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Magnetic mass in 4D AdS gravity

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We provide a fully covariant expression for the diffeomorphic charge in four-dimensional anti-de Sitter gravity, when the Gauss-Bonnet and Pontryagin terms are added to the action. The couplings of these topological invariants are such that the Weyl tensor and its dual appear in the on-shell variation of the action and such that the action is stationary for asymptotic (anti-)self-dual solutions in the Weyl tensor. In analogy with Euclidean electromagnetism, whenever the self-duality condition is global, both the action and the total charge are identically vanishing. Therefore, for such configurations, the *magnetic* mass equals the Ashtekhar-Magnon-Das definition.

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I. INTRODUCTION

The Maxwell Lagrangian for electromagnetism is the simplest gauge-invariant scalar that leads to second-order field equations. As is well known, gauge invariance is a consequence of using the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and not explicitly the gauge connection A_μ.

However, in four dimensions, the Maxwell term is not the only Lagrangian quadratic in F that can be considered in an electromagnetism action. We can always look at the physical implications which come from taking an action of the form

$$I = -\frac{1}{4}\int_M (F^{\mu\nu}F_{\mu\nu} + \gamma\,{}^*F^{\mu\nu}F_{\mu\nu})dtd^3x, \tag{1.1}$$

where the second contribution is given in terms of the field strength and its dual ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, and it is called Pontryagin density,

$$\mathcal{P}_4 = \frac{1}{4}{}^*F^{\mu\nu}F_{\mu\nu}. \tag{1.2}$$

For a given real coupling constant γ, the second part of the action (1.1) contributes just a surface term, such that it does not alter the bulk dynamics. Nevertheless, it may still modify the boundary conditions in the variational problem and, eventually, the Noether current of the theory.

In non-Abelian theories, Pontryagin is a topological term, which is added on top of Yang-Mills Lagrangian with a pseudoscalar coupling $\theta(x)$ (axion field) [1]. This θ term is responsible for violation of CP symmetry in QCD.

In a more recent context, $\mathcal{P}_4$ has been considered to account for properties of a new topological state in condensed matter physics known as topological insulators [2].

In the Euclidean sector of the theory (1.1), the electric field is defined as $E_i = F_{0i}$, in terms of derivatives with respect to the Euclidean time $x^0 = it$ and the spatial coordinates $\{x^i\}$. In turn, the magnetic field is the same as in the case of Lorentzian signature, that is, $B_i = \frac{1}{2}\epsilon^{0ijk}F_{jk}$. With this in mind, the Pontryagin invariant adopts the form

$$\mathcal{P}_4 = \mathbf{E}\cdot\mathbf{B}, \tag{1.3}$$

such that the Euclidean action $I^E = -iI$ reads

$$I^E = \frac{1}{2}\int_M (\mathbf{E}^2 + \mathbf{B}^2 + 2\gamma\mathbf{E}\cdot\mathbf{B})d^4x. \tag{1.4}$$

An arbitrary variation of this action produces

$$\delta I^E = \int_M (\partial_\mu F^{\mu\nu} + \gamma\partial_\mu{}^*F^{\mu\nu})\delta A_\nu d^4x - \int_{\partial M}(F^{\mu\nu} + \gamma\,{}^*F^{\mu\nu})\delta A_\nu d\Sigma_\mu, \tag{1.5}$$

where the bulk integral yields Maxwell equation and second term which vanishes due to the Bianchi identity, $\partial_\mu{}^*F^{\mu\nu} = 0$.

To have a well-defined action principle ($\delta I^E = 0$), it is necessary that field equations hold and that the surface term vanishes for a given boundary condition. Usually, one fixes the vector potential on the boundary, i.e., $\delta A_\mu = 0$. In particular, when the boundary is a surface separating two regions of the space, this Dirichlet condition defines the junction conditions for the electric and magnetic fields across the surface in terms of sources present on the boundary, i.e., the surface charge and current densities.

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Another way to achieve a well-posed variational principle is to demand that an asymptotic (anti-)self-duality condition holds at the boundary, that is,

$$F^{\mu\nu} = \pm{}^*F^{\mu\nu} \quad \text{at } \partial M, \tag{1.6}$$

such that this argument fixes the Pontryagin coupling as $\gamma = \mp 1$.

Self-duality is a global symmetry of the sourceless Maxwell equation, where the electric and magnetic degrees of freedom are interchanged. An extension to electromagnetism with sources should necessarily include a *magnetic* charge. In the Hamiltonian formulation of Maxwell theory, self-duality is an off-shell symmetry, as shown by Deser and Teitelboim in Ref. [3].

Using the identity

$$F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(F_{\mu\nu}F^{\mu\nu} + {}^*F_{\mu\nu}{}^*F^{\mu\nu}), \tag{1.7}$$

the Euclidean action can be rewritten as

$$I^E = \frac{1}{8}\int_M (F^{\mu\nu} \mp {}^*F^{\mu\nu})^2 d^4x. \tag{1.8}$$

It is worth noticing that for a global (anti-)self-duality condition, the action is identically zero. The solutions in this case are known as Euclidean instantons. The condition $I^E = 0$ defines a number of ground states of the theory, where $F_{\mu\nu} = 0$ is the simplest case of a globally self-dual solution.

Invariance under a $U(1)$ gauge transformation, where the gauge field changes as $\delta_\lambda A_\nu = \partial_\nu \lambda$, leads to a conservation law associated to this symmetry. Indeed, using the general on-shell variation of the action (1.5) in the Noether theorem (see Appendix A), a conserved charge can be constructed,

$$Q[\lambda] = -\int_{S^2} (F^{\mu\nu} \mp {}^*F^{\mu\nu})\lambda d\Sigma_{\mu\nu}, \tag{1.9}$$

where $d\Sigma_{\mu\nu}$ is the dual of the infinitesimal surface element in S^2. Since the gauge parameter λ is covariantly constant in the asymptotic region, it can be normalized as $\lambda = 1$. The first term is the contribution due to the Noether current for Maxwell electromagnetism, $J^\mu[\lambda] \sim F^{\mu\nu}\partial_\nu\lambda$, the conservation of which produces the electric charge. The second term is derived from a topological current, $\tilde{J}^\mu[\lambda] \sim {}^*F^{\mu\nu}\partial_\nu\lambda$, and it corresponds to the magnetic flux across the sphere S^2, i.e., magnetic charge [4]. Simply put, Eq. (1.9) identifies the Noether charge obtained from a topological term with a topological charge derived from the Bianchi identity.

It is evident from the formula (1.9) that any globally (anti-)self-dual solution will have a vanishing charge. This argument reinforces the idea that such a configuration can be regarded as a ground state of the theory and provides firmer ground for the extension of self-duality condition to anti-de Sitter (AdS) gravity discussed below.

