Instanton-monopole correspondence from M-branes on $\mathbb{S}^1$ and little string theory

Stefan Hohenegger,[1,*] Amer Iqbal,[2,†] and Soo-Jong Rey[3,‡]

[1]*Université Claude Bernard (Lyon 1), Institut de Physique Nucléaire, Bat. P. Dirac, 4 rue Enrico Fermi, F-69622-Villeurbanne, France*

[2]*Abdus Salam School of Mathematical Sciences Government College University, Lahore, Pakistan and Center of Mathematical Sciences and Applications, Harvard University 1 Oxford Street, Cambridge, Massachusetts 02138, USA*

[3]*School of Physics and Astronomy and Center for Theoretical Physics, Seoul National University, Seoul 08826, Korea and Fields, Gravity & Strings, Center for Theoretical Physics of the Universe, Institute for Basic Sciences, Daejeon 34047, Korea*

(Received 13 December 2015; published 31 March 2016)

We study Bogomol'nyi-Prasad-Sommerfield (BPS) excitations in M5-M2-brane configurations with a compact transverse direction, which are also relevant for type IIa and IIb little string theories. These configurations are dual to a class of toric elliptically fibered Calabi-Yau manifolds X_N with manifest $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ modular symmetry. They admit two dual gauge theory descriptions. For both, the nonperturbative partition function can be written as an expansion of the topological string partition function of X_N with respect to either of the two modular parameters. We analyze the resulting BPS-counting functions in detail and find that they can be fully constructed as linear combinations of the BPS-counting functions of M5-M2-brane configurations with noncompact transverse directions. For certain M2-brane configurations, we also find that the free energies in the two dual theories agree with each other, which points to a new correspondence between instanton and monopole configurations. These results are also a manifestation of T-duality between type IIa and IIb little string theories.

DOI: 10.1103/PhysRevD.93.066016

I. INTRODUCTION AND SUMMARY

In recent years, the interplay between M-theory/string theory, geometry, and superconformal gauge theories has been rigorously studied, leading to new and deep insights. At the focus of interest are configurations of N parallel M5-branes with multiple M2-branes stretched between them (see e.g. [1–6]). These brane configurations are known to be U dual to specific toric elliptically fibered Calabi-Yau threefolds X_N over an A_{N-1} base space. They can also be associated to six-dimensional non-Abelian supersymmetric field theories, which upon further compactification to four dimensions give rise to mass-deformed $\mathcal{N}=2^*$ gauge theories. All these six-dimensional systems exhibit very rich dynamics and contain extended Bogomol'nyi-Prasad-Sommerfield (BPS) degrees of freedom that are unfamiliar from a four-dimensional point of view.

Indeed, as was first pointed out in [1], the configuration of M2-branes stretched between M5-branes described above gives rise to one-dimensional dynamical objects at the brane intersections. When the M5-branes coincide, these so-called *M-strings* become tensionless, forming essential interacting degrees of freedom of the elusive (2,0) superconformal, local quantum field theory. When the M5-branes are separated, the M-strings become BPS string states with tension. Their BPS excitations, which are expected to elucidate the world sheet dynamics over the six-dimensional target space, are counted by the topological string partition function of the dual toric Calabi-Yau manifold X_N [1–3]. This partition function is efficiently computed by the refined topological vertex approach [7–9], and depends on two parameters, $\epsilon_{1,2}$, which are fugacities for the little group $SO(4)$ of massive particles in five dimensions in M-theory compactification on a Calabi-Yau threefold. From the viewpoint of the nonperturbative gauge theory partition function, these parameters correspond to putting the gauge theory on a curved spacetime, the so-called generalized Ω background [10].

Another manifestation of string degrees of freedom was discussed in [4,5]: upon compactification to five dimensions, the M-strings become (electrically charged) BPS particles which are related via five-dimensional S-duality to magnetically charged monopole strings. While the precise details of this duality map are somewhat intricate [11], we proposed in [5] that the degeneracies of certain M-string BPS configurations capture the elliptic genus (see [12] for its general definition) of the moduli space of monopole strings. This proposal applies to theories of $SU(N)$ gauge theories for any N and for general distributions of the constituent monopole strings. In [5], we successfully

*s.hohenegger@ipnl.in2p3.fr
†amer@alum.mit.edu
‡sjrey@snu.ac.kr

 © 2016 American Physical Society

checked this proposal for all known cases, namely, the Taub-NUT and Atiyah-Hitchin spaces. These spaces correspond to the moduli spaces of charge (1,1) monopoles in $SU(3)$ [13] and charge (2) monopoles in $SU(2)$ gauge theory [14], respectively. The elliptic genera of their respective moduli spaces were previously computed in [15,16]. In [5], we studied the elliptic genus of the moduli space of monopole strings for an arbitrary gauge group and for general distribution of constituent monopole strings.

The purpose of this paper is to expose new phenomena associated with a richer duality structure that arises when the above M5-M2 brane setup is extended to a configuration with a larger modular symmetry group. Such an extension appears in a variety of physical problems. We focus on a particularly interesting configuration that has to do with compactifying a direction transverse to the M5-branes to a circle. Concretely, the brane configuration studied in [5] consists of N parallel M5-branes which are separated along a noncompact direction. Here, we compactify this direction to $\mathbb{S}^1$. Geometrically, this modified brane configuration is again dual to a toric elliptically fibered Calabi-Yau threefold X_N. However, in contrast to the noncompact case, the base is now an *affine* A_{N-1} space, which in turn is a fibration over $\mathbb{P}^1$. As a consequence of this twofold fibration structure, this setup exhibits manifest $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ symmetry. This twofold $SL(2,\mathbb{Z})$ symmetry permits us to describe this theory by using two different approaches:

(i) The first approach relates the compact brane configuration to two different gauge theories: theory 1 is the Coulomb branch of a $U(N)$ gauge theory, while theory 2 is a circular quiver with N nodes of $U(1)$ gauge theories. At a generic value of the parameters, both are $[U(1)]^N$ circular quiver gauge theories. The difference is that, when the M5-branes are all separated, theory 1 has massive bifundamentals, while theory 2 has massless bifundamentals. The two gauge theories arise from the map of the M-theory brane configuration to type IIB brane configurations consisting of either one NS5-brane and N D5-branes or one D5-brane and N NS5-branes, intersecting in both cases on a torus. The former gives rise to theory 1, while the latter gives rise to theory 2. Therefore, the two gauge theories are related to each other by type IIB S-duality. On the other hand, in the description in terms of the toric Calabi-Yau manifold X_N, the two gauge theories are just two facets of topological string theory and are related to each other by an exchange of the base and the fiber in X_N. As such, the partition functions of these two gauge theories can be extracted from the topological string partition function of X_N by expanding in two different parameters. These correspond to the two modular parameters of $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ mentioned above.

(ii) The second approach relates the compact brane configuration to maximally supersymmetric little string theories in six dimensions [17–20].[1] These little strings are fundamental strings bound to NS5-branes that are decoupled from the ambient ten-dimensional spacetime. Therefore, descending from NS5-branes in type IIA and IIB string theories or asymptotically locally Euclidean (ALE) singularities in type IIB and IIA string theories, there are type IIb and IIa little string theories in six dimensions with (2,0) and (1,1) supersymmetries, respectively.[2] These little string theories are nonlocal theories since excitations contain "little strings" of finite tension. In the brane configuration description, the $\mathbb{S}^1$ compactification transverse to M5-branes renders the tension of these little strings. Moreover, one can see from U-duality of the brane configuration that gauge theory 1 and gauge theory 2 are related to type IIa and IIb little string theory, respectively. In the same way as the two gauge theories are related to each other by the exchange of the two coupling parameters, upon compactification on $\mathbb{S}^1$, the IIa and IIb little string theories are T dual to each other by the exchange of their $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ modular parameters.[3]

We analyze the modular properties of the partition functions of the two pairs of dual theories mentioned above and discover two remarkable properties. First of all, the functions capturing the degeneracies of single-particle BPS states of gauge theory 2 can be expressed by the analog functions of degeneracies of monopole strings in the noncompact M5-brane configuration as worked out in [5]. Roughly speaking, the free energy of compact monopole strings can be expressed as a linear combination of the free energies of noncompact monopole strings. Secondly, the generating functions of degeneracies for certain instanton configurations of theory 1 are equal to the generating functions of degeneracies for monopole strings of theory 2. The equivalence we observe is case specific in the sense that it maps configurations which are fully covariant under the respective $SL(2,\mathbb{Z})$ symmetries into each other. A more careful study of the relation of the remaining configurations (and thus a possible equivalence of the two partition functions) is currently under way [25].

This paper is organized as follows. In Sec. II, we discuss in detail the M-brane configuration and the dual Calabi-Yau

[1] See [21,22] for reviews of little string theories on $\mathbb{R}^{5,1}$ and [23] for little string theories on $AdS_5\times\mathbb{S}^1$.

[2] Our notations adhere to the convention that nonchiral string theories are labeled as A or a, while chiral string theories are labeled as B or b. We trust this will cause maximal confusion for the reader.

[3] We thank the authors of [24], communicated through Cumrun Vafa, for suggesting possible relations between these two approaches.

threefold X_N. We also describe the two distinct gauge theories associated with X_N, and relate them to type IIa and IIb little string theories. In Sec. III, we present the topological partition function $\mathcal{Z}_{X_N}$ of X_N and discuss in detail the manifest $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ modular symmetry, in particular the transformation properties of $\mathcal{Z}_{X_N}$. Furthermore, we extract the nonperturbative partition functions of the two gauge theories mentioned above by expanding them in the parameters associated with the two different $SL(2,\mathbb{Z})$'s. In Sec. IV, we find that the gauge theory free energies can be expressed in terms of their noncompact counterparts that we analyzed in the previous work [5]. In Sec. V, we exhibit remarkable relations between the free energies of the two different gauge theories. These relations are very nontrivial in that they relate quantities computed in the instanton moduli space with counting functions of multimonopole string configurations. In Sec. VI, following the conjecture in [5] for the noncompact case, we propose a concrete expression for the elliptic genus of the monopole moduli space of the affine A_{N-1} theory. From this, we extract the corresponding χ_y genus which encodes topological invariants of this moduli space. We conclude in Sec. VII and point out further directions for future research. The Appendix contains explicit series expansions of BPS-counting functions of various instanton and monopole configurations.

II. BRANE CONFIGURATION ON $\mathbb{S}^1$ AND DUAL THEORIES

A. M-brane configuration

Our starting point is a particular BPS configuration of M-branes in the 11-dimensional M-theory vacuum $\mathbb{T}^2\times\mathbb{R}^3_{\parallel}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}^4_{\perp}$ (with $\mathbb{T}^2\sim\mathbb{S}^1\times\mathbb{S}^1$), parametrized by the Cartesian coordinates $(x^0,\ldots,x^{10})$. Specifically, we consider N planar M5-branes, K open M2-branes stretched between M5-branes, and M M-waves on a two-dimensional intersection of M5-branes and M2-branes. The precise configuration is summarized in the following table:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
$M5$	=	=	=	=	=	=					
$M2$	=	=					=				
$M\sim$	=	=									
	$\mathbb{T}^2\sim\mathbb{S}^1\times\mathbb{S}^1$		$\mathbb{R}^3_{\parallel}$			$\mathbb{S}^1_{R_5}$	$\mathbb{S}^1_{R_6}$	$\mathbb{R}^4_{\perp}$			

(2.1)

This brane configuration is very similar to the one studied in [5], with the only difference that, in the present setup, in addition to $x^1\simeq x^1+2\pi R_1$, the direction $x^6\simeq x^6+2\pi R_6$ is compactified to a circle with radius R_6. The open M2-branes are extended along $\mathbb{S}^1_{R_0}\times\mathbb{S}^1_{R_1}\times\mathbb{S}^1_{R_6}$. We denote the geometric parameters of this $\mathbb{T}^3$ as

$$2\pi i R_1 := \tau \quad \text{and} \quad 2\pi i R_6 := \rho \tag{2.2}$$

and their respective fugacities as

$$Q_\tau = e^{2\pi i\tau} \quad \text{and} \quad Q_\rho = e^{2\pi i\rho}.$$

Along the x^6 direction, the M5-branes are placed at positions

$$0\le a_1\le a_2\le\cdots\le a_N\le 2\pi R_6, \tag{2.3}$$

thereby partitioning the x^6 direction into N intervals of length

$$\begin{aligned} t_{f_1} &= a_2-a_1,\\ t_{f_2} &= a_3-a_2,\\ &\vdots\\ t_{f_{N-1}} &= a_N-a_{N-1},\\ t_{f_N} &= 2\pi R_6-\sum_{i=1}^{N-1} t_{f_i}\\ &= 2\pi R_6-(a_N-a_1)\\ &= -i\rho-(a_N-a_1). \end{aligned} \tag{2.4}$$

For a fixed R_6, the brane configuration is specified by $(N-1)$ independent non-negative parameters. The fugacity associated with these independent parameters $t_{f_i}, (i=1,\ldots,N-1)$ are denoted as

$$Q_{f_1}=e^{-2\pi t_{f_1}},\quad Q_{f_2}=e^{-2\pi t_{f_2}},\quad \cdots,\quad Q_{f_{N-1}}=e^{-2\pi t_{f_{N-1}}}. \tag{2.5}$$

Thus, for meromorphic functions of the Q_{f_i}, we can view the complexified it_{f_i} as $(N-1)$ independent positions on a torus $\mathbb{T}^2(\rho)$ of complex structure ρ.

The K different M2-branes are stretched[4] between the M5-branes and distributed among these N intervals with multiplicities $(\{k_i\})=(k_1,k_2,\ldots,k_N)$ such that $K=\sum_{i=1}^N k_i$. In addition, there are M M-waves propagating along the intersections of M5- and M2-branes, i.e. the directions x^0 and x^1. Finally, all branes are pointlike and located at the origin with regards to $\mathbb{R}^4_{\perp}$.[5] Schematically, the whole setup is shown in Fig. 1.

The brane configuration saturates the BPS bound. Furthermore, the spacetime Poincaré and supersymmetry content is identical to that of the noncompact setting

[4]Since the transverse space $\mathbb{R}^4_{\perp}$ is topologically trivial, the M2-branes between any two M5-branes cannot be split but form a single stack (see [2,3]).

[5]We can also replace $\mathbb{R}^4_{\perp}$ by an affine A_{N-1} geometry, which is dual to the M5-branes on a circle [26–28].

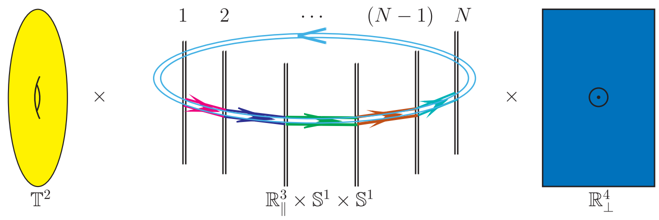


FIG. 1. Brane configuration. The M5-branes are all located at the origin in $\mathbb{R}^4_{\perp}$, wrapped around $\mathbb{T}^2$ and stretched along the (6) direction.

(i.e. with $\mathbb{R}^3_{\parallel} \times \mathbb{S}^1_{R_6}$ replaced by $\mathbb{R}^4_{\parallel}$), which we already extensively discussed in [5]. In Sec. III, we present the partition function of this configuration. However, in order to render the latter well defined, we need to regularize infrared divergences. To this end, we turn on various deformations of $\mathbb{R}^3_{\parallel} \times \mathbb{S}^1_{R_5} \times \mathbb{R}^4_{\perp}$, which can be described as a $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_m$ action with respect to the (0) direction. Specifically, for local coordinates $(z^1, z^2) = (x^2 + ix^3, x^4 + ix^5)$ and $(w^1, w^2) = (x^7 + ix^8, x^9 + ix^{10})$ of $\mathbb{R}^4_{\parallel}$, maximum deformations one can introduce with respect to x^0 are

$$\begin{aligned} &U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_m: \\ &\quad (z_1, z_2) \to (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2) \\ &\quad (w_1, w_2) \to (e^{2\pi i m - i\pi(\epsilon_1+\epsilon_2)} w_1, e^{-2\pi i m - \pi i(\epsilon_1+\epsilon_2)} w_2), \end{aligned} \tag{2.6}$$

with the parameters $\epsilon_{1,2}$ and m. From the perspective of the four-dimensional $\mathcal{N} = 2^*$ gauge theory, $\epsilon_{1,2}$ correspond to the deformation parameters of an Ω background [10,29,30],[6] while m can be associated with a mass deformation. In the present case, we are counting states on a partially compactified $\mathbb{R}^3_{\parallel} \times \mathbb{S}^1_{R_5}$. This space is not compatible with the above deformations. Therefore, in what follows, we shall take a suitable limit of the deformation that commutes with the isometries of $\mathbb{R}^3_{\parallel} \times \mathbb{S}^1_{R_5}$.

The Ω and mass deformations also affect the nature of the three torus $\mathbb{S}^1_{R_0} \times \mathbb{S}^1_{R_1} \times \mathbb{S}^1_{R_6}$. Among the three directions, the x^0 direction is twisted while the x^1 and x^6 directions remain untwisted. So, we should expect for the deformed brane configuration that the full U-duality group of the brane configuration is reduced by the deformations but that the $\mathbb{Z}_2$ exchange symmetry between $\mathbb{S}^1_{R_1}$ and $\mathbb{S}^1_{R_6}$, i.e. $\tau \leftrightarrow \rho$ in Eq. (2.2), is still intact.

[6]Several different string theoretic descriptions of the Ω background have been proposed in the literature (see for example [31–35].) In particular, a world-sheet approach based on physical scattering amplitudes has been proposed in [36–39] (see also [40]). Furthermore, in [41] its relation to topological gravity has been understood.

