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Entropy of extremal black holes: Horizon limits through charged thin shells in a unified approach

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Using a unified approach, we study the entropy of extremal black holes through the entropy of an electrically charged thin shell. We encounter three cases in which a shell can be taken to its own gravitational or horizon radius and become an extremal spacetime. In case 1, we use a nonextremal shell, calculate all the thermodynamic quantities including the entropy, take it to the horizon radius, and then take the extremal limit. In case 2, we take the extremal limit and the horizon radius limit simultaneously; i.e., as the shell approaches its horizon radius, it also approaches extremality. In case 3, we take first an extremal shell, and then take its horizon radius. We find that the thermodynamic quantities, in general, have different expressions in the three different cases. The entropy is the Bekenstein-Hawking entropy $S = A_+/4$ (where A_+ is the horizon area) in cases 1 and 2, and in case 3 it can be any well-behaved function of A_+. The contributions from the various thermodynamic quantities for the entropy in all three cases are distinct. Indeed, in cases 1 and 2, the limits agree in what concerns the entropy but they disagree in the behavior of all other thermodynamic quantities. Cases 2 and 3 disagree in what concerns the entropy but agree in the behavior of the local temperature and electric potential. Case 2 is, in a sense, intermediate between cases 1 and 3. Our approach sheds light on the extremal black hole entropy issue.

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I. INTRODUCTION

The fact that black holes possess thermodynamic properties [1–3] is arguably their brightest feature. Especially fascinating is that black holes have entropy. For the nonextremal black holes, it is known that its entropy S is the Bekenstein-Hawking entropy, equal to $A_+/4$, where A_+ is the horizon area. This has been put in firm ground in the works of York and collaborators [4–7] (see also a generalization in [8]), in a Hamiltonian formalism [9,10], and using quite generic matter fields [11], among other approaches. A special kind of matter field, thin shells have also been used in [12–15] to further probe the thermodynamic properties of black holes. In particular, in [13] (see also [14,15]), the results are based on the fact that a general thin shell can be taken to its gravitational radius where one must force its temperature to be equal to the Hawking temperature of a black hole; otherwise backreaction effects will destroy the shell. By doing so, the shell is seen to possess an entropy equal to the Bekenstein-Hawking entropy $S = A_+/4$ of the correspondent spacetime black hole, thus making it possible to calculate the entropy of an extremal black hole by using an extremal shell taken to its gravitational radius. Although several efforts have been made, it is still unclear what the microscopic explanation of this value is in the framework of a full quantum gravity theory.

Extremal black holes seem to be a different object from nonextremal ones. Indeed, for extremal black holes, not only the microscopic explanation of the entropy S is absent, but even the value S itself of the entropy is uncertain. Although some suggestions have been worked out that yield $S = 0$ [16–18], the entropy of an extremal black hole is still an open problem, as string theory claims that it is, in fact, given by the Bekenstein-Hawking entropy $S = A_+/4$ [19,20]; see also [21–35] on this discussion.

In Pretorius, Vollock and Israel [36], in [37] using matter fields, and in [38] using an extremal charged thin matter shell, an interesting solution to the debate was naturally

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deduced. It was found that the extremal black hole entropy could be any well-behaved function of A_+, $S = 0$ and $S = A_+/4$ included. Of course, one might also obtain the entropy of an extremal black hole by first calculating the entropy of a nonextremal charged thin shell [13] and then taking the extremal limit as a particular case, producing, as expected, $S = A_+/4$. There is even another case, an intermediate one, when one takes the extremal limit and the horizon radius limit simultaneously; i.e., as the shell approaches its horizon radius, it also approaches extremality. Therefore, it is particularly important to study the consistency of the thin shell approach in the various limits, to further strengthen the conclusions drawn in [38]. We use the results stated in [39,40] that the thermal stress energy tensor corresponding to a given temperature diverges in the horizon limit unless the temperature is the Hawking temperature.