II. FOUR-DIMENSIONAL ADS GRAVITY AND PONTRYAGIN INVARIANT

The addition of topological invariants, which modify the boundary dynamics of AdS gravity, was considered more than 15 years ago in Refs. [5,6]. Indeed, the regulation of the Noether current by the addition of the Euler density provides a generic expression for the mass and other charges for even-dimensional asymptotically AdS (AAdS) spaces. As this procedure was performed in first-order formalism, its relation to other approaches was not clear at that moment, even though the equivalence to Hamiltonian charges was given in Ref. [7]. In particular, its relevance within the framework of AdS/CFT correspondence [8] was certainly unknown. However, this approach was later translated into a metric formalism in Ref. [9] and understood as the addition of counterterms which depend on the extrinsic curvature. It was then extended to odd dimensions [10], giving rise to an alternative regularization scheme known as *Kounterterms*. Furthermore, the connection to holographic renormalization [11] in the context of AdS/CFT correspondence was shown in Refs. [12,13], as the asymptotic expansion of the extrinsic curvature reproduces the standard counterterm series [14,15].

The simplest example of regularization using topological invariants is the addition of Gauss-Bonnet term to the four-dimensional (4D) AdS action studied in Ref. [5],

$$I_4 = \frac{1}{16\pi G}\int_M d^4x\sqrt{g} \times \left[R + \frac{6}{\ell^2} + \frac{\ell^2}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\right], \tag{2.1}$$

where ℓ is the AdS radius and $g = |\det(g_{\mu\nu})|$. This is the same as the quadratic action given by MacDowell and Mansouri in four dimensions in Ref. [16] (see also Ref. [17]), which was later extended to higher dimensions by Vasiliev [18].

The Gauss-Bonnet coupling is such that the action is stationary for asymptotically locally AdS spaces, where the spacetime curvature tends to a constant, i.e., $R^{\mu\nu}_{\alpha\beta} \to -\frac{1}{\ell^2}\delta^{[\mu\nu]}_{[\alpha\beta]}$. This is evident from the on-shell variation of I_4,

$$\delta I_4 = \frac{\ell^2}{64\pi G}\int_{\partial M} d^3x\sqrt{h}n_{\mu_1}\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}g^{\nu_2\gamma}\delta\Gamma^{\nu_1}_{\gamma\mu_2} \times \left(R^{\nu_3\nu_4}_{\mu_3\mu_4} + \frac{1}{\ell^2}\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]}\right), \tag{2.2}$$

where n_{μ_1} is an outward pointing unit normal to the boundary with the induced metric h_{ij} and $h = |\det(h_{ij})|$.

Also, $\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}$ is the totally antisymmetric Kronecker delta defined as $\det[\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_4}_{\nu_4}]$. The key argument that supports the finiteness of the action principle is given by the fact that, for any solution of the Einstein equation $R_{\mu\nu} = -\frac{3}{\ell^2}g_{\mu\nu}$, the Weyl tensor is

$$W^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]}, \tag{2.3}$$

which is exactly the quantity that appears at the right-hand side of Eq. (2.2). The Weyl tensor is the only combination between the Riemann and Ricci tensors that has a suitable asymptotic behavior. A formal proof of the finiteness of the action, however, requires precise falloff conditions in the metric, valid for any AAdS spacetime [12].

The appearance of the Weyl tensor in the surface term coming from the variation of the total action (2.1) reflects the link to conformal mass definition in AAdS gravity [19]. Indeed, upon suitable expansion of the tensors involved, one can prove that the physical information on the conformal boundary is encoded in the electric part of the Weyl tensor [12,20].

Gauss-Bonnet is not the only possible topological invariant for the Lorentz group one can construct in four dimensions. Indeed, Pontryagin density in gravity [21], where the Riemann tensor plays the role of the field strength in Eq. (1.2), is given by

$$\mathcal{P}_4 = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}R^{\sigma\lambda}_{\mu\nu}R_{\sigma\lambda\alpha\beta}. \tag{2.4}$$

As the Pontryagin is a closed form, it can be written locally as the divergence of a Chern-Simons density current:

$$\mathcal{P}_4 = \partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\left(\Gamma^\sigma_{\nu\lambda}\partial_\alpha\Gamma^\lambda_{\beta\sigma} + \frac{2}{3}\Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\alpha\epsilon}\Gamma^\epsilon_{\beta\sigma}\right)\right]. \tag{2.5}$$

We consider the addition of the Pontryagin density on top of a finite AdS action given by Euclideanized version of Eq. (2.1), that is,

$$I = I_4 + \frac{\ell^2}{32\pi G}\gamma\int_M d^4x\mathcal{P}_4, \tag{2.6}$$

where γ is a coupling constant yet to be determined. We emphasize the fact that, in this case, γ is a given constant, not a function. As a consequence, the action in Eq. (2.6) does not describe the Chern-Simons modified gravity theory developed by Jackiw and Pi in Ref. [22], where, by analogy to dynamic couplings of electromagnetic Pontryagin, one is able to modify the gravitational field equation in the bulk.

It is direct to check that the addition of the Pontryagin density does not introduce divergences when evaluated on AAdS solutions. Indeed, $\mathcal{P}_4$ is zero for AdS black holes and, at most, finite for gravitational instantons, as it will be discussed below.

The addition of $\mathcal{P}_4$ produces a new surface term with respect to the one in Eq. (2.2), which is proportional to the dual of the Riemann tensor, i.e.,

$$\delta I = \frac{\ell^2}{64\pi G}\int_{\partial M} d^3x\sqrt{h}n_{\mu_1}\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}g^{\nu_2\gamma}\delta\Gamma^{\nu_1}_{\gamma\mu_2} \times\left(W^{\nu_3\nu_4}_{\mu_3\mu_4} - \frac{\gamma}{2\sqrt{g}}\epsilon^{\nu_3\nu_4\alpha\beta}R_{\alpha\beta\mu_3\mu_4}\right). \tag{2.7}$$

It is adequate to perform a shift in the curvature of the type $R_{\alpha\beta\mu_3\mu_4} \to R_{\alpha\beta\mu_3\mu_4} + \frac{1}{\ell^2}(g_{\alpha\mu_3}g_{\beta\mu_4} - g_{\beta\mu_3}g_{\alpha\mu_4})$, as the second term is identically zero due to the symmetry in the indices. In doing so, the variation of the total action can be rewritten as

$$\delta I = \frac{\ell^2}{64\pi G}\int_{\partial M} d^3x\sqrt{h}n_{\mu_1}\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}g^{\nu_2\gamma}\delta\Gamma^{\nu_1}_{\gamma\mu_2} \times(W^{\nu_3\nu_4}_{\mu_3\mu_4} - \gamma\,{}^*W^{\nu_3\nu_4}_{\mu_3\mu_4}), \tag{2.8}$$

in terms of the dual of the Weyl tensor

$${}^*W_{\alpha\beta\mu\nu} = \frac{1}{2}\sqrt{g}\epsilon_{\alpha\beta\sigma\lambda}W^{\sigma\lambda}_{\mu\nu}. \tag{2.9}$$

By analogy to the electromagnetism case, one can determine γ by demanding an asymptotic (anti-)self duality condition on the Weyl tensor,

$$W_{\alpha\beta\mu\nu} = \pm{}^*W_{\alpha\beta\mu\nu}. \tag{2.10}$$

The action is truly stationary if the field equations hold in the bulk and the surface term vanishes at the boundary. Therefore, a well-defined action principle for the boundary condition (2.10) implies that the Pontryagin coupling is $\gamma = \pm 1$ [12].

