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Classical and quantum cosmology of Born-Infeld type models

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We discuss Born-Infeld type fields (tachyon fields) in classical and quantum cosmology. We first partly review and partly extend the discussion of the classical solutions and focus in particular on the occurrence of singularities. For quantization, we employ geometrodynamics. In the case of constant potential, we discuss both Wheeler-DeWitt quantization and reduced quantization. We are able to give various solutions and discuss their asymptotics. For the case of general potential, we transform the Wheeler-DeWitt equation to a form where it leads to a difference equation. Such a difference equation was previously found in the quantization of black holes. We give explicit results for the cases of constant potential and inverse squared potential and point out special features possessed by solutions of the difference equation.

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I. INTRODUCTION

The recent discovery of cosmic acceleration [1] and the searches for dark energy, which can be responsible for such a phenomenon [2], have stimulated studies of different cosmological models, some of them including exotic types of fluids and fields. Among them are the so-called tachyon cosmological models [3–8], which arise as a byproduct of string theory [9]. The energy-momentum tensor of the tachyon field has a negative pressure component which can be used for the description of the cosmic acceleration. In spite of the somewhat misleading name, these tachyon fields represent a development of the old idea by Born and Infeld [10] that the kinetic term of a field Lagrangian is not necessarily a (quadratic) polynomial, but can contain a square root of fields and their derivatives.

In the framework of modern cosmology, even more general Lagrangians are employed, including some with arbitrary functions of the kinetic terms [11]; these models are known as *k*-essence models. From our point of view, however, the Born-Infeld type fields look more natural, because square-root Lagrangians arise in various parts of modern theoretical physics. The classical dynamics of tachyon dark energy models is rich and cannot only describe cosmic acceleration but can also lead to new types of future singularities which are of interest in themselves [7,12,13].

The quantization of a Born-Infeld type of model presents, however, some challenge. Let us address the most popular quantization method for cosmological models, which is the construction of the wave function of the Universe satisfying the Wheeler-DeWitt equation [14,15]. The main problem which one encounters there when applying this framework to tachyonic models is the appearance of the momentum operators under the square root [16]. When we represent these operators as partial derivatives of the field, we obtain nonlocal differential operators, and one has to invoke sophisticated methods for treating them. One possible method is to use the analogy with the quantum mechanics of black holes and thin shells developed in [17–20]. The corresponding Lagrangian contains the time derivatives of the observables under the square root, but the Hamiltonian depends on the momentum by a hyperbolic cosine. After quantization, this leads to a difference equation for the wave function, which displays interesting features. In the case of cosmology, the possible transition to difference equations does not arise automatically but can be achieved by means of an appropriate canonical transformation. We shall discuss such equations in our paper.

Besides the tachyonic field, there are also other Born-Infeld type fields called pseudotachyons [7] and quasi-tachyons [12]. Such fields arise in a natural way in some cosmological models. In the model considered in [7], a particular potential containing the square root of a trigonometrical function was chosen. The dynamical evolution of the model can bring the Universe to a point where the expressions inside of the two square roots, in the potential and in the kinetic term, change sign simultaneously. Thus, to provide a smooth cosmological evolution one is forced to change the Lagrangian of the Born-Infeld type tachyon field, transforming it into the pseudotachyon field. After having crossed this point, the Universe evolves towards a future cosmological singularity called "big brake." This

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singularity is characterized by a finite value of the cosmological radius of the Universe, by a vanishing Hubble parameter, and by an infinite value of the cosmic deceleration. Singularities of such kind are rather soft and can be passed through [21]; the details of the passage of the big brake singularity in the model [7] are described in [22]. In [23], it was also noticed that the presence of dust matter in these big brake models can create additional difficulties. In [12], it was shown that these difficulties can be overcome by means of another Born-Infeld type field—the quasi-tachyon, which will be briefly mentioned in the next section. Some global aspects of quantum cosmology similar to the ones here were recently investigated in [13,16,24–28]. These concern, in particular, the fate of classical singularities.

The structure of the paper is as follows. In Sec. II, we present the models for tachyonic and other Born-Infeld type fields and discuss their behavior in classical cosmology. In Sec. III, we address the quantum cosmology of these fields in the Wheeler-DeWitt framework. Section IV is devoted to the alternative approach of reduced quantization. In Sec. V, we rewrite the Wheeler-DeWitt equation in the form of a difference equation. We discuss various asymptotic forms and show ways towards its solution. The last section contains our conclusion.

II. TACHYONIC AND OTHER BORN-INFELD TYPE FIELDS IN CLASSICAL COSMOLOGY

We shall work with flat Friedmann models given by the metric

$$\mathrm{d}s^2 = N^2(t)\mathrm{d}t^2 - a^2(t)\mathrm{d}l^2, \tag{1}$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor; we choose a to have the dimension of a length.

The Lagrangian density for the spatially homogeneous tachyon field T is

$$L_T = -V(T)\sqrt{1-\frac{\dot{T}^2}{N^2}}, \tag{2}$$

where $V(T)$ is the tachyon potential and the dot means time derivative. The tachyon T has the dimension of a length (and thus is a geometric quantity), and V has the dimension of a mass (energy) density; we set $c = 1$.

The energy density of the tachyon field is

$$\rho = \frac{V(T)}{\sqrt{1-\frac{\dot{T}^2}{N^2}}}, \tag{3}$$

while the pressure is

$$p = -V(T)\sqrt{1-\frac{\dot{T}^2}{N^2}}. \tag{4}$$

We choose $V(T) \geq 0$ to have non-negative energy densities. We note that $p = -V^2(T)/\rho$.

The total (minisuperspace) action is given by

$$S = \mathcal{V}_0 \int \mathrm{d}t \left(-\frac{3a\dot{a}^2}{\kappa^2 N} - Na^3 V(T)\sqrt{1-\frac{\dot{T}^2}{N^2}} \right), \tag{5}$$

where $\kappa^2 \equiv 8\pi G$, G is the gravitational (Newton) constant. The volume of three-space is $\mathcal{V}_0 a^3$, where $\mathcal{V}_0$ is a pure number that is set equal to one below.

Choosing $N = 1$, the total Lagrangian then reads

$$L = -a^3 V(T)\sqrt{1-\dot{T}^2} - \frac{3a\dot{a}^2}{\kappa^2}. \tag{6}$$

We have $|\dot{T}| \leq 1$ for the square root to stay real. From (6), we get the equations of motion

$$\ddot{a} + \frac{\dot{a}^2}{2a} - \frac{a\kappa^2 V}{2}\sqrt{1-\dot{T}^2} = 0, \tag{7}$$

and

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V'(T)}{V(T)} = 0, \tag{8}$$

where $H = \dot{a}/a$ is the Hubble parameter, and a prime denotes a derivative with respect to the tachyon T.

The canonical momenta read

$$p_a = -\frac{6\dot{a}a}{\kappa^2}, \qquad p_T = \frac{a^3 V\dot{T}}{\sqrt{1-\dot{T}^2}}. \tag{9}$$

We note that both p_a and p_T have dimension of a mass. The usual Legendre transform then yields the Hamiltonian

$$\mathcal{H} = -\frac{\kappa^2}{12}\frac{p_a^2}{a} + \sqrt{p_T^2 + a^6 V^2}, \tag{10}$$

which is, in fact, a constraint, $\mathcal{H} = 0$.

If expressed in terms of the velocities, this Hamiltonian constraint gives the Friedmann equation

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3}\rho, \tag{11}$$

with ρ given by (3). Using (3) and (11) in (7), we get

$$\ddot{a} = -\frac{\kappa^2 aV}{2}\frac{3\dot{T}^2 - 2}{3\sqrt{1-\dot{T}^2}}. \tag{12}$$

With (3) and (4), this equation can be written in the standard form

$$\ddot{a} = -\frac{\kappa^2}{2} a(\rho + 3p). \tag{13}$$

Let us first consider the special case of constant potential, $V(T) = V_0 =$ constant. In this case, T is a cyclic variable and p_T thus a constant. It was noticed in [4] that the corresponding cosmological model is equivalent to a cosmological model with a Chaplygin gas [29], which has the equation of state

$$p = -\frac{V_0^2}{\rho}. \tag{14}$$

From (8) and (11), one can then find the following equation for $\dot{T}^2$:

$$\dot{T}^2 = \frac{1}{1 + (\frac{a}{a_*})^6}, \tag{15}$$

where a_* is an integration constant,

$$a_* := \left(\frac{p_T}{V_0}\right)^{\frac{1}{3}}.$$

We see that $\dot{T}^2$ vanishes for $a \to \infty$ and becomes equal to one for $a \to 0$ (big bang). Using again (11), (15) can be integrated to yield the curve in configuration space; one finds

$$T(a) = 2\kappa^{-1} \sqrt{\frac{1}{3V}} \left(\frac{a}{a_*}\right)^{3/2} {}_2\mathrm{F}_1\left(\frac{1}{4}, \frac{3}{4}; \frac{5}{4}; -\left(\frac{a}{a_*}\right)^6\right), \tag{16}$$

where ${}_2\mathrm{F}_1$ denotes a hypergeometric function; see e.g. [30], Chap. 15.