Thin shells are systems of great interest that have been used in a number of ways in classical general relativity, as a way to quantize gravitational systems, and concomitantly in a black hole context. Classically, we mention a variational principle found for dust shells [41], and the collapse of electrically charged thin shells to probe spacetime features and test cosmic censorship [42]. It has also been further used to understand in different ways the entropy of gravitational systems including black holes [43,44]. Quantically, thin shells have, for instance, been used in the understanding of quantum black hole states and Hawking radiation; see, e.g., [45–47].

The work is organized as follows. In Sec. II, we give the preliminaries necessary to discuss the various horizon limits. We display the first law of thermodynamics and give the expressions for the thermodynamic quantities that enter into it. In Sec. III, we define the two variables that are important to take the horizon limits, ε and δ. In Sec. IV, we define, through geometry, the three cases that appear in the horizon limit. In Sec. V, we see the expressions for the mass and electric charge in the three cases. In Sec. VI, we find the expressions for the surface pressure, the electric potential, and the temperature in the three cases. In Sec. VII, we put everything together into the first law and find the entropy in the three horizon limits. In Sec. VIII, we discuss the contribution of each thermodynamic quantity to the entropy and summarize these results in a table. In Sec. IX, a discussion on the backreaction issue is raised. In Sec. X, we conclude.

II. PRELIMINARIES

The study of the nonextremal charged thin shell developed in [13] involves three dynamical variables: the radius R of the shell, its rest mass M, and its charge Q. For thermodynamics, we also need the local temperature T, the surface pressure p, and the electric potential Φ, and then find the entropy S. Assuming that the shell is static, spherically symmetric, and has a well-defined temperature, the first law of thermodynamics is

$$TdS = dM + pdA - \Phi dQ, \tag{1}$$

where in all calculations we use natural units, i.e., the speed of light, the gravitational constant, the Planck constant, and the Boltzmann constant are set to one, $c = G = h = k_B = 1$, respectively.

There are two other particularly useful variables which can characterize the problem, namely the shell's radius R, the gravitational or horizon radius r_+, and its Cauchy radius r_-, which are functions of (R, M, Q) through

$$r_+(R, M, Q) = m + \sqrt{m^2 - Q^2}, \tag{2}$$

$$r_-(R, M, Q) = m - \sqrt{m^2 - Q^2}, \tag{3}$$

where m is the ADM mass, which can shown to be given by

$$m(R, M, Q) = M - \frac{M^2}{2R} + \frac{Q^2}{2R}. \tag{4}$$

It is quite interesting that the formula given in Eq. (4) can be obtained from quite different perspectives. In [6], it was obtained from the action formalism approach to black hole thermodynamics, but it has another meaning there since it applies to black holes, not to shells. In [7], it was rederived for bounded self-gravitating systems using the quasilocal energy formalism. In [13], probably for the first time, it was obtained (i) in a pure thermodynamic context, (ii) for thin shells, and (iii) using such general assumptions as the first law of thermodynamics and integrability conditions only. In [45], it had been derived from dynamics of shells.

Thus, inversely, the quantities M and Q can be written in terms of (R, r_+, r_-). Define, k as

$$k(R, r_+, r_-) = \sqrt{\left(1 - \frac{r_+}{R}\right)\left(1 - \frac{r_-}{R}\right)}, \tag{5}$$

usually called the redshift function. Then M is given by

$$M(R, r_+, r_-) = R(1 - k), \tag{6}$$

where we have chosen the solution that gives $M = m$ for R large. Also,

$$Q(R, r_+, r_-) = \sqrt{r_+ r_-}. \tag{7}$$

The area of the shell is

$$A(R, r_+, r_-) = 4\pi R^2, \tag{8}$$

and the gravitational area or horizon area is

$$A_+(R, r_+, r_-) = 4\pi r_+^2. \tag{9}$$

We have written explicitly the complete functional dependence (R, r_+, r_-), even though some quantities do not depend on one or two of these variables, in order to show that this is a thermodynamic system. Thus, $Q(R,r_+,r_-)$ only depends on (r_+,r_-), $A(R,r_+,r_-)$ only depends on (R), and $A_+(R,r_+,r_-)$ only depends on (r_+). It will prove useful to keep the generic functional dependence.