As the Weyl tensor carries information on the normalizable modes in AdS gravity, the above condition implies a nontrivial relation between different components of the Weyl tensor at a holographic order. Indeed, asymptotic self-duality for the Weyl tensor, which appears naturally at the boundary when one adds Gauss-Bonnet (parity-preserving) and Pontryagin (parity-violating) topological invariants, seems to be the ultimate reason behind holographic stress tensor/Cotton tensor duality, which arises when dealing with AdS instantons [23], hydrodynamic perturbations around AdS_4 black holes [24], and electric/magnetic duality in Riemann-Cartan-AdS gravity [25].

Only for the particular value of the Pontryagin coupling discussed above, the on-shell action adopts the compact form [12]

$$I = \frac{\ell^2}{512\pi G}\int_M d^4x\sqrt{g}\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}(W^{\nu_1\nu_2}_{\mu_1\mu_2} \pm {}^*W^{\nu_1\nu_2}_{\mu_1\mu_2}) \times (W^{\nu_3\nu_4}_{\mu_3\mu_4} \pm {}^*W^{\nu_3\nu_4}_{\mu_3\mu_4}), \tag{2.11}$$

in terms of the Weyl tensor and its dual, where we have used the identities

$$^*W^{\mu_1\mu_2}_{\nu_1\nu_2} = \frac{1}{4}\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}{}^*W^{\nu_3\nu_4}_{\mu_3\mu_4} \tag{2.12}$$

and

$$\delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}{}^*W^{\nu_1\nu_2}_{\mu_1\mu_2}{}^*W^{\nu_3\nu_4}_{\mu_3\mu_4} = \delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]}W^{\nu_1\nu_2}_{\mu_1\mu_2}W^{\nu_3\nu_4}_{\mu_3\mu_4}. \tag{2.13}$$

This action has been recently studied in the context of a search for a pure-spin connection formulation for general relativity [26].

In what follows, we compute the Noether charges for the gravity action I using Wald's method [27,28]. This is the fully covariant version of the boundary derivation which associates the addition of the Gauss-Bonnet term to the *electric* part of the Weyl tensor and the addition of Pontryagin to the *magnetic* part of the Weyl tensor (see Appendix B).

III. COVARIANT NOETHER CHARGES AND TOPOLOGICAL INVARIANTS

The Noether theorem provides a conserved current J^μ ($\partial_\mu(\sqrt{g}J^\mu) = 0$), for a given symmetry of an action. Indeed, global isometries in gravitational solutions imply the existence of a Noether charge defined as

$$Q = \int_{\partial M} d^3x\sqrt{h}n_\mu J^\mu. \tag{3.1}$$

When J^μ can be globally written as $J^\mu = \partial_\nu(\sqrt{h}Q^{\mu\nu})$ in ∂M, the Noether charge can be expressed as an integral on the two-dimensional surface $\partial\Sigma$ with the metric σ_{mn} and $\sigma = |\det(\sigma_{mn})|$,

$$Q = \int_{\partial\Sigma}\sqrt{\sigma}d^2x n_\mu u_\nu Q^{\mu\nu}, \tag{3.2}$$

where u_ν is a unit timelike vector, normal at every point to Σ (see Appendix A).

For the case under study here, we follow Wald's procedure defined in Refs. [27,28], which allows us to construct the Noether charges in an arbitrary gravity theory. We consider a Lagrangian density L, which depends on the metric, curvature, and covariant derivatives of the curvature,

$$L = L(g_{\mu\nu}, R_{\mu\nu\alpha\beta}, \nabla_{\gamma_1}R_{\mu\nu\alpha\beta}, \cdots \cdots, \nabla_{(\gamma_1\cdots}\nabla_{\gamma_m)}R_{\mu\nu\alpha\beta}, \psi, \nabla_{\gamma_1}\psi, \nabla_{(\gamma_1\cdots\gamma_l)}\psi). \tag{3.3}$$

One can also include matter fields, collectively denoted by ψ, and derivatives of them.

For this general class of theories, the conserved current corresponding to a set of Killing vectors $\{\xi^\mu\}$ is given by the expression (see Appendix A)

$$\sqrt{g}J^\mu = \Theta^\mu(\delta_\xi\Gamma) + \Theta^\mu(\delta_\xi g) + \xi^\mu L, \tag{3.4}$$

assuming that the surface term Θ^μ can be split in a part that contains variations of the Christoffel symbol and another that contains variations of the metric tensor. Because of the fact that $\Theta^\mu(\delta_\xi g)$ is proportional to the Lie derivative of the metric, using the Killing equation, this term can be set to zero. Then, the conserved current adopts the form

$$\sqrt{g}J^\mu = 2E^{\mu\nu\alpha\beta}g_{\alpha_\lambda}\delta_\xi\Gamma^\lambda_{\nu\beta} + \xi^\mu L, \tag{3.5}$$

where $E^{\mu\nu\alpha\beta}$ is the variation of L with respect to the Riemann tensor $R_{\mu\nu\alpha\beta}$. The diffeomorphism transformation of the Christoffel symbol is given by

$$\begin{aligned}\delta_\xi\Gamma^\lambda_{\nu\beta} &= -\frac{1}{2}g^{\lambda\rho}(\nabla_\beta \pounds_\xi g_{\rho\nu} + \nabla_\nu \pounds_\xi g_{\rho\beta} - \nabla_\rho \pounds_\xi g_{\beta\nu}) \\ &= -\frac{1}{2}(\nabla_\nu\nabla_\beta\xi^\lambda + \nabla_\beta\nabla_\nu\xi^\lambda) + \frac{1}{2}(R^\lambda{}_{\beta\nu\sigma} + R^\lambda{}_{\nu\beta\sigma})\xi^\sigma,\end{aligned} \tag{3.6}$$

which produces a current,

$$\sqrt{g}J^\mu = -E^{\mu\nu\alpha\beta}[2\nabla_\nu\nabla_\beta\xi_\alpha - (R_{\alpha\beta\nu\sigma} + 2R_{\alpha\nu\beta\sigma})\xi^\sigma] + \xi^\mu L, \tag{3.7}$$

where we have used the identity that involves the commutator of two covariant derivatives,

$$[\nabla_\beta, \nabla_\nu]\xi_\alpha = R_{\beta\nu\alpha\sigma}\xi^\sigma. \tag{3.8}$$

