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Spin-one matter fields

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Spin-one matter fields are relevant both for the description of hadronic states and as potential extensions of the Standard Model. In this work we present a formalism for the description of massive spin-one fields transforming in the $(1,0)\oplus(0,1)$ representation of the Lorentz group, based on the covariant projection onto parity eigenspaces and Poincaré orbits. The formalism yields a constrained dynamics. We solve the constraints and perform the canonical quantization accordingly. This formulation uses the recent construction of a parity-based covariant basis for matrix operators acting on the $(j,0)\oplus(0,j)$ representations. The algebraic properties of the covariant basis play an important role in solving the constraints and allowing the canonical quantization of the theory. We study the chiral structure of the theory and conclude that it is not chirally symmetric in the massless limit, hence it is not possible to have chiral gauge interactions. However, spin-one matter fields can have vector gauge interactions. Also, the dimension of the field makes self-interactions naively renormalizable. Using the covariant basis, we classify all possible self-interaction terms.

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I. INTRODUCTION

States transforming in the $(1,0)\oplus(0,1)$ representation have been shown to be appropriate for the description of low-energy interactions of the low-lying nonets of vector and axial-vector mesons [1]. The corresponding fields are written in tensor language (an antisymmetric second-rank tensor field is used to describe spin-one mesons) and the effective theory known as resonance chiral perturbation theory ($R\chi PT$) involves a nonlinear realization of chiral symmetry. Also, possible effects of spin-one matter particles described by tensor fields in physics beyond the standard model have been proposed in [2].

On the other hand, many alternatives for physics beyond the standard model have been proposed and although the first results of the Large Hadron Collider (LHC) showed no evidence of any of these possibilities up to energies of the order of 1.5 TeV [3–11], recently a series of excess of events in several searches of new spin-one bosons at the level of 2–3 standard deviations point to the possible existence of new spin-one resonances close to 2 TeV [12]. The simplest possibility for these resonances is some realization of the left-right symmetric models and the first possible explanations of the excess of events following this route have been already proposed in [13,14]. An alternative to the understanding of these events would be offered by spin-one matter fields. Indeed, it is intriguing that the standard model and most of the proposed nonsupersymmetric extensions use only the $(0,0)$, $(1/2,0)$, $(0,1/2)$ and $(1/2,1/2)$ representations of the Homogeneous Lorentz Group (HLG). The consistent formulation of a theory involving fields transforming in the chiral (1, 0) and (0, 1) representations of the HLG would certainly enlarge the possibilities for beyond the standard model theories.

Recently, an algorithm for the construction of a covariant basis for the matrix operators acting on the $(j,0)\oplus(0,j)$ representation space was put forth in Ref. [15]. This construction is based on the covariant properties of the parity operator, and the explicit form of the covariant matrices is given for $j=1/2$, 1, $3/2$. For $j=1/2$ the covariant basis reproduces the conventional basis acting on Dirac space, and the Dirac equation is recovered as the covariant projection onto parity eigenspaces. This alternative view of the Dirac equation, and the fact that the covariant basis for $(1,0)\oplus(0,1)$ has been already constructed in [15], leads us to explore the $j=1$ generalization of the structure of the Dirac theory. Since a chirality operator appears in a natural way in the covariant basis, chiral states can be constructed directly. This allows us to study alternatives for the formulation of chiral effective theories for hadrons using the Dirac-like theory for fields transforming in the $(1,0)\oplus(0,1)$ representation of the HLG.

In this work, we propose a theory for massive spin-one matter fields which is a direct generalization to $j=1$ of the structure of the Dirac theory for fermions. The formalism is based on the simultaneous projection onto invariant parity subspaces and appropriate Poincaré orbit. The formalism yields a constrained dynamics with second class constraints. We work out these constraints in the classical field theory, and show that sensible results are obtained upon quantization once we use the specific algebraic properties of the covariant basis. We study the chiral structure and classify the naively renormalizable self-interactions of the spin-one matter fields.

Our paper is organized as follows. In the next section we introduce the formalism and study the solutions and discrete symmetries at the classical level. The constraints and corresponding dynamics are analyzed in Sec. III. The canonical quantization of the free theory is discussed in Sec. IV. The chiral structure and naively renormalizable interactions are described in Sec. V. We give our conclusions in Sec. VI and close with two appendixes with technical details of the calculations.

II. PARITY-BASED FORMALISM FOR THE $(\mathbf{1},\mathbf{0})\oplus(\mathbf{0},\mathbf{1})$ REPRESENTATION

It was shown in [15] that the parity-based covariant basis for a general $(j,0)\oplus(0,j)$ operator space contains the following:

(1) Two Lorentz scalar operators, the unit matrix of dimension $2(2j+1)$ and the chirality operator χ.
(2) Six operators transforming in the $(1,0)\oplus(0,1)$ representation forming a rank-2 antisymmetric tensor, $M_{\mu\nu}$, whose components are the corresponding generators of the HLG.
(3) A pair of symmetric traceless matrix tensors transforming in the (j,j) representation, with the first one denoted $S^{\mu_1\mu_2\cdots\mu_{2j}}$ and the second one given by $\chi S^{\mu_1\mu_2\cdots\mu_{2j}}$.
(4) A series of tensor matrix operators with the appropriate symmetry properties such that they transform in the $(2,0)\oplus(0,2),(3,0)\oplus(0,3),\ldots,(2j,0)\oplus(0,2j)$ representations of the HLG.

The rest frame parity operator is the time component of the first symmetric traceless tensor, $\Pi=S^{00\ldots0}$. The boost operator can be explicitly constructed due to the simple representation form [in the chiral basis for the $(j,0)\oplus(0,j)$ space] of the boost generator $\mathbb{K}=-i\chi\mathbb{J}=-i\mathrm{diag}(\mathbf{J},-\mathbf{J})$. Using the boost operator, it is possible to construct explicitly the states (*j-spinors* or simply spinors in the following) in an arbitrary frame once we know them in the rest frame. Another important application of the boost operator is the construction of the covariant form of a given operator from its form in the rest frame. In particular, we can calculate the covariant form of the parity operator. A simple calculation yields

$$B(p)\Pi B^{-1}(p)=\frac{S^{\mu_1\mu_2\cdots\mu_{2j}}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_{2j}}}{m^{2j}}. \tag{1}$$

Let us briefly review the application to $j=1/2$. In this case the covariant basis is given by two scalar operators, $\mathbf{1}$ and χ, an antisymmetric tensor, $M_{\mu\nu}$, and two vector operators (the "symmetric" operators of rank $2j=1$)

$$\{\mathbf{1},\chi,S^\mu,\chi S^\mu,M^{\mu\nu}\}. \tag{2}$$

The algorithm outlined in [15] yields

$$S^\mu=\Pi(g^{0\mu}-2iM^{0\mu}). \tag{3}$$

This is the conventional set used in the literature up to a $1/2$ factor in $M_{\mu\nu}$, where the chirality operator is the conventional γ^5 Dirac matrix and $S^\mu=\gamma^\mu$. Boosting the rest frame parity operator we get

$$B(p)\Pi B^{-1}(p)=\frac{S^\mu p_\mu}{m}. \tag{4}$$

Since the rest frame projectors onto states of well-defined parity are

$$\tilde{\mathbb{P}}_\pm=\frac{1}{2}(1\pm\Pi), \tag{5}$$

the condition for well-defined parity in the rest frame is

$$\tilde{\mathbb{P}}_\pm u(0)=u(0), \tag{6}$$

and boosting this equation we get the following condition:

$$(S^\mu p_\mu\mp m)u(p)=0. \tag{7}$$

Transforming to configuration space the positive parity projection yields the Dirac equation

$$(iS^\mu\partial_\mu-m)\psi(x)=0, \tag{8}$$

where $\psi(x)=u(p)e^{-ip\cdot x}$.

A. The structure of the spin-one representation

In the case of spin-one, the basis of matrices with well-defined Lorentz transformation properties is

$$\{\mathbf{1},\chi,S^{\mu\nu},\chi S^{\mu\nu},M^{\mu\nu},C^{\mu\nu\alpha\beta}\}. \tag{9}$$

The symmetric tensor $S^{\mu\nu}$ is given by

$$S^{\mu\nu}=\Pi(g^{\mu\nu}-i(g^{0\mu}M^{0\nu}+g^{0\nu}M^{0\mu})-\{M^{0\mu},M^{0\nu}\}). \tag{10}$$

This tensor is traceless in the Lorentz indices

$$S^\mu{}_\mu=0, \tag{11}$$

which leaves nine independent components transforming in the (1, 1) representation of the HLG. These operators satisfy the following algebraic relations:

$$[S^{\mu\nu},S^{\alpha\beta}]=-i(g^{\mu\alpha}M^{\nu\beta}+g^{\nu\alpha}M^{\mu\beta}+g^{\nu\beta}M^{\mu\alpha}+g^{\mu\beta}M^{\nu\alpha}), \tag{12}$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = \frac{4}{3}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right) - \frac{1}{6}(C^{\mu\alpha\nu\beta} + C^{\mu\beta\nu\alpha}). \tag{13}$$

Finally the tensor transforming in the $(2,0) \oplus (0,2)$ representation is given by

$$C^{\mu\nu\alpha\beta} = 4\{M^{\mu\nu}, M^{\alpha\beta}\} + 2\{M^{\mu\alpha}, M^{\nu\beta}\} - 2\{M^{\mu\beta}, M^{\nu\alpha}\} - 8(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{14}$$

It has the following symmetries:

$$C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\nu\beta\alpha}, \qquad C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu}, \tag{15}$$

the contraction of any pair of indices vanishes and it satisfies the algebraic Bianchi identity

$$C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0. \tag{16}$$

These symmetries leave only 10 independent components out of the 256 components of a general four-index tensor.