In order for the nonextremal electric charged shell to remain static, its surface pressure must have a specific functional form, given by [13]

$$p(R,r_+,r_-) = \frac{R^2(1-k)^2 - r_+ r_-}{16\pi R^3 k}. \tag{10}$$

The electric potential Φ of the shell must also assume a specific form if the shell is to remain static. The integrability conditions out of the first law of thermodynamics assert that [13]

$$\Phi(R,r_+,r_-) = \frac{c(r_+,r_-) - \frac{1}{R}}{k}\sqrt{r_+ r_-}, \tag{11}$$

where $c(r_+,r_-)$ is an arbitrary function, which physically represents the electric potential of the shell multiplied by its charge, if it were located at infinity. Additionally, we need the nonextremal shell to have a well-defined electric potential in the horizon limit. This leads to

$$c(r_+,r_-) = \frac{1}{r_+}, \tag{12}$$

and, consequently,

$$\Phi(R,r_+,r_-) = \sqrt{\frac{r_-}{r_+}}\sqrt{\frac{1-\frac{r_+}{R}}{1-\frac{r_-}{R}}}. \tag{13}$$

Equation (13) formally coincides with the expression (4.15) of [6] derived for a black hole in a cavity (their ϕ coincides with our Φ). However, in our case, there is no black hole at all. Should the condition (12) on the function c be relaxed to an arbitrary function, we would obtain $\lim_{R\to r_+}\Phi(R,r_+,r_-) = \infty$, since the infinities inside the square root in the defining Eq. (11) would not be canceled.

Assuming that the shell has a well-defined temperature, the integrability conditions imposed from the first law of thermodynamics, Eq. (1), gives [13]

$$T(R,r_+,r_-) = \frac{T_0}{k}, \tag{14}$$

where T is the temperature at the shell and T_0 is the temperature seen from infinity.

Now, we impose

$$T_0 = T_H(r_+,r_-) = \frac{r_+ - r_-}{4\pi r_+^2}, \tag{15}$$

where T_H is the Hawking temperature of an electrically charged black hole. So $T(R,r_+,r_-) = \frac{T_H(r_+,r_-)}{k}$, i.e.,

$$T(R,r_+,r_-) = \frac{r_+ - r_-}{4\pi r_+^2 k}. \tag{16}$$

III. APPROACH TO THE EXTREMAL HORIZON: THE VARIABLES THAT DEFINE THE THREE EXTREMAL HORIZON LIMITS

To study independently the limit of an extremal shell and the limit of a shell being taken to its gravitational radius, it will prove fruitful to define the variables ε and δ through the equations

$$1 - \frac{r_+}{R} = \varepsilon^2, \tag{17}$$

$$1 - \frac{r_-}{R} = \delta^2. \tag{18}$$

It is clearly seen from Eqs. (17) and (18) that the variables ε and δ are the good ones to take the extremal limit. There are, however, different extremal limits depending on how ε and δ are taken to zero.

IV. GEOMETRY: THE THREE EXTREMAL HORIZON LIMITS

There are three physically relevant limits. Let us see them first through the geometry.

Case 1. In this case, we do $r_+ \neq r_-$ and $R \to r_+$, i.e.,

$$\delta = O(1), \qquad \varepsilon \to 0. \tag{19}$$

After all the calculations are finished and we have an expression for the entropy, we can then take the $\delta \to 0$ limit to get at the gravitational radius of an extremal shell. According to Eq. (17), this means bringing the shell to its gravitational radius. It follows from (18) that $r_+ \neq r_-$. Thus, there is the horizon limit, but there is no extremal limit; the shell remains nonextremal during the whole process.

Case 2. In this case, we do $R \to r_+$ and $r_+ \to r_-$, i.e.,

$$\delta = \frac{\varepsilon}{\lambda}, \qquad \varepsilon \to 0, \tag{20}$$

where it is assumed that the new parameter λ remains constant in the limiting process and that it must satisfy $\lambda \leq 1$ due to $r_+ \geq r_-$. The limit in which $\varepsilon \to 0$ means that simultaneously $R \to r_+$ and $r_+ \to r_-$ in such a way that $\delta \sim \varepsilon$. In other words, the horizon limit is accompanied with the extremal one.