The asymptotic solution for large a reads

$$T(a) = \frac{1}{\kappa\sqrt{3V_0}} \left(\frac{a_*}{a}\right)^3 + \text{constant}. \tag{17}$$

Figure 1 displays the configuration space trajectory (16).

From (3) and (4), we get the following expressions for density and pressure:

$$\rho(a) = V_0 \sqrt{1 + \left(\frac{a_*}{a}\right)^6}, \qquad p(a) = -\frac{V_0^2}{\rho(a)}. \tag{18}$$

For $a \to 0$, both expressions diverge (big bang), while for large a, they become constants with $p \approx -\rho$, thus mimicking dark energy. We mention that for small a the equation of state resembles the one for dust. This model thus encodes a transition from a matter to a vacuum dominated state, which is a feature observed in the real Universe.

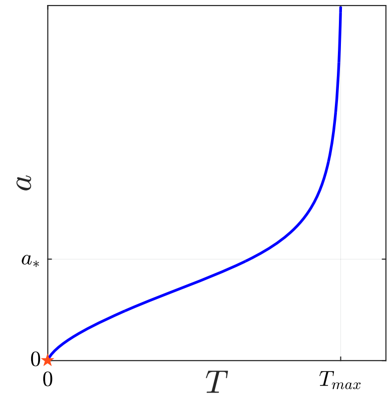

FIG. 1. Plot of the configuration space trajectory (16) for the tachyon model with constant potential.

Plugging the expression (18) into the Friedman equation (11) yields

$$\dot{a}^2 = \frac{\kappa^2 V_0}{3} \sqrt{a^4 + \frac{a_*^6}{a^2}}, \tag{19}$$

which is solved by

$$\sqrt{\frac{\kappa^2 V_0}{3}}(t - t_0) = \frac{2a^3}{3a_*^6}\left(a^4 + \frac{a_*^6}{a^2}\right)^{\frac{3}{4}} {}_2F_1\left(1, 1; \frac{5}{4}; -\left(\frac{a}{a_*}\right)^6\right) + \text{constant}. \tag{20}$$

The plot of this trajectory is displayed in Fig. 2.

Also of interest is the model of an inverse square potential, $V = V_1/T^2$; see, for example [8,31]. This model exhibits a particular solution with constant $\dot{T}$, describing a universe that undergoes a power-law inflation [5,6]. This

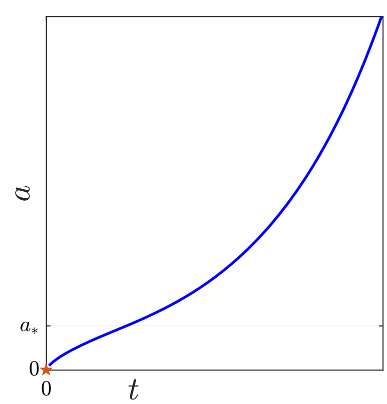

FIG. 2. Plot of the general solution $a(t)$ for the tachyon model with constant potential.

can be seen as follows. From (3), (8), and (11), one obtains the dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} T \\ s \end{pmatrix} = \begin{pmatrix} s \\ -(1-s^2)\frac{V'(T)}{V(T)} - \kappa s\sqrt{3V(T)}(1-s^2)^{\frac{3}{4}} \end{pmatrix}, \tag{21}$$

where $s \coloneqq \dot{T}$. The corresponding flow diagram is depicted in Fig. 3. One recognizes from the diagram that the particular solution with constant $\dot{T}$ serves as an attractor. For other tachyon potentials, see for example [8].

We shall now address another type of Born-Infeld type field called "pseudotachyon" which, as was explained in the Introduction, naturally arises in cosmological models [7]; it is less habitual than the tachyon model, but displays some interesting theoretical features. Its Lagrangian density reads

$$L_p = W(T)\sqrt{\dot{T}^2 - 1}. \tag{22}$$

The Friedmann equation and the Klein-Gordon-type equation are given by

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3}\frac{W(T)}{\sqrt{\dot{T}^2 - 1}} \tag{23}$$

and

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{W'(T)}{W(T)} = 0, \tag{24}$$

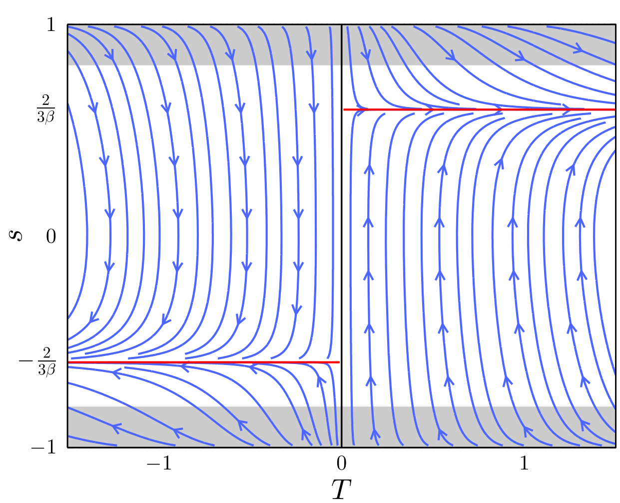

FIG. 3. Flow diagram of the tachyon model with the inverse square potential. Inside the grey shaded region, the universe undergoes a decelerated expansion, while it accelerates outside this region. The parameter β is given by $\beta = \frac{1}{9}\sqrt{2 + \sqrt{9\kappa^4 V_1^2 + 4}}$; T is in arbitrary units of time.

respectively. Energy density and pressure of the pseudotachyon read

$$\rho = \frac{W(T)}{\sqrt{\dot{T}^2 - 1}} \quad \text{and} \quad p = W(T)\sqrt{\dot{T}^2 - 1}. \tag{25}$$

In this case, both the energy density and the pressure are positive. When the potential is constant, $W(T) = W_0 =$ constant, this model coincides with a cosmological model containing an anti-Chaplygin gas [7,25]; this is a perfect fluid with the equation of state

$$p = \frac{W_0^2}{\rho}. \tag{26}$$

A universe with an anti-Chaplygin gas represents the simplest example of a cosmological evolution with a big brake singularity. It is interesting that the equation of state (26) arises in the theory of wiggly strings [32]. In Fig. 4, we depict the trajectory $a(t)$ and in Fig. 5 the trajectory in configuration space. We see explicitly the occurrence of the big brake singularity.

The Hamiltonian is readily obtained by a Legendre transform,

$$\mathcal{H} = -\frac{\kappa^2}{12}\frac{p_a^2}{a} + \sqrt{p_T^2 - a^6 W^2}. \tag{27}$$

In the following, we consider the case of the inverse square potential, $W = W_1/T^2$. This does not seem to have been discussed so far in this way. For simplicity, we set here $\kappa^2/3 = 1$. Similarly to the corresponding tachyon model we find solutions with constant $\dot{T}$. In this case, however, we have two solutions $\dot{T}_\pm = \frac{2}{3\beta_\pm}$, where $\beta_\pm^2 = \frac{1}{9}(2 \pm \sqrt{4 - 81W_1^2})$. In order to get real solutions, we have to demand $W_1 \leq \frac{2}{9}$. The scale factors for these two solutions are given by

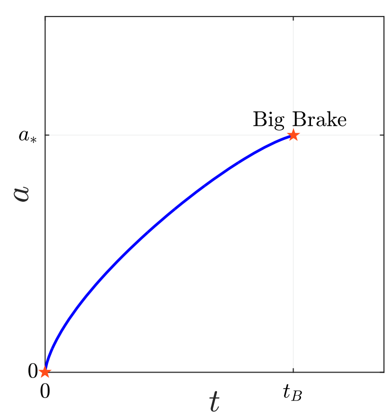

FIG. 4. Plot of the general solution $a(t)$ for the pseudotachyon model with constant potential.