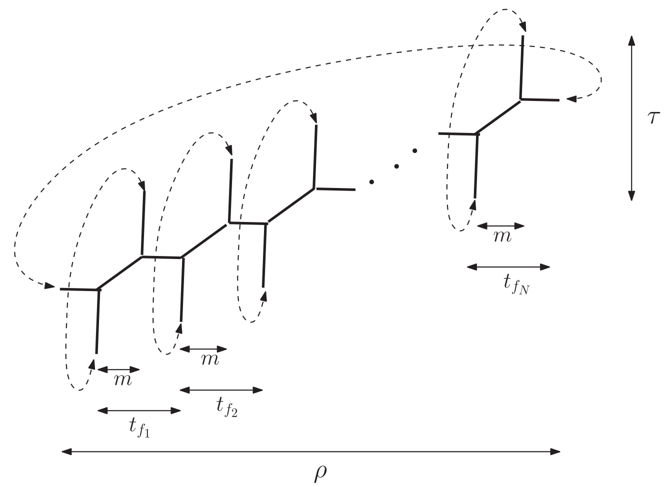


FIG. 2. The toric web diagram of the Calabi-Yau threefold X_N dual to the brane configuration. Both the horizontal and vertical directions are compactified on $\mathbb{S}^1$'s, defining the diagram on $\mathbb{T}^2$.

Finally, we can also connect this configuration to a setup of D-branes in string theory: indeed, by viewing $\mathbb{T}^2 \sim \mathbb{S}^1 \times \mathbb{S}^1$ (and particularly $x^0 \sim x^0 + 2\pi R_0$), we can interpret the direction x^0 as the M-theory circle and dimensionally reduce it to type IIA string theory. In this way, the M5-branes are reduced to D4-branes, whose world volume dynamics is described by five-dimensional $\mathcal{N} = 1^*$ gauge theory with coupling constant $g_5^2 = R_0$, the radius of the M-theory circle. The M2-branes become F1 strings with tension $T_2 R_1 R_6$, where T_2 is the M2-brane tension.

B. Calabi-Yau geometry

We can associate a toric Calabi-Yau threefold X_N to the brane configuration just discussed, whose web diagram is shown in Fig. 2. In the toric diagram, the compactification of the vertical direction reflects the fact that the brane configuration is compactified along the x^1 direction, while the compactification of the horizontal direction reflects the fact that the brane configuration is compactified along the x^6 direction. Therefore, the toric web of X_N is defined on a torus.

The new feature of this manifold in comparison to the noncompact configuration (i.e. $R_6 \to \infty$) discussed in [5], whose toric web is defined on a cylinder, is a twofold fibration structure: the X_N can be seen as an elliptic fibration over the affine A_{N-1} space, which itself is an elliptic fibration over $\mathbb{C}^1$. Thus, X_N is specified by three parameters, τ, ρ, m, together with $N-1$ parameters appearing from resolution of affine A_{N-1} singularities. The affine extension of A_{N-1} is a direct consequence of compactifying $x^6 \sim x^6 + 2\pi R_6$ in the brane setup. We will see below that this affine extension will play an important role in the gauge theory description.

This new structure can be made more transparent by using slightly different parameters than in the brane configuration. The latter is usually parametrized with the help of the distances between the M5-branes along x^6, i.e. by $(\tau, m, t_{f_1}, t_{f_2}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$ [see (2.4)]. We can replace one of these, i.e. t_{f_N}, by the size of the circle transverse to the M5-branes,

$$\rho = i \sum_{a=1}^{N} t_{f_a} = 2\pi i R_6, \tag{2.7}$$

and therefore use the parameters $(\tau, \rho, m, t_{f_1}, t_{f_2}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$ instead.

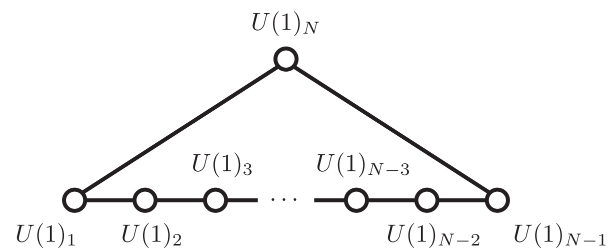
Recall that, in the toric web diagram, Fig. 2, the presence of two $\mathbb{S}^1$'s is associated with the twofold fibration structure in X_N. The exchange of these two $\mathbb{S}^1$'s in the toric web amounts to an exchange of the elliptic fiber and the elliptic base in X_N. This implies that, in the M5-M2-brane configuration picture, there is another configuration dual to the one discussed in Sec. II A: it is given by a single M5-brane wrapped on a circle with transverse space affine A_{N-1} geometry and N distinct M2-brane configurations. These two different brane configurations give rise to two dual gauge theory descriptions, as we shall discuss presently.[7]

C. Gauge theories with affine gauge group

As explained in [3], we can associate Ω- and mass-deformed supersymmetric gauge theories to the brane configuration discussed in Sec. II A. In fact, the brane setup can be related by a chain of U-dualities to two distinct (but dual) gauge theories, which will play an important role throughout this paper:

(a) Gauge theory 1: The first picture is to associate the Kähler parameter τ of the $\mathbb{T}^2$ with the coupling constant of a $U(N)$ gauge theory, while the Kähler parameters $t_{f_1}, \ldots, t_{f_N}$ are identified with the parameters of the Coulomb branch.[8] This theory is reduced to the $\mathcal{N} = 2^*$ supersymmetric gauge theory in four dimensions.

(b) Gauge theory 2: The second picture is to associate the Kähler parameters t_{f_a}'s of the base $\mathbb{P}^1$'s of X_N with the coupling constants of a $[U(1)]^N$ quiver gauge theory. It is important to notice that, because the x^6 direction of the brane configuration is compactified on a circle (which gives rise to the affine A_{N-1} structure of X_N), this quiver is circular rather than linear:



For the reader's convenience, we compiled below the identification of all Calabi-Yau parameters with gauge theory parameters from the above two different pictures.

Pm.	Brane configuration	Calabi-Yau	Gauge theory 1	Gauge theory 2
τ	size of $\mathbb{S}^1$ parallel to M5-branes	Kähler moduli of elliptic base	coupling constant	compact Coulomb branch parameter
ρ	size of $\mathbb{S}^1$ transverse to M5-branes	Kähler moduli of affine A_{N-1} fiber	compact Coulomb branch parameter	overall coupling constant
t_{fa}	separations between adjacent M5-branes	Kähler moduli of affine A_{N-1} fiber	compact Coulomb branch parameter	coupling constants $a = 1, \ldots, N-1$

When counting the number of parameters, note that $\rho = \sum_{a=1}^{N} t_{f_a}$ and thus $(t_{f_1}, \ldots, t_{f_N})$ and ρ are not independent of one another. In all cases, m and $\epsilon_{1,2}$ describe deformations.

D. IIa and IIb little string theories

The brane configuration discussed in Sec. II A can also be related to little string theories, which are six-dimensional nonlocal quantum theories with nongravitational string excitations [17–20]. We can associate type IIa and IIb little strings with type IIB and IIA NS5-branes in the decoupling limit

[7]In the situation where we have A_{N-1} geometry rather than affine A_{N-1} (i.e., $\rho \mapsto i\infty$), the two dual gauge theories were discussed in [5].

[8]It is known that, if the theory is coupled to g many massless adjoint hypermultiplets, the partition function is equal to the partition function of a two-dimensional topological field theory on a genus-g Riemann surface [42].

$$g_{\rm st} \to 0, \qquad \ell_{\rm st} = \text{finite} \tag{2.8}$$

for the string coupling and string length, respectively. At energies well below the string tension scale, the little string states are decoupled and the type IIa and IIb little string theories flow to the (1,1) super Yang-Mills theory and (2,0) superconformal theory, respectively. Notice that, since the limit (2.8) commutes with T-duality (which exchanges type IIA and IIB string theories), the type IIa and IIb little string theories are also related by T-duality. We discuss the precise relation in Sec. II D 2.

1. *Little string BPS excitations*

We first explain how the little string theories are related to the M-brane configuration discussed in Sec. II A. The BPS string excitations of little string theories are realized by the open M2-branes stretched between M5-branes. Since there are N such intervals on the $\mathbb{S}^1$ transverse to N M5-branes (i.e. along the direction x^6), these excitations carry $[U(1)]^N$ quantum numbers whose chemical potentials and fugacities are $t_{f_1}, \dots, t_{f_N}$ and $(Q_{f_1} \dots, Q_{f_N})$, respectively, in Eq. (2.5).

The crucial feature of the M-brane configuration that permits this identification with the little string states is the compactness of the x^6 direction: indeed, compared to the noncompact counterpart (as discussed in our previous paper [5]), the parameters (2.4) are modified in two important ways:

(1) There is one additional finite interval between the first and the last M5-brane, which we denoted as t_{f_N} in (2.4). Therefore, even in the limit that all the N M5-branes stack together and make the M-strings tensionless, there always exists a finite-tension string coming from the open M2-brane stretched around the compact $\mathbb{S}^1$ of the x^6 direction. This finite-tension string defines the little string. In our notation, the ground state of a single little string corresponds to the configuration $(k_1, \dots, k_N) = (1, \dots, 1)$, i.e. a closed M2-brane which pass through all N M5-branes on $\mathbb{S}^1$. Likewise, the ground state of k multiple little strings corresponds to the configuration $(k_1, \dots, k_N) = (k, \dots, k)$, which can be multiply wound.

(2) The intervals $t_{f_1}, \dots, t_{f_N}$ take values on a compact domain. More precisely, compared to the noncompact M-brane configuration, we have

$$0 \le t_{f_1} \le \cdots \le t_{f_N} < \infty \quad \to$$
$$0 \le t_{f_1} \le t_{f_2} \le \cdots \le t_{f_N} \le 2\pi R_6.$$

This implies that the tensions of M-strings and little strings can only take a finite maximum value. This property is imperative for the little string theories to retain stringy features such as T-duality, as we discuss in the following section.

To better explain the nature of the little string BPS excitations, we can compare the multiple M5-branes on a transverse circle with multiple Dp-branes on a transverse circle. In this comparison, we interpret the M-strings (i.e. open M2-branes) as noncritical counterparts of open fundamental strings, while a little string ground state [defined by the configuration $(k_1, \dots, k_N) = (1, \dots, 1)$] is the noncritical counterpart of a closed fundamental string. This analogy points to two very important facts: first, in the same way as multiple open fundamental strings on the Dp-branes can form a closed string and move freely in ambient ten-dimensional bulk spacetime, multiple open M2-branes ending on M5-branes can form a closed M2-brane and move freely in 11-dimensional spacetime. Secondly, while the open fundamental strings can carry a fractional winding number around the transverse circle, the M-strings also carry fractional winding numbers around the transverse circle. These are measured by the chemical potentials $(t_{f_1}, \dots, t_{f_N})$ and the fugacities $(Q_{f_1}, \dots, Q_{f_N})$. However, what makes the little strings very different from fundamental strings is that, in the decoupling limit Eq. (2.8), the little strings are confined inside the five-brane world volume, viz. the six-dimensional spacetime the little string theories live in.

2. *Relation to gauge theory and T-duality*

The above discussion establishes a connection between the little string theories and the M-brane configuration of Sec. II A. Therefore, we can also relate the former to the two gauge theories that we discussed in Sec. II C. To make this connection precise, we first need to discuss the moduli spaces of type IIa and IIb little string theories and explain their connection to gauge theory 1 and gauge theory 2, respectively.

To this end, we begin in six dimensions by first considering the direction x^1 in the M-brane configuration to be noncompact (i.e. $R_1 \to \infty$).[9] In this framework, the nonchiral type IIa and the chiral IIb little string theories are defined on the six-dimensional world volume of the N five-branes and preserve 16 supercharges each. Their respective moduli spaces of supersymmetric vacua are

$$\mathcal{M}^{\rm 6d}_{\rm IIa} = (\mathbb{R}^4)^N/S_N \qquad \text{and} \qquad \mathcal{M}^{\rm 6d}_{\rm IIb} = (\mathbb{R}^4 \times \mathbb{S}^1)^N/S_N. \tag{2.9}$$

The $\mathbb{S}^1$ in $\mathcal{M}^{\rm 6d}_{\rm IIb}$ can be understood from the definition of the IIb little string theory in terms of the world volume of M5-branes. In the brane configuration of Sec. II A, it corresponds

[9] The direction x^1 is singled out since it is untwisted with respect to the deformations (2.6).

to the $\mathbb{S}^1_{R_6}$ of the compact x^6 direction. Notice that the two spaces (2.9) cannot be related to each other by any duality transformation. Indeed, from the perspective of the type IIA and IIB string theories, the only compact direction that is not twisted by (2.6) (and would therefore lend itself to T-duality) is x^6, which, however, is transverse to the five-branes.

Next, we consider five-dimensional little string theories by taking the direction x^1 to be compact (i.e. R_1 to be finite). This compactification has very different impacts on the two moduli spaces (2.9): on the one hand, the IIb moduli space remains the same, since the six-dimensional tensor multiplet does not generate a scalar when reduced to five dimensions. On the other hand, the moduli space of IIa little string theory gets enlarged, since the six-dimensional vector multiplet generates a scalar in five dimensions. This scalar comes from the Wilson loop around the dual circle $\tilde{\mathbb{S}}^1_{1/R_1}$ and takes values over the interval $[0, R_1]$.[10] Therefore, the moduli spaces of the five-dimensional little string theories are

$$\mathcal{M}^{5d}_{\text{IIa}} = \frac{(\mathbb{R}^4 \times \mathbb{S}^1_{R_1})^N}{S_N}, \quad \text{and} \quad \mathcal{M}^{5d}_{\text{IIb}} = \frac{(\mathbb{R}^4 \times \mathbb{S}^1_{R_6})^N}{S_N}. \tag{2.10}$$

We see that parameters of circle-compactified IIa and IIb little string theories are mapped to each other by the exchange of the radii

$$R_1 \leftrightarrow R_6, \tag{2.11}$$

while the parameters originating from $\mathbb{R}^4_\perp$ are the same.

We stress that Eq. (2.11) is the manifestation of T-duality on the circle-compactified little string theories. Phrased differently, while from the perspective of the fundamental string theory the T-duality corresponds to the map $R_1 \leftrightarrow 1/R_1$, from the perspective of the circle-compactified five-branes the T-duality manifests as exchanging circle-wrapped IIA and IIB five-branes. This T-duality commutes with the decoupling limit Eq. (2.8), so the T-duality on the circle-compactified IIa and IIb little string theories is realized by Eq. (2.11).

Note also that, in the description in terms of the elliptically fibered Calabi-Yau manifold X_N, the exchange Eq. (2.11) corresponds to fiber-base duality, i.e. the exchange of the two Kähler parameters, τ and ρ of X_N.

With the moduli spaces identified for the circle-compactified little string theories, we are now ready to discuss their relation to the exact marginal couplings that specify the gauge theory descriptions introduced in Sec. II C. The U-duality map discussed in Sec. II A indicates that the IIa little string theory compactified on $\mathbb{S}^1_{R_1}$ is most naturally described by the Coulomb branch of the five-dimensional $U(N)$ gauge theory with the gauge coupling given by τ. At a generic point of the Coulomb branch, the theory is described by a $[U(1)]^N$ quiver gauge theory, and therefore the Coulomb branch is spanned by $t_{f_1}, \ldots, t_{f_N}$. Thus, we identify gauge theory 1 with the gauge theory description of the circle-compactified IIa little string theory.

Performing the T-duality $R_1 \to 1/R_1$, we obtain circle-compactified IIb little string theory, which is also described by a $[U(1)]^N$ quiver gauge theory. Since $\mathbb{S}^1_{R_1}$ spans part of the Coulomb branch [as becomes apparent from $\mathcal{M}^{5d}_{\text{IIa}}$ in (2.10)], the gauge coupling constants must be encoded by the brane configuration along the x^6 direction. Indeed, they are given by $t_{f_1}, \ldots, t_{f_N}$, while τ is the Coulomb branch parameter. That is, we can identify gauge theory 2 with the circle-compactified IIb little string theory.[11]

The T-duality (2.11) between the five-dimensional IIa and IIb little string theories suggests that their partition functions Z_{IIa} and Z_{IIb} are related to each other upon exchange of τ and ρ:

$$Z_{\text{IIa}}(\tau, \rho) = Z_{\text{IIb}}(\rho, \tau), \tag{2.12}$$

where we have only displayed the dependence on τ and ρ to save writing. Actually, the connection of the little string theories to the M-brane configuration discussed in Sec. II A and the dual Calabi-Yau threefold X_N suggests

$$Z_{\text{IIa}}(\tau, \rho) = \mathcal{Z}_{X_N}(\tau, \rho) \quad \text{and} \quad Z_{\text{IIb}}(\tau, \rho) = \mathcal{Z}_{X_N}(\rho, \tau), \tag{2.13}$$

where $\mathcal{Z}_{X_N}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}})$ is the topological string partition function associated with the elliptic Calabi-Yau threefold X_N. This makes (2.12) manifest.

Indeed, in Sec. V, we provide relations between BPS-counting functions of little string configurations with integer (i.e. nonfractional) winding and those with integer momentum, and find that they are in line with this proposal. A more careful study of (2.13) for general configuration and its implications is currently under way [25].

III. PARTITION FUNCTIONS

In this section, we obtain the partition function of BPS states corresponding to the brane configuration introduced in Sec. II A. The most efficient way to compute the partition function is to begin from the geometric perspective, i.e. with the toric Calabi-Yau threefold X_N introduced in Sec. II B. The topological string partition function on

[10] Here, we are invoking that, starting from the compact M-brane configuration as defining IIb little string theory on $\mathbb{S}^1_{R_1}$, compactification on the T-dual circle yields IIa little string theory on $\tilde{\mathbb{S}}^1_{1/R_1}$.

