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# Linear growth in power law $f(T)$ gravity

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We provide for the first time the growth index of linear matter fluctuations of the power law $f(T) \propto (-T)^b$ gravity model. We find that the asymptotic form of this particular $f(T)$ model is $\gamma \approx \frac{6}{11-6b}$, which obviously extends that of the $\Lambda$CDM model, $\gamma_\Lambda \approx 6/11$. Finally, we generalize the growth index analysis of $f(T)$ gravity in the case where $\gamma$ is allowed to vary with redshift.

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## I. INTRODUCTION

Over the last two decades the statistical analysis of cosmological data (see Refs. [1,2] and references therein) supports the idea that the Universe is spatially flat, and from the overall energy density, only ~30% consists of matter (luminous and dark). Despite the enormous progress made at the theoretical and observational levels, up to now we know almost nothing about the nature of the remaining energy (~70%), and for this reason, it is given the enigmatic name dark energy (DE). The discovery of the physical mechanism of dark energy, thought to be driving the late accelerated expansion of the Universe, is one of the main targets of theoretical physics and cosmology. In the literature one can find a plethora of cosmological scenarios that attempt to explain the accelerated expansion of the Universe. In general, the cosmological models are mainly classified in two large groups. The first category is the so-called scalar field DE models which adhere to general relativity, proposing however the existence of new fields in nature (for a review see [3]).

Alternatively, models of modified gravity provide an elegant mathematical treatment which points out that the present accelerating epoch appears as a sort of geometric effect [3]. In this context, the corresponding effective equation-of-state (EoS) parameter is allowed to take values in the phantom regime, namely $w < -1$ (for other possible explanations see [4] and [5]). This situation has been tested in Wilkinson microwave anisotropy probe (WMAP) observations, in combination with other observational data. The above feature did not completely disappear from the analysis of the Planck data which indicates that the value of $w$ can still be in the phantom region, within $1\sigma$ uncertainty [2]. For more details concerning the cosmological implications of modified gravity, we refer the reader to the review article of Clifton *et al.* [6].

Among the large body of nonstandard gravity theories, the so-called $f(T)$ gravity has been introduced in the literature on the basis of the old definition of the so-called teleparallel equivalent of general relativity (TEGR) [7–9]. In the TEGR framework one utilizes the corresponding four linearly independent vierbeins and the curvatureless Weitzenböck connection instead of the torsionless Levi-Civita of standard general relativity. Therefore, the properties of the gravitational field are included in the torsion tensor, and after performing the appropriate contractions, one can obtain the torsion scalar $T$ [8]. Subsequently, inspired by the notations of $f(R)$ modified gravity, if we allow the Lagrangian of the modified Einstein-Hilbert action to be a function of $T$ [10–12], then we provide a natural extension of TEGR, namely $f(T)$ gravity (for a recent review see [13]). The merit of $f(T)$ gravity with respect to $f(R)$ is related to the fact that the former produces second-order field equations, while the latter gives rise to fourth-order equations that may lead to problems, such as the well position and well formulation of the Cauchy problem [14].

But how can we distinguish modified gravity models from those of scalar field DE? In order to answer this question we need to test the models at the perturbation level (for a recent analysis see [15] and references therein). Specifically, the idea of utilizing the so-called growth index $\gamma$ (first introduced by [16]) of linear matter perturbations as a gravity tool is not new, and indeed there is a lot of work in the literature. There are plenty of studies available in which one can find the theoretical form of the growth index for various cosmological models, including scalar field DE [17–22], DGP [21,23–25], Finsler-Randers [26] and $f(R)$ [27,28].

Despite the fact that the $f(T)$ models have been investigated thoroughly at the background level (see Ref. [13] and references therein), to the best of our knowledge, we are unaware of any previous analysis concerning the $f(T)$ growth index, and thus, we believe that the present analysis can be of theoretical interest. In the current article, we wish to study the growth index of the power law $f(T) \propto (-T)^b$ model [10]. The layout of the manuscript is as follows: At the beginning of Sec. II we describe the main points of the $f(T)$ gravity, and then we focus our analysis on the power law $f(T) \propto (-T)^b$ model. In Sec. III we provide the growth

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index analysis and the corresponding predictions, using two functional forms of the growth index. Finally, we summarize our conclusions in Sec. IV.