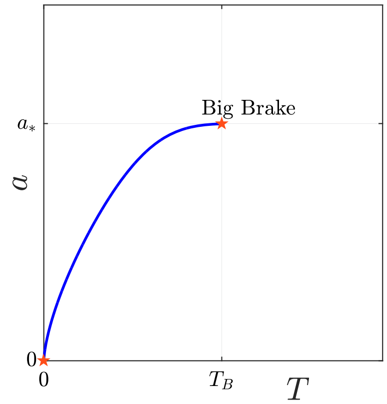

FIG. 5. Plot of configuration space trajectory for the pseudotachyon model with constant potential.

$$a_\pm(t) \propto t^{\frac{3\beta_\pm^2}{2}}. \tag{28}$$

In the limiting case $W_1 = \frac{2}{9}$, the two solutions merge into one. Analogously to the tachyon case, one can express the dynamics in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} T \\ s \end{pmatrix} = \begin{pmatrix} s \\ (s^2-1)\frac{W'(T)}{W(T)} + 3s\sqrt{W(T)}(s^2-1)^{\frac{3}{4}} \end{pmatrix}. \tag{29}$$

The flow chart for the case $W_1 < \frac{2}{9}$ is shown in Fig. 6. A closer inspection reveals that all solutions emerge from a big bang on the line determined by $s = 1$ or the single point $(T = 0, s = \frac{2}{3\beta_-})$. One explicitly sees that one of the particular solutions ($s = \frac{2}{3\beta_+}$) serves as an attractor, while the other one ($s < \frac{2}{3\beta_-}$) is repulsive. The solutions in the region $T > 0$, $s < \frac{2}{3\beta_-}$ are attracted to the particular solution with $s = \frac{2}{3\beta_+}$,

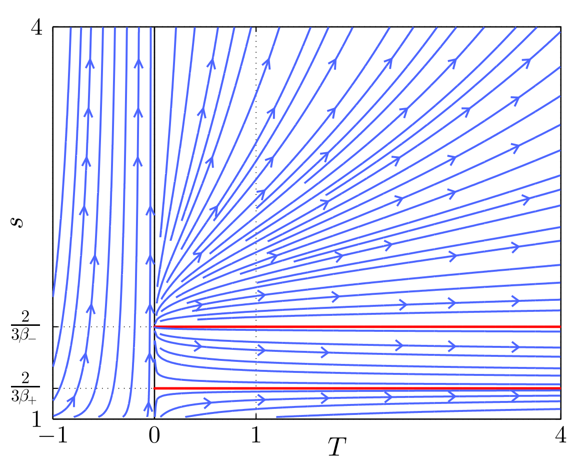

FIG. 6. Flow diagram of the pseudotachyon model with the inverse square potential for the case $W_1 < \frac{2}{9}$; T is in arbitrary units of time.

cf. (28). All other solutions end in a big brake. In the limiting case $W_1 = \frac{2}{9}$, the two particular solutions merge into one metastable solution. For $W_1 > \frac{2}{9}$ (not shown here), the particular solutions disappear, and all solutions end in a big brake.

In the following, we shall explicitly show the presence of the big brake singularity for a typical case. By defining $u := \ln(\sqrt{\dot{T}^2 - 1})$, one can find from (29),

$$\frac{\mathrm{d}u}{\mathrm{d}T} = \frac{1}{T}\left[3\,\mathrm{sgn}(T)\sqrt{2W_1\cosh(u)} - 2\right]. \tag{30}$$

Integration then yields a parametrization of T in terms of its time derivative $\dot{T}$,

$$T = T(\dot{T}_r)\exp\left[-\int_{\ln(\sqrt{\dot{T}_r^2-1})}^{\ln(\sqrt{\dot{T}^2-1})} \frac{\mathrm{d}u}{2\text{–}3\,\mathrm{sgn}(T)\sqrt{2W_1\cosh(u)}}\right], \tag{31}$$

where $T(\dot{T}_r)$ is the value of T at some reference value $\dot{T}_r$. This parametrization can now be used to prove the existence of the singularities.

To be specific, we consider the case $W_1 < \frac{2}{9}$ and the solutions in the region where $T > 0$ and $\dot{T} > \frac{2}{3\beta_-}$. We shall now show by using suitable estimates of the integral in (31) that $T \to 0$ as $\dot{T} \to \frac{2}{3\beta_-}$ and that T approaches a finite value T_∞ as $\dot{T} \to \infty$; here, it is convenient to use $\dot{T}_r = \infty$ as a reference value. Expression (31) then becomes

$$T = T_\infty \exp\left[\int_\infty^{\ln(\sqrt{\dot{T}^2-1})} \frac{\mathrm{d}u}{3\sqrt{2W_1\cosh(u)} - 2}\right]. \tag{32}$$

If we divide the function inside the integral by the function

$$\frac{1}{3\sqrt{W_1 e^u} - 2}, \tag{33}$$

the resulting function approaches 1 as $u \to \infty$. The integral of (33) over the same interval as in (32) is finite for $\dot{T} > \frac{2}{3\beta_-}$. Consequently, we can use the limit comparison test to conclude that the expression (32) is well defined and therefore T approaches a finite value T_∞ as $\dot{T} \to \infty$.

If we do the resubstitution $u = \ln(\sqrt{s^2 - 1})$, the integral in (32) assumes the form

$$\int_\infty^{\dot{T}} \frac{\mathrm{d}s}{(s^2-1)(3s\sqrt{\frac{W_1}{\sqrt{s^2-1}}} - 2)}. \tag{34}$$

If we now choose $s \in [s_-, s_- + \varepsilon)$, where $s_- := \frac{2}{3\beta_-}$ and $\varepsilon > 0$, we can estimate the integrand to be bigger than

$$\frac{1}{((s_- + \varepsilon)^2 - 1)^{\frac{3}{4}}(3\sqrt{W_1}s - 2(s_-^2 - 1)^{\frac{1}{4}})}. \tag{35}$$

By noting that s_- is a zero of the denominator, we conclude that the integral of this expression over the interval $[s_-, s_- + \varepsilon)$ blows up to $+\infty$. Therefore, the integral in (32) goes to $-\infty$ as $\dot{T} \to \frac{2}{3\beta_-}$ and thus $T \to 0$. By estimating that

$$T_\infty = \int_{t_0}^{t_\infty} \mathrm{d}t\dot{T} > t_\infty - t_0, \tag{36}$$

with t_∞ corresponding to T_∞ and t_0 corresponding to $T = 0$, we deduce that T grows from 0 to T_∞ in a finite amount of time. Later on, we show that this model possesses a constant of motion, see (111) below. This relation can be written as

$$a^3 = \frac{C}{\frac{3W_1\dot{T}}{T\sqrt{\dot{T}^2-1}} - 2H}, \tag{37}$$

where C is a positive constant. The considerations above now yield, on the one hand,

$$a \to 0, \qquad \rho \to \infty \quad \text{and} \quad p \to \infty \quad \text{as } \dot{T} \to \frac{2}{3\beta_-} \tag{38}$$

(big bang) and, on the other hand,

$$a \to \frac{C}{3W_1}T_\infty, \qquad \rho \to 0 \quad \text{and} \quad p \to \infty \quad \text{as} \quad \dot{T} \to \infty \tag{39}$$

(big brake). The limit $p \to \infty$ now implies that $\ddot{a} \to -\infty$, and we finally conclude that the solutions start from a big bang and end in a big brake.

In the case of constant potential, one can employ conformal diagrams to illustrate the behavior of solutions. Diagrams of this kind have been used in quantum cosmology before; see, for example, [33]. Figure 7 shows the case of the tachyon, while Fig. 8 shows the case of the pseudotachyon field. The blue lines mark the trajectories in the (a, p_T) phase space corresponding to lines with constant p_T. Here, $V_0 = W_0 = \frac{1}{2}$, respectively.

In some particular cases, yet another Born-Infeld type field can arise [12], with the Lagrangian density

$$L_q = U(T)\sqrt{\dot{T}^2 + 1}. \tag{40}$$

In this case, the energy density is negative, while the pressure is positive. This field is called "quasitachyon." When $U(T) = U_0 = \text{constant}$, the quasitachyon field behaves like a Chaplygin gas with negative energy density

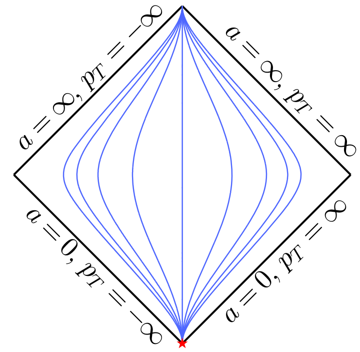

FIG. 7. Phase space trajectories of the constant potential tachyon field models. All solutions (except the one in the middle) evolve out of a big bang singularity marked by the red star and end in the point $(a = \infty, p_T)$.

and positive pressure. In this paper, however, we shall restrict attention to the tachyon and the pseudotachyon fields.