[11] Our identifications agree with the little string world-sheet description of [43], further discussed in [44].

X_N will be denoted by $\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$. It can subsequently be related to the partition function of the six-dimensional Ω-deformed field theories discussed in Sec. II C.

A. Topological string partition function

The refined topological vertex formalism [7,9] can be used to determine the topological string partition $\mathcal{Z}_{X_N}$ of a toric Calabi-Yau threefold X_N using its toric web diagram shown in Fig. 2. Recall the fact that X_N can be related to two dual gauge theories (as discussed in Sec. II C) corresponds geometrically to fiber-base duality [45]. At a computational level, it is related to the choice of a "preferred direction" in the refined topological vertex formalism [9]. Specifically, we need to choose a set of parallel edges in the web in Fig. 2 such that every vertex is one of the end points of one such edge. While the topological string partition function is independent of this choice (i.e. it is the same for each such choice), it leads to different gauge theory interpretations of the partition function. From Fig. 2, it is clear that there are two distinct choices for the preferred direction: vertical and horizontal.

Before we discuss the form of the refined topological string partition function for a specific choice of the preferred direction, let us recall that the refined topological string partition function captures the degeneracies of BPS states coming from M2-branes wrapping the holomorphic curves in the Calabi-Yau threefold X on which M-theory is compactified. Denote by $N_C^{(j_L,j_R)}$ the number of BPS states, with spin content (j_L, j_R) under the five-dimensional little group $SU(2)_L \times SU(2)_R$, coming from an M2-brane wrapped in the holomorphic curve C. Then, the refined topological string partition function is given by [8,46,47]

$$\mathcal{Z}_X = \mathrm{PExp}(F_X),$$
$$F_X = \sum_{C\in H_2(X,\mathbb{Z})} e^{-A(C)} F_C(\epsilon_1, \epsilon_2),$$

where PExp is the plethystic exponential, $A(C)$ is the complexified area of C, and F_C captures the degeneracies of single-particle states coming from M2-branes wrapping $C \subset X$,

$$F_C = \sum_{j_L,j_R} N_C^{(j_L,j_R)}(-1)^{2j_L+2j_R}\left[\left(\sqrt{\frac{t}{q}}\right)^{-j_R} + \ldots + \left(\sqrt{\frac{t}{q}}\right)^{+j_R}\right] \times [(\sqrt{tq})^{-j_L} + \cdots + (\sqrt{tq})^{+j_L}],$$

with $(q,t) = (e^{i\epsilon_1}, e^{-i\epsilon_2})$. For a generic Calabi-Yau threefold, $N^{(j_L,j_R)}$ can jump under complex structure deformations such that $\sum_{j_R}(-1)^{2j_R}N_C^{(j_L,j_R)}$ remains constant. Since toric Calabi-Yau threefolds do not admit any complex structure deformations, therefore $N_C^{(j_L,j_R)}$ are topological invariants captured by the refined topological string partition function. In subsequent sections, we will consider F_C for specific curve classes in the Calabi-Yau threefold X_N and refer to it as the degeneracy counting function or just the counting function.

1. *Vertical description*

If the preferred direction is chosen to be vertical, then the various partitions associated with the horizontal direction can be summed over completely to obtain $\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$ (see [3]):

$$\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2}) = Z_1(m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2})\tilde{\mathcal{Z}}_N^{(1)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2), \tag{3.1}$$

where $Z_1(m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2})$ is the part independent of τ and

$$\tilde{\mathcal{Z}}_N^{(1)} = \sum_{k\geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2}) \tag{3.2}$$

$$= \sum_{\alpha_1\cdots\alpha_N} Q_\tau^{|\alpha_1|+\cdots+|\alpha_N|} \prod_{a=1}^N \frac{\vartheta_{\alpha_a\alpha_a}(Q_m)}{\vartheta_{\alpha_a\alpha_a}(\sqrt{\frac{t}{q}})} \prod_{1\leq a<b\leq N} \frac{\vartheta_{\alpha_a\alpha_b}(Q_{ab}Q_m^{-1})\vartheta_{\alpha_a\alpha_b}(Q_{ab}Q_m)}{\vartheta_{\alpha_a\alpha_b}(Q_{ab}\sqrt{\frac{t}{q}})\vartheta_{\alpha_a\alpha_b}(Q_{ab}\sqrt{\frac{q}{t}})} \tag{3.3}$$

is the part that depends on τ through the fugacity Q_τ. In (3.2), we denote integer partitions as $\alpha_1, \ldots, \alpha_N$. We also use the notation

$$Q_m = e^{2\pi i m}, \qquad Q_\tau = e^{2\pi i\tau}, \qquad q = e^{i\epsilon_1}, \qquad t = e^{-i\epsilon_2}, \qquad Q_{ab} = \prod_{k=a}^{b-1} Q_{f_k}, \tag{3.4}$$

as well as

$$\vartheta_{\mu\nu}(x) = \prod_{(i,j)\in\mu} \theta_1(\rho; x^{-1} t^{-\nu_j^t+i-\frac{1}{2}} q^{-\mu_i+j-\frac{1}{2}}) \prod_{(i,j)\in\nu} \theta_1(\rho; x^{-1} t^{\mu_j^t-i+\frac{1}{2}} q^{\nu_i-j+\frac{1}{2}}). \tag{3.5}$$

Furthermore, $\theta_1(\tau; z)$ is one of the Jacobi theta functions (see [48] for further information),

$$\theta_1(\rho; x) = -iQ_\rho^{\frac{1}{8}}(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \prod_{k=1}^{\infty} (1 - Q_\rho^k)(1 - xQ_\rho^k)(1 - x^{-1}Q_\rho^k). \tag{3.6}$$

Recall that $\rho = 2\pi i R_6$ [see Eq. (2.7)] and $Q_\rho = e^{2\pi i \rho}$.

Associated with the partition function $\tilde{\mathcal{Z}}_N^{(1)}$, we also consider the free energy

$$\Sigma_N(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \mathrm{PLog} \tilde{\mathcal{Z}}_N^{(1)}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2), \tag{3.7}$$

defined in terms of the plethystic logarithm of a function f:

$$\mathrm{PLog} f(\omega, \epsilon_1, \epsilon_2) \coloneqq \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln f(k\omega, k\epsilon_1, k\epsilon_2), \tag{3.8}$$

where $\mu(k)$ is the Möbius function. Physically, the function Σ_N counts single-particle BPS bound states (see [49,50]). As in (3.2), we can equally introduce the fugacity expansion

$$\Sigma_N(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} Q_\tau^k \Sigma_{N,k}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2). \tag{3.9}$$

The coefficient functions can be further expanded in terms of the $N-1$ relative Kähler parameter fugacities $(Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{N-1}})$:

$$\Sigma_{N,k}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k_1,\ldots,k_{N-1}} Q_{f_1}^{k_1} \ldots Q_{f_{N-1}}^{k_{N-1}} \Sigma_{N,k}^{(k_1,\ldots,k_{N-1})}(\rho, m, \epsilon_1, \epsilon_2). \tag{3.10}$$

2. Horizontal description

If in Fig. 2 the preferred direction is chosen to be horizontal, then the topological string partition function $\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$ has the form

$$\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2}) = Z_2(N, \tau, m, \epsilon_{1,2}) \tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2), \tag{3.11}$$

where $Z_2(N, \tau, m, \epsilon_{1,2})$ is the part independent of t_{f_a}. In order to write $\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$, we recall [1] that the topological string partition function can be obtained by gluing together building blocks $W_{\nu_a\nu_{a+1}}$ labeled by the partitions of integers ν_a and ν_{a+1}. The $W_{\nu_a\nu_{a+1}}$'s are open topological string amplitudes but can also be considered as capturing the BPS degeneracies of M2-branes ending on a single M5-brane from either side. The web diagram corresponding to this situation is shown in Fig. 3 below.

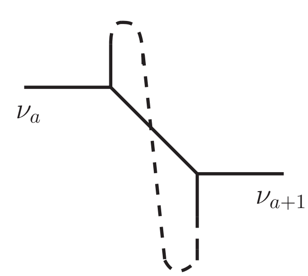


FIG. 3. The building block of a partition function of a configuration of M5-branes wrapping a circle.

The expression for $W_{\nu_a\nu_{a+1}}$ was calculated in [1] using the refined topological vertex formalism [8,9] and is given by

$$W_{\nu_a^t\nu_{a+1}}(\tau,m,t,q)=W_{\emptyset\emptyset}(\tau,m,t,q)D_{\nu_a^t\nu_{a+1}}(\tau,m,t,q), \tag{3.12}$$

where

$$W_{\emptyset\emptyset}(\tau,m,t,q)=\prod_{k=1}^{\infty}\left[(1-Q_\tau^k)^{-1}\prod_{i,j=1}^{\infty}\frac{(1-Q_\tau^kQ_m^{-1}q^{-j+\frac{1}{2}}t^{-i+\frac{1}{2}})(1-Q_\tau^{k-1}Q_mq^{-j+\frac{1}{2}}t^{-i+\frac{1}{2}})}{(1-Q_\tau^kq^{-j+1}t^{-i})(1-Q_\tau^kq^{-j}t^{-i+1})}\right] \tag{3.13}$$

and

$$\begin{aligned}D_{\nu_a^t\nu_{a+1}}(\tau,m,t,q)=&\left[t^{\frac{\|\nu_{a+1}^t\|^2}{2}}q^{\frac{\|\nu_a\|^2}{2}}Q_m^{\frac{|\nu_a|+|\nu_{a+1}|}{2}}\right]^{-1}\\&\times\prod_{k=1}^{\infty}\Bigg[\prod_{(i,j)\in\nu_a}\frac{(1-Q_\tau^kQ_m^{-1}q^{-\nu_{a,i}+j-\frac{1}{2}}t^{-\nu_{a+1,j}^t+i-\frac{1}{2}})(1-Q_\tau^{k-1}Q_mq^{\nu_{a,i}-j+\frac{1}{2}}t^{\nu_{a+1,j}^t-i+\frac{1}{2}})}{(1-Q_\tau^kq^{\nu_{a,i}-j}t^{\nu_{a,j}^t-i+1})(1-Q_\tau^{k-1}q^{-\nu_{a,i}+j-1}t^{-\nu_{a,j}^t+i})}\\&\times\prod_{(i,j)\in\nu_{a+1}}\frac{(1-Q_\tau^kQ_m^{-1}q^{\nu_{a+1,i}-j+\frac{1}{2}}t^{\nu_{a,j}^t-i+\frac{1}{2}})(1-Q_\tau^{k-1}Q_mq^{-\nu_{a+1,i}+j-\frac{1}{2}}t^{-\nu_{a,j}^t+i-\frac{1}{2}})}{(1-Q_\tau^kq^{\nu_{a+1,i}-j+1}t^{\nu_{a+1,j}^t-i})(1-Q_\tau^{k-1}q^{-\nu_{a+1,i}+j}t^{-\nu_{a+1,j}^t+i-1})}\Bigg].\end{aligned} \tag{3.14}$$

Here, our notation follows (3.4). Furthermore, for a partition ν of length $\ell(\nu)$ we define

$$|\nu|=\sum_{i=1}^{\ell(\nu)}\nu_i,\qquad \|\nu\|^2=\sum_{i=1}^{\ell(\nu)}\nu_i^2, \tag{3.15}$$

and ν^t denotes the transposed partition. From (3.12), the partition function can be calculated by gluing several $D_{\nu_a\nu_{a+1}}$'s together by summing over the partitions ν_a and ν_{a+1}. For example, the partition functions of X_2 which is dual to the brane configuration consisting of two M5-branes on the circle is given by

$$\mathcal{Z}_{X_2}=\sum_{\nu_1,\nu_2}(-Q_{f_1})^{|\nu_1|}(-Q_{f_2})^{|\nu_2|}W_{\nu_1\nu_2^t}(\tau,m,t,q)W_{\nu_2\nu_1^t}(\tau,m,q,t). \tag{3.16}$$

For general N, the toric web diagram of the Calabi-Yau threefold X_N that is dual to N M5-branes distributed on $\mathbb{S}^1$-compactified x^6 direction is given in Fig. 2. The latter encodes how various $W_{\nu_a\nu_{a+1}}$'s need to be glued together to compute the partition function. Specifically,

$$\mathcal{Z}_{X_N}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_1,\epsilon_2)=(W_{\emptyset\emptyset})^N\underbrace{\sum_{\nu_1,\ldots,\nu_N}\left(\prod_{a=1}^{N}(-Q_{f_a})^{|\nu_a|}\right)Z_{\nu_1\nu_2\cdots\nu_N}(\tau,m,\epsilon_1,\epsilon_2)}_{\tilde{\mathcal{Z}}_N^{(2)}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_1,\epsilon_2)}, \tag{3.17}$$

where the t_{f_a}-independent contribution in (3.11) is given by

$$Z_2(N,\tau,m,\epsilon_{1,2})=(W_{\emptyset\emptyset}(\tau,m,t,q))^N \tag{3.18}$$

and furthermore

$$Z_{\nu_1\nu_2\cdots\nu_N}=\begin{cases}D_{\nu_1\nu_2^t}(t,q)D_{\nu_2\nu_3^t}(q,t)D_{\nu_3\nu_4^t}(t,q)\cdots D_{\nu_N^t\nu_1}(q,t) & \text{if } N \text{ is even,}\\ D_{\nu_1\nu_2^t}(t,q)D_{\nu_2\nu_3^t}(q,t)D_{\nu_3\nu_4^t}(t,q)\cdots D_{\nu_N^t\nu_1}(t,q) & \text{if } N \text{ is odd.}\end{cases} \tag{3.19}$$

Using Eq. (3.12), the partition function can be written as

$$\tilde{\mathcal{Z}}_N^{(2)}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_1,\epsilon_2)=\sum_{\nu_1,\ldots,\nu_N}\left(\prod_{a=1}^{N}(-Q_{f_a})^{|\nu_a|}\right)\prod_{a=1}^{N}\prod_{(i,j)\in\nu_a}\frac{\theta_1(\tau;z_{ij}^a)\theta_1(\tau;v_{ij}^a)}{\theta_1(\tau;w_{ij}^a)\theta_1(\tau;u_{ij}^a)}, \tag{3.20}$$

where the sum runs over the set of N many integer partitions $\{\nu_1, \nu_2, \ldots, \nu_N\}$. For their arguments, we introduced the following shorthand notations:

$$\begin{aligned} z^a_{ij} &= -m + \epsilon_1\left(\nu_{a,i} - j + \frac{1}{2}\right) - \epsilon_2\left(\nu^t_{a+1,j} - i + \frac{1}{2}\right), \\ v^a_{ij} &= -m - \epsilon_1\left(\nu_{a,i} - j + \frac{1}{2}\right) + \epsilon_2\left(\nu^t_{a-1,j} - i + \frac{1}{2}\right), \\ w^a_{ij} &= \epsilon_1(\nu_{a,i} - j + 1) - \epsilon_2(\nu^t_{a,j} - i), \\ u^a_{ij} &= \epsilon_1(\nu_{a,i} - j) - \epsilon_2(\nu^t_{a,j} - i + 1). \end{aligned} \tag{3.21}$$

From the viewpoint of the brane configuration of Sec. II A, the partition function (3.20) captures BPS excitations of the stretched M2-branes. The fact that M5-branes are distributed on the $\mathbb{S}^1$-compactified x^6 direction is reflected in (3.20) through the identifications

$$\nu_{N+1} = \nu_1 \quad \text{and} \quad \nu_0 = \nu_N. \tag{3.22}$$

Again, associated with the partition function $\tilde{\mathcal{Z}}^{(2)}_N$, we may introduce the free energy

$$\begin{aligned} &\Omega_N(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) \\ &\quad = \mathrm{PLog}\tilde{\mathcal{Z}}^{(2)}_N(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2), \end{aligned} \tag{3.23}$$

where PLog is defined in (3.8). The free energy (3.23) in turn can be expanded in powers of the Kähler moduli $(t_{f_1}, t_{f_2}, \ldots, t_{f_N})$ [equivalently, $(t_{f_1}, t_{f_2}, \ldots, t_{f_{N-1}})$ and ρ]:

$$\begin{aligned} &\Omega_N(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) \\ &\quad = \sum_{k_1,\ldots,k_N \geq 0} Q^{k_1}_{f_1} \cdots Q^{k_N}_{f_N} G^{(k_1,\ldots,k_N)}(\tau, m, \epsilon_1, \epsilon_2), \end{aligned} \tag{3.24}$$

where $G^{(0,\ldots,0)} = 0$. Written in this form, the functions $G^{(k_1,\ldots,k_N)}$ encode the degeneracies of single-particle BPS bound states in configurations with N M5-branes distributed on a circle with k_i M2-branes stretched between the ith and the $(i+1)$th M5-brane for $i = 1, \ldots, N$.

3. Noncompact brane configuration

For completeness, we also present the topological string partition function for the case of a *noncompact* x^6 direction, i.e. for the case that the horizontal direction in Fig. 2 is decompactified to $\mathbb{R}^1$. From the brane configuration, this corresponds to the limit in which one of the distances t_{f_a} is taken to infinity.

