$, leading us to three constraints:

$$\hat{e}^a_i{R_{\tau a}}^{ki}(\omega) = 0. \tag{A10}$$

Next, let us take $I = 4$, $J = j$, $K = k$ in Eq. (A8) and notice that $\hat{e}^a_j\hat{e}^b_k e^{[4}_{[a}{R_{bc]}}^{jk]}(\omega) = 8\hat{e}^a_j{R_{ca}}^{4j}(\omega)$, leading us to three conditions:

$$\hat{e}^a_k{R_{ab}}^{4k}(\omega) = (e^l_b\hat{e}^a_i - \delta^a_b\delta^l_i)D_a(\bar{\omega})M^{il} = 0, \tag{A11}$$

where for the first step we have used ${R_{ab}}^{4k}(\omega) = D_{[a}(\omega)M^k_{b]} = D_{[a}(\bar{\omega})M^k_{b]} + \kappa^{kl}_{[a}M^l_{b]}$ and $\hat{e}^a_k\kappa^{kl}_a = 0$ and $M^{il} \equiv M^i_a\hat{e}^a_l = M^{li}$.

Finally taking $I = i$, $J = j$, $K = k$ in (A8) and using $\epsilon^{abc}\epsilon_{ijk}e^i_a{R_{bc}}^{jk} = 2e\hat{e}^b_j\hat{e}^c_k{R_{bc}}^{jk}$, we obtain the last condition as

$$\hat{e}^a_i\hat{e}^b_j{R_{ab}}^{ij}(\omega) = 0. \tag{A12}$$

Expanding ${\omega_a}^{ij} = {\bar{\omega}_a}^{ij}(e) + \epsilon^{ijk}N^{kl}e^l_a$, we find that

$$\begin{aligned} {R_{ab}}^{ij}(\omega) = {\bar{R}_{ab}}^{ij}(\bar{\omega}) - \epsilon^{ijk}e^l_{[a}D_{b]}(\bar{\omega})N^{kl} \\ - (M^{il}M^{jk} + N^{il}N^{jk})e^l_{[a}e^k_{b]}, \end{aligned} \tag{A13}$$

where ${\bar{R}_{ab}}^{ij}(\bar{\omega}) = \partial_{[a}{\bar{\omega}_{b]}}^{ij} + {\bar{\omega}_{[a}}^{il}{\bar{\omega}_{b]}}^{lj}$. Using this, the constraint (A12) can be recast as

$$\begin{aligned} \hat{e}^a_i\hat{e}^b_j{\bar{R}_{ab}}^{ij}(\bar{\omega}) + (M^{ij}M^{ji} - M^{ii}M^{jj}) \\ + (N^{ij}N^{ji} - N^{ii}N^{jj}) = 0, \end{aligned} \tag{A14}$$

where we have used the fact that matrix $N^{ij}$ is symmetric.

[1] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[2] S. Hawking, Nucl. Phys. **B144**, 349 (1978).
[3] M. Henneaux, Bull. Soc. Math. Bel., Ser. A **31**, 47 (1979); M. Henneaux, M. Pilati, and C. Teitelboim, Phys. Lett. **110B**, 123 (1982); M. Pilati, Phys. Rev. D **26**, 2645 (1982); **28**, 729 (1983).
[4] R. D'Auria and T. Regge, Nucl. Phys. **B195**, 308 (1982).
[5] A. A. Tseytlin, J. Phys. A **15**, L105 (1982).
[6] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).
[7] I. Bengtsson, Int. J. Mod. Phys. A **04**, 5527 (1989); S. Koshti and N. Dadhich, Classical Quantum Gravity **6**, L223 (1989); I. Bengtsson, Classical Quantum Gravity **7**, 27 (1990); **8**, 1847 (1991).
[8] M. Varadarajan, Classical Quantum Gravity **8**, L235 (1991).
[9] I. Bengtsson and T. Jacobson, Classical Quantum Gravity **14**, 3109 (1997); **15**, 3941(E) (1998).
[10] T. Jacobson and J. D. Romano, Classical Quantum Gravity **9**, L119 (1992); J. D. Romano, Phys. Rev. D **48**, 5676 (1993); M. P. Reisenberger, Nucl. Phys. **B457**, 643 (1995); G. Yoneda, H. Shinkai, and A. Nakamichi, Phys. Rev. D **56**, 2086 (1997).
[11] T. Jacobson, Classical Quantum Gravity **13**, L111 (1996); **13**, 3269(E) (1996); S. H. S. Alexander and G. Calcagni, Found. Phys. **38**, 1148 (2008); Phys. Lett. B **672**, 386 (2009).
[12] G. T. Horowitz, Classical Quantum Gravity **8**, 587 (1991).
[13] J. A. Wheeler, Ann. Phys. (N.Y.) **2**, 604 (1957); *Geometrodynamics* (Academic Press, New York, 1962).
[14] R. P. Geroch, J. Math. Phys. (N.Y.) **8**, 782 (1967).
[15] W. P. Thurston, Bull. A.M.S. **6**, 357 (1982); *The Geometry and Topology of 3-Manifolds* (Princeton University Lecture Notes, 1982).
[16] P. Scott, Bull. Lond. Math. Soc. **15**, 401 (1983).
[17] E. Molnar, Beit. Alg. Geom. **38**, 261 (1997).
[18] N. D. Sen Gupta, Nuovo Cimento **44**, 512 (1966).
[19] J.-M. Levy-Leblond, Ann. Inst. Henri Poincare **3**, 1 (1965).