The explicit form of the 6×6 matrix tensor operators in Eq. (9) can be found in [15], in the chiral basis of states diagonalizing the chirality operator, χ. For the purposes of this work it is convenient to work in the "parity" basis of states where the particle-antiparticle interpretation is easier. The matrix operators are related by $\mathcal{O} = F\mathcal{O}_\chi F^\dagger$ where F stands for the change of basis matrix

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}. \tag{17}$$

Here we will just need the explicit representation of $S^{\mu\nu}$, which in the parity basis is given by

$$S^{00} \equiv \Pi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad S^{0i} = \begin{pmatrix} 0 & -J^i \\ J^i & 0 \end{pmatrix},$$
$$S^{ij} = \begin{pmatrix} g^{ij} + \{J^i, J^j\} & 0 \\ 0 & -g^{ij} - \{J^i, J^j\} \end{pmatrix}, \tag{18}$$

where $J^i \equiv \frac{1}{2}\epsilon^{ijk}M_{jk}$ are the conventional spin one matrices.

B. The spin-one parity projection

The condition for a state transforming in $(1,0) \oplus (0,1)$ to have well-defined parity is given by Eq. (6), with the corresponding parity operator in this representation space. A similar procedure as the one used for the spin $1/2$ case yields the following equation:

$$(S^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi(x) = 0. \tag{19}$$

This equation was proposed long ago by Weinberg [16] following a different approach and several aspects of this theory have been studied in the literature [17–20]. The main drawback of this equation is that it contains unphysical solutions. In the parity-based covariant construction this is easily understood from the algebraic properties of the symmetric tensor in Eq. (13). Indeed, using this equation it is easy to show that

$$(S^{\mu\nu}\partial_\mu\partial_\nu)^2 \equiv (S(\partial))^2 = \partial^4, \tag{20}$$

and multipliying on the left Eq. (19) with $S(\partial) - m^2$ we obtain

$$(\partial^4 - m^4)\psi(x) = 0. \tag{21}$$

This equation has the conventional plane wave solutions with $p^2 = m^2$ but also solutions belonging to the $p^2 = -m^2$ Poincaré orbit. This problem can be traced back to the naive construction of the projectors in Eq. (5). It can be shown that the corresponding boosted operators

$$\tilde{\mathbb{P}}_\pm(\mathbf{p}) = \frac{1}{2}\left(1 \pm \frac{S(p)}{m^2}\right) \tag{22}$$

cease to be projectors as soon as we go off-shell. The correct parity projectors for the general off-shell case are

$$\mathbb{P}_\pm(\mathbf{p}) = \frac{1}{2}\left(1 \pm \frac{S(p)}{p^2}\right). \tag{23}$$

In addition to finding the right parity projection we must also take care of the projection on the desired Poincaré orbit. To this end we use the simultaneous mass and parity projector

$$\frac{p^2}{m^2}\mathbb{P}_\pm(\mathbf{p}) = \frac{1}{2m^2}(p^2 \pm S(p)). \tag{24}$$

This procedure yields the following equation in coordinate space:

$$(\Sigma^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi(x) = 0, \tag{25}$$

where

$$\Sigma^{\mu\nu} = \frac{1}{2}(g^{\mu\nu} + S^{\mu\nu}). \tag{26}$$

Using Eq. (20) and multiplying Eq. (25) on the left by $\frac{1}{2}(\partial^2 - S(\partial)) - m^2$ it is easy to show that the fields satisfy the Klein-Gordon equation

$$(\partial^2 + m^2)\psi(x) = 0, \tag{27}$$

whose solutions are of the form $\psi(x) = u_r(\mathbf{p})e^{-ip\cdot x}$ where r denotes the particle polarization. The theory for particles with negative parity can be constructed in a similar way; in the following we will focus on the positive parity case.

The formulation of wave equations for spinning particles is an old problem and as far as we know Eq. (25) was firstly considered in [21] following a different approach, including electromagnetic interactions at the classical level. Closely related work was also done in [22,23]. The present approach, based on the parity and Poincaré projections, permits us to identify all quantum numbers from first principles. Also, the algebraic structure of the $(1,0)\oplus(0,1)$ representation space will allow us to work out the constrained dynamics at the classical level and the proper quantization of this theory.

The spinors $u_r(\mathbf{p})$ have six components and satisfy the following equation:

$$(\Sigma^{\mu\nu}p_\mu p_\nu - m^2)u_r(\mathbf{p}) = 0. \tag{28}$$

Equivalently, since a free particle spinor must satisfy the Klein-Gordon condition, the spinor also satisfies

$$(S^{\mu\nu}p_\mu p_\nu - m^2)u_r(\mathbf{p}) = 0. \tag{29}$$

Let us first explore the free particle solutions of Eq. (25). Introducing the explicit form of the $S^{\mu\nu}$ matrices in Eq. (25) we get

$$\begin{pmatrix} \partial^2 + m^2 + (\mathbf{J}\cdot\nabla)^2 & -\mathbf{J}\cdot\nabla\partial_0 \\ \mathbf{J}\cdot\nabla\partial_0 & m^2 - (\mathbf{J}\cdot\nabla)^2 \end{pmatrix}\psi(x) = 0. \tag{30}$$

Writing ψ in terms of the "up" (φ) and "down" (ξ) three-component components we get

$$[\partial^2 + m^2 + (\mathbf{J}\cdot\boldsymbol{\nabla})^2]\varphi = \mathbf{J}\cdot\boldsymbol{\nabla}\partial_0\xi, \tag{31}$$

$$[m^2 - (\mathbf{J}\cdot\boldsymbol{\nabla})^2]\xi = -\mathbf{J}\cdot\boldsymbol{\nabla}\partial_0\varphi. \tag{32}$$

The second line yields the ξ field in terms of the time derivatives of the φ field, i.e. it is a constraint of the theory which leaves only the three complex components of φ required to describe a particle-antiparticle spin-one system as the physical degrees of freedom. The constraint equation reads

$$\xi = -\mathcal{O}^{-1}\mathbf{J}\cdot\boldsymbol{\nabla}\partial_0\varphi, \tag{33}$$

with $\mathcal{O} = m^2 - (\mathbf{J}\cdot\boldsymbol{\nabla})^2$ which is nonsingular.

The true equation of motion for the φ field is obtained multiplying the first equation by $\mathcal{O}$ and using the second one to get

$$([\partial^2 + m^2 + (\mathbf{J}\cdot\boldsymbol{\nabla})^2][m^2 - (\mathbf{J}\cdot\boldsymbol{\nabla})^2] + (\mathbf{J}\cdot\boldsymbol{\nabla})^2\partial_0^2)\varphi = 0. \tag{34}$$

Notice that this equation is second order in time derivatives and seemingly higher order in space derivatives. However, because of the algebraic properties of J_i matrices,

$$(\mathbf{J}\cdot\boldsymbol{\nabla})^3 = (\mathbf{J}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}^2, \tag{35}$$

and it is easy to show that this equation can be rewritten as

$$m^2[\partial^2 + m^2]\varphi = 0, \tag{36}$$

i.e., it is just the Klein-Gordon equation for the three complex degrees of freedom in φ.

In momentum space, writing $\varphi(x) = \phi_r(\mathbf{p})e^{-ip\cdot x}$ we find the following solutions to the equation of motion:

$$u_r(p) = N\begin{pmatrix} \phi_r(\mathbf{p}) \\ -\frac{\mathbf{J}\cdot\mathbf{p}}{E}\phi_r(\mathbf{p}) \end{pmatrix}, \tag{37}$$

where N is an appropriate normalization factor.

Our formalism is designed for massive particles. However, it has a soft $m \to 0$ limit which is worth exploring. In the massless limit, our equation reduces to the system

$$[\partial^2 + (\mathbf{J}\cdot\boldsymbol{\nabla})^2]\varphi - \mathbf{J}\cdot\boldsymbol{\nabla}\,\partial_0\xi = 0 \tag{38}$$

$$\mathbf{J}\cdot\boldsymbol{\nabla}\,\partial_0\varphi - (\mathbf{J}\cdot\boldsymbol{\nabla})^2\xi = 0. \tag{39}$$

Notice that now the operator $(\mathbf{J}\cdot\boldsymbol{\nabla})^2$ accompanying the ξ spinor is not invertible (in momentum space, it is the helicity operator, and it has a zero eigenvalue). In this case we expect to have a gauge invariance which reduces the degrees of freedom contained in the ψ spinor. In the next section, we will work out the Hamiltonian analysis of the constrained dynamics of the theory, and will show that in the massive case all constraints are second class. In the massless limit the characteristic matrix of the constraints has no inverse and first class constraints (gauge symmetries) appear. A straightforward calculation shows that the massless equation of motion (or the Lagrangian in the following section) is invariant under the following gauge transformations [24]:

$$\varphi_i \to \varphi_i + (\mathbf{J}\cdot\boldsymbol{\nabla})_{ij}\varepsilon_j, \tag{40}$$

$$\xi_i \to \xi_i + \partial^0\varepsilon_i + \partial_i f, \tag{41}$$

where $\varepsilon(x)$ is an arbitrary three component spinor, and $f(x)$ is an arbitrary scalar function. This reduces our six degrees of freedom to only two as expected.

Coming back to the massive theory which is the topic of this paper, the presence of nondynamical degrees of freedom in ψ makes clear that the quantization of the theory must proceed through a careful study of the constraints. Before elaborating on this point and in preparation for the particle interpretation necessary for the quantization of the theory, we study the charge conjugation operation.

C. Interacting theory and discrete symmetries

We use the gauge principle for the simplest case of a $U(1)$ gauge group. Gauging Eq. (25) we get

$$[\Sigma^{\mu\nu}(i\partial - qA)_\mu (i\partial - qA)_\nu - m^2]\psi = 0, \tag{42}$$

where q is the $U(1)$ charge of the particle. Complex conjugating Eq. (42) and multiplying on the left by a matrix in the $(1,0)\oplus(0,1)$ representation space denoted by Γ we obtain

$$[\Gamma(\Sigma^{\mu\nu})^*\Gamma^{-1}(i\partial + qA)_\mu (i\partial + qA)_\nu - m^2]\psi^c = 0, \tag{43}$$

with

$$\psi^c \equiv \mathcal{C}\psi = \Gamma\psi^*. \tag{44}$$

If we require ψ^c to satisfy the same equation as ψ but with the opposite $U(1)$ charge, $-q$, the symmetric tensor S must satisfy the following relation:

$$\Gamma(S^{\mu\nu})^*\Gamma^{-1} = S^{\mu\nu}. \tag{45}$$

The construction of the matrix Γ satisfying Eq. (45) can be done from first principles and we just quote the final result. Up to a phase this matrix is given by

$$\Gamma = \begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix}, \tag{46}$$

where U stands for the time reversal operator in the $(1,0)\oplus(0,1)$ representation space:

$$U = e^{-i\pi J_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{47}$$

A crucial difference with the Dirac theory is that for spin-one matter fields the charge conjugation operator commutes with the rest frame parity operator,

$$[\mathcal{C}, \Pi] = 0. \tag{48}$$

This relation defines the particle-antiparticle structure in the corresponding quantum field theory. In the rest frame, the "down" component of the spinors in Eq. (37) corresponds to negative parity as in the Dirac case. However, for spin-one matter particles, it is not connected with the antiparticle solutions. Indeed, as we can see from the explicit form of the spinors in Eq. (37), the "down" component vanishes in the rest frame, and for an arbitrary frame it is fixed by the kinematics.

