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Randall-Sundrum versus holographic cosmology

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We consider a model of a holographic braneworld universe in which a cosmological fluid occupies a $3+1$-dimensional brane located at the boundary of the asymptotic anti-de Sitter bulk. We combine the AdS/CFT correspondence and the second Randall-Sundrum (RSII) model to establish a relationship between the RSII braneworld cosmology and the boundary metric induced by the time dependent bulk geometry. In the framework of the Friedmann-Robertson-Walker cosmology, we discuss some physically interesting scenarios involving the RSII and holographic braneworlds.

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I. INTRODUCTION

The AdS/CFT correspondence establishes an equivalence of a four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory and string theory in a ten-dimensional $\mathrm{AdS}_5\times\mathrm{S}_5$ bulk [1–3]. In a wider context of gage-gravity duality, the AdS/CFT correspondence goes beyond pure string theory and links many other important theoretical and phenomenological issues. In particular, a simple physically relevant model related to AdS/CFT is the Randall-Sundrum (RS) model [4,5] and its cosmological applications. The model was originally proposed as a solution to the hierarchy problem in particle physics and as a possible mechanism for localizing gravity on the $3+1$-dimensional universe embedded in a $4+1$ spacetime without compactification of the extra dimension. Soon after the papers [4,5] appeared, it was realized that the Randall-Sundrum model is deeply rooted in a wider framework of AdS/CFT correspondence [6–12]. In the braneworld scheme, the RS brane provides a cutoff regularization for the infrared divergences of the on-shell bulk action.

Our purpose is to study in terms of the AdS/CFT correspondence a class of $3+1$ time dependent metrics induced on slices of the $4+1$-dimensional asymptotic anti-de Sitter (AdS_5) bulk. We consider two types of braneworld universes: the holographic braneworld in which a $3+1$-dimensional brane is located at the boundary of the AdS_5 bulk and the Randall-Sundrum (RSII) braneworld in which a single brane is located at a nonzero distance from the boundary. We combine the holographic map of Apostolopoulos, Siopsis, and Tetradis [13,14] and the homogeneous cosmology of the RSII model [5] to establish a mapping between the RSII braneworld cosmology and the Friedmann-Robertson-Walker (FRW) type cosmology on the holographic braneworld. We explicitly determine the functional relations between the two cosmologies in terms of cosmological scales, Hubble rates, and effective densities on the branes.

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Our approach is in a spirit similar to Brax and Peschanski [15], but we have included some salient features which were not sufficiently emphasized in the literature. In particular, in connection with the holographic map, we carefully analyze two versions of the RSII models: the so called "one-sided" and "two-sided" version. A general asymptotically AdS metric in Fefferman-Graham coordinates [16] is of the form

$$ds^2 = G_{ab}dx^a dx^b = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2), \qquad (1)$$

where we use the Latin alphabet for bulk indices and the Greek alphabet for $3+1$ spacetime indices. In the original RSII model, one assumes the Z_2 symmetry $z\leftrightarrow z_{\mathrm{br}}^2/z$, so the region $0<z\le z_{\mathrm{br}}$ is identified with $z_{\mathrm{br}}\le z<\infty$, with the observer brane at the fixed point $z=z_{\mathrm{br}}$. Hence, the braneworld is sitting between two patches of AdS_5, one on either side, and is therefore dubbed two sided [10,12]. In contrast, in the one-sided RSII model, the region $0\le z\le z_{\mathrm{br}}$ is simply cut off so the bulk is the section of spacetime $z_{\mathrm{br}}\le z<\infty$. These two versions are equivalent from the point of view of an observer at the braneworld. However, in the one-sided RSII model, as pointed out by Duff and Liu [10], by shifting the boundary in AdS_5 from $z=0$ to $z=z_{\mathrm{br}}$, the model is conjectured to be dual to a cutoff conformal field theory (CFT) coupled to gravity, with $z=z_{\mathrm{br}}$ providing the cutoff. This conjecture then reduces to the standard AdS/CFT duality as the boundary is pushed off to $z=0$. This connection involves a single CFT at the boundary of a single patch of AdS_5. In the two-sided RSII model, one would instead require two copies of the CFT, one for each of the AdS_5 patches. We shall demonstrate this explicitly in Sec. IV. The holographic mapping turns out to be unique for the two-sided RSII model, whereas in the one-sided model the mapping from the holographic to the RSII cosmology is a two-valued function.