We now turn to the quantum versions of the tachyon and pseudotachyon models.

III. QUANTUM COSMOLOGY FOR BORN-INFELD TYPE FIELDS

In spite of their apparently simple character, already the models with constant potentials are rather complicated from the point of view of quantum cosmology. In the following, we shall discuss various approaches for their quantization.

From (10), we get the following Wheeler-DeWitt equation for a universe filled with a tachyonic field [16],

$$\left(\sqrt{p_T^2 + a^6V^2} - \frac{p_a^2}{2a}\right)\Psi(T, a) = 0, \tag{41}$$

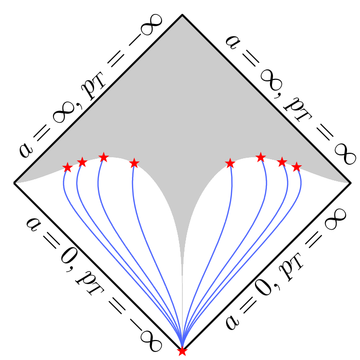

FIG. 8. Phase space trajectories of the constant potential pseudotachyon field models. All solutions evolve out of a big bang singularity marked by the red star and end in a big brake singularity at the edge of the grey shaded region, where the Hamiltonian (27) becomes ill defined. In the case of a contracting universe, the trajectories go along the same lines, starting from the big brake towards the big crunch singularity.

where $\Psi(T, a)$ is the quantum state of the universe and p_T, T, a, and p_a are now operators; here and in the following we set $\kappa^2 = 6$.

Equations such as (41) are plagued by the factor-ordering problem: there is no unique way to transform the classical configuration and momentum variables into operators [15]. Here, we shall adopt a pragmatic attitude and choose a simple factor ordering which facilitates the finding of explicit solutions.

Usually, one implements a and T as multiplication operators and p_a and p_T as derivative operators. In view of the square root in (41), this is, however, a delicate issue. But in the particular case of constant potential, we can use the fact that the field T does not enter explicitly into the Wheeler-DeWitt equation. Thus, we can use a momentum representation for the tachyon and consider instead of the wave function $\Psi(T, a)$ the wave function $\Psi(p_T, a)$ (using, for simplicity, the same letter). In this case, the operator p_T becomes multiplicative, and the Wheeler-DeWitt equation acquires the form

$$\left(\frac{\partial^2}{\partial a^2} + 2a\sqrt{p_T^2 + a^6 V_0^2}\right)\Psi(p_T, a) = 0. \tag{42}$$

We can look for a solution of (42) in the form

$$\Psi(p_T, a) = \psi(p_T, a)\chi(p_T), \tag{43}$$

where $\chi(p_T)$ denotes an arbitrary function of p_T. In this case, we arrive at

$$\left(\frac{\partial^2}{\partial a^2} + 2a\sqrt{p_T^2 + a^6 V_0^2}\right)\psi(p_T, a) = 0, \tag{44}$$

where p_T is a fixed parameter. The solutions of these equations do not seem to belong to known special functions, but we can consider some limiting cases. Namely, in the case when the cosmological radius is small, we have

$$\left(\frac{\partial^2}{\partial a^2} + 2a|p_T|\right)\psi(p_T, a) = 0. \tag{45}$$

The solution of this equation is known; it can be expressed by means of Airy functions:

$$\psi(p_T, a) = c_1 \mathrm{Ai}((-2|p_T|)^{1/3} a) + c_2 \mathrm{Bi}((-2|p_T|)^{1/3} a). \tag{46}$$

However, because (45) is valid only in the limit $a \to 0$, we need to take into account in the solution (46) only the leading terms and rewrite it as

$$\psi(p_T, a) = d_1 + d_2 a. \tag{47}$$

Because $a = 0$ corresponds in the classical model to the big bang, the question of singularity avoidance in quantum cosmology can be addressed. DeWitt has proposed the heuristic criterion that the wave function should vanish at the point of the classical singularity [14]. This criterion was implemented in the models discussed in [25–27]. If we adopt this criterion here, we have to demand that $\psi(p_T, 0) = 0$, that is, we have to choose $d_1 = 0$. For this choice, then, the big bang singularity would be avoided in the sense of DeWitt.

When a is very large, we get from (44) the following equation:

$$\left(\frac{\partial^2}{\partial a^2} + 2a^4 V_0\right)\psi(p_T, a) = 0. \tag{48}$$

Its solution can be expressed in terms of Bessel functions,

$$\psi(p_T, a) = f_1\sqrt{a} J_{-1/6}\left(\frac{\sqrt{2V_0}a^3}{3}\right) + f_2\sqrt{a} J_{1/6}\left(\frac{\sqrt{2V_0}a^3}{3}\right). \tag{49}$$

(Recall that $V_0 \geq 0$.) This is the quantum solution that corresponds to the asymptotic de Sitter phase of the classical solution, which is well known from the solution of the Wheeler-DeWitt equation with a cosmological constant ([15], Chap. 8).

Keeping only the leading terms at $a \to \infty$, this becomes

$$\psi(p_T, a) = g_1 \exp\left(\mathrm{i}\frac{\sqrt{2V_0}a^3}{3}\right) + g_2 \exp\left(-\mathrm{i}\frac{\sqrt{2V_0}a^3}{3}\right). \tag{50}$$

Let us now consider the pseudotachyon field with constant potential. In this case, the Wheeler-DeWitt equation has the following form:

$$\left(\frac{\partial^2}{\partial a^2} + 2a\sqrt{p_T^2 - a^6 W_0^2}\right)\Psi(p_T, a) = 0. \tag{51}$$

With an ansatz of the form (43), we get

$$\left(\frac{\partial^2}{\partial a^2} + 2a\sqrt{p_T^2 - a^6 W_0^2}\right)\psi(p_T, a) = 0. \tag{52}$$

At small values of a, this equation coincides with (45) and thus leads to the same solution in this limit.

The value of the scale factor

$$a_* \coloneqq \left(\frac{|p_T|}{W_0}\right)^{\frac{1}{3}} \tag{53}$$

corresponds to the big brake singularity. Let us consider the Wheeler-DeWitt equation (52) in the neighborhood of this point and write for this purpose

$$a =: a_* - \tilde{a}. \tag{54}$$

We then have

$$\left(\frac{\partial^2}{\partial \tilde{a}^2} + 2\sqrt{6}W_0(a_*)^{7/2}\sqrt{\tilde{a}}\right)\psi(p_T, \tilde{a}) = 0. \tag{55}$$

Its solution is

$$\psi(p_T, a) = c_1\sqrt{\tilde{a}}J_{-2/5}\left(\frac{4}{5}\sqrt{2\sqrt{6}W_0(a_*)^{7/2}}(\tilde{a})^{5/4}\right) + c_2\sqrt{\tilde{a}}J_{2/5}\left(\frac{4}{5}\sqrt{2\sqrt{6}W_0(a_*)^{7/2}}(\tilde{a})^{5/4}\right). \tag{56}$$

For small values of $\tilde{a}$, it behaves as

$$\psi(p_T, a) = d_1 + d_2\tilde{a}. \tag{57}$$

The self-adjointness of the Hamiltonian operator in the Wheeler-DeWitt equation is an open issue [15]. But if we demand this property to hold here, $\tilde{a}$ cannot be negative because otherwise the expressions under the square roots in (52) and (55) would become negative. We thus have to impose the boundary condition

$$\psi(p_T, a) = 0 \quad \text{at } \tilde{a} \leq 0. \tag{58}$$

According to the DeWitt criterion, the big brake singularity is then avoided, too. This is similar to the avoidance found in [25–27].

The question of the reality of the spectrum of the Hamiltonian in quantum cosmology was considered already in [34]. There, another approach to the construction of the wave function of the Universe called reduced quantization was discussed [15]. In this approach, a time parameter is chosen from the classical phase space variables, and a nonzero Hamiltonian appears, which depends on this time parameter and the physical degrees of freedom in the reduced phase space of the theory. Upon quantization, one arrives at a Schrödinger equation for the wave function of the Universe, depending on time and the physical degrees of freedom. In this case, the Hamiltonian almost unavoidably contains square roots, even if the initial Lagrangian does not contain them. This, together with other problems, makes the reduced approach untractable in most cases [15,35].

Later, the reduced approach was developed in great detail in [36], and its relation with Dirac quantization approach was analyzed. Its application to some rather simple cosmological models was presented in the recent paper [37]. However, considering the Born-Infeld type models, we encounter a more complicated problem, because here the square-root type Hamiltonians are present in the Wheeler-DeWitt equation defined on the full phase space. We shall apply the reduced approach to these models in the next section.