## II. BACKGROUND EXPANSION IN $f(T)$ COSMOLOGY

Let us briefly present the basic cosmological properties of $f(T)$ gravity. The overall action of $f(T)$ gravity is given by

$$I = \frac{1}{16\pi G_N}\int d^4xe[T + f(T) + L_m + L_r], \tag{1}$$

where the radiation and matter Lagrangians are associated with perfect fluids with pressures $P_r$, $P_m$ and densities $\rho_r$, $\rho_m$, respectively. Notice that $e = \det(e^A_\mu)$ and $\mathbf{e}_A(x^\mu)$ are the vierbein fields. Within this framework, the gravitational field is described by the torsion tensor [8,9] which produces the torsion scalar $T$. A similar situation holds in the case of the Riemann tensor which provides the Ricci scalar in standard general relativity.

Considering a spatially flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \tag{2}$$

the vierbien form becomes

$$e^A_\mu = \mathrm{diag}(1, a, a, a), \tag{3}$$

where $a(t)$ is the scale factor of the Universe. Now, if we vary the action (1) with respect to the vierbeins, then we obtain the modified Einstein equations

$$\begin{aligned} &e^{-1}\partial_\mu(ee^\rho_A S_\rho{}^{\mu\nu})[1 + f_T] + e^\rho_A S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT} \\ &\quad - [1 + f_T]e^\lambda_A T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu} + \frac{1}{4}e^\nu_A[T + f(T)] \\ &\quad = 4\pi G e^\rho_A \overset{\mathbf{em}}{T}{}_\rho{}^\nu, \end{aligned} \tag{4}$$

where $f_T = \partial f/\partial T$, $f_{TT} = \partial^2 f/\partial T^2$, and $\overset{\mathbf{em}}{T}{}_\rho{}^\nu$ corresponds to the standard energy-momentum tensor.

Substituting Eq. (3) into the field equations (4) we derive the Friedmann equations

$$H^2 = \frac{8\pi G_N}{3}(\rho_m + \rho_r) - \frac{f}{6} + \frac{Tf_T}{3}, \tag{5}$$

$$\dot{H} = -\frac{4\pi G_N(\rho_m + P_m + \rho_r + P_r)}{1 + f_T + 2Tf_{TT}}. \tag{6}$$

In the above set of equations, an overdot denotes a derivative with respect to time and $H \equiv \dot{a}/a$ is the Hubble parameter, given as a function of torsion $T$ through the following equation:

$$T = -6H^2. \tag{7}$$

This implies

$$E^2(a) \equiv \frac{H^2(a)}{H_0^2} = \frac{T(a)}{T_0}, \tag{8}$$

where $H_0$ is the Hubble constant and $T_0 \equiv -6H_0^2$.

If we look at the first Friedmann equation (5) then we realize that it is possible to obtain an effective dark energy component. Indeed, it has been shown in Ref. [12] that the effective dark energy density and pressure are given by

$$\rho_{\mathrm{DE}} \equiv \frac{3}{8\pi G_N}\left[-\frac{f}{6} + \frac{Tf_T}{3}\right], \tag{9}$$

$$P_{\mathrm{DE}} \equiv \frac{1}{16\pi G_N}\left[\frac{f - f_T T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}\right], \tag{10}$$

where the corresponding effective EoS parameter is

$$w = \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 - \frac{1}{3}\frac{d\ln T}{d\ln a}\frac{f_T + 2Tf_{TT}}{[(f/T) - 2f_T]}. \tag{11}$$

Combining Eqs. (7) and (8) we derive the logarithmic derivative of $T$ with respect to $d\ln a$,

$$\frac{d\ln T}{d\ln a} = 2T_0 E(a)\frac{d\ln E}{d\ln a}. \tag{12}$$

Following standard lines, namely $\rho_m = \rho_{m0}a^{-3}$ and $\rho_r = \rho_{r0}a^{-4}$, Eq. (5) is written as

$$E^2(a) = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{F0}y(a) \tag{13}$$

where

$$\Omega_{F0} = 1 - \Omega_{m0} - \Omega_{r0} \tag{14}$$

and $\Omega_{i0} = \frac{8\pi G\rho_{i0}}{3H_0^2}$. Obviously, $f(T)$ gravity affects the cosmic evolution via the function $y(z)$ (scaled to unity at the present time), which depends on the choice of $f(T)$ as well as on the usual cosmological parameters $(\Omega_{m0}, \Omega_{r0})$, and it is written as