The remainder of the paper is organized as follows. In Sec. II, we present a brief derivation of the cosmology on the RSII brane. In Sec. III, we discuss the cosmology on the holographic brane. The map from RSII to holographic cosmology is constructed in Sec. IV, where we confront two cosmological scenarios. We compare the corresponding effective energy densities and equations of state of the cosmological fluid and discuss a few physically interesting regimes. In the concluding section, Sec. V, we summarize our results and give conclusions. A brief review of the RSII model is presented in Appendix A, and a connection between RSII and AdS/CFT correspondence is demonstrated in Appendix B, where we derive the field equations on the boundary brane with matter and discuss the conformal anomaly.

II. RANDALL-SUNDRUM COSMOLOGY

Branewarld cosmology is based on the scenario in which matter is confined on a brane moving in the higher-dimensional bulk with only gravity allowed to propagate in the bulk [4,5,17,18]. The RS model was originally proposed as a possible mechanism for localizing gravity on the $3+1$ universe embedded in a $4+1$-dimensional spacetime without compactification of the extra dimension. The RSII model is a $4+1$-dimensional AdS_5 universe containing two 3-branes with opposite tensions separated in the fifth dimension: observers reside on the positive tension brane, and the negative tension brane is pushed off to infinity. The Planck mass scale is determined by the curvature of the AdS spacetime rather than by the size of the fifth dimension. Hence, the model provides an alternative to compactification [5].

As demonstrated in Appendix A, in this model, the fifth dimension can be integrated out to obtain a purely four-dimensional action with a well-defined value for Newton's constant in terms of the AdS curvature radius ℓ and the five-dimensional gravitational constant G_5,

$$G_{\rm N} = \frac{2G_5}{\gamma\ell}, \tag{2}$$

where we have introduced the *sidedness* constant γ to facilitate a joint description of the two versions of the RSII model: one-sided ($\gamma = 1$) and two-sided ($\gamma = 2$). In the following analysis, we shall consider $G_{\rm N}$ and ℓ as fixed basic physical parameters and G_5 as a derived quantity.

The classical $3+1$-dimensional gravity on the RSII brane is altered due to the extra dimension. It has been shown [19] that for $r \gg \ell$ the weak gravitational potential created by an isolated matter source on the brane is given by

$$\Phi(r) = \frac{G_{\rm N}M}{r}\left(1 + \frac{2\ell^2}{3r^2}\right). \tag{3}$$

Hence, the extra-dimension effects strengthen Newton's gravitational field. Table-top tests of Newton's laws [20] currently find no deviations of Newton's potential at distances greater than 0.1 mm, yielding the limit on the AdS_5 curvature

$$\ell < 0.1\ \text{mm}, \quad \text{or} \quad \ell^{-1} > 10^{-12}\ \text{GeV}. \tag{4}$$

Assuming (2), this yields a lower bound on the bulk scale parameter [21],

$$M_5 = G_5^{-1/3} > 10^8\ \text{GeV}. \tag{5}$$

Soon after Randall and Sundrum introduced their model [4,5], it was realized that the model, as well as any similar braneworld model, may have interesting cosmological implications [22–25]. In particular, the usual Friedmann equations are modified so the model has predictions different from the standard cosmology.

To study the braneworld cosmology, it is convenient to represent the bulk metric in Schwarzschild coordinates [26],

$$ds^2_{\rm ASch} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2_\kappa, \tag{6}$$

where

$$f(r) = \frac{r^2}{\ell^2} + \kappa - \mu\frac{\ell^2}{r^2}, \tag{7}$$

and

$$d\Omega^2_\kappa = d\chi^2 + \frac{\sin^2(\sqrt{\kappa}\chi)}{\kappa}(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{8}$$

is the spatial line element for a closed ($\kappa = 1$), open hyperbolic ($\kappa = -1$), or open flat ($\kappa = 0$) space. The dimensionless parameter μ is related to the black-hole mass via [27,28]

$$\mu = \frac{8G_5 M_{\rm bh}}{3\pi\ell^2}. \tag{9}$$

As shown in Appendix A, for a time dependent brane hypersurface defined by

$$r - a(t) = 0, \tag{10}$$

where $a = a(t)$ is an arbitrary function, the induced line element on the brane is given by

$$ds^2_{\rm ind} = n^2(t)dt^2 - a(t)^2 d\Omega^2_\kappa, \tag{11}$$

with the lapse function

$$n^2 = f(a) - \frac{(\partial_t a)^2}{f(a)}. \tag{12}$$

The effective Friedmann equation on the RSII brane derived in Appendix A reads

$$\mathcal{H}_{\rm RSII}^2 = \frac{(\sigma+\rho)^2}{\ell^2\sigma_0^2} - \frac{1}{\ell^2} + \frac{\mu\ell^2}{a^4}, \tag{13}$$

where

$$\mathcal{H}_{\rm RSII}^2 = H_{\rm RSII}^2 + \frac{\kappa}{a^2} = \frac{(\partial_t a)^2}{n^2a^2} + \frac{\kappa}{a^2}. \tag{14}$$