Note that in the case of the tachyon field discussed above, the demand for a self-adjoint Hamiltonian does not impose any restrictions on the wave function which is a solution of (44).

If we demanded the avoidance of both the big bang and the big brake singularities in the sense of the DeWitt criterion, we would have to impose the boundary conditions $\psi(p_T, 0) = 0$ *and* $\psi(p_T, (2|p_T|/W_0)^{1/3}) = 0$. The situation would then be analogous to that of a nonrelativistic particle in an infinite potential well, which is known to lead to a discrete spectrum. In our case, this would lead to discrete tachyon momenta $p_T = \pm|p_T|_n$, $n \in \mathbb{N}$.

For the case of a more general potential than the constant one, the quantization becomes complicated, for T and p_T appear simultaneously under the square root. In the following, we show one possibility how to deal with this problem. We first perform the canonical transformation

$$T \to \phi := \int V(T)dT, \qquad p_T \to p_\phi := \frac{p_T}{V(T)}. \tag{59}$$

The Hamiltonian constraint then takes the form

$$\mathcal{H} = -\frac{p_a^2}{2a} + \sqrt{p_\phi^2 + a^6}V(T(\phi)). \tag{60}$$

Thus the transformation enables us to move V out of the square root. In the following, we specialize to potentials of the form $V(T) = V_1 T^n$, where $n \neq -1$. According to (59), we obtain $\phi = \frac{V_1}{n+1}T^{n+1}$, and therefore the Hamiltonian constraint becomes

$$\mathcal{H} = -\frac{p_a^2}{2a} + V_1\left(\frac{(n+1)\phi}{V_1}\right)^{\frac{n}{n+1}}\sqrt{p_\phi^2 + a^6}. \tag{61}$$

After quantization and imposing a simple factor ordering, we get

$$\left[\frac{\partial^2}{\partial a^2} + \mu a\sqrt{p_\phi^2 + a^6}\left(\frac{\partial}{\partial p_\phi}\right)^{\frac{n}{n+1}}\right]\Psi(p_\phi, a) = 0, \tag{62}$$

where $\mu := 2V_1(\frac{\mathrm{i}(n+1)}{V_1})^{\frac{n}{n+1}}$. The Wheeler-DeWitt equation is thus a fractional partial differential equation, which can be well defined in the sense of fractional calculus; see, for example, [38]. Note, however, that fractional derivatives can be represented as integral (and thus nonlocal) operators.

For the special case of the inverse square potential ($n = -2$), the Wheeler-DeWitt equation becomes

$$\left[\frac{\partial^2}{\partial a^2} - \frac{2a}{V_1}\sqrt{p_\phi^2 + a^6}\frac{\partial^2}{\partial p_\phi^2}\right]\Psi(p_\phi, a) = 0. \tag{63}$$

This is a wave equation with variable coefficients. In the region where $a^6 \ll p_\phi^2$, it assumes the asymptotic form

$$\left[\frac{\partial^2}{\partial a^2} - \frac{2a|p_\phi|}{V_1}\frac{\partial^2}{\partial p_\phi^2}\right]\Psi(a, p_\phi) = 0. \tag{64}$$

The separation ansatz $\Psi(a, p_\phi) = \chi(a)\varphi(p_\phi)$ yields the solutions

$$\chi(a) = b_1 \mathrm{Ai}\left(\left[\frac{2\lambda^{\frac{1}{3}}}{V_1}a\right]\right) + b_2 \mathrm{Bi}\left(\left[\frac{2\lambda^{\frac{1}{3}}}{V_1}a\right]\right),$$
$$\varphi(p_\phi) = c_1\sqrt{p_\phi}I_1\left(2\sqrt{\lambda|p_\phi|}\right) + c_2\sqrt{p_\phi}K_1\left(2\sqrt{\lambda|p_\phi|}\right), \tag{65}$$

where $b_1, b_2, c_1, c_2 \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ is a separation constant; I_1 and K_1 are the modified Bessel functions. In the region where $a^6 \gg p_\phi^2$, the asymptotic form of the Wheeler-DeWitt equation is

$$\left[\frac{\partial^2}{\partial a^2} - \frac{2a^4}{V_1}\frac{\partial^2}{\partial p_\phi^2}\right]\Psi(a, p_\phi) = 0. \tag{66}$$

A separation ansatz of the form $\Psi(a, p_\phi) = \tilde{\chi}(a)\tilde{\varphi}(p_\phi)$ then yields

$$\tilde{\chi}(a) = d_1\sqrt{a}J_{\frac{1}{6}}\left(\sqrt{\frac{2\tilde{\lambda}}{V_1}}\frac{a^3}{3}\right) + d_2\sqrt{a}J_{-\frac{1}{6}}\left(\sqrt{\frac{2\tilde{\lambda}}{V_1}}\frac{a^3}{3}\right),$$
$$\tilde{\varphi}(p_\phi) = f_1 \exp\left(\sqrt{\tilde{\lambda}}p_\phi\right) + f_2 \exp\left(-\sqrt{\tilde{\lambda}}p_\phi\right), \tag{67}$$

where $d_1, d_2, f_1, f_2, \tilde{\lambda} \in \mathbb{C}$.

If applied to the case of the pseudotachyon, the above procedure leads to the Wheeler-DeWitt equation

$$\left[\frac{\partial^2}{\partial a^2} - \frac{2a}{W_1}\sqrt{p_\phi^2 - a^6}\frac{\partial^2}{\partial p_\phi^2}\right]\Psi(p_\phi, a) = 0. \tag{68}$$

In the region where $a^6 \ll p_\phi^2$, the asymptotic solutions are the same as in the tachyon case. An open question is the behavior of the wave function near the big brake singularity. Using the same method as in the constant W case does not work here, since p_ϕ cannot be treated as a fixed parameter anymore.

IV. REDUCED PHASE SPACE QUANTIZATION

In this section, we shall study the cosmological models with the Born-Infeld type fields with a constant potential, using the reduction to physical degrees of freedom approach [34,36,37]. In the case of tachyons, one can choose as a time parameter τ the cosmological radius a (such a choice is sometimes called an intrinsic time choice). Indeed, in this case the classical evolution of the Universe is such that the cosmological radius changes monotonically from the big bang to an infinite expansion or from an infinite contraction ending in the big crunch singularity. As a matter of fact, it is convenient to choose $\tau = a$ for the expansion and $\tau = -a$ for the contraction. (We choose here the letter τ to avoid confusion with the classical time parameter t.)

Let us first choose the case

$$\tau = a. \tag{69}$$

Then, the effective nonvanishing Hamiltonian in the reduced phase space of the physical degrees of freedom is given by the corresponding conjugate momentum p_a, taken with the inverse sign and expressed in terms of the physical degrees of freedom and τ,

$$\mathcal{H}_{\mathrm{red}} = -p_a = +\sqrt{2\tau\sqrt{p_T^2 + \tau^6 V_0^6}}. \tag{70}$$

The chosen negative sign of the momentum p_a corresponds to the expansion of the universe, cf. (9). The expression for this Hamiltonian is well defined for $0 \le \tau < \infty$. The general solution of the Schrödinger equation corresponding to (70) is

$$\psi(p_T, \tau) = \psi(p_T, 0)\exp\left(-\mathrm{i}\int_0^\tau \mathrm{d}\tilde{\tau}\sqrt{2\tilde{\tau}\sqrt{p_T^2 + \tilde{\tau}^6 V_0^6}}\right). \tag{71}$$

Here, we have used the fact that the time-dependent Hamiltonian (70) commutes with itself at different moments of time. Otherwise, it is necessary to use the chronological T-exponentiation, which makes the formalism more involved.

Analogously, to describe a contracting quantum universe, it is convenient to choose the time parameter as

$$\tau = -a. \tag{72}$$

Then the Hamiltonian is

$$\mathcal{H}_{\mathrm{red}} = -p_a = -\sqrt{-2\tau\sqrt{p_T^2 + \tau^6 V_0^6}}, \tag{73}$$

and it is well defined for $-\infty < \tau \le 0$. The solution of the Schrödinger equation is

$$\psi(p_T,\tau)=\psi(p_T,-\infty)\exp\left(+\mathrm{i}\int_0^\tau \mathrm{d}\tilde{\tau}\sqrt{-2\tilde{\tau}\sqrt{p_T^2+\tilde{\tau}^6V_0^6}}\right). \tag{74}$$

The only requirement which one should impose on these solutions is the normalizability of the wave functions $\psi(p_T,0)$ and $\psi(p_T,-\infty)$, prescribing the initial conditions. The time-dependent part of the solutions (71) and (74) is simply a phase factor, which behaves well at the values of time corresponding to both the singularities and the infinite volume of the universe.