Case 3. In this case, we do $r_+ = r_-$ and $R \to r_+$, i.e.,

$$\delta = \varepsilon, \qquad \varepsilon \to 0. \tag{21}$$

Then, $r_+ = r_-$ from the very beginning. This corresponds to the extremal shell. This case was analyzed in [38], so we will simply state the results and use them for comparison.

V. MASS AND ELECTRIC CHARGE: THE THREE EXTREMAL HORIZON LIMITS

Using Eqs. (17) and (18) in Eq. (5), we immediately get that the redshift function is

$$k(R,\varepsilon,\delta) = \varepsilon\delta. \tag{22}$$

In these variables, it depends on ε and δ and not on R.

Moreover, we immediately see that

$$M(R,\varepsilon,\delta) = R(1-\varepsilon\delta), \tag{23}$$

$$Q(R,\varepsilon,\delta) = R\sqrt{(1-\varepsilon^2)(1-\delta^2)}. \tag{24}$$

Then we can study the three cases already mentioned.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get from Eqs. (23)–(24)

$$M(r_+,\varepsilon,\delta) = r_+, \qquad Q(r_+,\varepsilon,\delta) = r_+, \tag{25}$$

where we have also sent δ to zero, $\delta \to 0$, in the end of the calculation.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, we get from Eqs. (23)–(24),

$$M(r_+,\varepsilon,\delta) = r_+, \qquad Q(r_+,\varepsilon,\delta) = r_+. \tag{26}$$

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, it is seen from Eqs. (23)–(24) that

$$M(r_+,\varepsilon,\delta) = r_+, \qquad Q(r_+,\varepsilon,\delta) = r_+. \tag{27}$$

The three limits here, not surprisingly, yield the same result, the mass-charge-radius extremal condition.

VI. PRESSURE, ELECTRIC POTENTIAL, AND TEMPERATURE: THE THREE EXTREMAL HORIZON LIMITS

A. Pressure limits

In order for the nonextremal electric charged shell to remain static, its surface pressure must have a specific functional form, given by Eq. (10) in terms of the variables ε and δ defined in Eqs. (17) and (18) can be readily written as

$$p(R,\varepsilon,\delta) = \frac{1}{16\pi R}\frac{(\delta-\varepsilon)^2}{\delta\varepsilon}. \tag{28}$$

Now, we will consider the behavior of pressure in all three cases.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get from Eq. (28)

$$p(r_+,\varepsilon,\delta) = \frac{\delta}{16\pi r_+\varepsilon} \sim \frac{1}{\varepsilon}. \tag{29}$$

So, the pressure is divergent in this case as $1/\varepsilon$.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, we get from Eq. (28), we put back the intermediate step,

$$p(r_+,\varepsilon,\delta) = \frac{1}{16\pi r_+}\frac{(1-\lambda)^2}{\lambda}. \tag{30}$$

Equation (30) means that the pressure will remain finite but nonzero in this horizon limit for the extremal shell.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, it is seen from Eq. (28) that

$$p(r_+,\varepsilon,\delta) = 0. \tag{31}$$

The result $p = 0$ holds, in fact, at any radius, including the horizon limit.

B. Electric potential limits

The electric potential Φ of the shell must also assume a specific form if the shell is to remain static. In terms of ε and δ defined in Eqs. (17) and (18), Eq. (13) gives

$$\Phi(R,\varepsilon,\delta) = \sqrt{\frac{1-\delta^2}{1-\varepsilon^2}}\frac{\varepsilon}{\delta}. \tag{32}$$

It is now straightforward to analyze the three limiting cases under discussion.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get from Eq. (32),

$$\Phi(r_+,\varepsilon,\delta) = 0. \tag{33}$$

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, we get from Eq. (32),

$$\Phi(r_+,\varepsilon,\delta) = \lambda, \tag{34}$$

with $0 \leq \lambda \leq 1$.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, it would seem from Eq. (32) that $\Phi(r_+,\varepsilon,\delta) = 1$. However, this case is special since, from the very beginning, we should proceed in a different way, so the form of the integrability condition (11) and Eq. (32) are no longer valid here. As is shown in [38], the calculations for this case lead to the inequality

$$\Phi(r_+, \varepsilon, \delta) \leq 1. \tag{35}$$

Thus, if we take an extremal shell from the very beginning, the electric potential in general differs from what is obtained by the extremal limit from the nonextremal state.