In the simplest case, for $N = 2$, if we take the limit $Q_{f_2} \mapsto 0$ in (3.16), we get the partition function of the Calabi-Yau threefold X_2 which is an A_1 space fibered over $\mathbb{T}^2$ and is dual to the brane configuration in which we have two M5-branes on a line, i.e. separated from each other by t_{f_1},

$$\mathcal{Z}^{\text{line}}_{X_2} = \sum_{\nu} (-Q_{f_1})^{|\nu|} W_{\emptyset\nu}(\tau, m, t, q) W_{\nu^t\emptyset}(\tau, m, q, t). \tag{3.25}$$

More generally, the partition functions of an $N \geq 2$ M5-brane separated along a noncompact direction x^6 can be obtained by restricting one of the partitions, say $\nu_N = \nu_0$, to be trivial:

$$\begin{aligned} &\mathcal{Z}^{\text{line}}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) \\ &\quad = (W_{\emptyset\emptyset})^N \sum_{\substack{\nu_1,\ldots,\nu_{N-1} \\ \nu_0 = \nu_N = \emptyset}} \left(\prod_{a=1}^{N} (-Q_{f_a})^{|\nu_a|}\right) \\ &\qquad \times \prod_{a=1}^{N} \prod_{(i,j)\in\nu_a} \frac{\theta_1(\tau; z^a_{ij})\theta_1(\tau; v^a_{ij})}{\theta_1(\tau; w^a_{ij})\theta_1(\tau; u^a_{ij})}. \end{aligned} \tag{3.26}$$

This is indeed the sole contribution to the partition function in the limit $Q_{f_N} = 0$, corresponding to the infinite volume limit of t_{f_N}, which sends the interval between the first and Nth M5-brane on $\mathbb{S}^1$ to infinity. The partition function $\mathcal{Z}^{\text{line}}_{X_N}$ has already been discussed in [1–3,5].

B. Gauge theory partition functions

Given the topological string partition function $\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \cdot, t_{f_N}, \epsilon_{1,2})$, we can extract the instanton partition functions of the two gauge theories associated with X_N as explained in Sec. II. 3. This depends on the identification of the parameters of the affine A_{N-1} fibration over $\mathbb{T}^2$ discussed earlier with the parameters of each gauge theory.

1. Gauge theory 1

We first discuss the reduction of the brane configuration (2.1) over $\mathbb{S}^1(x^1)$ to a five-dimensional $U(N)$ gauge theory. We identify the Kähler parameter τ of $\mathbb{T}^2$ with its gauge coupling constant, and extract the Nekrasov (instanton) partition function by dividing out the classical and one-loop contribution in the following manner:

$$\begin{aligned} &\tilde{\mathcal{Z}}^{(1)}_N(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) \\ &\quad = \frac{\mathcal{Z}_{X_N}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}})}{\lim_{\tau\mapsto i\infty}\mathcal{Z}_{X_N}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}})} \qquad (3.27) \\ &\quad = \sum_{k\geq 0} Q^k_\tau C_{N,k}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2). \qquad (3.28) \end{aligned}$$

Here, we have also identified the Kähler parameters of X_N (which we parametrize by $t_{f_1}, \ldots, t_{f_{N-1}}$ and ρ) with the gauge theory parameters of the configuration space

$(\mathbb{S}^1)^N/S_N$. The explicit expression for $\tilde{\mathcal{Z}}_N^{(1)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2})$ is given in Eq. (3.3).

Thus, the topological string partition function $\mathcal{Z}_{X_N}$ is the supersymmetric partition function of gauge theory 1 introduced in Sec. II C. The quantity $\tilde{\mathcal{Z}}_N^{(1)}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$ is its instanton contribution, i.e. the coefficient function $C_{N,k}$ in (3.27) encodes the charge k instanton contribution. Specifically, if $\mathcal{M}(N,k)$ denotes the moduli space of $SU(N)$ instantons of charge k, then the coefficient $C_{N,k}$ is the elliptic genus of $\mathcal{M}(N,k)$ (see [8,9]):

$$C_{N,k}(m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) = \phi_{\mathcal{M}(N,k)}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2), \tag{3.29}$$

with ρ being the elliptic parameter of the elliptic genus. Furthermore, $(t_{f_1}, \ldots, t_{f_{N-1}})$ are the equivariant deformation parameters associated with the Cartan $U(1)^{N-1}$ global symmetry and (ϵ_1, ϵ_2) are the equivariant parameters of the $U(1) \times U(1)$ action on $\mathcal{M}(N,k)$ coming from the Cartan of the $SO(4)$ action on $\mathbb{C}^2$.

Finally, in light of the discussion in Sec. II D, we see that the coefficients $\Sigma_{N,k}^{(k_1,\ldots,k_{N-1})}(\rho, m, \epsilon_1, \epsilon_2)$ defined in (3.10) encode the BPS degeneracies of type IIa little strings with charge configuration $(k_1 \cdots, k_{N-1})$.

2. Gauge theory 2

Upon T-dualizing along $\mathbb{S}^1(x^6)$, the M5-branes are mapped to an affine A_{N-1} geometry. This gives a five-dimensional $[U(1)]^N$ affine quiver gauge theory. We identify the Kähler parameters $t_{f_1}, t_{f_2}, \ldots, t_{f_N}$—equivalently, $(t_{f_1}, t_{f_2}, \ldots, t_{f_{N-1}})$ and ρ with the gauge coupling constants—and extract the BPS state partition function by dividing out the vacuum contribution

$$\begin{aligned}\tilde{\mathcal{Z}}_N^{(2)}&(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)\\ &= \frac{\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N})}{\lim_{Q_{f_1}\mapsto 0}\ldots\lim_{Q_{f_N}\mapsto 0}\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)},\\ &= \sum_{\nu_1,\ldots,\nu_N}\left(\prod_{a=1}^N(-Q_{f_a})^{|\nu_a|}\right)Z_{\nu_1\nu_2\cdots\nu_N}(\tau, m, \epsilon_1, \epsilon_2).\end{aligned} \tag{3.30}$$

The explicit form of $\tilde{\mathcal{Z}}_N^{(2)}$ is already given in Eq. (3.20), so the coefficient functions are

$$Z_{\nu_1\nu_2\cdots\nu_N}(\tau, m, \epsilon_1, \epsilon_2) = \prod_{a=1}^N\prod_{(i,j)\in\nu_a}\frac{\theta_1(\tau; z_{ij}^a)\theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a)\theta_1(\tau; u_{ij}^a)}. \tag{3.31}$$

Thus, the topological string partition function $\mathcal{Z}_{X_N}$ is the supersymmetric partition function of gauge theory 2 of Sec. II. 3, and the corresponding $\tilde{\mathcal{Z}}_N^{(2)}$ contains the contribution of BPS excitations. Since the gauge theory is $[U(1)]^N$ quiver gauge theory, therefore the pointlike instantons are labeled by $(k_1, k_2, \ldots, k_N)$ where k_a is the pointlike instanton charge for the ath factor. The corresponding instanton moduli space is $N_{k_1,\ldots,k_N} \coloneqq \mathrm{Hilb}^{k_1}[\mathbb{C}^2] \times \mathrm{Hilb}^{k_2}[\mathbb{C}^2] \times \cdots \times \mathrm{Hilb}^{k_N}[\mathbb{C}^2]$ where $\mathrm{Hilb}^k[\mathbb{C}^2]$ is the Hilbert scheme of k points on $\mathbb{C}^2$. The coefficient functions $Z_{k_1\cdots k_N}$ are given by an equivariant integral over $N_{k_1,\ldots,k_N}$ [1,3].

In light of the discussion in Sec. II D, we see that the coefficients $G^{(k_1,\ldots,k_N)}(\tau, m, \epsilon_1, \epsilon_2)$ encode degeneracies of type IIb little strings with charge configuration $(k_1, \ldots, k_N)$.

3. Noncompact partition function

For comparison, we also recall the instanton partition function in the limit $R_6 \to \infty$:

$$\begin{aligned}\tilde{\mathcal{Z}}_N^{\text{line}}&(\tau,m,t_{f_1},\ldots,t_{f_{N-1}},\epsilon_1,\epsilon_2)\\ &= \frac{\mathcal{Z}_{X_N}^{\text{line}}(\tau,m,t_{f_1},\ldots,t_{f_{N-1}})}{\lim_{Q_{f_1}\mapsto 0}\ldots\lim_{Q_{f_{N-1}}\mapsto 0}\mathcal{Z}_{X_N}^{\text{line}}(\tau,m,t_{f_1},\ldots,t_{f_{N-1}},\epsilon_1,\epsilon_2)},\\ &= \sum_{\substack{\nu_1,\ldots,\nu_{N-1}\\ \nu_0=\nu_N=\emptyset}}\left(\prod_{a=1}^N(-Q_{f_a})^{|\nu_a|}\right)\prod_{a=1}^N\prod_{(i,j)\in\nu_a}\frac{\theta_1(\tau;z_{ij}^a)\theta_1(\tau;v_{ij}^a)}{\theta_1(\tau;w_{ij}^a)\theta_1(\tau;u_{ij}^a)},\end{aligned}$$

where $\mathcal{Z}_{X_N}^{\text{line}}$ is introduced in (3.26). We can similarly define the free energy

$$\Omega_N^{\text{line}}(\tau, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \mathrm{PLog}\tilde{\mathcal{Z}}_N^{\text{line}}(\tau, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2), \tag{3.32}$$

which we can expand in counting the functions of single-particle BPS bound states:

$$\Omega_N^{\text{line}}(\tau, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k_1,\ldots,k_{N-1}\geq 0} Q_{f_1}^{k_1}\cdots Q_{f_{N-1}}^{k_{N-1}} F^{(k_1,\ldots,k_{N-1})}(\tau, m, \epsilon_1, \epsilon_2). \tag{3.33}$$

We discussed the properties of $\tilde{\mathcal{Z}}_N^{\text{line}}$ and $F^{(k_1,\ldots,k_{N-1})}$ in great detail in [5]. The latter counts the BPS bound states of configurations in which N M5-branes are distributed along a noncompact direction with k_i M2-branes stretched between the ith and the $(i+1)$th M5-brane.

For the reader's convenience, we provide the following overview of the notation for the three different theories

Quantity	Gauge theory 1	Gauge theory 2	Noncompact theory
variables	$\tau, \rho, m, t_{f_1}, \ldots t_{f_{N-1}}, \epsilon_{1,2}$	$\tau, m, t_{f_1}, \ldots t_{f_N}, \epsilon_{1,2}$	$\tau, m, t_{f_1}, \ldots t_{f_{N-1}}, \epsilon_{1,2}$
partition function	$\tilde{\mathcal{Z}}_N^{(1)}(\tau, \rho, m, t_{f_a}, \epsilon_{1,2})$	$\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_a}, \epsilon_{1,2})$	$\tilde{\mathcal{Z}}_N^{\mathrm{line}}(\tau, m, t_{f_a}, \epsilon_{1,2})$
free energy	$\Sigma_{N,k}(\rho, m, t_{f_a}, \epsilon_{1,2})$	$\Omega_N(\tau, m, t_{f_a}, \epsilon_{1,2})$	$\Omega_N^{\mathrm{line}}(\tau, m, t_{f_a}, \epsilon_{1,2})$
counting functions	$\Sigma_{N,k}^{(\{k_i\})}(\rho, m, \epsilon_{1,2})$	$G^{(\{k_i\})}(\tau, m, \epsilon_{1,2})$	$F^{(\{k_i\})}(\tau, m, \epsilon_{1,2})$

In [5] it was argued that $\lim_{\epsilon_2\to 0} F^{(k_1,\ldots,k_{N-1})}(\tau, m, \epsilon_1, \epsilon_2)$ are related to the equivariant elliptic genus of the moduli space of monopole strings with charge $(k_1, k_2, \ldots, k_{N-1})$. More precisely, if $\mathcal{M}_{k_1,\ldots,k_{N-1}}$ is the moduli space of charge $(k_1, \ldots, k_{N-1})$ monopoles, then its elliptic genus $\phi(\mathcal{M}_{k_1,\ldots,k_{N-1}})$ is given by

$$\phi(\mathcal{M}_{k_1,\ldots,k_{N-1}}) = \lim_{\epsilon_2\mapsto 0} \frac{F^{(k_1,\ldots,k_{N-1})}(\tau, m, \epsilon_1, \epsilon_2)}{F^{(1)}(\tau, m, \epsilon_1, \epsilon_2)}. \quad (3.34)$$

Let us define the analog of the right-hand side of the above equation for the compact brane configuration case,

$$P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1) := \lim_{\epsilon_2\mapsto 0} \frac{G^{(k_1,\ldots,k_N)}(\tau, m, \epsilon_1, \epsilon_2)}{G^{(1)}(\tau, m, \epsilon_1, \epsilon_2)}. \quad (3.35)$$

The function $P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1)$ have modular properties very similar to the right-hand side of Eq. (3.34),

$$\begin{aligned} P_{k_1,\ldots,k_N}(\tau+1, m, \epsilon_1) &= P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1) \\ P_{k_1,\ldots,k_N}\left(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_1}{\tau}\right) &= e^{\frac{2\pi i(m^2-\epsilon_1^2)}{\tau}(K-1)} \\ &\quad \times P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1), \\ P_{k_1,\ldots,k_N}(\tau, m+\ell\tau+r, \epsilon_1) &= e^{-2\pi i K\ell^2\tau + 4\pi i m K} \\ &\quad \times P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1), \end{aligned} \quad (3.36)$$

where $K = k_1 + \cdots + k_N$. These modular transformation properties together with relation between $F^{(k_1,\ldots,k_{N-1})}$ and $G^{(k_1,\ldots,k_{N-1})}$ lead us to conjecture that $P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1)$ is the equivariant elliptic genus of the moduli space of monopoles of charge $(k_1, \ldots, k_N)$. More specifically, if we denote the relative moduli space of affine A_{N-1} monopoles of charge $(k_1, \ldots, k_N)$ by $\mathcal{M}^{KK}_{k_1,\ldots,k_N}$, then

$$\phi(\mathcal{M}^{KK}_{k_1,\ldots,k_N}) = P_{k_1,\ldots,k_N}(\tau, m, \epsilon_1). \quad (3.37)$$

C. Modular properties

The topological string partition function $\mathcal{Z}_{X_N}$ depends on two different modular parameters, τ and ρ. These transform under the $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ modular group action in the following manner[12]:

$$\begin{aligned} &(\tau, \rho, m, t_{f_a}, \epsilon_1, \epsilon_2) \\ &\quad \mapsto \left(\frac{a\tau+b}{c\tau+d}, \rho, \frac{m}{c\tau+d}, t_{f_a}, \frac{\epsilon_1}{c\tau+d}, \frac{\epsilon_2}{c\tau+d}\right), \end{aligned} \quad (3.38)$$

$$\begin{aligned} &(\tau, \rho, m, t_{f_a}, \epsilon_1, \epsilon_2) \\ &\quad \mapsto \left(\tau, \frac{a\rho+b}{c\rho+d}, \frac{m}{c\rho+d}, \frac{t_{f_a}}{c\rho+d}, \frac{\epsilon_1}{c\rho+d}, \frac{\epsilon_2}{c\rho+d}\right), \end{aligned} \quad (3.39)$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$. The Calabi-Yau threefold X_N is an affine A_{N-1} space fibered over $\mathbb{T}^2$. In this geometric description, τ is the Kähler parameter of the base and therefore the fiber parameters are neutral under the modular transformation (3.38) (see [51]). The parameter ρ is the Kähler parameter of the elliptic fiber in the affine A_{N-1} space.

We will see that the topological string partition function $\mathcal{Z}_{X_N}(\tau, \rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$ is invariant (modulo a holomorphic anomaly [1–3,5] and nonperturbative corrections [51]) under the above transformations, i.e. $\mathcal{Z}_{X_N}$ is manifestly invariant under the $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ modular group action. The full invariance group might actually be larger, as in the case $N = 1$ for which the full invariance group is $Sp(2,\mathbb{Z})$ [8].

1. Transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$

To show that $\mathcal{Z}_{X_N}$ is invariant under Eq. (3.38), we use its form given by Eq. (3.17),

$$\begin{aligned} &\mathcal{Z}_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_{1,2}) \\ &\quad = (W_{\emptyset\emptyset})^N \tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2). \end{aligned} \quad (3.40)$$

The function $\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$ is a sum of a product of Jacobi theta functions $\theta_1(\tau, z)$ given by

[12] Here, we choose a convention in which we treat $(t_{f_1}, \ldots, t_{f_{N-1}}, \rho)$ as independent variables.

$$\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{\nu_1\cdots\nu_N} \prod_{a=1}^{N} (-Q_{f_a})^{|\nu_a|} \prod_{(i,j)\in\nu_a} \frac{\theta_1(\tau; z_{ij}^a)\theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a)\theta_1(\tau; u_{ij}^a)}, \qquad (3.41)$$

where $z_{ij}^a, v_{ij}^a, w_{ij}^a$, and u_{ij}^a are given in Eq. (3.21). The $\theta_1(\tau, z)$ transform under $\tau \mapsto -\frac{1}{\tau}$ in the following manner:

$$\frac{\theta_1(-\frac{1}{\tau}, \frac{z_1}{\tau})}{\theta_1(-\frac{1}{\tau}, \frac{z_2}{\tau})} = e^{\frac{i\pi}{\tau}(z_1^2 - z_2^2)} \frac{\theta_1(\tau, z_1)}{\theta_1(\tau, z_2)}. \qquad (3.42)$$

To understand the nontrivial phase factor, we recall that $\theta_1(\tau, z)$ can be expressed as

$$\theta_1(\tau, z) = \eta^3(\tau)(2\pi i z) \exp\left[\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi i z)^{2k}\right], \qquad (3.43)$$

where $\eta(\tau)$ is the Dedekind eta function, B_{2k} are the Bernoulli numbers, and $E_{2k}(\tau)$ are the Eisenstein series. Equation (3.43) in particular also contains $E_2(\tau)$, which is holomorphic but not a modular form. It is well known that by adding a term

$$E_2(\tau) \mapsto \hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}, \qquad (3.44)$$

it can be made into a modular form of weight 2. However, since the added term is not holomorphic in τ, $\hat{E}_2(\tau, \bar{\tau})$ is nonholomorphic. If we introduce

$$\hat{\theta}_1(\tau, z) = \eta^3(\tau)(2\pi i z) \exp\left[\frac{(2\pi i z)^2}{24} \hat{E}_2(\tau, \bar{\tau}) + \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi i z)^{2k}\right], \qquad (3.45)$$

then the replacement

$$\prod_{a=1}^{N} \prod_{(i,j)\in\nu_a} \frac{\theta_1(\tau; z_{ij}^a)\theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a)\theta_1(\tau; u_{ij}^a)} \mapsto \prod_{a=1}^{N} \prod_{(i,j)\in\nu_a} \frac{\hat{\theta}_1(\tau; z_{ij}^a)\hat{\theta}_1(\tau; v_{ij}^a)}{\hat{\theta}_1(\tau; w_{ij}^a)\hat{\theta}_1(\tau; u_{ij}^a)} \qquad (3.46)$$

in Eq. (3.41) makes $\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)$ modular invariant under Eq. (3.38). Similarly, since $W_{\emptyset\emptyset}$ is a ratio of products of $\theta_1(\tau, z)$, it too becomes modular invariant under a similar replacement.[13] Thus the complete partition function $\mathcal{Z}_{X_N}$ is invariant under modular transformation modulo the holomorphic anomaly, introduced by $\theta(\tau, z) \mapsto \hat{\theta}_1(\tau, z)$, and possible nonperturbative corrections.