$$y(a) = \frac{1}{T_0\Omega_{F0}}(f - 2Tf_T). \tag{15}$$

### A. Power law model

In this work we restrict our analysis to the power-law model of Bengochea and Ferraro [11], with

$$f(T) = \alpha(-T)^b, \tag{16}$$

where

$$\alpha = (6H_0^2)^{1-b}\frac{\Omega_{F0}}{2b-1}. \tag{17}$$

Inserting the above equations into Eqs. (11) and (15), we obtain

$$y(a,b) = E^{2b}(a,b) \tag{18}$$

and

$$w = -1 - \frac{2b}{3}\frac{d\ln E}{d\ln a} = -1 + \frac{2b}{3}(1+z)\frac{d\ln E}{dz}, \tag{19}$$

where for the latter equality we have used $a = 1/(1+z)$. In this case the normalized Hubble function (13) is given by

$$E^2(a,b) = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{F0}E^{2b}(a,b). \tag{20}$$

Clearly, for $b = 0$ the current $f(T)$ model boils down to $\Lambda$CDM cosmology,[1] namely $T + f(T) = T - 2\Lambda$ (where $\Lambda = 3\Omega_{F0}H_0^2$, $\Omega_{F0} = \Omega_{\Lambda 0}$), and thus we have

$$E^2(a,0) = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{F0} \equiv E_\Lambda^2(a). \tag{21}$$

Notice that in order to obtain an accelerating expansion which is consistent with the cosmological data, one needs $b \ll 1$ [12,30]. Within this framework, we can now follow the work of Nesseris *et al.* [30], in which they have shown that at the background level all the observationally viable $f(T)$ parametrizations can be expressed as perturbations deviating to $\Lambda$CDM cosmology. In particular, following the notations of [30] for the power law $f(T)$ model, we perform a Taylor expansion of $E^2(a,b)$ around $b = 0$,

$$E^2(a,b) = E^2(a,0) + \left.\frac{dE^2(a,b)}{db}\right|_{b=0} b + \cdots$$

or

$$E^2(a,b) = E_\Lambda^2(a) + \Omega_{F0}\left.\frac{dy(a,b)}{db}\right|_{b=0} b + \cdots, \tag{22}$$

where for the latter equality we have used Eq. (15). Now based on Eq. (18) we obtain

$$\frac{dy(a,b)}{db} = 2E(a,b)^{2b}\left\{\frac{b}{E(a,b)}\frac{dE(a,b)}{db} + \ln[E(a,b)]\right\}, \tag{23}$$

and evaluating the above equation for $b = 0$ we find

$$\left.\frac{dy(a,b)}{db}\right|_{b=0} = 2\ln[E(a,0)] = \ln[E_\Lambda^2(a)]. \tag{24}$$

Therefore, inserting Eq. (24) into Eq. (22) we provide the approximate normalized Hubble parameter for the current $f(T)$ model (see [30])

$$E^2(a,b) \simeq E_\Lambda^2(a) + \Omega_{F0}\ln[E_\Lambda^2(a)]b. \tag{25}$$

Implementing an overall likelihood analysis involving the latest cosmological data (Supernovae Type Ia (SNIa) [31], baryon acoustic oscillations (BAO) [32,33] and Planck cosmic microwave background (CMB) shift parameter [34]) and the appropriate Akaike information criterion [35], we can place constraints on the cosmological parameters $(\Omega_{m0}, b)$. Specifically, we find that the likelihood function peaks at $\Omega_{m0} = 0.286 \pm 0.012$, $b = -0.081 \pm 0.117$ with $\chi^2_{\rm min}(\Omega_{m0}, b) \simeq 563.6$ (AIC = 567.6), resulting in a reduced value of ~0.96.[2] In order to visualize the solution space in Fig. 1 we plot the $1\sigma$, $2\sigma$ and $3\sigma$ confidence contours in the $(\Omega_{m0}, b)$ plane. At this point we need to mention that the uncertainty of the $b$ parameter is quite large (see also [30]), as indicated in the relevant contour figure. Our statistical results are in agreement,