C. Temperature limits

Assuming that the shell has a well-defined temperature, the integrability conditions imposed from the first law of thermodynamics and in terms of ε and δ defined in Eqs. (17) and (18), Eq. (14) gives

$$T_H(R, \varepsilon, \delta) = \frac{\delta^2 - \varepsilon^2}{4\pi R(1 - \varepsilon^2)^2}, \tag{36}$$

and so the local temperature on the shell is thus

$$T(R, \varepsilon, \delta) = \frac{T_H}{k} = \frac{\delta^2 - \varepsilon^2}{4\pi R\delta\varepsilon(1 - \varepsilon^2)^2}. \tag{37}$$

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get from Eq. (37),

$$T(r_+, \varepsilon, \delta) = \frac{\delta}{4\pi r_+ \varepsilon} \sim \frac{1}{\varepsilon}. \tag{38}$$

Thus, it diverges.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, we get from Eq. (37),

$$T(r_+, \varepsilon, \delta) = \frac{1 - \lambda^2}{4\pi r_+ \lambda}. \tag{39}$$

It remains finite and nonzero. It is worth noting a simple formula that follows from Eqs. (30) and (39) and relates the pressure and temperature in this horizon limit, namely, $\frac{p}{T} = \frac{1}{4}\frac{1-\lambda}{1+\lambda}$.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, one can relax condition (15) in such a way that $T_0 \to 0$ but T remains finite (see [38] for details).

VII. ENTROPY: THE THREE EXTREMAL HORIZON LIMITS

To obtain the distinct limits for the entropy, one can express the first law of thermodynamics, Eq. (1), in terms of the variables (R, r_+, r_-), using the Eqs. (6), (7), (8), (10), (11), and (16). In turn, using Eqs. (17), (18), (22), (23), (24), (28), and (32), the first law of thermodynamics, Eq. (1), can be expressed in terms of the variables (R, ε, δ), in the quite general exact form $TdS = a_1 dR + a_2 d\varepsilon + a_3 d\delta$, where $a_1 = 1 - \delta\varepsilon + \frac{(\delta-\varepsilon)^2}{2\delta\varepsilon} + \frac{(1-\delta^2)(1-\varepsilon^2)}{\delta\varepsilon}(1 - Rc)$, $a_2 = -\delta R[1 + \frac{1-\delta^2}{\delta^2}(1 - Rc)]$, $a_3 = -\varepsilon R[1 + \frac{1-\varepsilon^2}{\varepsilon^2}(1 - Rc)]$. Imposing further that the electric potential must also assume the value of Eq. (12) enables us to simplify the coefficients a_1, a_2, and a_3, into $a_1 = \frac{\delta^2-\varepsilon^2}{2\delta\varepsilon}$, $a_2 = -\delta R[1 - \frac{\varepsilon^2}{\delta^2}(\frac{1-\delta^2}{1-\varepsilon^2})]$, $a_3 = 0$. Then, using Eq. (37), the differential for the entropy in the variables (R, ϵ, δ) becomes

$$dS(R, \epsilon, \delta) = 2\pi R(1 - \varepsilon^2)^2 dR - 4\pi R^2 \varepsilon(1 - \varepsilon^2) d\varepsilon. \tag{40}$$

This equation can be integrated to give

$$S(r_+, \epsilon, \delta) = \pi R^2(1 - \varepsilon^2)^2, \tag{41}$$

where we have put the integration constant to zero. Using Eq. (17), it gives

$$S(r_+) = \frac{A_+}{4}, \tag{42}$$

where A_+ is the gravitational radius area, or the horizon area when the shell is pushed into the gravitational radius; see Eq. (9). This is the Bekenstein-Hawking entropy. It is striking that all the other quantities, p, Φ, T, depend generically on ε and δ. The entropy does not; it only depends on r_+.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get from Eq. (42), $S(r_+) = \frac{A_+}{4}$. This is general for any nonextremal black hole. We can now take the extremal limit $\delta \to 0$ and obtain that the entropy of an extremal charged black hole is by continuity $S(r_+) = \frac{A_+}{4}$, the Bekenstein-Hawking entropy.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, we obtain from Eq. (42), $S(r_+) = \frac{A_+}{4}$. So, in the case that the shell achieves the gravitational radius simultaneously with the extremal limit, one also gets the Bekenstein-Hawking entropy.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, the entropy cannot be handled in this manner and should be considered separately. This has been done in [38] with the result that the entropy is not fixed unambiguously for a given r_+, it is any physical well behaved function of r_+, or if one prefers, of A_+, i.e.,