The charge conjugated spinor, given by

$$u_r^c(\mathbf{p}) = \Gamma u_r^*(\mathbf{p}), \tag{49}$$

also satisfies the equation

$$(\Sigma^{\mu\nu} p_\mu p_\nu - m^2) u_r^c(\mathbf{p}) = 0. \tag{50}$$

The adjoint spinors obey the adjoint equations

$$\bar{u}_r(\mathbf{p})(S^{\mu\alpha} p_\mu p_\alpha - m^2) = 0, \tag{51}$$

$$\bar{u}_r^c(\mathbf{p})(S^{\mu\alpha} p_\mu p_\alpha - m^2) = 0. \tag{52}$$

These spinors are normalized according to

$$\bar{u}_r^c(\mathbf{p}) u_s^c(\mathbf{p}) = \bar{u}_r(\mathbf{p}) u_s(\mathbf{p}) = \delta_{rs}. \tag{53}$$

The corresponding completeness relation is

$$\sum_r u_{ra}(\mathbf{p})\bar{u}_{rb}(\mathbf{p}) = \sum_r u_{ra}^c(\mathbf{p})\bar{u}_{rb}^c(\mathbf{p}) = \left(\frac{S(\mathbf{p}) + m^2}{2m^2}\right)_{ab}. \tag{54}$$

Now, the minimally coupled equation, Eq. (42), written in terms of the covariant derivative

$$D_\mu\psi = \partial_\mu\psi + iqA_\mu\psi, \tag{55}$$

and the parity components $\{\varphi, \xi\}$, is

$$\left(D^2 + m^2 + \frac{1}{2}D_i\{J_i, J_j\}D_j\right)\varphi - \frac{1}{2}J_i\{D_i, D_0\}\xi = 0, \tag{56}$$

$$\frac{1}{2}J_i\{D_i, D_0\}\varphi + \left(m^2 - \frac{1}{2}D_i\{J_i, J_j\}D_j\right)\xi = 0. \tag{57}$$

Again, Eq. (57) does not involve the time derivative of ξ and is therefore still a constraint. While the manipulation of this equation is complicated by the presence of the noncommuting differential operators D^μ, we can check in a calculation similar to the one leading to Eq. (34) that the true equation of motion has the form

$$\left[\left(D^2 + m^2 + \frac{1}{2} D_i \{J_i, J_j\} D_j\right) + \frac{1}{2} J_i \{D_i, D_0\} \mathcal{O}_c^{-1} \frac{1}{2} J_j \{D_j, D_0\}\right] \varphi = 0. \quad (58)$$

The operator $\mathcal{O}_c^{-1} = [m^2 - \frac{1}{2} D_i \{J_i, J_j\} D_j]^{-1}$ involves only the spatial components of D^μ, and therefore this is an equation containing only second time derivatives of the φ components. Therefore, the counting of degrees of freedom is unaltered from the free case.

III. CLASSICAL FIELD THEORY AND CONSTRAINTS

The equation of motion can be derived from the following Hermitian Lagrangian:

$$\mathcal{L} = \partial_\mu \bar{\Psi} \Sigma^{\mu\nu} \partial_\nu \Psi - m^2 \bar{\Psi} \Psi \quad (59)$$

where $\bar{\Psi} = \Psi^\dagger \Pi$. In order to exhibit the dynamical content we write the theory in terms of the "up" (φ) and "down" (ξ) components of the field and the corresponding conjugate momenta

$$\Psi = \begin{pmatrix} \varphi \\ \xi \end{pmatrix}, \qquad \zeta = \begin{pmatrix} \pi \\ \tau \end{pmatrix}. \quad (60)$$

In terms of these components the Lagrangian reads

$$\begin{aligned} \mathcal{L} = {} & \partial_0 \varphi^\dagger \partial_0 \varphi + \partial_i \varphi^\dagger \partial^i \varphi - \frac{1}{2} \partial_0 \varphi^\dagger J^i \partial_i \xi - \frac{1}{2} \partial_0 \xi^\dagger J^i \partial_i \varphi \\ & - \frac{1}{2} \partial_i \varphi^\dagger J^i \partial_0 \xi - \frac{1}{2} \partial_i \xi^\dagger J^i \partial_0 \varphi + \frac{1}{2} \partial_i \varphi^\dagger \{J^i, J^j\} \partial_j \varphi \\ & + \frac{1}{2} \partial_i \xi^\dagger \{J^i, J^j\} \partial_j \xi - m^2 (\varphi^\dagger \varphi - \xi^\dagger \xi). \end{aligned} \quad (61)$$

Notice that this Lagrangian does not contain second time derivatives in the "down" component ξ. The canonical conjugated momenta are given as

$$\begin{aligned} \pi_a &= \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi_a)} = \partial_0 \varphi_a^\dagger - \frac{1}{2} (\partial_i \xi^\dagger J^i)_a, \\ \pi_a^\dagger &= \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi_a^\dagger)} = \partial_0 \varphi_a - \frac{1}{2} (J^i \partial_i \xi)_a, \end{aligned} \quad (62)$$

$$\begin{aligned} \tau_a &= \frac{\delta \mathcal{L}}{\delta(\partial_0 \xi_a)} = -\frac{1}{2} (\partial_i \varphi^\dagger J^i)_a, \\ \tau_a^\dagger &= \frac{\delta \mathcal{L}}{\delta(\partial_0 \xi_a^\dagger)} = -\frac{1}{2} (J^i \partial_i \varphi)_a. \end{aligned} \quad (63)$$

Clearly, Eqs. (63) are (primary) constraints on the variables of the system

$$\rho_a = \tau_a + \frac{1}{2} (\partial_i \varphi^\dagger J^i)_a = 0, \qquad \rho_a^\dagger = \tau_a^\dagger + \frac{1}{2} (J^i \partial_i \varphi)_a = 0. \quad (64)$$

The Hamiltonian density is

$$\mathcal{H} = \pi_a \partial_0 \varphi_a + \partial_0 \varphi_a^\dagger \pi_a^\dagger + \tau_a \partial_0 \xi_a + \partial_0 \xi_a^\dagger \tau_a^\dagger - \mathcal{L}. \quad (65)$$

A straightforward calculation yields

$$\begin{aligned} \mathcal{H} = {} & \pi_a \pi_a^\dagger + \frac{1}{2} \pi_a (J^i \partial_i \xi)_a + \frac{1}{2} (\partial_i \xi^\dagger J^i)_a \pi_a^\dagger \\ & + \frac{1}{4} (\partial_i \xi^\dagger J^i)_a (J^j \partial_j \xi)_a - \partial_i \varphi_a^\dagger \partial^i \varphi_a \\ & - \frac{1}{2} \partial_i \varphi_a^\dagger \{J^i, J^j\}_{ab} \partial_j \varphi_b - \frac{1}{2} \partial_i \xi_a^\dagger \{J^i, J^j\}_{ab} \partial_j \xi_a \\ & + m^2 (\varphi_a^\dagger \varphi_a - \xi_a^\dagger \xi_a). \end{aligned} \quad (66)$$

Notice that this Hamiltonian density does not contain the τ momenta nor time derivatives of the "down" spinor. According to Dirac classic lectures [25] the time evolution of the system is given by the modified Hamiltonian $\mathcal{H}^*$ given by

$$H^* = \int d^3x \mathcal{H}^* \quad (67)$$

with the modified Hamiltonian density

$$\mathcal{H}^* = \mathcal{H} + \lambda_a \rho_a + \lambda_a^\dagger \rho_a^\dagger, \quad (68)$$

where λ_a and $\lambda_a^\dagger$ are the Lagrange multipliers.

The Hamilton equations read

$$\partial_0 \varphi_a = \frac{\delta H^*}{\delta \pi_a} = \pi_a^\dagger + \frac{1}{2} (J^i \partial_i \xi)_a, \quad (69)$$

$$\begin{aligned} \partial_0 \pi_a = -\frac{\delta H^*}{\delta \varphi_a} = {} & -\partial_i \partial^i \varphi_a^\dagger - \partial_j \partial_i (\varphi^\dagger J^i J^j)_a - m^2 \varphi_a^\dagger \\ & + \frac{1}{2} (\partial_i \lambda^\dagger J^i)_a, \end{aligned} \quad (70)$$

$$\partial_0 \xi_a = \frac{\delta H^*}{\delta \tau_a} = \lambda_a, \quad (71)$$

$$\partial_0 \tau_a = -\frac{\delta H^*}{\delta \xi_a} = \frac{1}{2} \partial_i (\pi J^i)_a - \frac{3}{4} (\partial_j \partial_i \xi^\dagger J^i J^j)_a + m^2 \xi_a^\dagger. \quad (72)$$

The corresponding equations for the adjoint phase space variables, not shown here, are given by the adjoint of these equations.

The time evolution of any observable can be written in terms of the Poisson brackets as

$$\dot{A} = \{A, H^*\}. \tag{73}$$

In our case the Poisson bracket is given by

$$\{A(\mathbf{x}), B(\mathbf{y})\} = \int d^3\mathbf{x}' \sum_a \left[\frac{\delta A(\mathbf{x})}{\delta \Psi_a(\mathbf{x}')}\frac{\delta B(\mathbf{y})}{\delta \zeta_a(\mathbf{x}')} - \frac{\delta B(\mathbf{y})}{\delta \Psi_a(\mathbf{x}')}\frac{\delta A(\mathbf{x})}{\delta \zeta_a(\mathbf{x}')}\right] \tag{74}$$

where the sum is over all the field components and their conjugate momenta.