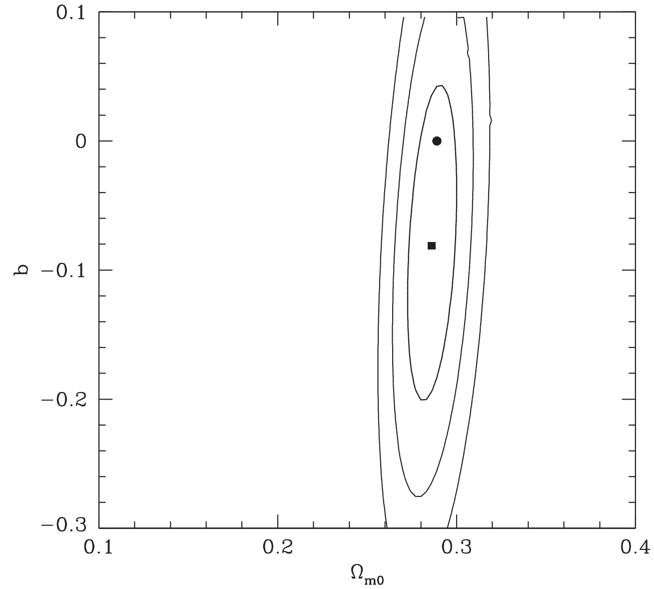


FIG. 1. The overall (SNIa/BAO/CMB$_{\rm shift}$) likelihood contours for $\Delta\chi^2 = \chi^2 - \chi^2_{\rm min}$ equal to $1\sigma$ (2.32), $2\sigma$ (6.18) and $3\sigma$ (11.83) confidence levels, in the $(\Omega_{m0}, b)$ plane. The solid square corresponds to the best-fit $f(T) \propto (-T)^b$ modified gravity model, namely $(\Omega_{m0}, b) = (0.286, -0.08)$. The solid point shows the best-fit solution for the concordance $\Lambda$CDM model.

[1] Notice that for $b = 1/2$ it reduces to the Dvali, Gabadadze and Porrati (DGP) ones [29].

[2] The total $\chi^2$ function is given by $\chi^2 = \chi^2_{\rm SNIa} + \chi^2_{\rm BAO} + \chi^2_{\rm CMB}$. For Gaussian errors, the Akaike information criterion (AIC) [35] is given by AIC $= \chi^2_{t,\rm min} + 2k$, where $k$ provides the number of free parameters.

within $1\sigma$ errors, with those of Nesseris *et al.* [30], who used a combined analysis of SNIa [31], BAO [32,33] and WMAP9 CMB shift parameters [36], and they found $(\Omega_{m0}, b) = (0.274 \pm 0.008, -0.017 \pm 0.083)$.

For the concordance $\Lambda$ cosmology ($b = 0$) we find $\Omega_{m0} = 0.289 \pm 0.012$, $\chi^2_{\rm min}(\Omega_{m0}) \simeq 564.6$ (AIC = 566.6). Since the difference $|\Delta{\rm AIC}| = |{\rm AIC}_\Lambda - {\rm AIC}_{f(T)}| < 2$ points to the fact that the power law $f(T)$ and $\Lambda$CDM models respectively fit the cosmological data equally well.

## III. LINEAR GROWTH IN $f(T)$ COSMOLOGY

In this section we present the linear matter fluctuations of $f(T)$ gravity in the matter dominated era (for details see Ref. [37]). Therefore, for the rest of the paper we neglect the radiation term from the cosmological expressions appearing in Sec. II. Based on standard treatment, the differential equation that describes the evolution of matter perturbations at the subhorizon scales takes on the form

$$\ddot{\delta}_m + 2\nu H\dot{\delta}_m - 4\pi G\mu\rho_m\delta_m = 0. \tag{26}$$

In the framework of modified gravity models the quantity $\mu = G_{\rm eff}/G_N$ depends on the scale factor, while for those dark energy models which are inside general relativity, $G_{\rm eff}$ reduces to Newton's constant as it should and thus $\mu = 1$. We refer the reader to Refs. [20,21,27,38–41] for full details of the calculation. One can show that $\delta_m \propto D(t)$ where $D(t)$ is the linear growth factor scaled to unity at the present epoch. Obviously, any modification to the gravity theory and to the Friedmann equation is reflected in the quantities $\nu$ and $\mu \equiv G_{\rm eff}/G_N$. As an example, in the framework of scalar field dark energy models which adhere to general relativity, one has $\nu = \mu = 1$. Moreover, for the concordance $\Lambda$ cosmology, one can solve (26) analytically in order to obtain the growth factor [16]

$$D_\Lambda(a) = \frac{5\Omega_{m0}E_\Lambda(a)}{2}\int_0^a \frac{du}{uE^3_\Lambda(u)}, \tag{27}$$

where

$$E_\Lambda(a) = (\Omega_{m0}a^{-3} + \Omega_{\Lambda0})^{1/2} \tag{28}$$

in the matter dominated era and $\Omega_{\Lambda0} = 1 - \Omega_{m0}$.