$$S(r_+) = \text{a physical well behaved function of } A_+. \tag{43}$$

Equations (40)–(42) work for cases 1 and 2. In case 3, the *ab initio* extremal shell with $\delta = \varepsilon$, one is led to the discussion given in [38].

VIII. DISCUSSION ON THE THREE EXTREMAL HORIZON LIMITS: WHERE DOES THE ENTROPY STEM FROM?

It is instructive to trace, in more detail, how the entropy arises from the first law. More precisely, we are interested in the question: Which contributions dominate for the three different cases?

TABLE I. The contributions of the pressure p, electric potential Φ, and temperature T to the extremal black hole entropy S, according to the first law of thermodynamics.

Case	Pressure p	Potential Φ	Local temperature T	Entropy	Contribution from (according to 1st law)
1	Divergent like ε^{-1}	1	Infinite	$A_+/4$	Pressure
2	Finite nonzero	any < 1	Finite nonzero	$A_+/4$	Mass, pressure and potential
3	0	any ≤ 1	Finite nonzero	a function of A_+	Mass and potential

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, let us, for simplcity, make $\varepsilon = \text{constant} \ll 1$. Then, in the first law, Eq. (1), and from Eq. (29), we can retain the term due to the pressure only. Taking also into account Eq. (38), we obtain the result (42). Thus, the pressure term gives the whole contribution to the entropy. See also [11].

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed according to Eq. (20), and $\varepsilon \to 0$, all three terms in the first law contribute to the entropy. Thus, the mass, pressure, and electric potential terms give contributions to the entropy.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, and according to Eq. (31), the first and third terms in Eq. (1) contribute to the entropy. Thus cases 1 and 3 are complementary to each other in what concerns the origin of the entropy.

It is convenient to present the results in a table, see Table I. It is implied that, in all three cases, the horizon limit is taken.

It is worth stressing that the results presented in the table refer, in general, not to black holes but to shells. Only in the horizon limit do these results apply to black holes. Usually, if one considers the extremal limit of a nonextremal black hole, it remains in the same topological class during the limiting transition, so it is not surprising that, in the extremal limit, one obtains the Bekenstein-Hawking value. However, in our case, we obtained something more: the fact that the exact value of the shell's entropy coincides with that of a black hole for a given r_+ independently of R. For an arbitrary self-gravitating matter system, this is not so; the entropy of the system is a function of r_+, R, and possibly other variables. Only in the horizon limit is the Bekenstein-Hawking value recovered [11].

IX. ROLE OF THE BACKREACTION

As is known, for a nonextremal spacetime, the thermal stress energy tensor corresponding to a temperature T_0 can be represented in the form [39,40]

$$T_\mu^\nu = \frac{T_0^4 - T_H^4}{(g_{00})^2} f_\mu^\nu, \tag{44}$$

where, f_μ^ν is some tensor finite on the horizon, with g_{00} being the 00 component of the metric in use. In the horizon limit, the requirement of the finiteness of T_μ^ν entails $T_0 = T_H$. For a nonextremal horizon, one has $T_H \neq 0$.

Now, in the extremal case, $T_H = 0$, where

$$T_\mu^\nu = \frac{T_0^4}{(g_{00})^2} f_\mu^\nu. \tag{45}$$

Thus, the attempt to put $T_0 \neq 0$ according to the prescriptions given in [16,17] leads to infinite stresses since $\frac{T_0^4}{(g_{00})^2}$ diverges as one approaches the horizon. This destroys the horizon [39,40].