From now on, a calligraphic $\mathcal{H}$ will always denote the Hubble rate H plus the corresponding curvature term κ/a^2. The quantity σ is the brane tension, and we have introduced a constant,

$$\sigma_0 = \frac{3\gamma}{8\pi G_5\ell} = \frac{3}{4\pi G_{\rm N}\ell^2}, \tag{15}$$

the value of which is restricted by

$$\sigma_0 > (10^3\ {\rm GeV})^4 \tag{16}$$

on account of the experimental constraint (4). Employing the RSII fine-tuning condition $\sigma = \sigma_0$ and (2), Eq. (13) may be expressed in the form

$$\mathcal{H}_{\rm RSII}^2 = \frac{8\pi G_{\rm N}}{3}\rho\left(1+\frac{\rho}{2\sigma_0}\right) + \frac{\mu\ell^2}{a^4}, \tag{17}$$

which differs from the standard Friedmann equation and is therefore subject to cosmological tests (see, e.g., Refs. [21,29]). The deviation proportional to ρ^2 poses no problem as it decays as a^{-8} in the radiation epoch and will rapidly become negligible after the end of the high-energy regime $\rho \simeq \sigma_0$ [21]. The last term on the right-hand side of (17), the so called "dark radiation," for positive μ should not exceed 10% of the total radiation content in the epoch of big bang (BB) nucleosynthesis whereas for negative μ could be as large as the rest of the radiation content [30,31]. As expected, both the one-sided and two-sided versions of the RSII model yield identical braneworld cosmologies.

Combining the time derivative of (17) with the energy conservation, one finds the second Friedmann equation (A53), which may be expressed as

$$\frac{1}{an}\frac{d}{dt}\left(\frac{1}{n}\frac{da}{dt}\right) + \mathcal{H}_{\rm RSII}^2 = \frac{4\pi G_{\rm N}}{3}(\rho-3p) - \frac{\rho}{\ell^2\sigma_0^2}(\rho+3p). \tag{18}$$

Note that the quadratic terms, i.e., the terms proportional to ρ^2 and ρp in (17) and (18), may be neglected in the low energy limit $\ell\mathcal{H}_{\rm RSII} \ll 1$. In that limit, Eqs. (17) and (18) reduce to the standard Friedmann equations for a two-component fluid consisting of dark radiation and the fluid obeying the equation of state $p = p(\rho)$.

For the purpose of comparison of the RSII and holographic cosmologies to be discussed in Sec. IV, it will be convenient to express the Friedman equation in terms of the metric in Fefferman-Graham coordinates for a brane placed at an arbitrary fixed $z = z_{\rm br}$. To this end, we transform the static bulk metric in Schwarzschild coordinates (r,t) to the time dependent metric in Fefferman-Graham coordinates (z,τ) in such a way that the time dependent brane position given by (10) is fixed at $z = z_{\rm br}$. Starting from (6), we make the coordinate transformation

$$t = t(\tau,z), \qquad r = r(\tau,z). \tag{19}$$

Then, the line element in new coordinates will have a general form,

$$ds_{(5)}^2 = \frac{\ell^2}{z^2}(\mathcal{N}^2(\tau,z)d\tau^2 - \mathcal{A}^2(\tau,z)d\Omega_\kappa^2 - dz^2), \tag{20}$$

where

$$\mathcal{A}^2(\tau,z) = \frac{z^2}{\ell^2}r^2(\tau,z). \tag{21}$$

To recover the induced metric (11) on the brane at $z = z_{\rm br}$, the functions $\mathcal{A}$ and $\mathcal{N}$ should satisfy the conditions

$$\frac{\ell^2}{z_{\rm br}^2}\mathcal{A}^2(\tau,z_{\rm br}) = a^2(t(\tau,z_{\rm br})), \tag{22}$$

$$\frac{\ell^2}{z_{\rm br}^2}\mathcal{N}^2(\tau,z_{\rm br}) = \dot{t}(\tau,z_{\rm br})^2 n^2(t(\tau,z_{\rm br})), \tag{23}$$