On the other hand, for nonstandard gravity models we have $\nu = 1$ and $\mu \neq 1$, and for the $f(T)$ gravity the quantity $\mu$ takes the following form [42,43]:

$$\mu = \frac{1}{1 + f_T}. \tag{29}$$

Inserting Eq. (16) into Eq. (29) we obtain

$$\mu(a) = \frac{1}{1 + \frac{b\Omega_{F0}}{(1-2b)E^{2(1-b)}}} \tag{30}$$

or

$$\mu(a) \simeq 1 - \frac{\Omega_{F0}}{E^2_\Lambda(a)}b + \cdots \tag{31}$$

where, as in Sec. II, for the latter expression we have utilized a Taylor expansion around $b = 0$.

In order to simplify the numerical calculations we provide the growth rate of clustering introduced by [16]

$$f(a) = \frac{d\ln\delta_m}{d\ln a} \simeq \Omega^\gamma_m(a), \tag{32}$$

based on which we can write the growth factor

$$D(a) = \exp\left[\int_1^a \frac{\Omega_m(x)^{\gamma(x)}}{x}dx\right], \tag{33}$$

with

$$\Omega_m(a) = \frac{\Omega_{m0}a^{-3}}{E^2(a)} \tag{34}$$

and from which we define

$$\frac{d\Omega_m}{da} = -3\frac{\Omega_m(a)}{a}\left(1 + \frac{2}{3}\frac{d\ln E}{d\ln a}\right). \tag{35}$$

The parameter $\gamma$ is the so-called growth index which can be used to distinguish between general relativity and modified gravity on cosmological scales (see Introduction). In this context, utilizing the first equality of (32) one can write Eq. (26) as follows:

$$a\frac{df}{da} + \left(2\nu + \frac{d\ln E}{d\ln a}\right)f + f^2 = \frac{3\mu\Omega_m}{2}. \tag{36}$$

Now differentiating Eq. (20) and utilizing Eq. (34) we find that

$$\frac{d\ln E}{d\ln a} = -\frac{3}{2}\frac{\Omega_m(a)}{[1 - bE^{2(b-1)}\Omega_{F0}]}. \tag{37}$$

For $b \ll 1$ the latter equation is well approximated by

$$\frac{d\ln E}{d\ln a} \simeq -\frac{3}{2}\Omega_m(a)\left[1 + \frac{\Omega_{F0}b}{E^2_\Lambda(a)} + \cdots\right] \tag{38}$$

Regarding the form of the growth index we consider the following two situations.

### A. Constant growth index

The simplest choice is to use the asymptotic value of the growth index, namely $\gamma_\infty$. Recently, Steigerwald *et al.* [41] proposed a general mathematical treatment which provides $\gamma_\infty$ analytically [see Eq. (8) in [41] and the discussion in [44]] for a large family of DE models. Based on the work of Steigerwald *et al.* [41] the asymptotic value of the growth index is given analytically by

$$\gamma_\infty = \frac{3(M_0 + M_1) - 2(H_1 + N_1)}{2 + 2X_1 + 3M_0} \tag{39}$$

where the relevant quantities are

$$M_0 = \mu|_{\omega=0}, \qquad M_1 = \frac{d\mu}{d\omega}\bigg|_{\omega=0} \tag{40}$$

and

$$N_1 = \frac{d\nu}{d\omega}\bigg|_{\omega=0}, \qquad H_1 = -\frac{X_1}{2} = \frac{d(d\ln E/d\ln a)}{d\omega}\bigg|_{\omega=0}. \tag{41}$$

We would like to point out that Steigerwald *et al.* [41] derived the basic cosmological functions in terms of the variable $\omega = \ln \Omega_{\rm m}(a)$, which implies that at $z \gg 1$ we have $\Omega_m(a) \to 1$ [or $\omega \to 0$].[3] For the $f(T)$ gravity the coefficient $N_1$ is strictly equal to zero since $\nu = 1$. Then, based on Eqs. (25), (31), (34), (35) and (38), we find after some algebra (for more details see the Appendix)

$$\{M_0, M_1, H_1, X_1\} \simeq \left\{1, b, -\frac{3(1-b)}{2}, 3(1-b)\right\},$$

and thus we calculate, for the first time (to the best of our knowledge), the asymptotic value of the growth index

$$\gamma_\infty \simeq \frac{6}{11-b} \approx \frac{6}{11}\left(1 + \frac{6}{11}b\right). \tag{42}$$