This is a good place to contrast the compact situation we presently consider with the noncompact situation. Without the replacement $\theta(\tau, z) \mapsto \hat{\theta}_1(\tau, z)$, the summand in Eq. (3.41),

$$Z_{\nu_1\cdots\nu_N}(\tau, m, \epsilon_1, \epsilon_2) = \prod_{a=1}^{N} \prod_{(i,j)\in\nu_a} \frac{\theta_1(\tau; z_{ij}^a)\theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a)\theta_1(\tau; u_{ij}^a)}, \qquad (3.47)$$

transforms by a phase factor

$$Z_{\nu_1,\ldots,\nu_N}\left(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_1}{\tau}, \frac{\epsilon_2}{\tau}\right) = e^{\frac{2\pi i r_{\vec{\nu}}}{\tau}} Z_{\nu_1,\ldots,\nu_N}(\tau, m, \epsilon_1, \epsilon_2). \qquad (3.48)$$

Here,

$$r_{\vec{\nu}}(m, \epsilon_1, \epsilon_2) = \frac{1}{2} \sum_{a=1}^{N} \sum_{(i,j)\in\nu_a} ((z_{ij}^a)^2 + (v_{ij}^a)^2 - (w_{ij}^a)^2 - (u_{ij}^a)^2), \qquad (3.49)$$

which depends explicitly on the shape of the partitions $\{\nu_1, \ldots, \nu_N\}$. This is in contrast to the noncompact case $\mathcal{Z}_N^{\mathrm{line}}$ in Eq. (3.26): as discussed in [5], for $\nu_N = \emptyset$, $r_{\nu_1,\ldots,\nu_{n-1},\emptyset}(m, \epsilon_1, \epsilon_2)$ depends on the size of the partitions $|\nu_1|, \ldots, |\nu_{N-1}|$ but not on their shape. Roughly speaking, this difference between noncompact and compact situations originates from whether the end point partitions ν_0, ν_N are trivial or not.

For a nontrivial ν_N, we can write $r_{\vec{\nu}}$ in the following suggestive form:

$$r_{\vec{\nu}}(m, \epsilon_1, \epsilon_2) = Km^2 + \left(p_{\vec{\nu}} - \frac{K}{2}\right)\epsilon_+^2 + \left(-p_{\vec{\nu}} - \frac{K}{2}\right)\epsilon_-^2 \quad \text{with} \quad K = \sum_{i=1}^{N} |\nu_i|, \qquad (3.50)$$

where only $p_{\vec{\nu}}$ depends on the form of the partitions. From the brane configuration point of view, K corresponds to the total number of M2-branes stretched between the N M5-branes. It is clear from Eq. (3.50) that the partition function (3.20) has interesting modular properties in the Nekrasov-Shatashvili (NS) limit $\epsilon_2 \to 0$ (see [52,53])[14] such that $\epsilon_+ = \epsilon_- = \epsilon_1/2$. Indeed, in this case, we have

[13] Since it is a product of an infinite number of theta functions its modular properties are better understood by writing it in terms of a double elliptic gamma function. In this way, one can show that it satisfies a nonperturbative modular transformation, i.e. that it is modular invariant up to nonperturbative corrections in Ω-deformation parameters [51].

[14] For a recent application of the NS limit to monopoles and vortices in the Higgs phase, see [54].

$$\lim_{\epsilon_2\to 0} r_{\tilde{\nu}}(m,\epsilon_1,\epsilon_2) = Km^2 - \frac{K}{4}\epsilon_1^2, \tag{3.51}$$

which depends only on $\{\nu_1,\ldots,\nu_N\}$ through K and hence can be absorbed in Q_{f_a}, making the partition function modular invariant without the holomorphic anomaly at the expense of making t_{f_a} transform as

$$t_{f_a} \mapsto t_{f_a} - \left(m^2 - \frac{\epsilon_1^2}{4}\right). \tag{3.52}$$

In our previous work [5], we gave a physical interpretation for the necessity of the NS limit when comparing BPS-counting functions of M- and monopole-string excitations (see also [4]).

2. Transformation $\rho \mapsto \frac{a\rho+b}{c\rho+d}$

Now let us consider the transformation with respect to ρ given by (3.39). To study it, we use the form of the topological string partition function of X_N given by Eq. (3.1),

$$\begin{aligned}&\mathcal{Z}_{X_N}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_{1,2})\\&\quad = Z_1(m,t_{f_1},\ldots,t_{f_N},\epsilon_{1,2})\tilde{\mathcal{Z}}_N^{(1)}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_1,\epsilon_2).\end{aligned} \tag{3.53}$$

We recall from (3.2) that the function $\tilde{\mathcal{Z}}_N^{(1)}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_1,\epsilon_2)$ is given by

$$\begin{aligned}\tilde{\mathcal{Z}}_N^{(1)} &= \sum_{\alpha_1\cdots\alpha_N} Q_\tau^{|\alpha_1|+\cdots+|\alpha_N|}\prod_{a=1}^{N}\frac{\vartheta_{\alpha_a\alpha_a}(Q_m)}{\vartheta_{\alpha_a\alpha_a}(\sqrt{\frac{t}{q}})}\\&\quad\times\prod_{1\le a<b\le N}\frac{\vartheta_{\alpha_a\alpha_b}(Q_{ab}Q_m^{-1})\vartheta_{\alpha_a\alpha_b}(Q_{ab}Q_m)}{\vartheta_{\alpha_a\alpha_b}(Q_{ab}\sqrt{\frac{t}{q}})\vartheta_{\alpha_a\alpha_b}(Q_{ab}\sqrt{\frac{q}{t}})},\end{aligned} \tag{3.54}$$

whose building blocks are the product of $\theta_1(\tau,z)$ functions:

$$\begin{aligned}\vartheta_{\mu\nu}(x) &= \prod_{(i,j)\in\mu}\theta_1(\rho;x^{-1}t^{-\nu_j^t+i-\frac{1}{2}}q^{-\mu_i+j-\frac{1}{2}})\\&\quad\times\prod_{(i,j)\in\nu}\theta_1(\rho;x^{-1}t^{\mu_j^t-i+\frac{1}{2}}q^{\nu_i-j+\frac{1}{2}}).\end{aligned} \tag{3.55}$$

Since it is a sum over products of $\theta_1(\tau,z)$, as discussed in Sec. (III C 1), it too can be made modular invariant at the expense of introducing a holomorphic anomaly. The function Z_1 in Eq. (3.53) has many properties similar to $W_{\emptyset\emptyset}$. (In a recent study [55], it was shown that Z_1 is modular invariant up to nonperturbative corrections in Ω-deformation parameters in the refined topological string setup.) Thus the complete partition function $\mathcal{Z}_{X_N}(\tau,m,t_{f_1},\ldots,t_{f_N},\epsilon_{1,2})$ is invariant under the modular transformation equation (3.39).

So far, we showed that the topological string partition function $\mathcal{Z}_{X_N}$ can be made fully modular invariant with respect to ρ or τ. These two Kähler parameters are independent, so it is expected that $\mathcal{Z}_{X_N}$ can be made simultaneously modular invariant with respect to both ρ and τ. We will not discuss technical details of the construction here except for remarking that a closely parallel question was answered affirmatively positive in the context of topological string amplitudes of type II string theory compactified on a two-parameter model of elliptically fibered Calabi-Yau threefolds [56,57].

IV. COMPACT VERSUS NONCOMPACT FREE ENERGIES

We start by searching for relations between the BPS-counting functions $F^{(\{k_i\})}(\tau,m,\epsilon_1,\epsilon_2)$ of the noncompact theory and $G^{(\{k_i\})}(\tau,m,\epsilon_1,\epsilon_2)$ of gauge theory 2. We first consider the special class of configurations $\{k_i\}=\{1,\ldots,1\}$ and conjecture the relation for the generic case based on an emergent pattern. We also comment on implications of this pattern on the little string theories.

A. Examples of compact free energies $G^{(k_1,\ldots,k_N)}$

The simplest configuration in the compact case corresponds to a single M2-brane starting and ending on the same M5-brane. In our notation, this corresponds to $N=1$ and $\{k_i\}=(1)$. The BPS bound states of this configuration are counted by

$$G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) = \frac{\theta_1(\tau;m+\epsilon_-)\theta_1(\tau;m-\epsilon_-)}{\theta_1(\tau;\epsilon_1)\theta_1(\tau;\epsilon_2)}. \tag{4.1}$$

On the other hand, the simplest configuration in the noncompact case corresponds to a single M2-brane stretched between two M5-branes, for which the corresponding BPS-counting function is given by

$$F^{(1)}(\tau,m,\epsilon_1,\epsilon_2) = \frac{\theta_1(\tau;m+\epsilon_+)\theta_1(\tau;m-\epsilon_+)}{\theta_1(\tau;\epsilon_1)\theta_1(\tau;\epsilon_2)}. \tag{4.2}$$

Comparing (4.1) with (4.2), we notice the following relation:

$$\begin{aligned}G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) &= -F^{(1)}(\tau,m,\epsilon_1,-\epsilon_2)\\&= -F^{(1)}(\tau,m,-\epsilon_1,\epsilon_2).\end{aligned} \tag{4.3}$$

Most importantly, while both $F^{(1)}$ and $G^{(1)}$ have a first order pole for $\epsilon_2=0$, we find in the NS limit the relation

$$\lim_{\epsilon_2\to 0}\epsilon_2 G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)=\lim_{\epsilon_2\to 0}\epsilon_2 F^{(1)}(\tau,m,\epsilon_1,\epsilon_2), \tag{4.4}$$

which we will use later on.

The next, more complicated configuration is $G^{(1,1)}$, which corresponds to two M5-branes with two M2-branes stretched between them. Their BPS-counting function is given by

$$G^{(1,1)}=\left(\frac{\theta_1(\tau;m+\epsilon_-)\theta_1(\tau;m-\epsilon_-)}{\theta_1(\tau;\epsilon_1)\theta_1(\tau;\epsilon_2)}\right)^2-\left(\frac{\theta_1(\tau;m+\epsilon_+)\theta_1(\tau;m-\epsilon_+)}{\theta_1(\tau;\epsilon_1)\theta_1(\tau;\epsilon_2)}\right)^2. \tag{4.5}$$

Further configurations can be worked out in the same manner. However, their free energies are generically very complicated and we will not display them here in full generality.

Following the reasoning in our previous paper [5] for the noncompact free energies $F^{(\{k_i\})}$, we will consider the NS limit together with a series expansion in the remaining deformation parameter ϵ_1:

$$\lim_{\epsilon_2\to 0}\frac{G^{(\{k_i\})}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)}=\sum_{n=0}^{\infty}\epsilon_1^{2n}g^{n,(\{k_i\})}(\tau,m), \tag{4.6}$$

$$\lim_{\epsilon_2\to 0}\frac{F^{(\{k_i\})}(\tau,m,\epsilon_1,\epsilon_2)}{F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)}=\sum_{n=0}^{\infty}\epsilon_1^{2n}f^{n,(\{k_i\})}(\tau,m). \tag{4.7}$$

Dividing by $F^{(1)}$ and $G^{(1)}$, respectively, removes the ϵ_2^{-1} pole and yields a finite NS limit. Furthermore, the coefficient functions $g^{n,(\{k_i\})}$ and $f^{n,(\{k_i\})}$ are quasimodular Jacobi forms of weight $2n$ and index $K=\sum_a k_a$, i.e. they can be written in the following form:

$$g^{n,(\{k_i\})}(\tau,m)=\sum_{a=0}^{K}s_{2a+2n}^{(n,\{k_i\})}(\tau)(\varphi_{0,1}(\tau,m))^{K-a}(\varphi_{-2,1}(\tau,m))^{a}, \tag{4.8}$$

$$f^{n,(\{k_i\})}(\tau,m)=\sum_{a=0}^{K}t_{2a+2n}^{(n,\{k_i\})}(\tau)(\varphi_{0,1}(\tau,m))^{K-a}(\varphi_{-2,1}(\tau,m))^{a}. \tag{4.9}$$

Here, $s_m^{(n,\{k_i\})}$ and $t_m^{(n,\{k_i\})}$ are quasimodular forms of weight m, which can be written as polynomials in the Eisenstein series [including $E_2(\tau)$]. The explicit expressions for a few $f^{n,(\{k_i\})}$'s and $g^{n,(\{k_i\})}$'s for simple configurations $(\{k_i\})$ are given in Appendix A 1.

Equation (4.4) shows that the free energies of the simplest compact and noncompact configurations of M5-branes agree in the NS limit. In the following, we address the question of whether there are further relations between $G^{(\{k_i\})}$ and $F^{(\{k_i\})}$ for more complicated configurations $\{k_i\}$.

B. Configurations $(1,\ldots,1)$

In [5], we have seen that the free energies for configurations $(\{k_i\})=(\underbrace{1,\ldots,1}_{N-1\ \text{times}})$, i.e. for N parallel M5-branes with a single M2-brane between each of them in the noncompact case are proportional to $F^{(1)}$. Specifically, they can be written in the form

$$F^{(1,\ldots,1)}(\tau,m,\epsilon_1,\epsilon_2)=F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)W(\tau,m,\epsilon_1,\epsilon_2)^{N-2}, \tag{4.10}$$

with

$$W(\tau,m,\epsilon_1,\epsilon_2)=\frac{\theta_1(\tau;m+\epsilon_-)\theta_1(\tau;m-\epsilon_-)-\theta_1(\tau;m+\epsilon_+)\theta_1(\tau;m-\epsilon_+)}{\theta_1(\tau;\epsilon_1)\theta_1(\tau;\epsilon_2)}. \tag{4.11}$$

We therefore expect that the counting function for configurations with N M5-branes on a circle with a single M2-brane between each of them should also simplify in the NS limit. The first nontrivial such configuration is $G^{(1,1)}$ introduced in (4.5). It can be written in the following manner:

$$\begin{aligned}G^{(1,1)}(\tau,m,\epsilon_1,\epsilon_2)&=(G^{(1)}(\tau,m,\epsilon_1,\epsilon_2))^2-(G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2))^2\\&=W(\tau,m,\epsilon_1,\epsilon_2)[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)+G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2)]\\&=W(\tau,m,\epsilon_1,\epsilon_2)[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)+F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)],\end{aligned} \tag{4.12}$$

where in the last line we have used (3.26). Using furthermore (4.4), this relation simplifies in the NS limit,

$$\lim_{\epsilon_2\to 0}\frac{G^{(1,1)}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)}=2W(\tau,m,\epsilon_1,\epsilon_2=0). \tag{4.13}$$

The next, more complicated, configuration is (1,1,1) for which we find

$$\begin{aligned} G^{(1,1,1)}(\tau,m,\epsilon_1,\epsilon_2) &= (G^{(1)}(\tau,m,\epsilon_1,\epsilon_2))^3 - 3G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)(G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2))^2 \\ &\quad + 2(G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2))^3 \\ &= W(\tau,m,\epsilon_1,\epsilon_2)^2[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) + 2G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2)] \\ &= W(\tau,m,\epsilon_1,\epsilon_2)^2[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) + 2F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)]. \end{aligned} \tag{4.14}$$

Generalizing the two examples (4.12) and (4.14) we conjecture the general pattern,

$$\begin{aligned} G^{(1,\ldots,1)}(\tau,m,\epsilon_1,\epsilon_2) &= W(\tau,m,\epsilon_1,\epsilon_2)^{N-1}[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) + (N-1)G^{(1,0)}(\tau,m,\epsilon_1,\epsilon_2)] \\ &= W(\tau,m,\epsilon_1,\epsilon_2)^{N-1}[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2) + (N-1)F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)]. \end{aligned} \tag{4.15}$$

Thus, the counting of a circular M2-brane over N intervals can be generated from the counting of a circular M2-brane over $(N-1)$ intervals via the two-term recursion relation:

$$\begin{aligned} G^{\overbrace{(1,\ldots 1)}^{N\,\text{times}}}(\tau,m,\epsilon_1,\epsilon_2) &= W(\tau,m,\epsilon_1,\epsilon_2)G^{\overbrace{(1,\ldots 1)}^{(N-1)\,\text{times}}}(\tau,m,\epsilon_1,\epsilon_2) \qquad (N>1) \\ &\quad + W(\tau,m,\epsilon_1,\epsilon_2)^{(N-1)}F^{(1)}(\tau,m,\epsilon_1,\epsilon_2). \end{aligned} \tag{4.16}$$

We checked (4.16) explicitly up to $N=5$. A different way to express the recursion relation in Eq. (4.16) is the following:

$$G^{\overbrace{(1,\ldots 1)}^{N\,\text{times}}}(\tau,m,\epsilon_1,\epsilon_2) = W(\tau,m,\epsilon_1,\epsilon_2)(G^{\overbrace{(1,\ldots 1)}^{(N-1)\,\text{times}}}(\tau,m,\epsilon_1,\epsilon_2) + F^{\overbrace{(1,\ldots 1)}^{(N-1)\,\text{times}}}(\tau,m,\epsilon_1,\epsilon_2)).$$

C. General configurations

While the counting functions $G^{(1,\ldots,1)}$ reduce to a universal structure in the NS limit, more general configurations show more involved relations to the noncompact $F^{(\{k_i\})}$'s. To study these configurations, we may work perturbatively in ϵ_1. Indeed, in the NS limit, we have worked out several $G^{(\{k_i\})}$'s and $F^{(\{k_i\})}$'s in Appendix A 2 to various orders in ϵ_1. Built upon these examples, we conjecture a general pattern for these relations.