A straightforward calculation yields

$$\{\varphi_a(\mathbf{x}), \pi_b(\mathbf{y})\} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}), \quad \{\xi_a(\mathbf{x}), \tau_b(\mathbf{y})\} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}), \tag{75}$$

and the corresponding adjoint relations.

The dynamics generated by H^* must preserve the constraints hence the following relations must hold:

$$\partial_0\rho_a = \{\rho_a, H^*\} = 0, \qquad \partial_0\rho^\dagger_a = \{\rho^\dagger_c, H^*\} = 0. \tag{76}$$

In our system this produces new (secondary) constraints

$$\kappa_a = \partial_i(\pi J^i)_a - \frac{1}{2}(\partial_j\partial_i\xi^\dagger J^i J^j)_a + m^2\xi^\dagger_a = 0, \tag{77}$$

$$\kappa^\dagger_a = \partial_i(J^i\pi^\dagger)_a - \frac{1}{2}(\partial_j\partial_i J^i J^j \xi)_a + m^2\xi_a = 0. \tag{78}$$

Requiring that the new constraints be preserved by the dynamics we get

$$\lambda^\dagger_a - \partial_i(\varphi^\dagger J^i)_a = 0, \qquad \lambda_a - \partial_i(J^i\varphi)_a = 0. \tag{79}$$

These relations just define the Lagrange multipliers but do not generate new constraints.

In total, we have 24 degrees of freedom in the Hamiltonian description, 12 coming from the $\{\varphi^\dagger, \varphi\}$ fields and their associated momenta $\{\pi^\dagger, \pi\}$, and another 12 from the $\{\xi^\dagger, \xi\}$ fields and their momenta $\{\tau^\dagger, \tau\}$. On the other hand, we have the set of 12 constraints $\{f_a\} = \{\rho_a, \rho^\dagger_a, \chi_a, \chi^\dagger_a\}$. This leaves us with 12 degrees of freedom in phase space, that correspond to 3 complex degrees of freedom obeying a second-degree equation of motion, as expected for a particle-antiparticle field with 3 degrees of freedom.

Following the procedure outlined by Dirac in Ref. [25] we calculate now the matrix of the Poisson brackets of the constraints

$$\Delta_{ab}(\mathbf{x}, \mathbf{y}) = \{f_a(\mathbf{x}), f_b(\mathbf{y})\}. \tag{80}$$

A straightforward calculation yields the following block matrix form:

$$\Delta(\mathbf{x}, \mathbf{y}) = m^2\delta^3(\mathbf{x}-\mathbf{y}) \begin{pmatrix} 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & \mathbf{-1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix}. \tag{81}$$

This is a nonsingular matrix thus all the obtained constraints are *second class constraints*. The inverse of this matrix is given by

$$\Delta^{-1}(\mathbf{y}, \mathbf{z}) = \frac{1}{m^2}\delta^3(\mathbf{y}-\mathbf{z}) \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & 0 & 0 \\ \mathbf{-1} & 0 & 0 & 0 \end{pmatrix}. \tag{82}$$

To proceed with the quantization we need the Dirac bracket, defined as

$$\{A, B\}_D = \{A, B\} - \int d^3\mathbf{z} d^3\mathbf{z}' \{A, f_a(\mathbf{z})\}\Delta^{-1}_{ab}(\mathbf{z}, \mathbf{z}')\{f_b(\mathbf{z}'), B\}. \tag{83}$$

For the canonical variables the inverse matrix in Eq. (82) simplifies the calculation. For example

$$\begin{aligned} \{\varphi_a(\mathbf{x}), \pi_b(\mathbf{y})\}_D &= \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) - \frac{1}{m^2}\int d^3\mathbf{z}\{\varphi_a(\mathbf{x}), \rho_c(\mathbf{z})\}\{\kappa^\dagger_c(\mathbf{z}), \pi_b(\mathbf{y})\} \\ &- \frac{1}{m^2}\int d^3\mathbf{z}\{\varphi_a(\mathbf{x}), \rho^\dagger_c(\mathbf{z})\}\{\kappa_c(\mathbf{z}), \pi_b(\mathbf{y})\} \\ &+ \frac{1}{m^2}\int d^3\mathbf{z}\{\varphi_a(\mathbf{x}), \kappa_c(\mathbf{z})\}\{\rho^\dagger_c(\mathbf{z}), \pi_b(\mathbf{y})\} \\ &+ \frac{1}{m^2}\int d^3\mathbf{z}\{\varphi_a(\mathbf{x}), \kappa^\dagger_c(\mathbf{z})\}\{\rho_c(\mathbf{z}), \pi_b(\mathbf{y})\}, \end{aligned}$$

and similar expressions hold for the remaining pairs of conjugate variables. A straightforward calculation yields

$$\{\varphi_a(\mathbf{x}), \pi_b(\mathbf{y})\}_D = \left[1 - \frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}\right]_{ab}\delta^3(\mathbf{x}-\mathbf{y}), \tag{84}$$

$$\{\varphi_a(\mathbf{x}), \tau_b(\mathbf{y})\}_D = 0, \tag{85}$$

$$\{\xi_a(\mathbf{x}), \pi_b(\mathbf{y})\}_D = 0, \tag{86}$$

$$\{\xi_a(\mathbf{x}), \tau_b(\mathbf{y})\}_D = \frac{(\mathbf{J}\cdot\nabla)^2_{ab}}{2m^2}\delta^3(\mathbf{x}-\mathbf{y}). \tag{87}$$

We can rewrite these relations in compact spinor notation

$$\{\Psi_a(\mathbf{x}),\zeta_b(\mathbf{y})\}_D = \left[\Sigma^{00} - \frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}S^{00}\right]_{ab}\delta^3(\mathbf{x}-\mathbf{y}). \quad (88)$$

The quantization of the theory must be done replacing the Dirac bracket by the quantum commutator $-i[,]$, and we expect the quantum commutator of the canonical conjugate fields to be

$$[\Psi_a(\mathbf{x}),\zeta_b(\mathbf{y})] = i\left[\Sigma^{00} - \frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}S^{00}\right]_{ab}\delta^3(\mathbf{x}-\mathbf{y}). \quad (89)$$

To end this section we would like to remark that the coupling to an external $U(1)$ field can spoil the quantization procedure rendering the commutation relations of the canonical variables ill defined for some values of the external field [26]. We do not expect this to be the case here as pointed by the coupled true equation of motion (58) but in order to ensure this, we performed the analogous calculations for the coupled theory finding the very same canonical commutation relations. The calculations are rather long, and we defer the details to Appendix A.

IV. CANONICAL QUANTIZATION OF SPIN 1 MATTER FIELDS

Under an infinitesimal transformation

$$\Psi \to \Psi' = \Psi + \delta\Psi \quad (90)$$

the Lagrangian changes as

$$\delta\mathcal{L} = \partial_\mu[\partial_\alpha\bar{\Psi}\Sigma^{\alpha\mu}\delta\Psi + \delta\bar{\Psi}\Sigma^{\mu\alpha}\partial_\alpha\Psi]. \quad (91)$$

Invariance under a given transformation yields conserved currents. First, our Lagrangian is invariant under the global $U(1)$ transformations $\Psi' = e^{iq\lambda}\Psi$. The corresponding conserved current is given by

$$J^\alpha = iq((\partial_\mu\bar{\Psi})\Sigma^{\mu\alpha}\Psi - \bar{\Psi}\Sigma^{\alpha\nu}(\partial_\nu\Psi)). \quad (92)$$

Invariance under space-time translations yields the following stress tensor:

$$T^\mu{}_\nu = \partial_\nu\bar{\Psi}\Sigma^{\mu\alpha}\partial_\alpha\Psi + \partial_\alpha\bar{\Psi}\Sigma^{\alpha\mu}\partial_\nu\Psi - \eta^\mu_\nu(\partial_\alpha\bar{\Psi}\Sigma^{\alpha\beta}\partial_\beta\Psi - m^2\bar{\Psi}\Psi). \quad (93)$$

The angular momentum density is similarly obtained as

$$\mathcal{M}^{0ij} = T^{0j}x^i - T^{0i}x^j + i(\bar{\Psi}\epsilon_{ijk}J_k\Sigma^{0\nu}\partial_\nu\Psi) - i(\partial_\mu\bar{\Psi}\Sigma^{\mu0}\epsilon_{ijk}J_k\Psi). \quad (94)$$

The field and its adjoint are expanded in the conventional Fourier series

$$\Psi(x) = \sum_{\mathbf{p},r}\alpha(\mathbf{p})[c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^+(\mathbf{p})u_r^c(\mathbf{p})e^{ipx}], \quad (95)$$

$$\bar{\Psi}(x) = \sum_{\mathbf{p},r}\alpha(\mathbf{p})[c_r^+(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx} + d_r(\mathbf{p})\bar{u}_r^c(\mathbf{p})e^{-ipx}], \quad (96)$$

where $\alpha(\mathbf{p}) = 1/\sqrt{2E(\mathbf{p})V}$ and r denotes the polarization of the one-particle states. The particle (antiparticle) creation (annihilation) operators satisfy the following commutation relations:

$$[c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')] = \delta_{rs}\delta_{\mathbf{p}\mathbf{p}'}, \qquad [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')] = \delta_{rs}\delta_{\mathbf{p}\mathbf{p}'}. \quad (97)$$

A. Commutation relations

The conjugated momenta are given by

$$\bar{\zeta}_d = \frac{\partial\mathcal{L}}{\partial\bar{\Psi}_{d,0}} = \Sigma^{0\mu}_{da}(\partial_\mu\Psi)_a, \quad (98)$$

$$\zeta_d = \frac{\partial\mathcal{L}}{\partial\Psi_{d,0}} = (\partial_\mu\bar{\Psi})_a\Sigma^{\mu0}_{ad}. \quad (99)$$