However, when we deal with a shell instead of a black hole, an intermediate case can be realized. Namely, $T_0 \to 0$ and $g_{00} \to 0$ simultaneously in such a way that T is kept bounded. This is realized in case 2 according to Eq. (39). It is also realized in case 3.

X. CONCLUSIONS

We found what happens in calculating the entropy and other thermodynamic quantities when different limiting transitions for a shell are taken. We also found how they are related to each other when the radius of the shell approaches the horizon radius, i.e., the shell spacetime turns into a black hole spacetime.

It happens that the limits in cases 1 and 2 agree in what concerns the entropy but disagree in the behavior of all other quantities. Cases 2 and 3 disagree in what concerns the entropy but agree in the behavior of the local temperature and electric potential. Case 2 is intermediate between 1 and 3.

The results obtained showed how careful one should be in the calculations when a system approaches the horizon which, in turn, is close to the extremal state. It will be interesting to trace whether, and how, these subtleties can affect calculations in quantum field theory, including string theory.

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[1] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).

[2] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).

[3] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).

[4] J. W. York, Black-hole thermodynamics and the Euclidean Einstein action, Phys. Rev. D **33**, 2092 (1986).

[5] E. A. Martinez and J. W. York, Additivity of the entropies of black holes and matter in equilibrium, Phys. Rev. D **40**, 2124 (1989).

[6] H. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Charged black hole in a grand canonical ensemble, Phys. Rev. D **42**, 3376 (1990).

[7] J. D. Brown and J. W. York, Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D **47**, 1407 (1993).

[8] O. B. Zaslavskii, Canonical ensemble for arbitrary configurations of self-gravitating systems, Phys. Lett. A **152**, 463 (1991).

[9] J. Louko and B. F. Whiting, Hamiltonian thermodynamics of the Schwarzschild black hole, Phys. Rev. D **51**, 5583 (1995).

[10] G. A. S. Dias and J. P. S. Lemos, Hamiltonian thermodynamics of d-dimensional ($d \geq 4$) Reissner-Nordström–anti-de Sitter black holes with spherical, planar, and hyperbolic topology, Phys. Rev. D **79**, 044013 (2009).

[11] J. P. S. Lemos and O. B. Zaslavskii, Entropy of quasiblack holes, Phys. Rev. D **81**, 064012 (2010).

[12] E. A. Martinez, Fundamental thermodynamical equation of a self-gravitating system, Phys. Rev. D **53**, 7062 (1996).

[13] J. P. S. Lemos, G. M. Quinta, and O. B. Zaslavskii, Entropy of a self-gravitating electrically charged thin shell and the black hole limit, Phys. Rev. D **91**, 104027 (2015).

[14] J. P. S. Lemos and G. M. Quinta, Entropy of thin shells in a $(2+1)$-dimensional asymptotically AdS spacetime and the BTZ black hole limit, Phys. Rev. D **89**, 084051 (2014).

[15] J. P. S. Lemos, F. J. Lopes, M. Minamitsuji, and J. V. Rocha, Thermodynamics of rotating thin shells in the BTZ spacetime, Phys. Rev. D **92**, 064012 (2015).

[16] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Entropy, area, and black hole pairs, Phys. Rev. D **51**, 4302 (1995).

[17] C. Teitelboim, Action and entropy of extreme and nonextreme black holes, Phys. Rev. D **51**, 4315 (1995).

[18] G. W. Gibbons and R. E. Kallosh, Topology, entropy and Witten index of dilaton black holes, Phys. Rev. D **51**, 2839 (1995).

[19] A. Strominger and C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy, Phys. Lett. B **379**, 99 (1996).

[20] A. Sen, Microscopic and macroscopic entropy of extremal black holes in string theory, Gen. Relativ. Gravit. **46**, 1711 (2014).

[21] A. Ghosh and P. Mitra, Entropy for extremal Reissner-Nordström black holes, Phys. Lett. B **357**, 295 (1995).

[22] O. B. Zaslavskii, Extreme state of a charged black hole in a grand canonical ensemble, Phys. Rev. Lett. **76**, 2211 (1996).