For a general BPS configuration $G^{(\{k_i\})}$, labeled by the sequence of positive integers $(\{k_i\}) = (k_1,\ldots,k_\ell)$ with $\sum_{i=1}^{\ell} k_i = K$ and $k_{a=1,\ldots,\ell} \neq 0$, we find that

$$\begin{aligned} &G^{(\{k_i\})}(\tau,m,\epsilon_1,0) \\ &\quad = d_{(\{k_i\})} \sum_{\sum m_i = K} a_{(\{m_i\})} F^{(\{m_i\})}(\tau,m,\epsilon_1,0) \end{aligned} \tag{4.17}$$

is compatible with all cases worked out in Appendix A 2. Here, the summation is over all sequences of positive integers $(\{m_i\}) = (m_1,\ldots,m_p)$ such that $\sum_{a=1}^{p} m_a = K$ and $a_{(\{m_i\})}$ and $d_{(\{k_i\})}$ are integer-valued coefficients that depend on the combinatorics of $(\{k_i\})$ and $(\{m_i\})$, respectively.

Specifically, the prefactor $d_{(\{k_i\})}$ is nontrivial (i.e. it differs from 1) if the corresponding $(\{k_i\})$ can be written as an iteration of a smaller (elementary) building block $\{k_j\}_m = (k_1,\ldots,k_m)$ with $m < \ell$ and $n = \frac{\ell}{m} \in \mathbb{N}$:

$$d_{(\{k_i\})} = \begin{cases} n = \frac{\ell}{m} & \text{if } (\{k_i\}) = (\underbrace{\{k_j\}_m,\ldots,\{k_j\}_m}_{n\,\text{times}}) \\ 1 & \text{else} \end{cases}. \tag{4.18}$$

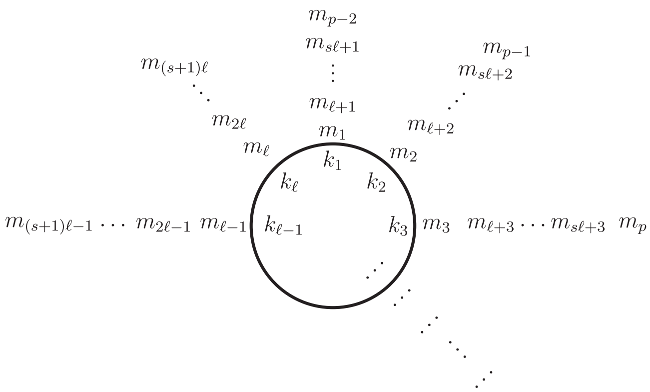
For example, the configuration (2,1,2,1) is the double repetition of the elementary block (2,1) and (1,1,1) is the threefold repetition of the elementary block (1), while (2,2,1) cannot be written as the iteration of a more elementary block:

$$d_{(2,1,2,1)} = 2, \qquad d_{(1,1,1)} = 3, \qquad d_{(2,2,1)} = 1. \tag{4.19}$$

The relative coefficients $a_{(\{m_i\})}$ single out specific configurations $(\{m_i\})$,

$$a_{(\{m_i\})} = \begin{cases} 1 & \text{if } k_i = \sum_{r=0}^{\infty} m_{i+r\ell} \quad (i=1,\ldots,\ell) \\ 0 & \text{else} \end{cases}. \tag{4.20}$$

We can describe this prescription in a more intuitive way: the idea is to construct the compact sequence $(\{k_i\}) = (k_1, \ldots, k_\ell)$ by "tape wrapping" the noncompact sequence $(\{m_i\}) = (m_1, \ldots, m_p)$ multiple times around a circle of circumference ℓ:



The coefficient $a_{(\{m_i\})}$ is nonzero only if the overlapping BPS excitations of the noncompact $(\{m_i\})$ add up to the k_i of the compact BPS excitations

$$\begin{aligned} k_1 &= m_1 + m_{\ell+1} + \ldots, \\ k_2 &= m_2 + m_{\ell+2} + \ldots, \text{etc.}, \\ k_\ell &= m_\ell + m_{2\ell} + \ldots. \end{aligned} \tag{4.21}$$

In the figure above, this corresponds to summing up all multiplicities along the radial directions.

We note that this "wrapping" prescription also reproduces the correct relation between $G^{(\{k_i\})}$ and $F^{(\{m_i\})}$ if one of the k_i vanishes. Due to the cyclic symmetry of the partition k_i, we can without loss of generality choose $k_\ell = 0$. In this case, the conditions we obtain from the wrapping procedure are

$$\begin{aligned} k_1 &= m_1 + m_{\ell+1} + \ldots, \\ k_2 &= m_2 + m_{\ell+2} + \ldots, \text{etc.}, \\ 0 &= m_\ell + m_{2\ell} + \ldots. \end{aligned} \tag{4.22}$$

This in particular indicates that $m_\ell = 0$, which means that the noncompact configuration $\{m_i\}$ has only $\ell - 1$ entries (i.e. it does not fully wrap around the compact configuration). Therefore, the only configuration contributing is

$$m_i = k_i, \ \forall\, i = 1, \ldots, \ell - 1. \tag{4.23}$$

The coefficient in this case, however, is always 1:

$$G^{(\{k_1,\ldots,k_{\ell-1},0\})} = F^{(\{k_1,\ldots,k_{\ell-1}\})}. \tag{4.24}$$

Finally, let us illustrate the procedure with an example: consider $G^{(3,1)}$ (with $\ell = 2$). According to (4.18), we have $d_{(3,1)} = 1$. Furthermore, there are eight compact $F^{(\{m_i\})}$'s with $\sum_i m_i = 4$:

$$\begin{aligned} &F^{(4)}, \qquad F^{(3,1)} = F^{(1,3)}, \\ &F^{(2,2)}, \qquad F^{(2,1,1)} = F^{(1,1,2)}, \\ &F^{(1,2,1)}, \qquad F^{(1,1,1,1)}. \end{aligned} \tag{4.25}$$

For each of these $\{m_i\}$'s, we can compute the sum $\sum_r m_{i+2r}$, which we can tabulate as follows:

$\{m_i\}$	$\{\sum_r m_{i+2r}\}$	$a_{(\{m_i\})}$
(4)	(4)	0
(3,1)	(3,1)	1
(2,2)	(2,2)	0
(2,1,1)	(3,1)	1

$\{m_i\}$	$\{\sum_r m_{i+2r}\}$	$a_{(\{m_i\})}$
(1,3)	(1,3)	1
(1,1,2)	(3,1)	1
(1,2,1)	(2,2)	0
(1,1,1,1)	(2,2)	0

These are indeed the coefficients we find in the genus expansion in (A12).

As another example consider the configuration (2,1,2,1). From (4.18) and (4.20) it follows that $m = 2$ and

$$\begin{aligned} G^{(2,1,2,1)} &= 2(F^{(2,1,2,1)} + F^{(1,2,1,2)} + F^{(1,1,2,1,1)}), \\ &= 2(2F^{(2,1,2,1)} + F^{(1,1,2,1,1)}), \end{aligned} \tag{4.26}$$

where the second equation follows from the fact that $F^{(1,2,1,2)} = F^{(2,1,2,1)}$.

V. MONOPOLE VERSUS INSTANTON FREE ENERGIES

In this section, we discover remarkable relations between the counting function of gauge theory 1 and the counting function of gauge theory 2.

A. Connection between monopole and instanton free energies

The moduli $(t_{f_1}, \ldots, t_{f_{N-1}})$ transform in a nontrivial fashion with respect to (3.39). Therefore, the coefficients $\Sigma_{N,k}^{(k_1,\ldots,k_{N-1})}(\rho, m, \epsilon_1, \epsilon_2)$ [see (3.10)] generically do not transform nicely under (3.39). In the NS limit $\epsilon_2 \to 0$, the function $\Sigma_{N,k}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_{1,2})$ in (3.10) transforms with index k for each t_{f_a} under (3.39) and hence can be reexpressed as an expansion in terms of a basis[15] of theta functions of index k. Thus,

$$\lim_{\epsilon_2\to 0}\epsilon_2\Sigma_{N,k}(\rho,m,t_{f_1},\dots,t_{f_{N-1}},\epsilon_1,\epsilon_2) = \sum_{m_1=0}^{2k-1}\dots\sum_{m_{N-1}=0}^{2k-1}\vartheta_{k,m_1}(\rho,t_{f_1})\dots\vartheta_{k,m_{N-1}} \times(\rho,t_{f_{N-1}})h_{m_1,\dots,m_{N-1}}(\rho,m,\epsilon_1), \quad (5.1)$$

and the coefficients $h_{m_1,\dots,m_{N-1}}$ will transform as vector-valued modular forms under the $SL(2,\mathbb{Z})$ transformation generated by

$$(\rho,m,\epsilon_1)\mapsto\left(\frac{a\rho+b}{c\rho+d},\frac{m}{c\rho+d},\frac{\epsilon_1}{c\rho+d}\right). \quad (5.2)$$

However, they may transform covariantly under certain congruence subgroups. Therefore, the coefficients $h_{m_1,\dots,m_{N-1}}$ have the properties that allow them to be compared with the free energies of certain monopole-string configurations. To check this, we extract the simplest coefficient, $h_{0,\dots,0}(\rho,m,\epsilon_1)=\lim_{\epsilon_2\to 0}\epsilon_2\sigma_{N,k}(\rho,m,\epsilon_1,\epsilon_2)$. through

$$\sigma_{N,k}(\rho,m,\epsilon_1,\epsilon_2) := \Sigma_{N,k}^{(0,\dots,0)}(\rho,m,\epsilon_1,\epsilon_2) = \oint_0\frac{dQ_{f_1}}{Q_{f_1}}\dots\oint_0\frac{dQ_{f_{N-1}}}{Q_{f_{N-1}}}\Sigma_{N,k}(\rho,m,t_{f_1},\dots,t_{f_{N-1}},\epsilon_1,\epsilon_2), \quad (5.3)$$

where the contour integrals[16] are just to extract the constant term of $\Sigma_{N,k}(\rho,m,t_{f_1},\dots,t_{f_{N-1}},\epsilon_{1,2})$ in an expansion of the fugacities $Q_{f_1},\dots,Q_{f_{N-1}}$, as defined in Eq. (3.10).

Remarkably, in the NS limit, we find evidence that the quotients $(\sigma_{N,k}/\sigma_{1,1})$ are related to the free energies of specific configurations $(\{k_i\})$ of the monopole strings with $k_1=k_2=\dots=k_N=k$. Indeed, based on the examples discussed below, we conjecture

$$\lim_{\epsilon_2\to 0}\frac{\sigma_{N,k}(t,m,\epsilon_1,\epsilon_2)}{\sigma_{1,1}(t,m,\epsilon_1,\epsilon_2)} = \lim_{\epsilon_2\to 0}\frac{G^{\overbrace{(k,\dots,k)}^{N\text{times}}}(t,m,\epsilon_1,\epsilon_2)}{G^{(1)}(t,m,\epsilon_1,\epsilon_2)}. \quad (5.4)$$

Interpreting this conjecture from the point of view of IIa and IIb little strings (see Sec. II D) we notice that $\sigma_{N,k}$ and $G^{(k,\dots,k)}$ are the free energies of configurations of little strings with momentum and winding number k in types IIa and IIb, respectively. Thus, we believe our conjecture is in line with the T-duality property between type IIa and IIb little strings. Indeed, under T-duality, the momentum quantum number k, weighed with the fugacity Q_τ^k, is mapped to the winding quantum number k, weighed with the fugacity Q_ρ^k.

B. Checks and series expansions

In this subsection, we provide support for the conjecture (5.4): we give an analytic proof for the case $k=1$ (and N generic) and provide additional checks for $k>1$ by comparing the power series expansion of the left- and right-hand sides of (5.4).

1. The case $C_{N,1}$

To simplify the notation, we introduce the shorthand for the individual building blocks in the partition function (3.3):

$$E(\rho,t,m,\epsilon) := \frac{\theta_1(\rho;t+m)\theta_1(\rho;t-m)}{\theta_1(\rho;t+\epsilon)\theta_1(\rho;t-\epsilon)}. \quad (5.5)$$

Using these building blocks, we can write

$$\lim_{\epsilon_2\to 0}\frac{C_{N,1}(\rho,m,t_{f_1},\dots,t_{f_{N-1}},\epsilon_1,\epsilon_2)}{C_{1,1}(\rho,m,\epsilon_1,\epsilon_2)} = \sum_{k=1}^{N}\prod_{a=1}^{k-1}E(\rho,t_{ak}-\epsilon_+,m,\epsilon_+)\times\prod_{b=k+1}^{N}E(\rho,t_{kb}+\epsilon_+,m,\epsilon_+) = \sum_{k=1}^{N}\prod_{a=1}^{k-1}E(\rho,\hat{t}_{ak},m,\epsilon_+)\times\prod_{b=k+1}^{N}E(\rho,\hat{t}_{kb},m,\epsilon_+), \quad (5.6)$$

where we introduced the shorthand

$$\hat{t}_{ak}=t_{ak}-\epsilon_+,\quad \text{for } a=1,\dots,k-1,$$
$$\hat{t}_{kb}=t_{kb}+\epsilon_+,\quad \text{for } b=k+1,\dots,N. \quad (5.7)$$

Following (5.3) and (5.4), we are interested in the terms that are independent in Q_{f_a}, which are extracted by contour integration:

$$\oint\frac{dQ_{f_1}\cdots dQ_{f_{N-1}}}{Q_{f_1}\cdots Q_{f_{N-1}}}\times\left(\lim_{\epsilon_2\to 0}\frac{C_{N,1}(\rho,m,t_{f_1},\dots,t_{f_{N-1}},\epsilon_1,\epsilon_2)}{C_{1,1}(\rho,m,\epsilon_1,\epsilon_2)}\right). \quad (5.8)$$

Using the definition $Q_{f_a}=e^{-2\pi t_{f_a}}$, we can perform the following change of variables:

[15] For a definition of $\vartheta_{s,m}(\rho,t_{f_i})$, we refer readers to our previous paper [5].

[16] The ith contour is defined as a small circle around the point $Q_{f_i}=0$, as was previously prescribed in a noncompact situation in [5] (see also similar considerations in [58]).

$$\oint \frac{dQ_{f_1}\cdots dQ_{f_{N-1}}}{Q_{f_1}\cdots Q_{f_{N-1}}} = (-1)^{N-1}\int dt_{12}dt_{23}\cdots dt_{N-1N} = (-1)^{N-1}\int \prod_{a=1}^{k-1} dt_{ak}\prod_{b=k+1}^{N} dt_{kb}, \forall k. \tag{5.9}$$

Shifting the individual t_{ak} and t_{kb}, we then find

$$\begin{aligned}\oint \frac{dQ_{f_1}\cdots dQ_{f_{N-1}}}{Q_{f_1}\cdots Q_{f_{N-1}}}\left(\lim_{\epsilon_2\to 0}\frac{C_{N,1}(\rho,m,t_{f_1},\ldots,t_{f_{N-1}},\epsilon_1,\epsilon_2)}{C_{1,1}(\rho,m,\epsilon_1,\epsilon_2)}\right) &= (-1)^{N-1}N\left(\int dtE(\rho,t,m,\epsilon_1)\right)^{N-1}\\ &= N(\lim_{\epsilon_2\mapsto 0}W(\rho,m,\epsilon_1,\epsilon_2))^{N-1},\end{aligned} \tag{5.10}$$

where we have used the relation ($x = e^{2\pi i t}$),

$$\begin{aligned}\oint \frac{dx}{2\pi i x}\frac{\theta_1(\rho;t+m)\theta_1(\rho;t-m)}{\theta_1(\rho;t+\epsilon_1)\theta_1(\rho;t-\epsilon_1)} &= \lim_{\epsilon_2\mapsto 0}W(\rho,m,\epsilon_1,\epsilon_2)\\ &= \frac{\theta_1(\rho;m-\frac{\epsilon_1}{2})\theta_1'(\rho;m+\frac{\epsilon_1}{2})-\theta_1(\rho;m+\frac{\epsilon_1}{2})\theta_1'(\rho;m-\frac{\epsilon_1}{2})}{\theta_1(\rho;\epsilon_1)\theta_1'(\rho;0)}.\end{aligned} \tag{5.11}$$

We have additionally checked (5.10) up to $N = 4$ through an explicit computation of $C_{N,1}$.