The commutators of the fields with the canonical conjugated momenta are given by

$$[\zeta_d,\Psi_b] = (\partial_\mu\bar{\Psi})_a\Sigma^{\mu0}_{ad}\Psi_b - \Psi_b(\partial_\mu\bar{\Psi})_a\Sigma^{\mu0}_{ad}, \quad (100)$$

$$[\bar{\zeta}_d,\bar{\Psi}_b] = \Sigma^{0\mu}_{da}(\partial_\mu\Psi)_a\bar{\Psi}_b - \bar{\Psi}_b\Sigma^{0\mu}_{da}(\partial_\mu\Psi)_a. \quad (101)$$

Inserting the Fourier series in Eq. (100) we get

$$[\zeta_d(x_1),\Psi_b(x_2)] = \sum_{\mathbf{p},r}\frac{-ip_\mu}{2Vp_0}[\bar{u}_{ra}(\mathbf{p})u_{rb}(\mathbf{p})\Sigma^{\mu0}_{ad}e^{ip(x_1-x_2)} + \bar{u}^c_{ra}(\mathbf{p})u^c_{rb}(\mathbf{p})\Sigma^{\mu0}_{ad}e^{ip(x_2-x_1)}]. \quad (102)$$

For equal time $x_1^0 = x_2^0 = 0$, using Eq. (54) we get

$$[\zeta_d(\mathbf{x}_1),\Psi_b(\mathbf{x}_2)]_{x^0_{1,2}=0} = -i\sum_{\mathbf{p}}\frac{p_\mu}{2Vp_0}\left(\frac{S(\mathbf{p})+m^2}{2m^2}\right)_{ba}\Sigma^{\mu0}_{ad}e^{ip_i(x_1^i-x_2^i)} - i\sum_{\mathbf{p}}\frac{p_\mu}{2Vp_0}\left(\frac{S(\mathbf{p})+m^2}{2m^2}\right)_{ba}\Sigma^{\mu0}_{ad}e^{-ip_i(x_1^i-x_2^i)}. \quad (103)$$

Changing $p_i \to -p_i$ in the second term we get

$$[\zeta_d(\mathbf{x}_1),\Psi_b(\mathbf{x}_2)]_{x^0_{1,2}=0} = -i\sum_{\mathbf{p}}\frac{e^{ip_i(x_1^i-x_2^i)}}{V} \times\left(\frac{\Sigma^{00}p_0p_0+(2\Sigma^{0i}\Sigma^{0j}+\Sigma^{ij}\Sigma^{00})p_ip_j}{m^2}\right)_{bd}. \quad (104)$$

This equation can be further reduced using the algebra satisfied by S. Indeed, using Eq. (13) it is possible to show that

$$(2\Sigma^{0i}\Sigma^{0j}+\Sigma^{ij}\Sigma^{00})p_ip_j=\frac{1}{2}(\Sigma^{ij}p_ip_j-\mathbf{p}^2\Sigma^{00}). \quad (105)$$

Using this relation we can further reduce our commutator to

$$[\zeta_d(\mathbf{x}_1),\Psi_b(\mathbf{x}_2)]_{x^0_{1,2}=0} = -i\sum_{\mathbf{p}}\frac{e^{ip_i(x_1^i-x_2^i)}}{V} \times\left(\Sigma^{00}+\frac{(S^{ij}-g^{ij}S^{00})p_ip_j}{4m^2}\right)_{bd}. \quad (106)$$

Finally, using the explicit representation of the $S^{\mu\nu}$ matrices it can be shown that

$$(S^{ij}-g^{ij}S^{00})p_ip_j=2(\mathbf{J}\cdot\mathbf{p})^2S^{00}, \quad (107)$$

and putting it all together we obtain

$$[\zeta_d(\mathbf{x}_1),\Psi_b(\mathbf{x}_2)]_{x^0_{1,2}=0} = -i\left(\Sigma^{00}-\frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}S^{00}\right)_{bd} \times\delta(\mathbf{x}_1-\mathbf{x}_2). \quad (108)$$

A similar calculation yields

$$[\bar{\zeta}_d(\mathbf{x}_1),\bar{\Psi}_b(\mathbf{x}_2)]_{x^0_{1,2}=0} = -i\left(\Sigma^{00}-\frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}S^{00}\right)_{bd} \times\delta(\mathbf{x}_1-\mathbf{x}_2). \quad (109)$$

This is exactly the result expected from our classical analysis of the constrained dynamics in the previous section summarized in Eq. (88).

B. Energy and momentum of the field

The energy density of the field is defined as

$$\mathcal{H}=T^{00}=\partial_0\bar{\Psi}\Sigma^{00}\partial_0\Psi-\partial_i\bar{\Psi}\Sigma^{ij}\partial_j\Psi+m^2\bar{\Psi}\Psi. \quad (110)$$

After a straightforward algebra, integrating the normal product of T^{00} we get the following expression for the total energy of the field:

$$\begin{aligned}H=(2\pi)^3\sum_{p,r}\sum_{r'}\alpha(\mathbf{p})^2[&c_r^+(\mathbf{p})c_{r'}(\mathbf{p})\bar{u}_r(\mathbf{p})(\Sigma^{00}p_0p_0-\Sigma^{ij}p_ip_j+m^2)u_{r'}(\mathbf{p}')\\&+d_r(\mathbf{p})c_{r'}(-\mathbf{p})\bar{u}^c_r(\mathbf{p})(-\Sigma^{00}p_0p_0-\Sigma^{ij}p_ip_j+m^2)u_{r'}(-\mathbf{p})e^{-2ip_0x^0}\\&+c_r^+(\mathbf{p})d^+_{r'}(-\mathbf{p})\bar{u}_r(\mathbf{p})(-\Sigma^{00}p_0p_0-\Sigma^{ij}p_ip_j+m^2)u^c_{r'}(-\mathbf{p})e^{2ip_0x^0}\\&+d_r(\mathbf{p})d^+_{r'}(-\mathbf{p})\bar{u}^c_r(\mathbf{p})(\Sigma^{00}p_0p_0-\Sigma^{ij}p_ip_j+m^2)u^c_{r'}(-\mathbf{p})].\end{aligned} \quad (111)$$

Next, we use

$$-\Sigma^{00}p_0p_0-\Sigma^{ij}p_ip_j=2\Sigma^{0i}p_0p_i-\Sigma(\mathbf{p}), \quad (112)$$

and the equations of motion in Eqs. (28), (50), (51), (52) to obtain

$$\begin{aligned}H=(2\pi)^3\sum_{p,r}\sum_{r'}\alpha(\mathbf{p})^2[&c_r^+(\mathbf{p})c_{r'}(\mathbf{p})\bar{u}_r(\mathbf{p})(2\Sigma^{0\mu}p_0p_\mu)u_{r'}(\mathbf{p})\\&+d_r(\mathbf{p})c_{r'}(-\mathbf{p})\bar{u}^c_r(\mathbf{p})(2\Sigma^{0i}p_0p_i)u_{r'}(-\mathbf{p})e^{-2ip_0x^0}\\&+c_r^+(\mathbf{p})d^+_{r'}(-\mathbf{p})\bar{u}_r(\mathbf{p})(2\Sigma^{0i}p_0p_i)u^c_{r'}(-\mathbf{p})e^{2ip_0x^0}\\&+d^+_{r'}(\mathbf{p})d_r(\mathbf{p})\bar{u}^c_r(\mathbf{p})(2\Sigma^{0\mu}p_0p_\mu)u^c_{r'}(\mathbf{p})].\end{aligned} \quad (113)$$

With the aid of Eqs. (12), (13) it is possible to show that

$$\bar{u}_r(\mathbf{p})(\Sigma^{0\mu}p_\mu)u_s(\mathbf{p})=p^0\delta_{rs}, \quad (114)$$

$$\bar{u}_r(\mathbf{p})(\Sigma^{0i}p_i)u^c_{r'}(-\mathbf{p})=\bar{u}^c_r(\mathbf{p})(\Sigma^{0i}p_0p_i)u_{r'}(-\mathbf{p})=0. \quad (115)$$

Using these results we obtain the expected total energy of the field:

$$H = \frac{(2\pi)^3}{V}\sum_{p,r} p_0[c_r^+(\mathbf{p})c_r(\mathbf{p}) + d_r^+(\mathbf{p})d_r(\mathbf{p})]. \quad (116)$$

The total momentum of the field is

$$P_i = \int N\{T_i^0\}d^3x$$
$$= \int N\{\partial_i\bar{\Psi}\Sigma^{0\nu}\partial_\nu\Psi + \partial_\mu\bar{\Psi}\Sigma^{\mu 0}\partial_i\Psi\}d^3x. \quad (117)$$

Inserting the Fourier expansion of the fields in Eqs. (95), (96) a little algebra yields

$$P_i = (2\pi)^3\sum_{p,r}\sum_{r'}\alpha(\mathbf{p})^2$$
$$\times [c_r^+(\mathbf{p})c_{r'}(\mathbf{p})\bar{u}_r(\mathbf{p})\Sigma^{0\nu}p_\nu u_{r'}(\mathbf{p})(2p_i)$$
$$- c_r^+(\mathbf{p})d_{r'}^+(-\mathbf{p})\bar{u}_r(\mathbf{p})\Sigma^{0j}p_j u_{r'}^c(-\mathbf{p})(-2p_i)e^{2ip^0x_0}$$
$$- d_r(\mathbf{p})c_{r'}(-\mathbf{p})\bar{u}_r^c(\mathbf{p})\Sigma^{0j}p_j u_{r'}(-\mathbf{p})(-2p_i)e^{-2ip^0x_0}$$
$$+ d_r^+(\mathbf{p})d_{r'}(\mathbf{p})\bar{u}_r^c(\mathbf{p})\Sigma^{0\nu}p_\nu u_{r'}^c(\mathbf{p})(2p_i)]. \quad (118)$$

The terms appearing here are similar to the previous calculation and we simply give the final result

$$P_i = \frac{(2\pi)^3}{V}\sum_{p,r} p_i[c_r^+(\mathbf{p})c_r(\mathbf{p}) + d_r^+(\mathbf{p})d_r(\mathbf{p})]. \quad (119)$$