[23] A. Ghosh and P. Mitra, Comment on "Extreme state of a charged black hole in a grand canonical ensemble," Phys. Rev. Lett. **77**, 4848 (1996).

[24] A. Ghosh and P. Mitra, Understanding the area proposal for extremal black hole entropy, Phys. Rev. Lett. **78**, 1858 (1997).

[25] O. B. Zaslavskii, Geometry of nonextreme black holes near the extreme state, Phys. Rev. D **56**, 2188 (1997).

[26] O. B. Zaslavskii, Entropy of quantum fields for nonextreme black holes in the extreme limit, Phys. Rev. D **57**, 6265 (1998).

[27] R. Mann and S. N. Solodukhin, Universality of quantum entropy for extreme black holes, Nucl. Phys. **B523**, 293 (1998).

[28] O. B. Zaslavskii, Comment on "Understanding the area proposal for extremal black hole entropy," Phys. Rev. Lett. **80**, 3412 (1998).

[29] P. Mitra, Entropy of extremal black holes in asymptotically anti-de Sitter spacetime, Phys. Lett. B **441**, 89 (1998).

[30] C. Kiefer and J. Louko, Hamiltonian evolution and quantization for extremal black holes, Ann. Phys. **8**, 67 (1999).

[31] B. Wang, E. Abdalla, and R.-K. Su, Entropy, topology of two-dimensional extreme black holes, Phys. Rev. D **59**, 104006 (1999).

[32] B. Wang, E. Abdalla, and R.-K. Su, Geometry and topology of two kinds of extreme Reissner-Nordström-anti de Sitter black holes, Phys. Rev. D **62**, 047501 (2000).

[33] S. Hod, Evidence for a null entropy of extremal black holes, Phys. Rev. D **61**, 084018 (2000).

[34] S. M. Carroll, M. C. Johnson, and L. Randall, Extremal limits and black hole entropy, J. High Energy Phys. 11 (2009) 109.

[35] A. Edery and B. Constantineau, Extremal black holes, gravitational entropy and nonstationary metric fields, Classical Quantum Gravity **28**, 045003 (2011).

[36] F. Pretorius, D. Vollick, and W. Israel, An operational approach to black hole entropy, Phys. Rev. D **57**, 6311 (1998).

[37] J. P. S. Lemos and O. B. Zaslavskii, Entropy of extremal black holes from entropy of quasiblack holes, Phys. Lett. B **695**, 37 (2011).

[38] J. P. S. Lemos, G. M. Quinta, and O. B. Zaslavskii, Entropy of an extremal electrically charged thin shell and the extremal black hole, Phys. Lett. B **750**, 306 (2015).

[39] P. R. Anderson, W. A. Hiscock, and D. J. Loranz, Semiclassical stability of the extreme Reissner-Nordström black hole, Phys. Rev. Lett. **74**, 4365 (1995).

[40] D. J. Loranz, W. A. Hiscock, and P. R. Anderson, Thermal divergences on the event horizons of two-dimensional black holes, Phys. Rev. D **52**, 4554 (1995).

[41] V. D. Gladush, On the variational principle for dust shells in general relativity, J. Math. Phys. (N.Y.) **42**, 2590 (2001).

[42] S. Gao and J. P. S. Lemos, Collapsing and static thin massive charged dust shells in a Reissner-Nordström black hole background in higher dimensions, Int. J. Mod. Phys. A **23**, 2943 (2008).

[43] P. C. W. Davies, L. H. Ford, and D. N. Page, Gravitational entropy: Beyond the black hole, Phys. Rev. D **34**, 1700 (1986).
[44] W. A. Hiscock, Gravitational entropy of nonstationary black holes and spherical shells, Phys. Rev. D **40**, 1336 (1989).
[45] V. A. Berezin, Quantum black hole model and Hawking's radiation, Phys. Rev. D **55**, 2139 (1997).
[46] A. Yu. Neronov, Quasiclassical mass spectrum of the black hole model with self-gravitating dust shell, Phys. Rev. D **59**, 044023 (1999).
[47] M. Ambrus and P. Hájíček, Quantum superposition principle and gravitational collapse: Scattering times for spherical shells, Phys. Rev. D **72**, 064025 (2005).