For constant potential, the classical momentum conjugated to the tachyon field is constant, because then the Hamiltonian (10) is T-independent. Thus, it seems that it is impossible to describe the dynamics in terms of this momentum, which is the only observable on which the wave functions (71) and (74) depend. However, in quantum theory there is no strong causal relation between the geometric characteristics of the Universe and the quantities that characterize the matter content. In our case, one can say that the cosmological radius is not a geometric characteristic, but a time parameter. Hence, the cosmological singularity can be associated with a state of the system corresponding to infinite energy density. In our case, this means that the time derivative of the tachyon field tends to one, see (3), and the conjugate momentum tends to infinity. It is clear that the requirement of the normalizability of the wave function of the Universe implies the rapid vanishing of this function at $|p_T|\to\infty$ and this fact could be interpreted as a suppression of the big bang–big crunch singularity [16].

The case of the cosmological model with a pseudotachyon field is more complicated, because the Universe begins its evolution from the big bang singularity, then expands until the occurrence of the big brake singularity, after which it contracts to the big crunch singularity. In such a situation, it is preferable to employ an extrinsic instead of an intrinsic time [15,35]; an extrinsic time is one that depends on the extrinsic curvature; see [37] and the references therein. In fact, we cannot use the cosmological radius or a function of it as a time parameter, because it changes nonmonotonically during the evolution.

It is convenient to perform a canonical transformation, which leads to the new coordinate

$$q\coloneqq\frac{p_a}{a^2}. \tag{75}$$

It is easy to check that q is equal to the Hubble parameter $H=\dot{a}/a$, taken with the opposite sign. If we identify this new coordinate with the extrinsic time parameter, $\tau\equiv q$, the latter is defined in the interval $-\infty<\tau<+\infty$. The conjugate momentum to q is

$$p_q=-\frac{a^3}{3}. \tag{76}$$

The reduced Hamiltonian depends on the physical degree of freedom p_T and on τ and is given by

$$\mathcal{H}_{\rm red}=-p_q=\frac{a^3}{3}=\frac{|p_T|}{3\sqrt{\frac{1}{4}\tau^4+W_0^2}}. \tag{77}$$

Here, we have used the Hamiltonian constraint to express the momentum p_q in terms of the physical variable p_T and τ. Note that this constraint represents a simple quadratic equation with respect to a^3, but because of the nonnegativity of the cosmological radius a we should take the positive square root. This means that, in contrast to the preceding case of the tachyon field, we have only one possibility for the choice of the Hamiltonian and, hence, there exists only one branch of the physical wave function of the Universe,

$$\psi(p_T,\tau)=\psi(p_T,-\infty)\exp\left(-\mathrm{i}\int_{-\infty}^\tau \mathrm{d}\tau\frac{|p_T|}{3\sqrt{\frac{1}{4}\tau^4+W_0^2}}\right). \tag{78}$$

The existence of only one branch for the wave function in the reduced approach is in agreement with the fact that the wave function satisfying the Wheeler-DeWitt equation, as described in the preceding section, should obey a boundary condition that guarantees the self-adjointness of the super-Hamiltonian. Note that the form of the Hamiltonian (77) is automatically self-adjoint as it should be. The presence of two branches in the Wheeler-DeWitt wave function or of two different wave functions, with two different Hamiltonians in the case of the tachyon field model, is connected with the fact that there are two different classical cosmological evolutions: expansion and contraction. In the model with a pseudotachyon field we have only one Hamiltonian for the Schrödinger equation in the reduced space and an additional boundary condition for the solution of the Wheeler-DeWitt equation. This corresponds to only one type of cosmological evolution in this model—from the big bang to the big crunch, passing through the point of maximal expansion where the universe crosses the big brake singularity.

Speaking about singularities, we can say that the same arguments which we have used in analyzing the relation between the wave function and the big bang singularity in the model with the tachyon field can be applied to the case of the pseudotachyon field as well. The situation with the big brake singularity is different. Its appearance is not connected with some particular behavior of the momentum p_T and it is not suppressed by the wave function of the Universe. This seems natural, because classically a universe can pass through this singularity without any difficulty; see e.g. [12,13] and the references therein.

We finally note that the reduced approach becomes rather complicated in the general situation of a nonconstant potential.

V. QUANTUM COSMOLOGY AND DIFFERENCE EQUATIONS

In this section, we shall perform a canonical transformation in such a way that the square root in the Hamiltonian disappears, making the problem more tractable. Here, then, we can use the analogy that was already mentioned in the Introduction; this analogy concerns the quantum mechanics of a collapsing (expanding) thin shell [17–20]. Identifying the Hamiltonian of the system under consideration with the physical mass, one obtains there an expression for the Hamiltonian which contains a hyperbolic function of the momentum operator. Because the exponent of the momentum operator is the generator of spatial translation, one then arrives at finite difference equations for the wave function.

In cosmology, we can try to follow this analogy and perform a transition to new canonical variables and momenta such that the new Hamiltonian will be free of square roots and will instead contain a combination of translation operators. Hence, the Wheeler-DeWitt equation will become a finite difference equation; more precisely, a mixed difference-differential equation. Such difference equations are common in loop quantum cosmology [39,40], but have so far not been discussed in the framework of standard quantum cosmology.

To be concrete, we introduce a new canonical momentum $\mathcal{P}$ by

$$p_T =: a^3 V(T) \sinh \mathcal{P}. \tag{79}$$

The reason for this choice is that it turns the square root in (10) into the expression $a^3 V \cosh \mathcal{P}$. Such a form for the kinetic term has been found in the above-mentioned papers [17–20].

We now have to construct the corresponding new canonical coordinate $\mathcal{Q}$ such that

$$\{\mathcal{Q}, \mathcal{P}\} = 1. \tag{80}$$

Generally, the tachyon field T can depend on $\mathcal{Q}$ as well as on $\mathcal{P}$. Then,

$$\begin{aligned}\{T, p_T\} = {}& \frac{\partial T}{\partial \mathcal{Q}} \times a^3 V(T) \cosh \mathcal{P} \\ & + \frac{\partial T}{\partial \mathcal{Q}} \times a^3 \frac{dV(T)}{dT} \frac{\partial T}{\partial \mathcal{P}} \sinh \mathcal{P} \\ & - \frac{\partial T}{\partial \mathcal{P}} \times a^3 \frac{dV(T)}{dT} \frac{\partial T}{\partial \mathcal{Q}} \sinh \mathcal{P} = 1,\end{aligned} \tag{81}$$

which gives the condition

$$a^3 V(T) \cosh \mathcal{P} \frac{\partial T}{\partial \mathcal{Q}} = 1. \tag{82}$$

The last equation can be rewritten as

$$V(T) \mathrm{d}T = \frac{d\mathcal{Q}}{a^3 \cosh \mathcal{P}}. \tag{83}$$

The canonical coordinate $\mathcal{Q}$ can thus be written as

$$\mathcal{Q} = a^3 \cosh \mathcal{P} \int V(T) \mathrm{d}T. \tag{84}$$

What about the modification of a and p_a? It is convenient to keep the scale factor as the configuration variable. Unfortunately, this is not possible for its momentum. The reason is that the Poisson brackets between p_a and T and between p_a and p_T must vanish. This is not the case for an unmodified p_a. One can easily see that the transformation

$$p_a \to \tilde{p}_a \coloneqq p_a - \frac{3\mathcal{Q} \tanh \mathcal{P}}{a} \tag{85}$$

leads to the vanishing of those brackets; that is, the new variables a, $\mathcal{Q}$, $\tilde{p}_a$, $\mathcal{P}$ arise from the old ones a, T, p_a, p_T by a canonical transformation.

In the terminology of [41], Sec. 9. 1, the generator of this canonical transformation reads

$$F_2(a, \tilde{p}_a; T, \mathcal{P}) = a^3 \sinh \mathcal{P} \int V(T) \mathrm{d}T + a\tilde{p}_a, \tag{86}$$

where

$$p_T = \frac{\partial F_2}{\partial T}, \qquad p_a = \frac{\partial F_2}{\partial a}, \tag{87}$$

and

$$\mathcal{Q} = \frac{\partial F_2}{\partial \mathcal{P}}, \qquad a = \frac{\partial F_2}{\partial \tilde{p}_a}. \tag{88}$$

In principle, with the help of this generating function, one may use the method discussed in [42] to relate the corresponding wave functions.