A minor arrangement can be performed in the above expression for the current, as the tensor $E^{\mu\nu\alpha\beta}$ inherits a given symmetry in the indices which is derived from the first Bianchi identity, that is,

$$0 = R_{\alpha\beta\nu\sigma} + R_{\beta\nu\alpha\sigma} + R_{\nu\alpha\beta\sigma}, \tag{3.9}$$

which implies

$$E^{\mu\nu\alpha\beta}(R_{\alpha\beta\nu\sigma} - 2R_{\alpha\nu\beta\sigma}) = 0. \tag{3.10}$$

Finally, the formula for the Noether current in a generic gravity theory is given by

$$\sqrt{g}J^{\mu} = 2E^{\mu\nu\alpha\beta}(\nabla_{\nu}\nabla_{\alpha}\xi_{\beta} + R_{\alpha\beta\nu\sigma}\xi^{\sigma}) + \xi^{\mu}L. \quad (3.11)$$

For the case under study, we can see that the Noether current associated to the Einstein-Hilbert (EH) Lagrangian plus Gauss-Bonnet (GB) term in Eq. (2.1) yields

$$J_{4}^{\mu}[\xi] = \frac{\ell^2}{64\pi G}\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]}W_{\lambda\sigma}^{\gamma\delta}\nabla_{\nu}\nabla^{\alpha}\xi^{\beta}, \quad (3.12)$$

which, using the second Bianchi identity in the indices $\nu\lambda\sigma$, can be written down as a total derivative,

$$J_{4}^{\mu}[\xi] = \frac{\ell^2}{64\pi G}\nabla_{\nu}(\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]}W_{\lambda\sigma}^{\gamma\delta}\nabla^{\alpha}\xi^{\beta}). \quad (3.13)$$

Here, we have used the field equations and permutational identities in order to eliminate additional terms in the curvature, which are coming from the Lie derivative acting on the Christoffel symbol (3.6). Integrated on $\partial\Sigma$, the above expression produces the charge

$$Q_{4}[\xi] = \frac{\ell^2}{64\pi G}\int_{\partial\Sigma} d^2x\sqrt{\sigma}n_{\mu}u_{\nu}\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]}\nabla^{\alpha}\xi^{\beta}W_{\lambda\sigma}^{\gamma\delta}. \quad (3.14)$$

Taking now the Lagrangian density corresponding to the Pontryagin term, we have

$$E_{\mathcal{P}_4}^{\mu\nu\alpha\beta} = \mp\frac{\ell^2}{64\pi G}\epsilon^{\mu\nu\lambda\sigma}R_{\lambda\sigma}^{\alpha\beta}, \quad (3.15)$$

which determines the current associated to this term as

$$J_{\mathcal{P}_4}^{\mu} = \mp\frac{\ell^2}{64\pi G}\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]}\nabla_{\nu}(\nabla^{\alpha}\xi^{\beta *}W_{\lambda\sigma}^{\gamma\delta}). \quad (3.16)$$

As a consequence, the total Noether charge computed for the AdS gravity action with the addition of topological invariants is

$$Q[\xi] = \frac{\ell^2}{64\pi G}\int_{\partial\Sigma} d^2x\sqrt{\sigma}n_{\mu}u_{\nu}\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]}\nabla^{\alpha}\xi^{\beta}(W_{\lambda\sigma}^{\gamma\delta}\mp{}^{*}W_{\lambda\sigma}^{\gamma\delta}). \quad (3.17)$$

It is then that the analogy with self-dual electromagnetism becomes evident: self-dual or anti-self-dual solutions in AdS gravity have mass (and other conserved quantities) identically zero. Such a configuration is a vacuum state, which reaches a minimum of the Euclidean action.

A. Taub-NUT/Bolt AdS solutions

For static black hole and even Kerr-AdS solutions, the magnetic part of the Weyl tensor is zero, such that there is no contribution to the current (3.16). Therefore, nontrivial examples to evaluate the above expressions for the conserved quantities are Taub-Newman-Unti-Tamburino (NUT) and Taub-Bolt AdS solutions. These spaces are Euclidean gravitational solutions to the Einstein equations characterized by a line element [29–31]

$$ds^2 = f(r)(d\tau + 2n\cos\theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.18)$$

where the function $f(r)$ is given by $(G = 1)$

$$f(r) = \frac{r^2 - 2Mr + n^2 - \frac{3}{\ell^2}(n^4 + 2n^2r^2 - \frac{r^4}{3})}{r^2 - n^2}. \quad (3.19)$$

Here, n is a parameter, and M is identified as the solution mass [5,32]. The Taub-NUT-AdS solution is defined by the condition $f(|n|) = 0$, but one still has to eliminate the conical singularities that appear at $r = |n|$. By imposing a regularity condition, which is given by $f'(|n|) = 1/2n$, the electric mass takes the particular value

$$Q_{4}^{\mathrm{NUT}}[\partial_{\tau}] = M_{\mathrm{NUT}} = \pm n(1 - 4\ell^{-2}n^2). \quad (3.20)$$

This value of M is the exact point where the Weyl tensor becomes globally (anti-)self-dual [33,34]. As a consequence, the total Noether charge (3.17) vanishes for any isometry, that is,

$$Q^{\mathrm{NUT}}[\xi] = 0, \quad (3.21)$$

as the electric mass is equal to the magnetic mass. This solution can be regarded as a family of ground states labeled by N.

On the other hand, the Taub-Bolt AdS solution is found for $r = r_b > |n|$ and $f(r = r_b) = 0$. In this case, the electric mass is

$$Q_{4}^{\mathrm{Bolt}}[\partial_{\tau}] = M_{\mathrm{Bolt}} = \frac{r_b^2 + n^2}{2r_b} - \frac{3}{2\ell^2}\left(\frac{n^4}{r_b} + 2n^2r_b - \frac{r_b^3}{3}\right). \quad (3.22)$$

In turn, the magnetic mass for the Bolt solution remains the same as in the NUT case, such that the total mass and angular momentum are

$$Q_{\mathrm{Bolt}}[\partial_{\tau}] = M_{\mathrm{Bolt}} \pm M_{\mathrm{NUT}}, \quad (3.23)$$

$$Q_{\mathrm{Bolt}}[\partial_{\phi}] = 0. \quad (3.24)$$

The anti-self-dual case in Eq. (3.23) corresponds to the mass calculated in Ref. [32] following a background-dependent procedure.

IV. CONCLUSIONS

A fully covariant expression for the conserved quantities for 4D AdS gravity supplemented by Gauss-Bonnet and Pontryagin terms has been obtained *à la* Wald.

By analogy with electromagnetism, all the charges are identically zero for globally self-dual solutions.