Obviously, for $b = 0$ we recover the ΛCDM value $6/11$, as we should. On the other hand, utilizing the aforementioned best-fit value $b = -0.081$ and the corresponding $1\sigma$ $b$-uncertainty $\sigma_b = 0.117$, we find that $\gamma_\infty$ lies in the interval [0.492, 0.556] (see upper panel of Fig. 2). In the lower panel of Fig. 2 we show the relative deviation of the $f(T)$ growth index with respect to $\gamma_\Lambda \approx 6/11$. The relative difference can reach $\sim -9\%$ when we approach the aforesaid theoretical lower $1\sigma$ bound of $b \simeq -0.2$. For the best-fit value $b = -0.081$ we have $\gamma = 0.5223$, which gives a $\sim -4\%$ difference from $6/11$. We also see that for positive values of $b$ the asymptotic value of the growth index becomes greater than $6/11$, while the opposite holds for negative values of $b$.

### B. Varying growth index

The second possibility is to consider that $\gamma$ evolves with redshift. Therefore, in this scenario we need to generalize the original Polarski and Gannouji [45] method for the

[3]For Λ cosmology ($b = 0$) Eq. (38) becomes $\frac{d\ln E_\Lambda}{d\ln a} = -\frac{3\Omega_m^{(\Lambda)}(a)}{2}$, where $\Omega_m^{(\Lambda)}(a) = \frac{\Omega_{m0}a^{-3}}{E_\Lambda^2(a)}$. Of course at large redshifts $z \gg 1$ we have $\Omega_m^{(\Lambda)}(a) \to 1$ and thus $\frac{d\ln E_\Lambda}{d\ln a} \to -\frac{3}{2}$.

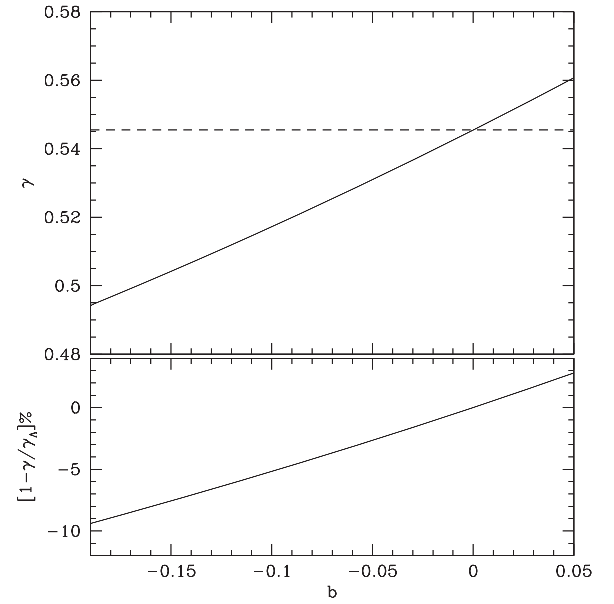


FIG. 2. *Upper panel:* We show the asymptotic value of the growth index as a function of $b$ (solid line). The dashed curve corresponds to $\gamma_\Lambda \approx 6/11$. *Lower panel:* We plot the relative difference $[1 - \gamma/\gamma_\Lambda]\%$ versus $b$.

$f(T)$ gravity. Specifically, upon substituting $f(a) = \Omega_m(a)^{\gamma(a)}$ into Eq. (36) and using Eq. (35), we are led to

$$a\ln(\Omega_m)\frac{d\gamma}{da} + \Omega_m^\gamma - 3\left(\gamma - \frac{1}{2}\right)\left(1 + \frac{2}{3}\frac{d\ln E}{d\ln a}\right) + \frac{1}{2} = \frac{3}{2}\mu\Omega_m^{1-\gamma}. \tag{43}$$

Writing the above equation at the present time ($a = 1$) we simply have

$$-\gamma'(1)\ln(\Omega_{m0}) + \Omega_{m0}^{\gamma(1)} - 3\left[\gamma(1) - \frac{1}{2}\right]\left(1 + \frac{2}{3}\frac{d\ln E}{d\ln a}\right)_{a=1} + \frac{1}{2} = \frac{3}{2}\mu_0\Omega_{m0}^{1-\gamma(1)}, \tag{44}$$

where a prime denotes a derivative with respect to the scale factor and

$$\mu_0 = \mu(1) \simeq 1 - \Omega_{F0}b,$$

$$\frac{d\ln E}{d\ln a}\bigg|_{a=1} \simeq -\frac{3}{2}\Omega_{m0}(1 + \Omega_{F0}b).$$

For the latter two expressions we have used Eqs. (31) and (38).