C. U(1) charge

The total current of the field is given by

$$j^\alpha = \int d^3xN(J^\alpha)$$
$$= \int d^3xN(iq((\partial_\mu\bar{\Psi})\Sigma^{\mu\alpha}\Psi - \bar{\Psi}\Sigma^{\alpha\nu}(\partial_\nu\Psi))), \quad (120)$$

which after substitution of the Fourier expansion of the fields and some manipulation yields

$$j^\alpha = q\int d^3xN\sum_{\mathbf{p},r}\sum_{\mathbf{p}',r} i^2\alpha(\mathbf{p}')\alpha(\mathbf{p})$$
$$\times ([p'_\mu + p_\mu]c_{r'}^+(\mathbf{p}')c_r(\mathbf{p})\bar{u}_{r'}(\mathbf{p}')\Sigma^{\mu\alpha}u_r(\mathbf{p})e^{i(p'-p)x}$$
$$+ [p'_\mu - p_\mu]c_{r'}^+(\mathbf{p}')d_r^+(\mathbf{p})u_r(\mathbf{p}')\Sigma^{\mu\alpha}u_r^c(\mathbf{p})e^{i(p'+p)x}$$
$$- [p'_\mu - p_\mu]d_{r'}(\mathbf{p}')c_r(\mathbf{p})\bar{u}_{r'}^c(\mathbf{p}')\Sigma^{\mu\alpha}u_r(\mathbf{p})e^{-i(p'+p)x}$$
$$- [p'_\mu + p_\mu]d_{r'}(\mathbf{p}')d_r^+(\mathbf{p})\bar{u}_{r'}^c(\mathbf{p}')\Sigma'^{\mu\alpha}u_r^c(\mathbf{p})e^{i(p-p')x}). \quad (121)$$

For $\alpha = 0$ we get the charge associated with the $U(1)$ invariance as

$$Q = \sum_{\mathbf{p},r}\sum_{r'}\frac{i^2q(2\pi)^3}{2Vp_0}2p_\mu c_{r'}^+(\mathbf{p})c_r(\mathbf{p})\bar{u_{r'}}(\mathbf{p})\Sigma^{\mu 0}u_r(\mathbf{p})$$
$$-\sum_{\mathbf{p},r}\sum_{r'}\frac{i^2q(2\pi)^3}{2Vp_0}2p_\mu d_r^+(\mathbf{p})d_{r'}(\mathbf{p})\bar{u}_{r'}^c(\mathbf{p})\Sigma^{\mu 0}u_r^c(\mathbf{p}), \quad (122)$$

and using again Eq. (114) we get

$$Q = \frac{(2\pi)^3}{V}q\sum_{\mathbf{p},r}(-c_r^+(\mathbf{p})c_r(\mathbf{p}) + d_r^+(\mathbf{p})d_r(\mathbf{p})). \quad (123)$$

D. Propagator

The propagator is the expectation value of the time-ordered product of the fields

$$i\Gamma_F(x-y)_{ab} = \langle 0|T(\Psi_a(x)\bar{\Psi}_b(y))|0\rangle. \quad (124)$$

Substituting the Fourier expansion of the fields we get

$$i\Gamma_F(x-y)_{ab} = \begin{cases} \sum_{\mathbf{p}}\frac{1}{2V\omega_{\mathbf{p}}}\left(\frac{S(\mathbf{p})+m^2}{2m^2}\right)_{ab}e^{-ip(x-y)} & x_0 > y_0 \\ \sum_{\mathbf{p}}\frac{1}{2V\omega_{\mathbf{p}}}\left(\frac{S(\mathbf{p})+m^2}{2m^2}\right)_{ab}e^{ip(x-y)} & y_0 > x_0 \end{cases}, \quad (125)$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and we used the polarization sum relations in Eq. (54). We can rewrite this equation with the help of the step function and in the continuum limit as

$$i\Gamma_F(x-y)$$
$$= \theta(x_0 - y_0)\int\frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}}\left(\frac{S(\mathbf{p}) + m^2}{2m^2}\right)e^{-ip(x-y)}$$
$$+ \theta(y_0 - x_0)\int\frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}}\left(\frac{S(\mathbf{p}) + m^2}{2m^2}\right)e^{ip(x-y)}. \quad (126)$$

Writing $\Gamma_F(x-y)$ in a four-dimensional integral representation we expect to connect with the classical Green's function, $G(x-y)$, obtained solving the wave equation in Eq. (25) in the presence of sources. The Fourier transform of the Green's function, $\tilde{G}(p)$, satisfies

$$(\Sigma^{\mu\nu}p_\mu p_\nu - m^2)\tilde{G}(p) = I. \quad (127)$$

Using

$$[S(p)]^2 = p^4, \quad (128)$$

it is easy to show that

$$\tilde{G}(p) = \frac{\Delta(p)}{p^2 - m^2 + i\epsilon} \quad (129)$$

where

$$\Delta(p) = \frac{S(p) - p^2 + 2m^2}{2m^2}. \quad (130)$$

Notice that we are distinguishing $S(\mathbf{p})$ from $S(p)$ here. We use $S(\mathbf{p})$ when the momentum p is on-shell whereas if p is off-shell as in Eq. (130) we use $S(p)$. Also notice that on-shell

$$\Delta(p)|_{p^2=m^2} = \frac{S(\mathbf{p}) + m^2}{2m^2} = \sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}). \quad (131)$$

This result suggests that the appropriate four-dimensional integral representation of the two-point QFT Green's function in Eq. (126) is not just the direct generalization of the polarization sum; rather, it can incorporate terms proportional to $p^2 - m^2$.

In coordinates space the classical Green's function reads

$$iG(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{\Delta(p)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \quad (132)$$

In order to connect with (126) it is convenient to write the above equation as

$$iG(x-y) = i\int \frac{d^3\mathbf{p}}{(2\pi)^3} I(\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \quad (133)$$

where $I(\mathbf{p})$ is the integral with respect to $p^0 = \omega$

$$I(\mathbf{p}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta(\omega, \mathbf{p}) e^{-i\omega(x-y)^0}}{\omega^2 - \mathbf{p}^2 - m^2 + i\epsilon} d\omega. \quad (134)$$

This integral can be solved by the residue theorem in the conventional way. However, since we are working with an unconventional extension off-shell of the polarization sum and at the end we obtain additional terms, we give some details of the calculation in Appendix B. The final result for the relation of the two-point correlation function in Eq. (126) and the integral in Eq. (132) is

$$i\Gamma_F(x-y) = iG(x-y) + \frac{S^{00} - 1}{2m^2}\delta^4(x-y). \quad (135)$$

In conclusion, the two-point function in Eq. (126) is noncovariant and differs from the covariant four-dimensional integral representation in Eq. (132) by the term proportional to $\delta^4(x-y)$. The noncovariance of the two-point correlation function in the canonical quantization is a generic property of $s > 1/2$ field theories. This point has been discussed in detail by Weinberg in [16] and we refer the reader to this reference for further details. Concerning the calculation of the covariant S-matrix elements, the conclusion there is that the correct Feynman rules are obtained just skipping the noncovariant terms like the term proportional to $\delta^4(x-y)$ in Eq. (135), i.e., in the calculations we must use

$$i\Gamma_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{\Delta(p)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \quad (136)$$

We remark that this four-dimensional integral representation of the propagator incorporates terms proportional to $p^2 - m^2$ to the naive off-shell generalization of the polarization sum projector

$$\Delta(p) = \frac{S(p) + m^2}{2m^2} - \frac{p^2 - m^2}{2m^2}. \quad (137)$$

This point is crucial when we incorporate interactions via the gauge principle. Indeed, for the simplest case of interactions with $U(1)$ massless vector fields, the three-point function in momentum space is given by

$$\Gamma^\mu(p, p') = \Sigma^{\mu\nu}(p' + p)_\nu. \quad (138)$$

It is easy to show that the Ward identity due to gauge invariance is satisfied by this vertex with the propagator in Eq. (136) but not with the propagator constructed only with the first term in Eq. (137).

Before ending this section we would like to remark that the algebraic structure of the symmetric traceless symmetric tensor in Eqs. (12), (13) is crucial in obtaining all the results presented in this section.

V. CHIRAL DECOMPOSITION AND SELF-INTERACTIONS

The parity-based covariant basis in Eq. (1) includes the chirality operator χ with the properties

$$\{\chi, S^{\mu\nu}\} = 0, \qquad \chi^2 = 1, \qquad [\chi, \mathcal{O}] = 0 \quad (139)$$

with $\mathcal{O}$ denoting any other member of the covariant basis.

Chiral fields transforming in the (1, 0) ("right" fields) and (0, 1) ("left" fields) representations are defined as

$$\psi_R = P_R\psi \quad \text{and} \quad \psi_L = P_L\psi, \quad (140)$$

where the projectors onto well-defined chirality subspaces are given by

$$P_R = \frac{1}{2}(1+\chi), \qquad P_L = \frac{1}{2}(1-\chi). \quad (141)$$

These operators have the following projector properties:

$$P_R + P_L = 1, \qquad P_R P_L = 0, \qquad P_R^2 = P_R, \qquad P_L^2 = P_L, \tag{142}$$

which together with the commutation relations in Eqs. (139) imply

$$\mathcal{O} P_{R,L} = P_{R,L} \mathcal{O}, \qquad S^{\mu\nu} P_{R,L} = P_{L,R} S^{\mu\nu}. \tag{143}$$

The Lagrangian in Eq. (59) can be decomposed in terms of the chiral fields as

$$\mathcal{L} = \frac{1}{2}[\bar{\psi}_R (i\partial)^2 \psi_L + \bar{\psi}_R S(i\partial)\psi_R + \bar{\psi}_L S(i\partial)\psi_L] - m^2[\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R]. \tag{144}$$

The first term in the kinetic part couples left and right fields; hence, the Lagrangian is not chirally symmetric in the massless limit. Spin 1 matter fields cannot have chiral gauge interactions. Concerning possible applications to hadron physics, it is not possible to realize chiral symmetry linearly and our theory can be useful only with formalisms realizing chiral symmetry nonlinearly. As for possible applications to model building for theories beyond the standard model, the only possibilities for the interactions of spin-one matter fields in this context are (i) vector gauge interactions connected or not with the standard model group, and (ii) self-interactions.