A similar expression for the Noether charges has been worked out in Refs. [35,36] in first-order formalism. In this Riemann-Cartan approach, the parity-violating sector appears enlarged by the Holst and Nieh-Yang terms, which are identically vanishing in Riemannian gravity [37]. As a consequence, a contribution associated to this new topological invariant enters in the expression of the Noether charges with an arbitrary coupling. For a such a case, the surface term is not proportional to the dual of the Weyl tensor, which implies that no considerations about the self-duality condition can be made.

We understand that, in Riemann-Cartan theory, a sensible choice of the Holst coupling is the one that produces the dual of the Weyl tensor at the boundary for asymptotically AdS spaces, in a similar fashion that only for the Gauss-Bonnet coupling in Eq. (2.1) the surface term is proportional to the Weyl tensor [38].

Implications of the addition of Pontryagin term and self-duality condition for the Weyl tensor at the level of the Euclidean action and thermodynamics of AAdS gravitational objects will be discussed elsewhere.

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APPENDIX A: NOETHER THEOREM

The Noether theorem states that, for any action invariant under a continuous transformation, there is a conserved current which leads to a conserved charge.

Let $I[\phi]=\int d^4xL(\phi,\partial\phi)$ be an action for a set of the fields $\phi(x)$, where the Lagrangian L may contain boundary terms added to the action. By varying the form of fields, $\delta\phi(x)=\phi'(x)-\phi(x)$, an extremum on the action is reached for the Euler-Lagrange equations,

$$\frac{\delta I[\phi]}{\delta\phi}=\frac{\partial L}{\partial\phi}-\partial_\mu\frac{\partial L}{\partial\partial_\mu\phi}=0. \tag{A1}$$

The surface term in a general variation of the action,

$$\delta I[\phi]=\text{e.o.m.}+\int d^4x\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi}\delta\phi\right)\equiv\int d^4x\partial_\mu\Theta^\mu(\phi,\delta\phi), \tag{A2}$$

must vanish upon suitable boundary conditions on the field ϕ, in order to have a well-posed action principle.

Let us assume that the action $I[\phi]$ is invariant under the continuous transformations

$$x^\mu\to x'^\mu=x^\mu+\delta x^\mu,\qquad \phi(x)\to\phi'(x')=\phi(x)+\delta_T\phi(x), \tag{A3}$$

where the variation of the form of the field, $\delta\phi$, is related to the total variation of the field, $\delta_T\phi$, as

$$\delta_T\phi(x)=\delta\phi(x)+\partial_\mu\phi\delta x^\mu. \tag{A4}$$

Transformations (A3) are a symmetry of the theory if the action is off-shell invariant,

$$\delta I[\phi]=\int d^4x'L'(x')-\int d^4xL(x)=0. \tag{A5}$$

The Noether current is obtained by rewriting the invariance condition (A5) and identifying the equations of motion. Using the Euler-Lagrange equations (A1), the Lagrangian changes as

$$\delta L=\frac{\partial L}{\partial\phi}\delta\phi+\frac{\partial L}{\partial\partial_\mu\phi}\partial_\mu\delta\phi=\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi}\delta\phi\right)=\partial_\mu\Theta^\mu(\phi,\delta\phi), \tag{A6}$$

and the volume element changes by the Jacobian, $|\frac{\partial x'}{\partial x}|\approx 1+\partial_\mu\delta x^\mu$. Therefore, the total change in the Lagrangian is

$$L'(x')=L(x)+\partial_\mu\Theta^\mu(\phi,\delta\phi)+\partial_\mu L\delta x^\mu. \tag{A7}$$

The relations (A4)–(A7) imply that the symmetry transformations change the action as a total derivative,

$$\delta I[\phi]=\int d^4x\partial_\mu(\Theta^\mu(\phi,\delta\phi)+L\delta x^\mu)=\int d^4x\partial_\mu(\sqrt{g}J^\mu). \tag{A8}$$

Furthermore, the invariance condition (A5) leads to the conservation law

$$\partial_\mu(\sqrt{g}J^\mu)=0. \tag{A9}$$

The Noether current is then given by

$$\sqrt{g}J^\mu = \Theta^\mu(\phi,\delta\phi) + L\delta x^\mu. \tag{A10}$$

On the contrary to the situation described in Eq. (A5), if the action does change by a boundary term $\int d^4x\partial_\mu(\sqrt{g}\Omega^\mu)$, the conserved current is modified as $\tilde{J}^\mu = J^\mu - \Omega^\mu$.

The computation of the conserved charge requires us to specify the boundary. The spacetime has topology $M \simeq \mathbb{R}\times\Sigma$, where Σ is the spatial section with a unit normal vector $u_\mu = (-\tilde{N},0,0,0)$. To define the invariant volume element, we use the relation $\sqrt{g} = N\sqrt{h} = N\tilde{N}\sqrt{\sigma}$. The conserved charge reads

$$Q = \int_\Sigma d^3x\sqrt{\sigma}Nu_\mu J^\mu. \tag{A11}$$

If, in turn, the Noether current can be written as a total derivative,

$$\sqrt{g}J^\mu = \partial_\nu(\sqrt{h}Q^{\mu\nu}), \tag{A12}$$

then the charge becomes

$$Q = \int_{\partial\Sigma} d^2x\sqrt{\sigma}u_\mu n_\nu Q^{\mu\nu}. \tag{A13}$$

Here, $n_\mu = (0,N,0,0)$ is a normal to the boundary $\partial M \simeq \mathbb{R}\times\partial\Sigma$. The quantity $d^2x\sqrt{\sigma}n_\mu u_\nu$ is the dual surface element of $\partial\Sigma$ that is antisymmetric, such that $Q^{\mu\nu} = -Q^{\nu\mu}$.

1. Electromagnetic charge

Maxwell electrodynamics with the Pontryagin term is invariant under $U(1)$ gauge transformations, $\delta_\lambda A_\nu = \partial_\nu\lambda$. This implies $\delta_\lambda F_{\mu\nu} = 0$, so that the invariance condition (A5) of the action is fulfilled. This is an internal symmetry ($\delta x^\mu = 0$), and the Noether current (A10) reads

$$J^\mu = \frac{\partial L}{\partial\partial_\mu A_\nu}\partial_\nu\lambda. \tag{A14}$$

We take $\sigma = 1$. Differentiating the Lagrangian $L = \frac{1}{4}(F^{\alpha\beta}F_{\alpha\beta} + \gamma^* F^{\alpha\beta}F_{\alpha\beta})$ leads to

$$\begin{aligned} J^\mu &= (F^{\mu\nu} + \gamma^* F^{\mu\nu})\partial_\nu\lambda \\ &= \partial_\nu[(F^{\mu\nu} + \gamma^* F^{\mu\nu})\lambda], \end{aligned} \tag{A15}$$

where the last line is obtained using the Maxwell equations and the Bianchi identity in order to obtain the charge tensor (A12) as

$$Q^{\mu\nu} = (F^{\mu\nu} + \gamma^* F^{\mu\nu})\lambda. \tag{A16}$$

In spherical coordinates, the boundary manifold $\partial M = \mathbb{R}\times S^2$ has a radial normal $n_\mu = \delta_\mu^r$ and the timelike normal $u_\nu = -\delta_\nu^t$, and the parameter λ is constant on $\partial\Sigma$, such that it can be set to 1. This enables us to compute the electromagnetic charge as in Eq. (A13).