In this work we consider the most popular $\gamma(a)$ parametrization that has appeared in the literature (see [45–49]), which is a Taylor expansion around $a(z) = 1$,

$$\gamma(a) = \gamma_0 + \gamma_1(1 - a), \tag{45}$$

with the asymptotic value becoming $\gamma_\infty \simeq \gamma_0 + \gamma_1$ where we have set $\gamma_0 = \gamma(1)$.

Utilizing Eqs. (44) and (45), and the above notations, we can easily obtain $\gamma_1$ in terms of $\gamma_0$:

$$\gamma_1 = \frac{\Omega_{m0}^{\gamma_0} - 3(\gamma_0 - \frac{1}{2})[1 - \Omega_{m0}(1 + \Omega_{F0}b)] - \frac{3}{2}\mu_0\Omega_{m0}^{1-\gamma_0} + \frac{1}{2}}{\ln \Omega_{m0}}. \tag{46}$$

As expected, for the $\Lambda$ cosmology ($b = 0$) the above formula reduces to its standard expression [45–49]. Lastly, inserting $\gamma_0 = \gamma_\infty - \gamma_1$ into Eq. (46) and utilizing $\gamma_\infty \approx \frac{6}{11-6b}$, we can derive the constants $\gamma_{0,1}$ as a function $(\Omega_{m0}, b)$. For example, if we use the fitting values $(\Omega_{m0}, b) = (0.286, -0.081)$, then we estimate $(\gamma_0, \gamma_1) \simeq (0.541, -0.019)$, while for the concordance $\Lambda$ cosmological model with $(\Omega_{m0}, b) = (0.289, 0)$, we find $(\gamma_0, \gamma_1) \simeq (0.557, -0.011)$.

In order to check the variants of the $f(T) \propto (-T)^b$ model from the $\Lambda$CDM case at the perturbation level, we present in Fig. 3 a comparison of the evolution of the growth index $\gamma(z)$ (upper panel) and the evolution of the $\mu(z) \equiv G_{\rm eff}(z)/G_N$ (lower panel). The solid and the dashed curves correspond to $f(T)$ and $\Lambda$CDM models, respectively. Also, the thin-line error bars correspond to $1\sigma$ $b$-uncertainties which affect the growth index and $\mu$ via Eqs. (31) and (46). As expected, at large redshifts $f(T)$ tends to general relativity, namely $\mu \to 1$, while as we approach the present epoch $\mu$ starts to deviate from unity. Of course, due to large $1\sigma$ $b$-uncertainties we cannot exclude the value $b = 0$ which corresponds to the concordance $\Lambda$ cosmology. Therefore, in order to test possible departures from general relativity we need to place tight constraints on the $b$ parameter and thus on $\gamma$. Hopefully, using the next generation of surveys (like *Euclid*—see discussion in [50]) we expect to be able to constrain the $b$ parameter.

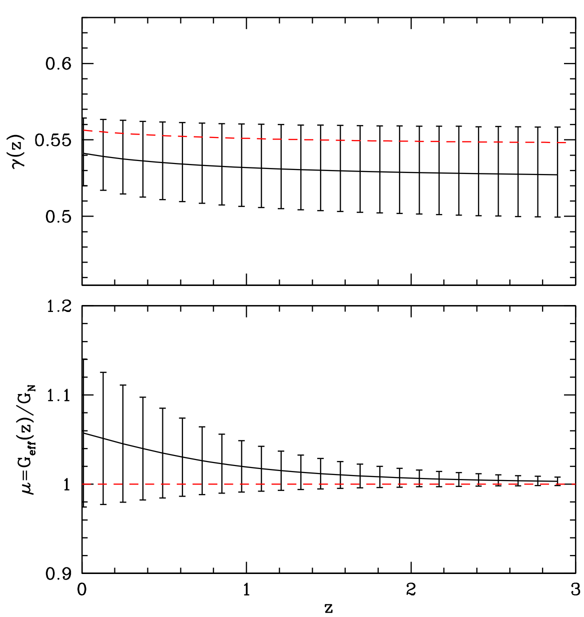


FIG. 3. In the upper panel we show the growth index as a function of redshift for the $f(T) \propto (-T)^b$ gravity model (solid line). In the lower panel we plot the evolution of the $\mu(z) \equiv G_{\rm eff}/G_N$ [see Eq. (31)]. Notice, that the thin-line error bars correspond to $1\sigma b$-uncertainties which affect the growth index and $\mu$ via Eqs. (46) and (31). For comparison, the dashed line corresponds to the traditional $\Lambda$CDM model.