Concerning interactions, we remark that the spin-one matter field has mass dimension one, thus self-interactions are naively renormalizable. We can use the covariant basis to classify all naively renormalizable terms in the corresponding Lagrangian. These terms must be constructed from the following operators bilinear in the field

$$\bar{\psi}\psi, \quad \bar{\psi}\chi\psi, \quad \bar{\psi}S_{\mu\nu}\psi, \quad \bar{\psi}\chi S_{\mu\nu}\psi, \quad \bar{\psi}M_{\mu\nu}\psi, \quad \bar{\psi}C_{\mu\nu\alpha\beta}\psi, \quad \bar{\psi}\chi M_{\mu\nu}\psi, \quad \bar{\psi}\chi C_{\mu\nu\alpha\beta}\psi. \tag{145}$$

The last two bilinears arise from the contractions of the previous two with the Levi-Civita tensor (contractions with the metric tensor vanish) which can be rewritten in terms of the chirality operators using the relations

$$\tilde{M}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} M_{\rho\sigma} = -i\chi M_{\mu\nu}, \qquad \tilde{C}_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\epsilon_{\mu_\nu}{}^{\rho\sigma} C_{\rho\sigma\alpha\beta} = -i\chi C_{\mu\nu\alpha\beta}. \tag{146}$$

There are ten independent nonvanishing Lorentz invariant terms that can be built from the products of these bilinears. The most general naively renormalizable self-interaction Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{self}} = {} & c_1(\bar{\psi}\psi)^2 + c_2(\bar{\psi}\chi\psi)^2 + c_3(\bar{\psi}S_{\mu\nu}\psi)^2 \\ & + c_4(\bar{\psi}\chi S_{\mu\nu}\psi)^2 + c_5(\bar{\psi}M_{\mu\nu}\psi)^2 + c_6(\bar{\psi}C_{\mu\nu\alpha\beta}\psi)^2 \\ & + c_7(\bar{\psi}\psi)(\bar{\psi}\chi\psi) + c_8(\bar{\psi}S_{\mu\nu}\psi)(\bar{\psi}\chi S^{\mu\nu}\psi) \\ & + c_9(\bar{\psi}M_{\mu\nu}\psi)(\bar{\psi}\chi M^{\mu\nu}\psi) \\ & + c_{10}(\bar{\psi}C_{\mu\nu\alpha\beta}\psi)(\bar{\psi}\chi C^{\mu\nu\alpha\beta}\psi).\end{aligned} \tag{147}$$

Some of these terms violate discrete symmetries and it would be interesting to explore the consequences of the existence of spin-one matter particles in physics beyond the standard model, in particular if it could play a role in resolving the dark matter enigma. If the massless limit of our formalism is a sensible theory, all these coefficient must vanish since all these terms violate the gauge invariance in Eqs. (41).

VI. SUMMARY AND CONCLUSIONS

In this work we introduced a Dirac-like formalism for the description of spin 1 massive fields transforming in the $(1,0) \oplus (0,1)$ representation of the HLG. The formalism is based on the simultaneous projection on parity eigenspaces and on the appropriate Poincaré orbit. This projection is done using the parity-based covariant basis for the matrix operators acting on the $(1,0) \oplus (0,1)$ representation space constructed in [15]. We constructed the charge conjugation operator and showed that it commutes with parity. An explicit construction of the solutions using the representation of operators in the basis of well-defined parity for $(1,0) \oplus (0,1)$ shows that the "down" component of the solutions are suppressed as v/c with respect to the "up" part of the solutions in the nonrelativistic limit. More importantly, the "down" part of the solutions are fixed by the kinematics, as a consequence of the constrained dynamics. We worked out the constraints at the classical field theory level, showed that the system has only second class constraints, and obtained the Dirac bracket of the canonical conjugate variables. We carried out the canonical quantization of the theory, and calculated commutator relations for the canonical variables consistent with the classical Dirac brackets. Sensible results were obtained for the relevant physical quantities: energy, momentum, $U(1)$ charge, and the propagator. The algebraic properties of the covariant basis were instrumental in obtaining these results. With the aid of the chirality operator which naturally appears in the construction of the covariant basis, we analyzed the chiral structure of the theory finding that spin-one matter fields cannot have chiral gauge interactions, but admit vector gauge interactions. Spin-one matter fields have mass-dimension one therefore self-interactions are naively renormalizable. Using the covariant basis, we classified all renormalizable self-interaction terms.

Although the formalism is designed for massive particles, the classical theory has a soft $m \to 0$ limit, in whose case first class constraints (gauge symmetries) appear. It would be interesting to explore if a sensible quantum field theory can be obtained in this case.

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APPENDIX A: CONSTRAINED DYNAMICS WITH A $U(1)$ COUPLING

The Lagrangian with $U(1)$ coupling is

$$\mathcal{L} = D_\mu \bar{\Psi} \Sigma^{\mu\nu} D_\nu \Psi - m^2 \bar{\Psi}\Psi, \tag{A1}$$

$$D_\mu \Psi = \partial_\mu \Psi + iqA_\mu \Psi, \tag{A2}$$

$$D^\dagger_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} - iqA_\mu \bar{\Psi}. \tag{A3}$$

In terms of the "up" φ and "down" ξ fields, the Lagrangian reads

$$\begin{aligned}\mathcal{L} = {} & D^\dagger_0 \varphi^\dagger D_0 \varphi - \frac{1}{2}(D^\dagger_0 \varphi^\dagger D_i J^i \xi + D^\dagger_0 \xi^\dagger D_i J^i \varphi) \\ & - \frac{1}{2}(D^\dagger_i \varphi^\dagger J^i D_0 \xi + D^\dagger_i \xi^\dagger J^i D_0 \varphi) \\ & + D^\dagger_i \varphi^\dagger D^i \varphi + \frac{1}{2} D^\dagger_i \varphi^\dagger \{J^i, J^j\} D_j \varphi \\ & + \frac{1}{2} D^\dagger_i \xi^\dagger \{J^i, J^j\} D_j \xi - m^2(\varphi^\dagger \varphi - \xi^\dagger \xi).\end{aligned} \tag{A4}$$

For the purposes of quantization, it is instructive to analyse the Dirac bracket. The canonical momenta in the presence of a $U(1)$ coupling are

$$\pi_a = \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi_a)} = D^\dagger_0 \varphi^\dagger_a - \frac{1}{2}(D^\dagger_i \xi^\dagger J^i)_a, \tag{A5}$$

$$\pi^\dagger_a = \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi^\dagger_a)} = D_0 \varphi_a - \frac{1}{2}(J^i D_i \xi)_a, \tag{A6}$$

and

$$\tau_a = \frac{\delta \mathcal{L}}{\delta(\partial_0 \xi_a)} = -\frac{1}{2}(D_i \varphi^\dagger J^i)_a, \tag{A7}$$

$$\tau^\dagger_a = \frac{\delta \mathcal{L}}{\delta(\partial_0 \xi^\dagger_a)} = -\frac{1}{2}(J^i D_i \varphi)_a, \tag{A8}$$

which imply constraints even in the presence of electromagnetic interactions.

The Hamiltonian density, incorporating the constraints as Lagrange multipliers, is

$$\begin{aligned}\mathcal{H}^* = {} & \pi_a \pi^\dagger_a + \frac{1}{2}\pi_a (J^i D_i \xi)_a + \frac{1}{2}(D^\dagger_i \xi^\dagger J^i)_a \pi^\dagger_a + \frac{1}{4}(D^\dagger_i \xi^\dagger J^i)_a (J^j D_j \xi)_a \\ & - D^\dagger_i \varphi^\dagger D^i \varphi - \frac{1}{2} D^\dagger_i \varphi^\dagger \{J^i, J^j\} D_j \varphi - \frac{1}{2} D^\dagger_i \xi^\dagger \{J^i, J^j\} D_j \xi + m^2(\varphi^\dagger \varphi - \xi^\dagger \xi) \\ & + ieA_0[\pi_a \varphi_a - \varphi^\dagger_a \pi^\dagger_a] + \frac{ie}{2} A_0[\xi^\dagger_a (J^i D_i \varphi)_a - (D^\dagger_i \varphi^\dagger J^i)_a \xi_a] \\ & + \lambda_a \rho_a + \lambda^\dagger_a \rho^\dagger_a,\end{aligned} \tag{A9}$$

and the Hamilton equations are

$$\partial_0 \varphi_a = \frac{\delta H^*}{\delta \pi_a} = \pi^\dagger_a + \frac{1}{2}(J^i D_i \xi)_a + ieA_0 \varphi_a, \tag{A10}$$

$$\begin{aligned}\partial_0 \pi^\dagger_a = -\frac{\delta H^*}{\delta \varphi^\dagger_a} = {} & -D_i D^i \varphi_a - \frac{1}{2} D_i D_j (\{J^i, J^j\}\varphi)_a \\ & - m^2 \varphi_a + \frac{1}{2}(J^i D^\dagger_i \lambda)_a + ie\left[A_0 \pi^\dagger - \frac{1}{2}(J^i D_i A_0 \xi)\right]_a\end{aligned} \tag{A11}$$

$$\partial_0 \xi_a = \frac{\delta H^*}{\delta \tau_a} = \lambda_a, \tag{A12}$$

$$\begin{aligned}\partial_0 \tau^\dagger_a = -\frac{\delta H^*}{\delta \xi^\dagger_a} = {} & \frac{1}{2} D_i (J^i \pi^\dagger)_a + \frac{1}{4}(J^i J^j D_i D_j \xi)_a \\ & - \frac{1}{2} D_i D_j (\{J^i, J^j\}\xi)_a + m^2 \xi_a - \frac{ie}{2} A_0 (J^i D_i \varphi)_a.\end{aligned} \tag{A13}$$

The temporal evolution of any dependent dynamic variable fields and momenta can be written as

$$\dot{B} = \frac{\partial B}{\partial t} + \{B, H^*\}, \tag{A14}$$

and again they have the same Poisson brackets between fields and canonical momenta, Eqs. (75).