2. Diffeomorphic current

An action for Riemmanian gravity, with the metric as the only fundamental field, is invariant under an infinitesimal change of coordinates $\delta x^\mu = \xi^\mu(x)$, where the metric transforms as a Lie derivative,

$$\delta_\xi g_{\mu\nu} = -\pounds_\xi g_{\mu\nu} = -(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu). \tag{A17}$$

Since the action depends on $g_{\mu\nu}$ and its derivatives combined in the Cristoffel symbol $\Gamma^\lambda_{\alpha\beta}$, it is convenient to separate the boundary term (A2) which depends on $\delta g_{\mu\nu}$ from the one that depends on $\delta\Gamma^\lambda_{\alpha\beta}$, so that the Noether current (A10) can be written as

$$\sqrt{g}J^\mu = \Theta^\mu(g,\delta_\xi\Gamma) + \Theta^\mu(g,\delta_\xi g) + L\xi^\mu. \tag{A18}$$

Note that $\Theta^\mu(g,\delta_\xi g) = 0$ as a consequence of the asymptotic Killing equation, $\pounds_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$, which describes isometries of the spacetime.

APPENDIX B: ASYMPTOTICALLY ADS SPACETIMES

We first consider a radial foliation of the spacetime, given by the normal coordinates

$$ds^2 = N^2(\rho)d\rho^2 + h_{ij}(\rho,x)dx^i dx^j, \tag{B1}$$

where h_{ij} is the induced metric on a boundary ∂M defined at $\rho = \text{Const}$ and parametrized by the coordinate set $\{x^i\}$. In this frame, the only nonvanishing components of the Christoffel symbol are

$$\begin{aligned} \Gamma^\rho_{ij} &= \frac{1}{N}K_{ij}, \qquad \Gamma^i_{\rho j} = -NK^i_j, \\ \Gamma^\rho_{\rho\rho} &= \frac{d(\ln N)}{dr}, \qquad \Gamma^i_{jl}(g) = \Gamma^i_{jl}(h), \end{aligned} \tag{B2}$$

where $K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}$ is the extrinsic curvature.

This spacetime foliation implies the Gauss-Codazzi relations

$$\begin{aligned} R^{i\rho}_{jl} &= \frac{1}{N}(\nabla_l K^i_j - \nabla_j K^i_l), \\ R^{i\rho}_{j\rho} &= \frac{1}{N}(K^i_j)' - K^i_n K^n_j, \\ R^{ik}_{jl} &= \mathcal{R}^{ik}_{jl}(h) - K^i_j K^k_l + K^i_l K^k_j, \end{aligned} \tag{B3}$$

where $\nabla_j = \nabla_j(h)$ is the covariant derivative defined with respect to the boundary metric and $\mathcal{R}^{ik}_{jl}(h)$ is the intrinsic curvature of ∂M.

1. Asymptotic falloff of boundary tensors

A suitable choice of the the lapse function and induced metric in Eq. (B1) as $N = \frac{\ell}{2\rho}$ and $h_{ij} = \frac{1}{\rho} g_{ij}$, that is,

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{ij} dx^i dx^j, \tag{B4}$$

makes it easier to work out an asymptotic [Fefferman-Graham (FG)] form of the boundary fields for AAdS spaces [39]. The metric defined at the asymptotic boundary ($\rho = 0$) can be seen as a power-series expansion. In particular, in four spacetime dimensions,

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^{3/2} g_{(3/2)ij}(x) + \mathcal{O}(\rho^2). \tag{B5}$$

The coefficient $g_{(3/2)ij}$ cannot be determined from the field equations, as it corresponds to the response to the boundary source $g_{(0)ij}$; i.e., it is proportional to the stress tensor. Because of the fact that there is no Weyl anomaly at the boundary of 4D AdS gravity, $g_{(3/2)ij}$ is traceless.

In the FG coordinate frame, the expansion for the relevant boundary quantities leads to the expression

$$\sqrt{h} = \frac{\sqrt{g}}{\rho^{3/2}} = \frac{\sqrt{g_{(0)}}}{\rho^{3/2}} + \mathcal{O}\left(\frac{1}{\rho}\right), \tag{B6}$$

and for the extrinsic curvature,

$$K^i_j(h) = \frac{1}{\ell}\delta^i_j - \rho\ell S^i_j(g) + \mathcal{O}(\rho^2), \tag{B7}$$

where $S^i_j(g)$ is the Schouten tensor defined as

$$S^i_j(g) = \mathcal{R}^i_j(g) - \frac{1}{4}\delta^i_j \mathcal{R}(g), \tag{B8}$$

in terms of the boundary Ricci tensor and the Ricci scalar.

For the intrinsic curvature, the asymptotic expansion gives

$$\mathcal{R}^{ik}_{jl}(h) = \rho\mathcal{R}^{ik}_{jl}(g) = \rho\mathcal{R}^{ik}_{jl}(g_{(0)}) + \mathcal{O}(\rho^2), \tag{B9}$$

which is also valid for traces of the boundary Riemann tensor. That means that Eq. (B7) can be rewritten in terms of curvatures of h_{ij} in the next-to-leading order,

$$K^i_j(h) = \frac{1}{\ell}\delta^i_j - \ell S^i_j(h) + \mathcal{O}(\rho^2). \tag{B10}$$

Equipped with the asymptotic form of the tensorial quantities involved, we can expand the variation of the total action (2.6) and work out the holographic version of the electric and magnetic parts of the Weyl tensor.

2. Holographic stress tensor in AdS_4 gravity

The projection in the radial foliation (B1) of the variation of gravity action in Eq. (2.2) can be written as

$$\delta I_4 = \frac{\ell^2}{32\pi G}\int_{\partial M} d^3x\sqrt{h}\delta^{[ikl]}_{[jmn]}\left[\left(\delta K^j_i + \frac{1}{2}K^j_q h^{qs}\delta h_{si}\right)W^{mn}_{kl} + N h^{mq}\delta\Gamma^j_{qi}(h) W^{\rho n}_{kl}\right]. \tag{B11}$$

The expansion of the Weyl tensor in the FG frame up to quadratic order in ρ is given by

$$\begin{aligned} W^{i\rho}_{jl} &= \mathcal{O}(\rho^2), \\ W^{i\rho}_{j\rho} &= -\frac{3}{2}\frac{\rho^{3/2}}{\ell^2} g^{ik}_{(0)} g_{(3/2)kj} + \mathcal{O}(\rho^2), \\ W^{ik}_{jl} &= \rho\mathcal{W}^{ik}_{jl}[g_{(0)}] + \frac{3}{2}\frac{\rho^{3/2}}{\ell^2} g^{[im}_{(0)} g_{(3/2)m[j}\delta^{k]}_{l]} + \mathcal{O}(\rho^2), \end{aligned} \tag{B12}$$

where the first term in the last relation is the Weyl tensor of the metric at the conformal boundary $g_{(0)ij}$.