2. Case $C_{N,k>1}$

For $k > 1$, the quantities $\Sigma_{N,k}(\rho, m, t_{f_1}, \ldots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$ become complicated quotients of θ_1 functions and we therefore only study their series expansions. Concretely, to compare with (4.6), we introduce

$$\lim_{\epsilon_2\to 0}\frac{\sigma_{N,k}(\rho,m,\epsilon_1,\epsilon_2)}{\sigma_{1,1}(\rho,m,\epsilon_1,\epsilon_2)} = \sum_{n=0}^{\infty}\epsilon_1^{2n}\sigma_{N,k}^n(\rho,m). \tag{5.12}$$

Starting with $(N, k) = (2, 2)$, we have for the cases $n = 1$, 2

$$\begin{aligned}\sigma_{2,2}^0(\rho,m) = 2 &+ Q_\rho\left(-4Q_m^3 - \frac{4}{Q_m^3} + 38Q_m^2 + \frac{38}{Q_m^2} - 124Q_m - \frac{124}{Q_m} + 180\right)\\ &+ Q_\rho^2\left(38Q_m^4 + \frac{38}{Q_m^4} - 448Q_m^3 - \frac{448}{Q_m^3} + 2012Q_m^2 + \frac{2012}{Q_m^2} - 4640Q_m - \frac{4640}{Q_m} + 6076\right)\\ &+ Q_\rho^3\left(2Q_m^6 + \frac{2}{Q_m^6} - 124Q_m^5 - \frac{124}{Q_m^5} + 2012Q_m^4 + \frac{2012}{Q_m^4} - 12892Q_m^3 - \frac{12892}{Q_m^3}\right.\\ &\left.+ 43350Q_m^2 + \frac{43350}{Q_m^2} - 86568Q_m - \frac{86568}{Q_m} + 108440\right) + \mathcal{O}(Q_\rho^4),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\sigma_{2,2}^1(\rho,m) = Q_\rho&\left(2Q_m^3 + \frac{2}{Q_m^3} - 32Q_m^2 - \frac{32}{Q_m^2} + 158Q_m + \frac{158}{Q_m} - 264\right)\\ &+ Q_\rho^2\left(-32Q_m^4 - \frac{32}{Q_m^4} + 800Q_m^3 + \frac{800}{Q_m^3} - 4824Q_m^2 - \frac{4824}{Q_m^2} + 12944Q_m + \frac{12944}{Q_m} - 17792\right)\\ &+ Q_\rho^3\left(158Q_m^5 + \frac{158}{Q_m^5} - 4824Q_m^4 - \frac{4824}{Q_m^4} + 42366Q_m^3 + \frac{42366}{Q_m^3} - 169920Q_m^2\right.\\ &\left.- \frac{169920}{Q_m^2} + 372708Q_m + \frac{372708}{Q_m} - 481008\right) + \mathcal{O}(Q_\rho^4),\end{aligned} \tag{5.14}$$

which indeed agree with the expansion of (A4).

For $(N,k)=(2,3)$, we find

$$\begin{aligned}\sigma^0_{2,3}(\rho,m) &= 2+Q_\rho\left(2Q_m^4+\frac{2}{Q_m^4}-60Q_m^3-\frac{60}{Q_m^3}+360Q_m^2+\frac{360}{Q_m^2}-944Q_m-\frac{944}{Q_m}+1284\right)\\ &\quad+Q_\rho^2\left(2Q_m^6+\frac{2}{Q_m^6}-200Q_m^5-\frac{200}{Q_m^5}+3010Q_m^4+\frac{3010}{Q_m^4}-18396Q_m^3-\frac{18396}{Q_m^3}\right.\\ &\quad\left.+60284Q_m^2+\frac{60284}{Q_m^2}-118840Q_m-\frac{118840}{Q_m}+148280\right)+\mathcal{O}(Q_\rho^3), \end{aligned} \tag{5.15}$$

$$\sigma^1_{2,3}(\rho,m)=Q_\rho\frac{3(3Q_m^6-28Q_m^5+103Q_m^4-158Q_m^3+103Q_m^2-28Q_m+3)}{2Q_m^3}+\mathcal{O}(Q_\rho^2), \tag{5.16}$$

which indeed agrees with the corresponding expansions of $g^{0,(3,3)}$ and $g^{1,(3,3)}$, respectively.

Finally, for $(N,k)=(3,2)$, we have

$$\begin{aligned}\sigma^0_{3,2}(\rho,m) &= 3+Q_\rho\left(3Q_m^4+\frac{3}{Q_m^4}-36Q_m^3-\frac{36}{Q_m^3}+195Q_m^2+\frac{195}{Q_m^2}-516Q_m-\frac{516}{Q_m}+708\right)\\ &\quad+Q_\rho^2\left(3Q_m^6+\frac{3}{Q_m^6}-144Q_m^5-\frac{144}{Q_m^5}+1572Q_m^4+\frac{1572}{Q_m^4}-8304Q_m^3-\frac{8304}{Q_m^3}+25479Q_m^2\right.\\ &\quad\left.+\frac{25479}{Q_m^2}-48864Q_m-\frac{48864}{Q_m}+60516\right)+\mathcal{O}(Q_\rho^3), \end{aligned} \tag{5.17}$$

which matches with a corresponding expansion of $g^{0,(2,2,2)}$.

These very nontrivial checks lend strong support to our conjecture (5.4).

VI. ELLIPTIC GENERA AND TOPOLOGICAL INVARIANTS

In the previous sections, we studied the properties of the NS limit of the free energy of M-strings with a compact transverse direction. We found evidence that these functions are related to the affine A_{N-1} relative monopole-string moduli space $M_{k_1,\ldots,k_N}$ with charges $(k_1,\ldots,k_N)$. Here, following [4,5], we conjecture a concrete relation between the NS limit of the free energy and the elliptic genus $\chi_{\rm ell}(M_{k_1,\ldots,k_N})$ of $M_{k_1,\ldots,k_N}$ as

$$\chi_{\rm ell}(M_{k_1\cdots k_N}) = \begin{cases}\frac{1}{N}\lim_{\epsilon_2\mapsto 0}\frac{G^{(k_1,\ldots,k_N)}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)} & \text{for } k_1=k_2=\ldots=k_N\\ \lim_{\epsilon_2\mapsto 0}\frac{G^{(k_1,\ldots,k_N)}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)} & \text{else}\end{cases}. \tag{6.1}$$

A. The case of charges $(k_1,\ldots,k_N)=(1,\ldots,1)$

For the charge configuration $(1,1,\ldots,1)$, we see from Eq. (4.15) that

$$\begin{aligned}G^{(1,\ldots,1)}(\tau,m,\epsilon_1,\epsilon_2) &= W(\tau,m,\epsilon_1,\epsilon_2)^{N-1}[G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)\\ &\quad+(N-1)F^{(1)}(\tau,m,\epsilon_1,\epsilon_2)]. \end{aligned} \tag{6.2}$$

In the NS limit, the above expression simplifies due to Eq. (4.3):

$$\lim_{\epsilon_2\to 0}\frac{G^{(1,\ldots,1)}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)} = NW(\tau,m,\epsilon_1,\epsilon_2=0)^{N-1}. \tag{6.3}$$

Therefore, the elliptic genus is given by

$$\chi_{\rm ell}(M_{1,\ldots,1}) = W(\tau,m,\epsilon_1,\epsilon_2=0)^{N-1}. \tag{6.4}$$

B. χ_y genus for $M_{k_1,\ldots,k_N}$

In the limit $\tau\mapsto i\infty$, the elliptic genus reduces to the χ_y genus

$$\begin{aligned}\chi_y(M_{k_1,\ldots,k_N}) &:= \lim_{\tau\mapsto i\infty}\chi_{\rm ell}(M_{k_1,\ldots,k_N})\\ &= \lim_{\tau\mapsto i\infty}\lim_{\epsilon_2\mapsto 0}\frac{G^{(k_1,\ldots,k_N)}(\tau,m,\epsilon_1,\epsilon_2)}{G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)}. \end{aligned} \tag{6.5}$$

This $\tau\mapsto i\infty$ limit can easily be computed for the partition function using the results of [3]. It is given by

$$\lim_{\tau\mapsto i\infty}\mathrm{PLog}\tilde{\mathcal{Z}}^{(2)}(\tau,m,t_{f_1},\dots,t_{f_N},\epsilon_{1,2}) = \frac{N(Q_m+Q_\rho Q_m^{-1})-Q_\rho(\sqrt{qt}+\frac{1}{\sqrt{qt}}+(N-1)(\sqrt{\frac{t}{q}}+\sqrt{\frac{q}{t}}))}{(1-Q_\rho)(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(t^{\frac{1}{2}}-t^{-\frac{1}{2}})} + \sum_{1\leq a<b\leq N}\frac{(Q_{ab}+Q_\rho Q_{ab}^{-1})(Q_m+Q_m^{-1})-(Q_{ab}+Q_\rho Q_{ab}^{-1})(\sqrt{\frac{t}{q}}+\sqrt{\frac{q}{t}})}{(1-Q_\rho)(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(t^{\frac{1}{2}}-t^{-\frac{1}{2}})}, \tag{6.6}$$

where we recall the definitions $Q_{ab}=Q_{f_a}Q_{f_{a+1}}\dots Q_{f_{b-1}}$ and $Q_\rho=e^{2\pi i\rho}$. Following (6.5), we further need to divide by $\lim_{\tau\to i\infty}G^{(1)}(\tau,m,\epsilon_1,\epsilon_2)$ and obtain

$$\lim_{\epsilon_2\mapsto 0}\lim_{\tau\mapsto i\infty}\frac{\mathrm{PLog}\tilde{\mathcal{Z}}^{(2)}(\tau,m,t_{f_1},\dots,t_{f_N},\epsilon_{1,2})}{G^{(1)}(\tau,m,\epsilon_{1,2})} = N\frac{Q_m^2}{(1-Q_m q^{\frac{1}{2}})(1-Q_m q^{-\frac{1}{2}})}+N\sum_{k\geq 1}Q_\rho^k+\sum_{1\leq a<b\leq N}\frac{(Q_{ab}+Q_\rho Q_{ab}^{-1})}{(1-Q_\rho)}. \tag{6.7}$$

From this, it follows that

$$\chi_y(M_{k_1\cdots k_N})=\begin{cases}1, & (k_1,\dots,k_N)=(k,\dots,k), \quad k\geq 1\\ 1, & (k_1,\dots,k_N)=(k,\dots,k,k+1,\cdots k+1,k\cdots k), \quad k\geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{6.8}$$

This implies that

$$\sum_{q=0}^{d}(-1)^q\dim_{\mathbb{C}}H^{p,q}(M_{k_1\cdots k_N})=\begin{cases}\delta_{p,0}, & (k_1,\dots,k_N)=(k,\dots,k), \quad k\geq 1\\ \delta_{p,0}, & (k_1,\dots,k_N)=(k,\dots,k,k+1,\cdots k+1,k\cdots k), \quad k\geq 0\\ 0, & \text{otherwise,}\end{cases} \tag{6.9}$$

where $d=\dim_{\mathbb{C}}M_{k_1,\dots,k_N}$. The cases where some of the k_i's are zero capture the χ_y genus of noncompact configurations that we studied in [5]. For nonzero k_i's, to the best knowledge of the authors, the above results for the χ_y genus are new. It would be interesting to confirm them by a direct computation of the multimonopole moduli space.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we have studied aspects of BPS excitations in M5-M2-brane configurations where a transverse direction is compactified. Following our previous work [5], these configurations allow two dual descriptions, namely, in terms of M-strings and monopole strings. A key feature of this compact setup is a manifest $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ symmetry (which reduces to a single $SL(2,\mathbb{Z})$ in the decompactification limit). These two modular symmetries are associated with two dual gauge theories whose partition functions we have presented explicitly. The BPS excitations in these two five-dimensional theories can physically be interpreted as instanton particles and monopole strings, respectively. Comparing the compact partition functions to their noncompact counterparts studied in [5] we found an interesting relationship. Indeed, the counting function of compact BPS configurations can fully be constructed as a linear superposition of the noncompact ones. The result, as summarized by Eq. (4.16), points to interesting implications for the little string theories: For IIA and IIB string theories, open and closed fundamental strings are distinct states. In particular, the closed string is not treated as a composite of open strings. However, for IIa and IIb little string theories, our wrapping prescription equation (4.16) implies that the little strings can be viewed as bound states of M-strings. Stated differently, for the purpose of BPS counting of IIb little strings, one only needs to know BPS excitations of the (2,0) superconformal field theory, which is just the low-energy limit of the IIb little string theory.

Furthermore, by carefully studying specific expansions of the two gauge theory partition functions mentioned above, we also discovered remarkable relations between their BPS state counting. Physically, this implies new relations between specific instanton and monopole configurations, respectively, which have not been observed in the literature so far. It will be interesting, both from physics and mathematics aspects, to further explore this observation: phrased more concretely, the question is how instantons on $\mathbb{R}^4$ are related to monopoles on $\mathbb{R}^3$ and what its physical reason is. Another concrete question is to understand whether the relations discussed here can be generalized to instanton configurations whose contribution to the partition function depends explicitly on t_{f_a}.

Generalizing our previous work [5], we have proposed that the compact gauge theory partition function allows us to extract the elliptic genus of the relative moduli space of affine A_{N-1} monopole strings. Based on this conjecture, by computing the corresponding χ_y genus we have extracted topological data of this moduli space. The latter are not yet known in the mathematics literature. It would be very interesting to confirm our conjectures by independent methods.

Finally, consequences and implications of our results on the BPS excitations in type IIa and IIb little string theories in six dimensions is a very interesting topic, which we will relegate to a forthcoming paper [25].

We believe that a further exploration of M5-M2-brane configurations along the lines we have advocated in this work will shed further light on the role of tensionless strings in the elusive six-dimensional superconformal field theories.

ACKNOWLEDGMENTS

We thank Ofer Aharony, Dongsu Bak, Andreas Gustavsson, Babak Haghighat, David Kutasov, Sameer Murthy, and Cumrun Vafa for the helpful discussions. S. H. and S. J. R. acknowledge the "Liouville, Integrability and Branes (10) and (11)" Focus Program at the Asia-Pacific Center for Theoretical Physics for the excellent collaboration environment. A. I. acknowledges the "2015 Simons Summer Workshop on Mathematics and Physics" for its hospitality during this work. S. J. R. was supported in part by National Research Foundation of Korea Grants No. 2005-0093843, No. 2010-220-C00003, and No. 2012K2A1A9055280. The work of S. H. is partly supported by the BQR Accueil EC 2015. A. I. was supported in part by Higher Education Commission Grant No. HEC-20-2518.

APPENDIX: COMPACT AND NONCOMPACT FREE ENERGIES

1. Series expansion

We begin with the compact free energies:

$$\begin{aligned} g^{0,(2)} &= \frac{1}{12}[(2E_2(2\tau) - E_2(\tau))\varphi_{-2,1} + \varphi_{0,1}], \\ g^{1,(2)} &= \frac{1}{288}[4(E_2(\tau) - E_2(2\tau))\varphi_{0,1} - (E_2(\tau)^2 - 4E_2(2\tau)^2 + 15E_4(\tau) - 12E_4(2\tau))\varphi_{-2,1}], \end{aligned} \tag{A1}$$

$$\begin{aligned} g^{0,(3)} &= \frac{1}{1440}[10\varphi_{0,1}^2 + 10(3E_2(3\tau) - E_2(\tau))\varphi_{-2,1}\varphi_{0,1} + (37E_4(\tau) - 27E_4(3\tau))\varphi_{-2,1}^2], \\ g^{1,(3)} &= \frac{1}{60480}[105(E_2(\tau) - E_2(3\tau))\varphi_{0,1}^2 - 7(5E_2(\tau)^2 - 45E_2(3\tau)^2 + 157E_4(\tau) - 117E_4(3\tau))\varphi_{0,1}\varphi_{-2,1} \\ &\quad + [2592E_6(3\tau) + 1496E_6(\tau) + 7E_2(\tau)(37E_4(\tau) + 270E_4(3\tau)) - 6237E_2(3\tau)E_4(3\tau)]\varphi_{-2,1}^2], \end{aligned} \tag{A2}$$

$$\begin{aligned} g^{0,(2,1)} &= \frac{1}{144}[\varphi_{0,1}^2 + 2E_2(\tau)\varphi_{0,1}\varphi_{-2,1} + (3E_4(\tau) - 2E_2(\tau))\varphi_{-2,1}^2], \\ g^{1,(2,1)} &= \frac{\varphi_{-2,1}}{432}[2(E_2(\tau)^2 - E_4(\tau))\varphi_{0,1} - (E_2(\tau)^3 + 5E_2(\tau)E_4(\tau) - 6E_6(\tau))\varphi_{-2,1}], \end{aligned} \tag{A3}$$

$$g^{0,(2,2)} = \frac{1}{90720}[(546E_2(\tau)E_4(\tau) - 672E_2(2\tau)E_4(2\tau) - 601E_6(\tau) + 832E_6(2\tau))\varphi^3_{-2,1} - 21(10E_2(\tau)^2 - 40E_2(2\tau)^2 - 9E_4(\tau) + 24E_4(2\tau))\phi_{0,1}\phi^2_{-2,1} + 105(E_2(\tau) + 2E_2(2\tau))\phi^2_{0,1}\phi_{-2,1} + 105\phi^3_{0,1}],$$

$$g^{1,(2,2)} = \frac{2}{3628800}[(2730E_4(\tau)E_2(\tau)^2 + 5875E_6(\tau)E_2(\tau) - 9893E_4(\tau)^2 + 64E_4(2\tau)(242E_4(2\tau) - 105E_2(2\tau)^2) - 40E_2(2\tau)(3E_6(\tau) + 184E_6(2\tau)))\phi^3_{-2,1} - 50(28E_2(\tau)^3 + 287E_4(\tau)E_2(\tau) - 224E_2(2\tau)(E_2(2\tau)^2 + E_4(2\tau)) - 219E_6(\tau) + 352E_6(2\tau))\phi_{0,1}\phi^2_{-2,1} + 35(85E_2(\tau)^2 - 100E_2(2\tau)^2 - 133E_4(\tau) + 148E_4(2\tau))\phi^2_{0,1}\phi_{-2,1} + 700(E_2(\tau) - E_2(2\tau))\phi^3_{0,1}]. \quad (A4)$$

Here, $E_{2k}(\tau)$ is the Eisenstein series defined as

$$E_{2k}(\tau) := 1 + \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)Q^n_\tau, \quad (A5)$$

and $\varphi_{-2,1}(\tau,z)$ and $\varphi_{0,1}(\tau,z)$ are the standard Jacobi forms of index 1 and weight -2 and 0, respectively,

$$\varphi_{0,1}(\tau,m) = 4\sum_{i=2}^{4}\frac{\theta_i(\tau;m)^2}{\theta_i(\tau;0)} \quad \text{and} \quad \varphi_{-2,1}(\tau,m) = -\frac{\theta_1^2(\tau;m)}{\eta(\tau)^6}, \quad (A6)$$

where $\theta_i(\tau,z)$ are the Jacobi theta functions and $\eta(\tau)$ the Dedekind eta function (see [48] for further information).