After the canonical transformation, the Hamiltonian constraint (10) assumes the form

$$\mathcal{H} \equiv -\frac{1}{2a}\left(\tilde{p}_a + \frac{3\mathcal{Q} \tanh \mathcal{P}}{a}\right)^2 + a^3 V \cosh \mathcal{P} = 0. \tag{89}$$

In a particular factor ordering, the Wheeler-DeWitt equation can now be written as

$$-\frac{1}{2a}\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial a}+\frac{3\mathcal{Q}}{a}\tanh\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\mathcal{Q}}\right)\right)^2\psi(a,\mathcal{Q})$$
$$+a^3V\cosh\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\mathcal{Q}}\right)\psi(a,\mathcal{Q})=0, \quad (90)$$

where $V=V(T(a,\mathcal{Q},\mathcal{P}))$. This equation has a rather complicated form. We note that $\tanh(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\mathcal{Q}})$ is not a suitable operator, since the series expansion of tanh has a finite radius of convergence. But since $\mathcal{H}$ is a constraint, we can multiply it by an arbitrary factor and the resulting quantity will be a constraint as well. If we define $\tilde{\mathcal{H}}\coloneqq 2a^3\cosh^2(\mathcal{P})\mathcal{H}$ and perform the transformation $a\to\alpha=\ln a$, $\tilde{p}_a\to\tilde{p}_\alpha=a\tilde{p}_a$, we obtain the new constraint

$$\tilde{\mathcal{H}}\coloneqq-\tilde{p}_\alpha^2\cosh^2\mathcal{P}+9\mathcal{Q}^2\sinh^2\mathcal{P}+6\mathcal{Q}\tilde{p}_\alpha\sinh\mathcal{P}\cosh\mathcal{P}$$
$$+2e^{6\alpha}V\cosh^3\mathcal{P}. \quad (91)$$

We emphasize that the hyperbolic functions with the momentum as argument generate a translation in the argument; we have, for example,

$$\cosh\left(-\mathrm{i}\frac{\partial}{\partial\mathcal{Q}}\right)\psi(\tilde{p}_\alpha,\mathcal{Q})=\frac{1}{2}(e^{-\mathrm{i}\frac{\partial}{\partial\mathcal{Q}}}+e^{\mathrm{i}\frac{\partial}{\partial\mathcal{Q}}})$$
$$=\frac{1}{2}(\psi(\tilde{p}_\alpha,\mathcal{Q}-\mathrm{i})+\psi(\tilde{p}_\alpha,\mathcal{Q}+\mathrm{i})). \quad (92)$$

After naive factor ordering, setting $\hbar=1$, and returning to the case of constant potential, the Wheeler-DeWitt equation assumes the following form:

$$\frac{V_0e^{6\alpha}}{4}\psi(\alpha,\mathcal{Q}+3\mathrm{i})-\frac{1}{4}(\tilde{p}_\alpha-3\mathcal{Q})^2\psi(\alpha,\mathcal{Q}+2\mathrm{i})$$
$$+\frac{3V_0e^{6\alpha}}{4}\psi(\alpha,\mathcal{Q}+\mathrm{i})+\frac{-\tilde{p}_\alpha^2+9\mathcal{Q}^2}{2}\psi(\alpha,\mathcal{Q})$$
$$+\frac{3V_0e^{6\alpha}}{4}\psi(\alpha,\mathcal{Q}-\mathrm{i})-\frac{1}{4}(\tilde{p}_\alpha+3\mathcal{Q})^2\psi(\alpha,\mathcal{Q}-2\mathrm{i})$$
$$+\frac{V_0e^{6\alpha}}{4}\psi(\alpha,\mathcal{Q}-3\mathrm{i})=0. \quad (93)$$

This is a mixed difference-differential equation (or partial difference equation, if we use the momentum representation for α).

In the asymptotic limit of large a, (90) reads

$$\frac{\partial^2\psi}{\partial a^2}+2a^4V_0\cosh\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\mathcal{Q}}\right)\psi=0. \quad (94)$$

Apart from the cosh-term, this coincides with the earlier form (48), which guarantees the consistency of the formalism. Employing the product ansatz

$$\psi(a,\mathcal{Q})=\phi(\mathcal{Q})\chi(a), \quad (95)$$

we find

$$\frac{d^2\chi}{da^2}=\frac{12}{\kappa^2}a^4V_0\lambda\chi(a), \quad (96)$$

$$\cosh\left(-\mathrm{i}\hbar\frac{d}{d\mathcal{Q}}\right)\phi(\mathcal{Q})=-\lambda\phi(\mathcal{Q}), \quad (97)$$

where we take λ to be a real constant. Introducing for convenience $\Lambda\coloneqq 2V_0\lambda$ (recall $V_0>0$), we find for the solutions of (96) a combination of Bessel functions. For $\Lambda>0$, we find the solutions $\sqrt{a}I_{1/6}(\sqrt{\Lambda}a^3/3)$ and $\sqrt{a}K_{1/6}(\sqrt{\Lambda}a^3/3)$, while for $\Lambda<0$, we find $\sqrt{a}J_{1/6}(\sqrt{-\Lambda}a^3/3)$ and $\sqrt{a}J_{-1/6}(\sqrt{-\Lambda}a^3/3)$. We note that for $\lambda=1$ this corresponds to the solutions (49).

In order to make a selection amongst these Bessel functions, we inspect their asymptotic behavior. Let us first consider the case $\Lambda>0$. The solution $\sqrt{a}I_{1/6}(\sqrt{\Lambda}a^3/3)$ increases exponentially with large a and is thus not normalizable; it must be excluded. The solution $\sqrt{a}K_{1/6}(\sqrt{\Lambda}a^3/3)$ decreases exponentially and is thus normalizable. The solutions for $\Lambda<0$, on the other hand, are oscillatory and thus both allowed; they correspond to (49) with $\lambda=-1$.

The second equation (97) can be rewritten in the form of the following difference equation:

$$\frac{1}{2}[\phi(\mathcal{Q}+\mathrm{i})+\phi(\mathcal{Q}-\mathrm{i})]=-\lambda\phi(\mathcal{Q}). \quad (98)$$

Making the ansatz

$$\phi(\mathcal{Q})=e^{\alpha\mathcal{Q}}, \quad (99)$$

one finds

$$\mathrm{i}\alpha_{1,2}=\ln\left(-\lambda\pm\sqrt{\lambda^2-1}\right). \quad (100)$$

Inserting this into (99) and writing $-\lambda\eqqcolon\cosh\mathcal{P}_0$, one gets for $\lambda<0$,

$$\phi(\mathcal{Q})=d_1e^{-\mathrm{i}\mathcal{P}_0\mathcal{Q}}+d_2e^{\mathrm{i}\mathcal{P}_0\mathcal{Q}}, \quad (101)$$

with constants d_1 and d_2. (This is also expected from the momentum representation of the Wheeler-DeWitt equation.) Taking all this together, the most general allowed asymptotic solution for $\Lambda<0$ is given by

$$\psi(a,\mathcal{Q})=(d_1e^{-\mathrm{i}\mathcal{P}_0\mathcal{Q}}+d_2e^{\mathrm{i}\mathcal{P}_0\mathcal{Q}})\times[c_1\sqrt{a}J_{1/6}(\sqrt{-\Lambda}a^3/3)+c_2\sqrt{a}J_{-1/6}(\sqrt{-\Lambda}a^3/3)].$$

For $\Lambda > 0$, one obtains

$$\psi(a,\mathcal{Q}) = (e_1 e^{-i\mathcal{P}_0\mathcal{Q}} + e_2 e^{i\mathcal{P}_0\mathcal{Q}})e^{-\pi\mathcal{Q}}\sqrt{a}K_{1/6}(\sqrt{\Lambda}a^3/3).$$

Can we say something about the general equation (90)? In the limit of small a, this equation assumes the form

$$(a\tilde{p}_a)\cosh\mathcal{P}\psi(a,\mathcal{Q}) + 3\mathcal{Q}\sinh\mathcal{P}\psi(a,\mathcal{Q}) = 0. \quad (102)$$

This leads to the difference equation

$$(a\tilde{p}_a)[\psi(a,\mathcal{Q}+i) + \psi(a,\mathcal{Q}-i)] - 3\mathcal{Q}[\psi(a,\mathcal{Q}+i) - \psi(a,\mathcal{Q}-i)] = 0. \quad (103)$$

After switching to the variable α and going to momentum space, it reads

$$(\tilde{p}_\alpha - 3\mathcal{Q})\psi(\tilde{p}_\alpha,\mathcal{Q}+i) + (\tilde{p}_\alpha + 3\mathcal{Q})\psi(\tilde{p}_\alpha,\mathcal{Q}-i) = 0.$$

A particular set of solutions is

$$\psi(\tilde{p}_\alpha,\mathcal{Q}) = \mu(\tilde{p}_\alpha,\mathcal{Q})\frac{\Gamma(-\frac{i}{2}[\frac{\tilde{p}_\alpha}{3} + \mathcal{Q} + i])}{\Gamma(\frac{i}{2}[\frac{\tilde{p}_\alpha}{3} - \mathcal{Q} - i])}, \quad (104)$$

where $\mu(\tilde{p}_\alpha,\mathcal{Q})$ is a function that is an arbitrary 2i-periodic function in the second argument, that is, $\mu(\tilde{p}_\alpha,\mathcal{Q}) = \mu(\tilde{p}_\alpha,\mathcal{Q}+2i)$.