The dynamics generated by the modified Hamiltonian must preserve the restrictions

$$\partial_0 \rho_a = \frac{\partial \rho_a}{\partial t} + \{\rho_a, H^*\} = 0. \tag{A15}$$

This leads to secondary constraints

$$\kappa_a = D_i^\dagger (\pi J^i)_a - \frac{1}{2}(D_j^\dagger D_i^\dagger \xi^\dagger J^j J^i)_a + m^2 \xi_a^\dagger + \frac{ie}{2} F_{0i} (\varphi^\dagger J^i)_a = 0. \tag{A16}$$

and, again, for consistency, requires

$$\partial_0 \kappa_a = \frac{\partial \kappa_a}{\partial t} + \{\kappa_a, H^*\} = 0. \tag{A17}$$

This condition yields

$$\begin{aligned}\dot{\kappa} = {} & -D_k^\dagger [D_j^\dagger D^{\dagger j} \varphi + \frac{1}{2} D_j^\dagger D_i^\dagger (\varphi^\dagger \{J^i, J^j\})] J^k - m^2 D_k^\dagger \varphi^\dagger J^k \\ & + \lambda^\dagger \left(\frac{ie}{2} F_{ki} J^k J^i + m^2\right) - ie D_i^\dagger \left[A_0 \pi - \frac{1}{2}(D_j^\dagger (A_0 \xi^\dagger) J^j)\right] J^i \\ & + \frac{ie}{2} F_{0k} \left(\pi - ie A_0 \varphi^\dagger + \frac{1}{2}(D_i^\dagger \xi^\dagger J^i)\right) J^k + \frac{ie}{2} \dot{F}_{0i} \varphi^\dagger J^i \\ & - \frac{ie}{2}(\dot{A}_j D_i^\dagger \xi^\dagger J^j J^i + D_j^\dagger \dot{A}_i \xi^\dagger J^j J^i) + ie \dot{A}_i \pi J^i = 0.\end{aligned} \tag{A18}$$

Although this is a complicated equation, it just defines $\lambda^\dagger$ and does not give rise to additional secondary constraints.

Now we write the Poisson brackets between the constraints. It is straightforward to see that

$$\{\rho_a(\mathbf{x}), \rho_b(\mathbf{y})\} = 0, \tag{A19}$$

$$\{\rho_a(\mathbf{x}), \rho_b^\dagger(\mathbf{y})\} = 0, \tag{A20}$$

$$\{\rho_a(\mathbf{x}), \kappa_b(\mathbf{y})\} = 0. \tag{A21}$$

However, for $\{\rho_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\}$ we get

$$\begin{aligned}\{\rho_a(\mathbf{x}), \kappa_b^\dagger(\mathbf{y})\} = {} & \left\{\left[\tau_a + \frac{1}{2}(D_k^\dagger \varphi^\dagger J^k)_a\right](\mathbf{x}), \left[D_i (J^i \pi^\dagger)_b - \frac{1}{2}(D_i D_j J^i J^j \xi)_b + m^2 \xi_b\right](\mathbf{y})\right\} \\ & - \frac{ie}{2}\left\{\left[\tau_a + \frac{1}{2}(D_k^\dagger \varphi^\dagger J^k)_a\right](\mathbf{x}), [F_{0i}(J^i \varphi)_a](\mathbf{y})\right\},\end{aligned} \tag{A22}$$

and since the last line of this equation vanishes,

$$\begin{aligned}\{\rho_a(\mathbf{x}), \kappa_b^\dagger(\mathbf{y})\} = {} & -\frac{1}{2}\{\tau_a(\mathbf{x}), (D_i D_j J^i J^j \xi)_b(\mathbf{y})\} + m^2 \{\tau_a(\mathbf{x}), \xi_b(\mathbf{y})\} \\ & + \frac{1}{2}\{(D_k^\dagger \varphi^\dagger J^k)_a(\mathbf{x}), D_i (J^i \pi^\dagger)_b(\mathbf{y})\}.\end{aligned} \tag{A23}$$

This can be written as

$$\{\rho_a(\mathbf{x}), \kappa_b^\dagger(\mathbf{y})\} = \frac{1}{2}[D_{\mathbf{y}i} D_{\mathbf{y}j} + D_{\mathbf{y}i} D_{\mathbf{x}j}^\dagger] \delta^3(\mathbf{x} - \mathbf{y}) (J^j J^i)_{ba} - m^2 \delta_{ab} \delta^3(\mathbf{x} - \mathbf{y}). \tag{A24}$$

In this expression, we can change ∂_x by $-\partial_y$ and $A_j(\mathbf{x})$ by $A_j(\mathbf{y})$ in $D_{\mathbf{x}j}^\dagger \delta^3(\mathbf{x} - \mathbf{y})$ to get $-D_{\mathbf{y}j} \delta^3(\mathbf{x} - \mathbf{y})$. This allows us to conclude that

$$\{\rho_a(\mathbf{x}), \kappa_b^\dagger(\mathbf{y})\} = -m^2\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}). \tag{A25}$$

These are equal to the free field Poisson brackets.

APPENDIX B: INTEGRAL REPRESENTATION OF THE PROPAGATOR

For the calculation of the integral in Eq. (134) we split it into the real axis and the semicircle contributions

$$\frac{1}{2\pi}\oint_C \frac{\Delta(\omega,\mathbf{p})e^{-i\omega(x-y)^0}}{\omega^2-\mathbf{p}^2-m^2+i\epsilon}d\omega = I(\mathbf{p}) + \frac{1}{2\pi}\int_{C_R}\frac{\Delta(\omega,\mathbf{p})e^{-i\omega(x-y)^0}}{\omega^2-\mathbf{p}^2-m^2+i\epsilon}d\omega. \tag{B1}$$

Causality requires us to close the contour C with a (counterclockwise) semicircle on the upper complex plane for $(x-y)^0 < 0$ and with a (clockwise) semicircle on the lower plane for $(x-y)^0 > 0$. In the case $(x-y)^0 > 0$, C encloses the pole $\omega_\epsilon = \sqrt{\mathbf{p}^2+m^2-i\epsilon}$ and we get

$$I(\mathbf{p}) = \frac{-i\Delta(\omega_\epsilon,\mathbf{p})e^{-i\omega_\epsilon(x-y)^0}}{2\omega_\epsilon} - \frac{1}{2\pi}\int_{C_R^-}\frac{\Delta(\omega,\mathbf{p})e^{-i\omega(x-y)^0}}{\omega^2-\mathbf{p}^2-m^2+i\epsilon}d\omega. \tag{B2}$$

Similarly, for $(x-y)^0 < 0$ we obtain

$$I(\mathbf{p}) = \frac{-i\Delta(-\omega_\epsilon,\mathbf{p})e^{i\omega_\epsilon(x-y)^0}}{2\omega_\epsilon} - \frac{1}{2\pi}\int_{C_R^+}\frac{\Delta(\omega,\mathbf{p})e^{-i\omega(x-y)^0}}{\omega^2-\mathbf{p}^2-m^2+i\epsilon}d\omega. \tag{B3}$$

Next we parametrize $\omega \in C_R^\pm$ as $\omega = Re^{i\theta}$ with $0 \le \theta \le \pi$ for C_R^+ and $\pi \le \theta \le 2\pi$ for C_R^-. For large R we get

$$\lim_{R\to\infty}\frac{\Delta(Re^{i\theta},\mathbf{p})}{R^2e^{2i\theta}-\mathbf{p}^2-m^2+i\epsilon} = \frac{1}{2m^2}(S^{00}-1) \neq 0, \tag{B4}$$

and unlike the scalar and fermion case, the integrals over $C_R^\pm$ do not vanish

$$\lim_{R\to\infty}\int_{C_R^\pm}\frac{\Delta(\omega,\mathbf{p})e^{-i\omega(x-y)^0}}{\omega^2-\mathbf{p}^2-m^2+i\epsilon}d\omega = \frac{(S^{00}-1)}{2m^2}\int_{C_R^\pm}e^{-i\omega(x-y)^0}d\omega. \tag{B5}$$

The integral on the right-hand side of Eq. (B5) is readily obtained as

$$\int_{C_R^\pm}e^{-i\omega(x-y)^0}d\omega = -2\pi\delta(x^0-y^0). \tag{B6}$$

Using Eqs. (B5), (B6) we can rewrite Eqs. (B2), (B3) as

$$I(\mathbf{p}) = \frac{-i\Delta(\omega_\epsilon,\mathbf{p})e^{-i\omega_\epsilon(x-y)^0}}{2\omega_\epsilon} + \frac{(S^{00}-1)}{2m^2}\delta(x^0-y^0); \quad (x-y)^0 > 0, \tag{B7}$$

$$I(\mathbf{p}) = \frac{-i\Delta(-\omega_\epsilon,\mathbf{p})e^{i\omega_\epsilon(x-y)^0}}{2\omega_\epsilon} + \frac{(S^{00}-1)}{2m^2}\delta(x^0-y^0); \quad (x-y)^0 < 0, \tag{B8}$$

and Eq. (132) reads

$$\begin{aligned} iG(x-y) = {} & \frac{\theta(x^0-y^0)}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2\omega_\epsilon}\Delta(\omega_\epsilon,\mathbf{p})e^{-i\omega(x-y)^0}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \\ & + \frac{\theta(y^0-x^0)}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2\omega_\epsilon}\Delta(-\omega_\epsilon,\mathbf{p})e^{+i\omega(x-y)^0}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} + \frac{S^{00}-1}{2m^2}\delta^4(x-y). \end{aligned} \tag{B9}$$

Changing $\mathbf{p}$ by $-\mathbf{p}$ in the second line of Eq. (B9), taking the $\epsilon \to 0$ limit and using

$$\Delta(\omega_{\mathbf{p}}, p) = \frac{S(\mathbf{p}) + m^2}{2m^2}, \tag{B10}$$

we finally obtain

$$i\Gamma_F(x - y) = iG(x - y) + \frac{S^{00} - 1}{2m^2}\delta^4(x - y). \tag{B11}$$

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