where the overdot denotes a derivative with respect to τ. Besides, from (10), it follows that

$$r(\tau,z_{\rm br}) = a(t(\tau,z_{\rm br})). \tag{24}$$

Using (22), the quantity $\mathcal{H}_{\rm RSII}$ may be expressed in terms of $\mathcal{A}_{\rm br}(\tau) = \mathcal{A}(\tau,z_{\rm br})$ and $\mathcal{N}_{\rm br}(\tau) = \mathcal{N}(\tau,z_{\rm br})$:

$$\frac{\ell^2}{z_{\rm br}^2}\mathcal{H}_{\rm RSII}^2 = \mathcal{H}_{\rm br}^2 = \frac{\dot{\mathcal{A}}_{\rm br}^2}{\mathcal{A}_{\rm br}^2\mathcal{N}_{\rm br}^2} + \frac{\kappa}{\mathcal{A}_{\rm br}^2}. \tag{25}$$

Then, the effective Friedmann equation (13) on the $z_{\rm br}$-brane takes the form

$$\mathcal{H}_{\rm br}^2 = \frac{(\sigma+\rho)^2}{z_{\rm br}^2\sigma_0^2} - \frac{1}{z_{\rm br}^2} + \frac{\mu z_{\rm br}^2}{\mathcal{A}_{\rm br}^4}. \tag{26}$$

This expression will be exploited in Sec. IV in the mapping between the RSII and holographic cosmologies.

III. HOLOGRAPHIC COSMOLOGY

Here, we outline a derivation of the Friedmann equations on the holographic brane following Apostolopoulos *et al.* [13]. Consider the line element (1) for a general asymptotically AdS_5 spacetime in Fefferman-Graham coordinates. The four-dimensional metric $g_{\mu\nu}$ near the boundary at $z=0$ can be expanded as [32]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + z^6 g^{(6)}_{\mu\nu} + \ldots. \tag{27}$$

By plugging this expansion into bulk Einstein's equations (A7) and solving thus obtained equations order by order in z, the tensors $g^{(n)}_{\mu\nu}$, $n>0$ may be found in terms of the metric $g^{(0)}_{\mu\nu}$ and its curvature tensor $R_{\mu\nu}$. The explicit expressions for $g^{(2)}_{\mu\nu}$ and $g^{(4)}_{\mu\nu}$ are found in the Appendix A of Ref. [32]. In particular, we will need

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6} R g^{(0)}_{\mu\nu}\right) \tag{28}$$

and the relation

$$\mathrm{Tr} g^{(4)} = -\frac{1}{4}\mathrm{Tr}(g^{(2)})^2, \tag{29}$$

where the trace of a tensor $A_{\mu\nu}$ is defined as

$$\mathrm{Tr} A = A^{\mu}_{\mu} = g^{(0)\mu\nu} A_{\mu\nu}. \tag{30}$$

We assume now that the time dependent bulk metric is of the form (20) such that

$$\mathcal{N}(\tau,0) = 1, \qquad \mathcal{A}(\tau,0) = a_0(t). \tag{31}$$

The boundary geometry is then described by a general FRW spacetime metric:

$$ds^2_{(0)} = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = d\tau^2 - a_0^2(\tau) d\Omega^2_\kappa. \tag{32}$$

Using effective Einstein equations (B13) derived in Appendix A, we obtain the holographic Friedmann equation

$$\frac{\dot{a}_0}{a_0} + \frac{\kappa}{a_0^2} = \frac{8\pi G_{\mathrm{N}}}{3}(\gamma\langle T^{\mathrm{CFT}}_{00}\rangle + T^{\mathrm{matt}}_{00}), \tag{33}$$

where $T^{\mathrm{matt}}_{\mu\nu}$ is the energy-momentum tensor associated with matter on the holographic brane and $T^{\mathrm{CFT}}_{\mu\nu}$ the energy-momentum tensor of the CFT on the boundary. According to the AdS/CFT prescription, the expectation value $\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle$ is obtained by functionally differentiating the renormalized on-shell bulk gravitational action with respect to the boundary metric $g^{(0)}_{\mu\nu}$. With this procedure, referred to as *holographic renormalization*, one finds [32]

$$\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle = -\frac{\ell^3}{4\pi G_5}\left\{ g^{(4)}_{\mu\nu} - \frac{1}{8}[(\mathrm{Tr} g^{(2)})^2 - \mathrm{Tr}(g^{(2)})^2] g^{(0)}_{\mu\nu} - \frac{1}{2}(g^{(2)})^2_{\mu\nu} + \frac{1}{4}\mathrm{Tr} g^{(2)} g^{(2)}_{\mu\nu}\right\}. \tag{34}$$