## IV. CONCLUSIONS

We studied the power-law $f(T) \propto (-T)^b$ model at the linear perturbation level. Applying the technique of Steigerward *et al.* [41] in the framework of the current $f(T)$ model, we derive (for the first time) the asymptotic value of the growth index of matter perturbations, namely $\gamma \approx \frac{6}{11-6b}$. Evidently, for $b = 0$ the latter formula reduces to that of the usual $\Lambda$CDM model, $\gamma_\Lambda \approx \frac{6}{11}$. It is interesting to mention that Nesseris *et al.* [30] proved that the power-law $f(T)$ model can be seen as a perturbation around $\Lambda$CDM at the expansion level. Here we extended the latter work, by writing the asymptotic value of the $f(T)$ growth index as a perturbation around that of $\Lambda$CDM, namely $\gamma \approx \frac{6}{11}(1 + \frac{6}{11}b)$. Finally, we generalized the analysis in the regime where the growth index is allowed to vary with redshift, and we found that an accurate determination of $b$ is needed in order to test the range of validity of the $f(T) \propto (-T)^b$ modified gravity model.

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## APPENDIX: BASIC COEFFICIENTS

Here we provide some calculations concerning the coefficients $M_0, M_1, H_1$ and $X_1$ which appear in Eq. (39). As we have already discussed in Sec. II A, these quantities are given in terms of the variable $\omega = \ln \Omega_m$ which means that as long as $a \to 0$ ($z \gg 1$) we have $\Omega_m \to 1$ (or $\omega \to 0$) and thus $E^2(a) \gg 1$. Therefore, from Eq. (31) we simply find

$$M_0 = \mu|_{\omega=0} \simeq 1.$$

Now, $M_1$ is defined as

$$M_1 = \left.\frac{d\mu}{d\omega}\right|_{\omega=0} = \Omega_m \left.\frac{d\mu}{d\Omega_m}\right|_{\Omega_m=1}.$$

Using Eqs. (28), (31), (34), and (35) we obtain, after some calculations,

$$\Omega_m(a)\frac{d\mu}{d\Omega_m} \simeq \Omega_m(a)\frac{b\Omega_{F0}}{E_\Lambda^2(a)\Omega_\Lambda(a)} = \Omega_m(a)\frac{b\Omega_{F0}}{\Omega_{\Lambda 0}}.$$

Notice that for the latter equality we use the well-known formula $E_\Lambda^2(a)\Omega_\Lambda(a) = \Omega_{\Lambda 0}$. Under these conditions $M_1$ becomes

$$M_1 \simeq \frac{b\Omega_{F0}}{\Omega_{\Lambda 0}} \simeq b,$$

where we have set $\Omega_{F0} = \Omega_{\Lambda 0}$ [see the corresponding discussion before Eq. (21)].

Finally, the coefficient $H_1$ (or $X_1$) is given by

$$H_1 = -\frac{X_1}{2} = \frac{d(d\ln E/d\ln a)}{d\omega}\Big|_{\omega=0} = \Omega_m \frac{d(d\ln E/d\ln a)}{d\Omega_m}\Big|_{\Omega_m=1}.$$

Again, utilizing Eqs. (28), (31), (34), (35) and (38) we find

$$\Omega_m \frac{d(d\ln E/d\ln a)}{d\Omega_m} \simeq -\frac{3\Omega_m}{2}\left[1 + \frac{b\Omega_{F0}}{E_\Lambda^2(a)} + \frac{2b\Omega_{F0}}{3E_\Lambda^2(a)\Omega_\Lambda(a)}\frac{d\ln E_\Lambda}{d\ln a}\right].$$

Therefore, in the context of the aforementioned limitations ($\Omega_m \to 1$), $H_1$ (and thus $X_1$) takes the form

$$H_1 = -\frac{X_1}{2} \simeq -\frac{3}{2}(1-b).$$


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