We emphasize that the occurrence of the Gamma function in (104) is not an accident. The Gamma function obeys the perhaps most famous difference equation, $\Gamma(x+1) = x\Gamma(x)$, and it is known that it does not satisfy any algebraic differential equation whose coefficients are rational functions; the latter property is known as Hölder's theorem (see e.g. [43,44]). Thus, as emphasized in [44], one gets from difference equations transcendental functions of a very different kind than from differential equations.

Let us now consider the case of the inverse T-squared potential,

$$V = \frac{V_1}{T^2}, \quad (105)$$

whose classical behavior was discussed in Sec. II. For this potential, one can perform the canonical transformation

$$p_T = a^3 V_1 \sinh\mathcal{P}, \quad (106)$$

$$T = -\frac{a^3 V_1\cosh\mathcal{P}}{\mathcal{Q}}, \quad (107)$$

$$p_a = \tilde{p}_a + \frac{3\mathcal{Q}\tanh\mathcal{P}}{a}. \quad (108)$$

The Hamiltonian constraint becomes

$$\mathcal{H} = -\frac{1}{2a}\left(\tilde{p}_a + \frac{3\mathcal{Q}\tanh\mathcal{P}}{a}\right)^2 + \frac{\mathcal{Q}^2}{a^3 V_1\cosh\mathcal{P}}. \quad (109)$$

After another canonical transformation $a \to \alpha = \ln a$, $\tilde{p}_a \to \tilde{p}_\alpha = a\tilde{p}_a$, the Hamiltonian assumes the form

$$\mathcal{H} = e^{-3\alpha}\left[-\frac{1}{2}(\tilde{p}_\alpha^2 + 3\mathcal{Q}\tanh\mathcal{P})^2 + \frac{\mathcal{Q}^2}{V_1\cosh\mathcal{P}}\right]. \quad (110)$$

We observe that $\frac{\partial\mathcal{H}}{\partial\alpha} = -3\mathcal{H} = 0$. Thus the canonical momentum $\tilde{p}_\alpha = p_\alpha - 3\mathcal{Q}\tanh\mathcal{P}$ is a constant of motion. If we reinsert the old coordinates, this leads to

$$ap_a + 3p_T T = \text{constant} =: C. \quad (111)$$

As a side remark, we want to mention that the canonical transformation for the pseudotachyon case looks similar. One just has to replace $\sinh \to \cosh$, $\cosh \to \sinh$ and $\tanh \to \coth$. The constant of motion is then also present (with the same value) in the pseudotachyon model with inverse T-squared potential.

If we employ in the tachyon case the same procedure as for the constant V model, we obtain

$$\tilde{\mathcal{H}} = -\tilde{p}_\alpha^2\cosh^2\mathcal{P} - 6\mathcal{Q}\tilde{p}_\alpha\sinh\mathcal{P}\cosh\mathcal{P} - 9\mathcal{Q}^2\sinh^2\mathcal{P} + \frac{2\mathcal{Q}^2}{V_1}\cosh\mathcal{P}. \quad (112)$$

With naive factor ordering, the Wheeler-DeWitt equation reads

$$-\frac{1}{4}(\tilde{p}_\alpha - 3\mathcal{Q})^2\psi(\tilde{p}_\alpha,\mathcal{Q}+2i) + \frac{\mathcal{Q}^2}{V_1}\psi(\tilde{p}_\alpha,\mathcal{Q}+i) + \frac{-\tilde{p}_\alpha^2 + 9\mathcal{Q}^2}{2}\psi(\tilde{p}_\alpha,\mathcal{Q}) + \frac{\mathcal{Q}^2}{V_1}\psi(\tilde{p}_\alpha,\mathcal{Q}-i) - \frac{1}{4}(\tilde{p}_\alpha + 3\mathcal{Q})^2\psi(\tilde{p}_\alpha,\mathcal{Q}-2i) = 0. \quad (113)$$

This is an analytic difference equation. The foundations for such equations are presented in the book by Nø rlund [44]. In order to achieve conformity with Nø rlund's notation, we define $z := -i\mathcal{Q}$ and $u(z) := \psi(\mathcal{Q}(z))$. The function u then fulfills the difference equation

$$\sum_{k=0}^{4} P_k(z)u(z+k) = 0, \quad (114)$$

where the polynomials P_k are given by

$$P_0(z) = \frac{1}{4}(\tilde{p}_\alpha - 3\mathrm{i}z)^2, \qquad P_1(z) = \frac{z^2}{V_1},$$
$$P_2(z) = \frac{\tilde{p}_\alpha^2 + 9z^2}{2}, \qquad P_3(z) = \frac{z^2}{V_1},$$
$$P_4(z) = \frac{1}{4}(\tilde{p}_\alpha + 3\mathrm{i}z)^2.$$

The problem of solving the difference equation (114) is equivalent to the problem of finding a fundamental system of solutions. The solution space is a subset of the meromorphic functions and the fundamental system will be a vector space over the field of 1-periodic meromorphic functions. Following Nørlund, we expect the fundamental system to be composed of four linearly independent functions. Note that $\deg P_k = 2$ for all $k = 0, 1, \ldots, 4$, and thus the difference equation fulfills the first criterion to belong to a certain class which Nø rlund calls normal difference equations. To check the second criterion we have to rewrite

$$P_k(z) = \sum_{l=0}^{2} c_{k,l} \prod_{n=0}^{l-1} (z + k + n). \qquad (115)$$

The second criterion demands the nondegeneracy of the zeros a_n of the polynomial $f_2(s)$, where

$$f_l(s) \coloneqq \sum_{k=0}^{4} c_{k,l} s^k. \qquad (116)$$

The zeros are given by complicated expression. However, they are nondegenerate. Nørlund shows explicitly that the solutions to normal difference equations can be written in terms of products of Gamma functions and a series of rational functions (Chap. 11, Sec. II in [44]). In our case, the solutions will be of the form

$$u_n(z) = a_n^z \frac{\Gamma(z)}{\Gamma(-\beta_n)\Gamma(z + \beta_n + 1)} \Omega(z, \beta_n), \qquad (117)$$

where

$$\Omega(z, \beta_n) = \sum_{\nu=0}^{\infty} A_\nu \frac{(\beta_n + 1)(\beta_n + 2)\ldots(\beta_n + \nu)}{(z + \beta_n + 1)\ldots(z + \beta_n + \nu)} \qquad (118)$$

and $n = 1, 2, 3, 4$. The coefficients A_ν can in principle be determined by plugging the solutions into (114). The coefficients β_n, however, depend on the explicit form of the singular solutions of a certain differential equation in a neighborhood of the a_n's and cannot be determined easily. Nevertheless, the above discussion shows how difference equations occurring in quantum cosmology can in principle be dealt with.

We finally mention that, with the Hamiltonian (110), wet get in the momentum representation an ordinary differential equation.

VI. CONCLUSION

The purpose of our paper is to investigate a certain class of cosmological models with Born-Infeld (tachyon) type of fields. This is of interest for (at least) two reasons. First, such models have been encountered in the study of models for dark energy. Second, the Lagrangian is of a square-root type and thus poses challenges for quantization, which we have discussed here in detail. In the classical part, we have in particular obtained new results concerning the occurrence of a big brake singularity for the inverse square potential in the pseudotachyon case. Concerning quantization, we have managed to get nontrivial results for two models: for the model with a constant tachyon (pseudotachyon) potential, which is equivalent to the Chaplygin (anti-Chaplygin) gas and for the model where the potential is inversely proportional to the tachyon field squared. We have derived and discussed the Wheeler-DeWitt equation for these models. For constant potential, it is convenient to use the momentum representation for the quantum state and to consider the reduced phase space of physical variables. This becomes quite complicated for nonconstant potentials. We have thus pointed out a general method to transform the Wheeler-DeWitt equation into a form without square roots. This leads to a difference equation, which requires new methods for its solution. We have discussed these methods and pointed out that difference equations lead in general to solutions of a different type than solutions from differential equations. We have outlined a general procedure to finding such solutions. The methods may also be of use in loop quantum cosmology [39,40].

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