At the same time, we can see the contribution coming from the quantities that involve variations in the Eq. (B11), that is,

$$\begin{aligned} \delta K^j_i &= \mathcal{O}(\rho), \\ h^{qs}\delta h_{si} &= g^{qs}_{(0)}\delta g_{(0)si} + \mathcal{O}(\rho), \\ \delta\Gamma^j_{qi}(h) &= \mathcal{O}(1). \end{aligned} \tag{B13}$$

A simple power-counting argument applied to the expansion of Eq. (B11) shows that the first and the last terms are subleading and actually go to zero as one approaches the boundary $\rho \to 0$. As expected, the finite of the variation of EH action plus GB term is the holographic stress tensor [12,40]

$$\delta I_4 = \frac{1}{2}\int_{\partial M} d^3x\sqrt{g_{(0)}}\left(-\frac{3}{16\pi G\ell} g^{im}_{(0)} g_{(3/2)mn} g^{nj}_{(0)}\right)\delta g_{(0)ij}, \tag{B14}$$

where we have used the fact that any trace of the boundary Weyl tensor is zero.

We can covariantize back the above expression in terms of tensorial quantities related to the full boundary metric h_{ij} and prove that, up to the relevant order, the variation of I_4 can be cast in the form

$$\delta I_4 = -\frac{\ell}{16\pi G}\int_{\partial M} d^3x\sqrt{h} W^{ik}_{jk}(h^{-1}\delta h)^j_i. \tag{B15}$$

Using the fact that a single trace of the Weyl tensor is zero, we have that

$$W^{ik}_{jk} = -W^{i\rho}_{j\rho}, \tag{B16}$$

and it is easy to show that the quantity that appears at the boundary is the electric part of the Weyl tensor

$$E^i_j = W^{i\mu}_{j\nu} n_\mu n^\nu, \tag{B17}$$

as we can rewrite Eq. (B15) in the form

$$\delta I_4 = \frac{\ell}{16\pi G}\int_{\partial M} d^3x\sqrt{h}E^i_j(h^{-1}\delta h)^j_i. \tag{B18}$$

This makes manifest the link between the concept of conformal mass [19] and the addition of the Gauss-Bonnet term in 4D AdS gravity [20].

3. Holographic Cotton tensor

The Pontryagin term, written as a boundary term in the coordinate frame (B1), is expressed as

$$\begin{aligned}\int_M d^4x\mathcal{P}_4 &= \int_{\partial M} d^3x\frac{n_\mu}{N}\epsilon^{\mu\nu\alpha\beta} \\ &\quad\times\left(\Gamma^\sigma_{\nu\lambda}\partial_\alpha\Gamma^\lambda_{\beta\sigma} + \frac{2}{3}\Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\alpha\epsilon}\Gamma^\epsilon_{\beta\sigma}\right) \\ &= \int_{\partial M} d^3x\epsilon^{ijk}\left[-\Gamma^l_{im}\left(\partial_j\Gamma^m_{kl} + \frac{2}{3}\Gamma^m_{jn}\Gamma^n_{kl}\right)\right. \\ &\quad\left. + 2K^l_i\nabla_jK_{kl}\right].\end{aligned} \tag{B19}$$

Using the asymptotic form of the fields in FG expansion, the last term reads

$$\begin{aligned}2\epsilon^{ijk}K^l_i\nabla_jK_{kl} &= 2\epsilon^{ijk}\left(\frac{1}{\ell}\delta^l_i - \rho\ell S^l_i + \mathcal{O}(\rho^2)\right) \\ &\quad\times(-\ell\nabla_jS_{kl} + \mathcal{O}(\rho)),\end{aligned} \tag{B20}$$

in terms of a Schouten tensor and the covariant derivative defined with respect the conformal metric $g_{(0)ij}$. Manipulating the last relation, we see that

$$2\epsilon^{ijk}K^l_i\nabla_jK_{kl} = -2\epsilon^{ijk}\nabla_iS_{jk} + \mathcal{O}(\rho) = \mathcal{O}(\rho), \tag{B21}$$

because S_{jk} is symmetric.

Therefore, using $\delta\Gamma^l_{im} = \frac{1}{2}h^{ln}(\nabla_i\delta h_{nm} + \nabla_m\delta h_{ni} - \nabla_n\delta h_{im})$, the variation of the Pontryagin invariant takes the form

$$\begin{aligned}\delta P_4 &= -\int_{\partial M} d^3x\epsilon^{ijk}\delta\Gamma^l_{im}\mathcal{R}^m_{ljk}(h) \\ &= \int_{\partial M} d^3x\epsilon^{ijk}(h^{-1}\delta h)^l_i\nabla_m\mathcal{R}^m_{ljk}.\end{aligned} \tag{B22}$$

As the boundary is three dimensional, its Weyl tensor vanishes,

$$\begin{aligned}0 = \mathcal{W}^{ml}_{jk}(h) &= \mathcal{R}^{ml}_{jk}(h) - \delta^m_jS^l_k(h) + \delta^l_jS^m_k(h) \\ &\quad + \delta^m_kS^l_j(h) - \delta^l_kS^m_j(h),\end{aligned} \tag{B23}$$

such that

$$\delta\int_M d^4x\mathcal{P}_4 = \int_{\partial M} d^3x\epsilon^{ijk}(h^{-1}\delta h)^l_i\nabla_m(2\delta^m_jS_{kl} - 2h_{lj}S^m_k), \tag{B24}$$

where the second term in the first line identically vanishes due to the symmetry of the indices. In doing so, the variation is written as

$$\delta\int_M d^4x\mathcal{P}_4 = 2\int_{\partial M} d^3x\sqrt{h}(h^{-1}\delta h)^l_iC^i_l, \tag{B25}$$

where C^i_l is the Cotton-York tensor,

$$C^i_l = \frac{1}{\sqrt{h}}\epsilon^{ijk}\nabla_jS_{kl}. \tag{B26}$$

Finally, putting together the holographic stress tensor in Eq. (B14) and by rescaling the Cotton tensor in Eq. (B25), we see that the finite part of the variation of the total action is

$$\delta I = \frac{1}{2}\int_{\partial M} d^3x\sqrt{g_{(0)}}\left(T^{ij}\mp\frac{\ell^2}{8\pi G}C^{ij}(g_{(0)})\right)\delta g_{(0)ij}, \tag{B27}$$

where T^{ij} is the holographic stress tensor.