Similarly, we can write for the noncompact coefficient functions

$$f^{0,(2)} = \frac{E_2(2\tau) - E_2(\tau)}{6}\varphi_{-2,1},$$
$$f^{1,(2)} = \frac{12E_4(2\tau) - 13E_4(\tau) - 3E_2(\tau)^2 + 4E_2(2\tau)^2}{288}\varphi_{-2,1} - \frac{E_2(2\tau) - E_2(\tau)}{72}\varphi_{0,1}, \quad (A7)$$

$$f^{0,(3)} = \left[\frac{20E_2(\tau)^2 + 7E_4(\tau) - 27E_4(3\tau)}{1440}\varphi_{-2,1} + \frac{E_2(3\tau) - E_2(\tau)}{48}\varphi_{0,1}\right]\varphi_{-2,1},$$
$$f^{1,(3)} = \frac{1}{60480}[105[E_2(\tau) - E_2(3\tau)]\varphi^2_{0,1} - 63[5(E_2(\tau)^2 - E_2(3\tau)^2) + 13(E_4(\tau) - E_4(3\tau))]\varphi_{0,1}\varphi_{-2,1} + [140E_2(\tau)^3 - 6237E_2(3\tau)E_4(3\tau) + 7E_2(\tau)(137E_4(\tau) + 270E_4(3\tau)) + 656E_6(\tau) + 2592E_6(3\tau)]\varphi^2_{-2,1}], \quad (A8)$$

$$f^{0,(2,1)} = \frac{E_4(\tau) - E_2(\tau)^2}{96}\varphi^2_{-2,1},$$
$$f^{1,(2,1)} = -\frac{\varphi_{-2,1}}{576}[[E_4(\tau) - E_2(\tau)^2]\varphi_{0,1} + [E_2(\tau)^3 + 3E_4(\tau)E_2(\tau) - 4E_6(\tau)]\varphi_{-2,1}]. \quad (A9)$$

2. Relations between compact and noncompact coefficient functions

With the expressions above (and several others which we do not display to save space) the compact and noncompact coefficients are as follows.

(a) Case $K = 2$:

$$g^{n,(2)} = f^{n,(2)} + f^{n,(1,1)}, \quad g^{n,(1,1)} = 2f^{n,(1,1)}, \quad \forall n = 0,1,2,3. \quad (A10)$$

(b) Case $K = 3$:

$$\begin{aligned} g^{n,(3)} &= f^{n,(3)} + 2f^{n,(2,1)} + f^{n,(1,1,1)}, \\ g^{n,(2,1)} &= 2f^{n,(2,1)} + f^{n,(1,1,1)}, \\ g^{n,(1,1,1)} &= 3[f^{n,(1,1,1)}], \quad \forall n = 0, 1, 2. \end{aligned} \tag{A11}$$

(c) Case $K = 4$:

$$\begin{aligned} g^{n,(4)} &= f^{n,(4)} + 2f^{n,(3,1)} + f^{n,(2,2)} + 2f^{,(2,1,1)} + f^{n,(1,2,1)} + f^{n,(1,1,1,1)}, \\ g^{n,(3,1)} &= 2f^{n,(2,1,1)} + 2f^{n,(3,1)}, \\ g^{n,(2,2)} &= 2[f^{n,(1,1,1,1)} + f^{n,(1,2,1)} + f^{n,(2,2)}], \\ g^{n,(2,1,1)} &= f^{n,(1,1,1,1)} + 2f^{n,(2,1,1)} + f^{n,(1,2,1)}, \\ g^{n,(1,1,1,1)} &= 4[f^{n,(1,1,1,1)}], \quad \forall n = 0, 1. \end{aligned} \tag{A12}$$

(d) Case $K = 5$:

$$\begin{aligned} g^{0,(5)} &= f^{0,(5)} + 2f^{0,(4,1)} + 2f^{0,(3,2)} + f^{0,(2,1,2)} + 2f^{0,(2,2,1)} + f^{0,(1,3,1)} \\ &\quad + 2f^{0,(3,1,1)} + 2f^{0,(1,2,1,1)} + 2f^{0,(2,1,1,1)} + f^{0,(1,1,1,1,1)}, \\ g^{0,(2,1,1,1)} &= f^{0,(1,1,1,1,1)} + 2f^{0,(1,2,1,1)} + 2f^{0,(2,1,1,1)}, \\ g^{0,(3,1,1)} &= f^{0,(1,3,1)} + 2f^{0,(2,1,1,1)} + 2f^{0,(3,1,1)}, \\ g^{0,(2,2,1)} &= f^{0,(1,1,1,1,1)} + 2f^{0,(1,2,1,1)} + 2f^{0,(2,2,1)} + f^{0,(2,1,2)}, \\ g^{0,(3,2)} &= f^{0,(1,1,1,1,1)} + 2f^{0,(1,2,1,1)} + f^{0,(1,3,1)} + 2f^{0,(2,1,1,1)} + 2f^{0,(2,2,1)} + 2f^{0,(3,2)}, \\ g^{0,(4,1)} &= f^{0,(2,1,2)} + 2f^{0,(3,1,1)} + 2f^{0,(4,1)}, \\ g^{0,(1,1,1,1,1)} &= 5[f^{0,(1,1,1,1,1)}]. \end{aligned} \tag{A13}$$

(e) Case $K = 6$:

$$\begin{aligned} g^{0,(3,3)} &= 2[f^{0,(1,1,1,1,1,1)} + 2f^{0,(1,2,1,1,1)} + f^{0,(1,2,2,1)} + f^{0,(2,1,1,2)} + 2f^{0,(2,2,1,1)}] + 4f^{0,(2,3,1)} + 2f^{0,(3,3)}, \\ g^{0,(2,1,2,1)} &= 2[2f^{0,(2,1,2,1)} + f^{0,(1,1,2,1,1)}]. \end{aligned} \tag{A14}$$

As we can see, to each order, we can express the compact free energies as particular linear combinations of the noncompact ones. However, these relations are not invertible, due to the fact that the compact $g^{n,(\{k_i\})}$'s are invariant under cyclic rotations of the k_i, while the noncompact ones $f^{n,(\{k_i\})}$ are only invariant under mirror reflection.

---

[1] B. Haghighat, A. Iqbal, C. Kozcaz, G. Lockhart, and C. Vafa, M-strings, Commun. Math. Phys. **334**, 779 (2015).
[2] B. Haghighat, C. Kozcaz, G. Lockhart, and C. Vafa, Orbifolds of M-strings, Phys. Rev. D **89**, 046003 (2014).
[3] S. Hohenegger and A. Iqbal, M-strings, elliptic genera and $\mathcal{N} = 4$ string amplitudes, Fortschr. Phys. **62**, 155 (2014).
[4] B. Haghighat, From strings in 6d to strings in 5d, J. High Energy Phys. 01 (2016) 062.

[5] S. Hohenegger, A. Iqbal, and S. J. Rey, M-strings, monopole strings, and modular forms, Phys. Rev. D **92**, 066005 (2015).

[6] B. Haghighat, S. Murthy, C. Vafa, and S. Vandoren, F-theory, spinning black holes and multi-string branches, J. High Energy Phys. 01 (2016) 09.

[7] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, The topological vertex, Commun. Math. Phys. **254**, 425 (2005).

[8] T. J. Hollowood, A. Iqbal, and C. Vafa, Matrix models, geometric engineering and elliptic genera, J. High Energy Phys. 03 (2008) 069.

[9] A. Iqbal, C. Kozcaz, and C. Vafa, The refined topological vertex, J. High Energy Phys. 10 (2009) 069.

[10] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. **7**, 831 (2003).

[11] M. R. Douglas, On $D = 5$ super Yang-Mills theory and (2,0) theory, J. High Energy Phys. 02 (2011) 011.

[12] E. Witten, Elliptic genera and quantum field theory, Commun. Math. Phys. **109**, 525 (1987); A. N. Schellekens and N. P. Warner, Anomalies, characters and strings, Nucl. Phys. **B287**, 317 (1987); W. Lerche, B. E. W. Nilsson, A. N. Schellekens, and N. P. Warner, Anomaly cancelling terms from the elliptic genus, Nucl. Phys. **B299**, 91 (1988).

[13] S. A. Connell, The dynamics of the SU(3) charge (1,1) magnetic monopole (unpublished); see also J. P. Gauntlett and D. A. Lowe, Dyons and S duality in $N = 4$ supersymmetric gauge theory, Nucl. Phys. **B472**, 194 (1996); K. M. Lee, E. J. Weinberg, and P. Yi, The moduli space of many BPS monopoles for arbitrary gauge groups, Phys. Rev. D **54**, 1633 (1996).

[14] M. Atiyah and N. Hitchin, Low energy scattering of non-Abelian monopoles, Phys. Lett. **107A**, 21 (1985); Low-energy scattering of non-Abelian magnetic monopoles [and discussion], Phil. Trans. R. Soc. A **315**, 459 (1985); *The Geometry and Dynamics of Magnetic Monopoles* (Princeton University Press, Princeton, NJ, 1988).

[15] J. A. Harvey, S. Lee, and S. Murthy, Elliptic genera of ALE and ALF manifolds from gauged linear sigma models, J. High Energy Phys. 02 (2015) 110.

[16] D. Bak and A. Gustavsson, Elliptic genera of monopole strings, J. High Energy Phys. 01 (2015) 097.

[17] M. Berkooz, M. Rozali, and N. Seiberg, Matrix description of M theory on T^4 and T^5, Phys. Lett. B **408**, 105 (1997).

[18] N. Seiberg, New theories in six-dimensions and matrix description of M theory on T^5 and T^5/Z_2, Phys. Lett. B **408**, 98 (1997).

[19] A. Losev, G. W. Moore, and S. L. Shatashvili, M & m's, Nucl. Phys. **B522**, 105 (1998).

[20] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg, Linear dilatons, NS five-branes and holography, J. High Energy Phys. 10 (1998) 004.

[21] O. Aharony, A brief review of "little string theories", Classical Quantum Gravity **17**, 929 (2000).

[22] D. Kutasov, in *Proceedings of Superstrings and Related Matters, Spring School, Trieste, 2001*, edited by C. Bachas, J. M. Maldacena, K. S. Narain, and S. Randjbar-Daemi (International Centre for Theoretical Physics, Trieste, 2002), p. 165.

[23] O. Aharony, M. Berkooz, and S. J. Rey, Rigid holography and six-dimensional $\mathcal{N} = (2,0)$ theories on AdS_5 $\mathbb{S}^1$, J. High Energy Phys. 03 (2015) 121.

[24] L. Bhardwaj, M. Del Zotto, J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, F-theory and the classification of little strings, arXiv:1511.05565.

[25] S. Hohenegger, A. Iqbal, and S.-J. Rey (to be published).

[26] D. Anselmi, M. Billo, P. Fre, L. Girardello, and A. Zaffaroni, ALE manifolds and conformal field theories, Int. J. Mod. Phys. A **09**, 3007 (1994).

[27] H. Ooguri and C. Vafa, Two-dimensional black hole and singularities of CY manifolds, Nucl. Phys. **B463**, 55 (1996).

[28] D. Kutasov, Orbifolds and solitons, Phys. Lett. B **383**, 48 (1996).

[29] G. W. Moore, N. Nekrasov, and S. Shatashvili, Integrating over Higgs branches, Commun. Math. Phys. **209**, 97 (2000).

[30] A. Losev, N. Nekrasov, and S. L. Shatashvili, in *Strings, Branes and Dualities*, edited by L. Baulieu, P. Di Francesco, M. Douglas, V. Kazakov, M. Picco, and P. Windey, *NATO Advanced Study Institutes, Ser. C*, Vol. 520 (Springer, New York, 1998), p. 359.

[31] M. Billó, M. Frau, F. Fucito, and A. Lerda, Instanton calculus in R-R background and the topological string, J. High Energy Phys. 11 (2006) 012.

[32] K. Ito, H. Nakajima, T. Saka, and S. Sasaki, $N = 2$ instanton effective action in Ω-background and D3/D(-1)-brane system in R-R background, J. High Energy Phys. 11 (2010) 093.

[33] M. X. Huang and A. Klemm, Direct integration for general Ω backgrounds, Adv. Theor. Math. Phys. **16**, 805 (2012).

[34] M. X. Huang, A. K. Kashani-Poor, and A. Klemm, The Ω deformed B-model for rigid $\mathcal{N} = 2$ theories, Ann. Inst. Henri Poincaré **14**, 425 (2013).

[35] S. Hellerman, D. Orlando, and S. Reffert, String theory of the omega deformation, J. High Energy Phys. 01 (2012) 148.

[36] I. Antoniadis, S. Hohenegger, K. S. Narain, and T. R. Taylor, Deformed topological partition function and Nekrasov backgrounds, Nucl. Phys. **B838**, 253 (2010).

[37] I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain, and A. Zein Assi, Worldsheet realization of the refined topological string, Nucl. Phys. **B875**, 101 (2013).

[38] I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain, and A. Zein Assi, Non-perturbative Nekrasov partition function from string theory, Nucl. Phys. **B880**, 87 (2014).

[39] I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain, and A. Z. Assi, Probing the moduli dependence of refined topological amplitudes, Nucl. Phys. **B901**, 252 (2015).

[40] Y. Nakayama and H. Ooguri, Comments on worldsheet description of the omega background, Nucl. Phys. **B856**, 342 (2012).

[41] J. Bae, C. Imbimbo, S.-J. Rey, and D. Rosa, New supersymmetric localizations from topological gravity, arXiv:1510.00006.

[42] A. Iqbal, A. Z. Khan, B. A. Qureshi, K. Shabbir, and M. A. Shehper, Topological field theory amplitudes for A_{N-1} fibration, J. High Energy Phys. 12 (2015) 017.

[43] O. Aharony and M. Berkooz, IR dynamics of $D = 2$, $N = (4,4)$ gauge theories and DLCQ of "little string theories", J. High Energy Phys. 10 (1999) 030.

[44] J. Kim, S. Kim, and K. Lee, Little strings and T-duality, arXiv:1503.07277.

[45] S. Katz, P. Mayr, and C. Vafa, Mirror symmetry and exact solution of 4D $N = 2$ gauge theories. I, Adv. Theor. Math. Phys. **1**, 53 (1998).
[46] R. Gopakumar and C. Vafa, M theory and topological strings. I, arXiv:hep-th/9809187.
[47] R. Gopakumar and C. Vafa, M theory and topological strings. II, arXiv:hep-th/9812127.
[48] M. Eichler and D. Zagier, *The Theory of Jacobi Forms* (Birkhäuser, Basel, Switzerland, 1985).
[49] B. Sundborg, The Hagedorn transition, deconfinement and $N = 4$ SYM theory, Nucl. Phys. **B573**, 349 (2000).
[50] A. M. Polyakov, Gauge fields and space-time, Int. J. Mod. Phys. A **17**, 119 (2002).
[51] G. Lockhart and C. Vafa, Superconformal partition functions and non-perturbative topological strings, arXiv:1210.5909.
[52] N. A. Nekrasov and S. L. Shatashvili, in *Proceedings of the 16th International Congress on Mathematical Physics (ICMP09), Prague, 2009*, edited by P. Exner (World Scientific, Singapore, 2010), p. 265.
[53] A. Mironov and A. Morozov, Nekrasov functions and exact Bohr-Zommerfeld integrals, J. High Energy Phys. 04 (2010) 040.
[54] D. Tong and C. Turner, Vortices and monopoles in a harmonic trap, J. High Energy Phys. 12 (2015) 098.
[55] K. Shabbir, On non-perturbative modular transformation, arXiv:1510.03332.
[56] B. de Wit, G. Lopes Cardoso, D. Lust, T. Mohaupt, and S. J. Rey, Higher order gravitational couplings and modular forms in $N = 2$, $D = 4$ heterotic string compactifications, Nucl. Phys. **B481**, 353 (1996).
[57] B. de Wit, G. Lopes Cardoso, D. Lust, T. Mohaupt, and S. J. Rey, Higher derivative couplings and duality in $N = 2$, $D = 4$ string theories, Nucl. Phys. B, Proc. Suppl. **56B**, 108 (1997).
[58] H. C. Kim, S. Kim, E. Koh, K. Lee, and S. Lee, On instantons as Kaluza-Klein modes of M5-branes, J. High Energy Phys. 12 (2011) 031.