In Ref. [41], the holographic reconstruction of gravity is performed for perfect-Cotton geometries, where the Cotton tensor of the boundary geometry is proportional to the energy-momentum tensor. A corresponding gravity theory in the bulk is characterized by the self-duality condition for the Weyl tensor.

[1] F. Wilczek, Two Applications of Axion Electrodynamics, Phys. Rev. Lett. **58**, 1799 (1987).

[2] J. Maciejko, X.-L. Qi, H. D. Drew, and S.-C. Zhang, Topological Quantization in Units of the Fine Structure Constant, Phys. Rev. Lett. **105**, 166803 (2010).

[3] S. Deser and C. Teitelboim, Duality transformations of Abelian and non-Abelian gauge fields, Phys. Rev. D **13**, 1592 (1976).

[4] M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics, Bristol, UK, 1991).

[5] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Conserved Charges for Gravity with Locally AdS Asymptotics, Phys. Rev. Lett. **84**, 1647 (2000).

[6] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Conserved charges for even dimensional asymptotically AdS gravity theories, Phys. Rev. D **62**, 044002 (2000).

[7] R. Aros, Analyzing charges in even dimensions, Classical Quantum Gravity **18**, 5359 (2001).

[8] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[9] R. Olea, Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes, J. High Energy Phys. 06 (2005) 023.

[10] R. Olea, Regularization of odd-dimensional AdS gravity: Kounterterms, J. High Energy Phys. 04 (2007) 073.

[11] K. Skenderis, Lecture notes on holographic renormalization, Classical Quantum Gravity **19**, 5849 (2002).

[12] O. Miskovic and R. Olea, Topological regularization and self-duality in four-dimensional anti-de Sitter gravity, Phys. Rev. D **79**, 124020 (2009).

[13] O. Miskovic, R. Olea, and M. Tsoukalas, Renormalized AdS action and critical gravity, J. High Energy Phys. 08 (2014) 108.

[14] R. Emparan, C. V. Johnson, and R. C. Myers, Surface terms as counterterms in the AdS/CFT correspondence, Phys. Rev. D **60**, 104001 (1999).

[15] P. Kraus, F. Larsen, and R. Siebelink, The gravitational action in asymptotically AdS and flat spacetimes, Nucl. Phys. **B563**, 259 (1999).

[16] S. W. MacDowell and F. Mansouri, Unified Geometric Theory of Gravity and Supergravity, Phys. Rev. Lett. **38**, 739 (1977).

[17] H. Garcia-Compean, O. Obregon, and C. Ramirez, Gravitational duality in MacDowell-Mansouri gauge theory, Phys. Rev. D **58**, 104012 (1998).

[18] M. A. Vasiliev, Cubic interactions of bosonic higher spin gauge fields in AdS(5), Nucl. Phys. **B616**, 106 (2001); **B652**, 407 (2003).

[19] A. Ashtekar and A. Magnon, Asymptotically anti-de Sitter space-times, Classical Quantum Gravity **1**, L39 (1984); A. Ashtekar and S. Das, Asymptotically anti-de Sitter space-times: Conserved quantities, Classical Quantum Gravity **17**, L17 (2000).

[20] D. P. Jatkar, G. Kofinas, O. Miskovic, and R. Olea, Conformal mass in AdS gravity, Phys. Rev. D **89**, 124010 (2014).

[21] S. Deser, M. J. Duff, and C. J. Isham, Gravitationally induced *CP* effects, Phys. Lett. B **93**, 419 (1980).

[22] R. Jackiw and S. Pi, Chern-Simons modification of general relativity, Phys. Rev. D **68**, 104012 (2003).

[23] S. de Haro, Dual gravitons in AdS(4)/CFT(3) and the holographic cotton tensor, J. High Energy Phys. 01 (2009) 042.

[24] I. Bakas, Energy-momentum/Cotton tensor duality for AdS(4) black holes, J. High Energy Phys. 01 (2009) 003.

[25] D. S. Mansi, A. C. Petkou, and G. Tagliabue, Gravity in the $3+1$-split formalism II: Self-duality and the emergence of the gravitational Chern-Simons in the boundary, Classical Quantum Gravity **26**, 045009 (2009).

[26] T. Basile, X. Bekaert, and N. Boulanger, A note about a pure spin-connection formulation of general relativity and spin-2 duality in (A)dS, arXiv:1512.09060.

[27] V. Iyer and R. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994).

[28] V. Iyer and R. M. Wald, A comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes, Phys. Rev. D **52**, 4430 (1995).

[29] E. Newman, L. Tamburino, and T. Unti, Empty-space generalization of the Schwarzschild metric, J. Math. Phys. (N.Y.) **4**, 915 (1963).

[30] D. N. Page and C. N. Pope, Inhomogeneous Einstein metrics on complex line bundles, Classical Quantum Gravity **4**, 213 (1987).

[31] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[32] S. W. Hawking, C. J. Hunter, and D. N. Page, Nut charge, anti-de Sitter space and entropy, Phys. Rev. D **59**, 044033 (1999).

[33] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. A **362**, 425 (1978).

[34] G. W. Gibbons and C. N. Pope, CP^2 as a gravitational instanton, Commun. Math. Phys. **61**, 239 (1978).

[35] R. Durka and J. Kowalski-Glikman, Gravity as a constrained BF theory: Noether charges and Immirzi parameter, Phys. Rev. D **83**, 124011 (2011).

[36] R. Durka, Immirzi parameter and Noether charges in first order gravity, J. Phys. Conf. Ser. **343**, 012032 (2012).

[37] For asymptotically flat gravity, the influence of topological terms in first-order formalism has been studied in A. Corichi, I. Rubalcava-García, and T. Vukašinac, Hamiltonian and Noether charges in first order gravity, Gen. Relativ. Gravit. **46**, 1813 (2014).

[38] The Holst coupling is related to the Barbero-Immirzi parameter, which is relevant for computations in the context of Loop Quantum Gravity. For the self-dual choice $\gamma = \pm i$, which simplifies and clarifies the description of black hole entropy, see S. Carlip, Four-Dimensional Entropy from Three-Dimensional Gravity, Phys. Rev. Lett. **115**, 071302 (2015).

[39] C. Fefferman and C. Robin Graham, *Élie Cartan et les Mathématiques d'Aujourd'Hui* (European Mathematical Society, Astérisque, Paris, 1985), p. 95.

[40] V. Balasubramanian and P. Kraus, A Stress tensor for Anti-de Sitter gravity, Commun. Math. Phys. **208**, 413 (1999).

[41] A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, and K. Siampos, Holographic perfect fluidity, Cotton energy-momentum duality and transport properties, J. High Energy Phys. 04 (2014) 136.