This expression is an explicit realization of the AdS/CFT correspondence: the vacuum expectation value of a boundary CFT operator is obtained solely in terms of the geometrical quantities of the bulk. The components of the tensors $g^{(2)}_{\mu\nu}$ and $g^{(4)}_{\mu\nu}$ may be calculated either by applying the explicit expressions from Ref. [32] to the metric (27) or by expanding the metric (20) near $z=0$ and comparing the z^2 and z^4 terms with the corresponding ones in the expansion (27). Then, from (34), one obtains

$$\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle = t_{\mu\nu} + \frac{1}{4}\langle T^{\mathrm{CFT}\,\alpha}{}_{\alpha}\rangle g^{(0)}_{\mu\nu}. \tag{35}$$

The first term on the right-hand side is a traceless tensor, the nonvanishing components of which are

$$t_{00} = -3t^i_i = \frac{3\ell^3}{64\pi G_5}\left(\mathcal{H}_0^4 + \frac{4\mu}{a_0^4} - \frac{\ddot{a}_0}{\dot{a}_0}\mathcal{H}_0^2\right), \tag{36}$$

where

$$\mathcal{H}_0^2 = H_0^2 + \frac{\kappa}{a_0^2}, \tag{37}$$

and $H_0 = \dot{a}_0/a_0$ is the Hubble expansion rate on the $z=0$ boundary. The second term on the right-hand side of (35) corresponds to the conformal anomaly

$$\langle T^{\mathrm{CFT}\,\alpha}{}_{\alpha}\rangle = \frac{3\ell^3}{16\pi G_5}\frac{\ddot{a}_0}{a_0}\mathcal{H}_0^2. \tag{38}$$

Hence, the CFT dual to the time dependent asymptotically AdS_5 bulk metric (20) is a conformal fluid with the equation of state $p_{\mathrm{CFT}} = \rho_{\mathrm{CFT}}/3$, where $\rho_{\mathrm{CFT}} = t_{00}$, $p_{\mathrm{CFT}} = -t^i_i$. In a static case, i.e., when $\dot{a}_0 = 0$, the fluid is dual to the AdS_5 black hole with the energy density related to the black-hole mass M_{bh} defined in (9) as

$$\rho_{\mathrm{CFT}} = \frac{M_{\mathrm{bh}}}{V} + \frac{3\kappa^2}{64\pi G_5 \ell}, \tag{39}$$

where $V = 2\pi^2\ell^3$ is the volume of the three-dimensional space for a spherical geometry. If the boundary geometry is FRW, the dual conformal fluid behaves as radiation, the so called dark radiation.

So far in our consideration, the cosmological scale $a_0(\tau)$ at the boundary is assumed to be an arbitrary function of τ. In order to satisfy the appropriate boundary condition for a given $a_0(\tau)$, we place a brane on the boundary with matter described by the energy-momentum tensor

$$T_{00}^{\rm matt} = \rho_0, \qquad T_{ij}^{\rm matt} = p_0 g_{ij}^{(0)}, \tag{40}$$

where ρ_0 and p_0 are the total density and pressure, respectively, including the brane tension $\sigma_{\rm br}$,

$$\rho_0 = \rho_{\rm matt} + \sigma_{\rm br}, \qquad p_0 = p_{\rm matt} - \sigma_{\rm br}. \tag{41}$$

The Einstein equations (B13) together with (40), (35), and (36) yield the holographic Friedmann equation [13,33]

$$\mathcal{H}_0^2 = \frac{\ell^2}{4}\left(\mathcal{H}_0^4 + \frac{4\mu}{a_0^4}\right) + \frac{8\pi G_{\rm N}}{3}\rho_0. \tag{42}$$

Note that the coefficient of the quartic term does not depend on whether one is using a one-sided or a two-sided regularization. Equation (42) was derived by Kiritsis [33] and independently by Apostolopoulos *et al.* [13], albeit they disagree in the coefficient of the quartic term.[1] Solving the quadratic equation (42), one finds $\mathcal{H}_0$ expressed as an explicit function of ρ_0,

$$\mathcal{H}_0^2 = \frac{2}{\ell^2}\left(1 + \epsilon\sqrt{1 - \frac{2\rho_0}{\sigma_0} - \frac{\mu\ell^4}{a_0^4}}\right), \tag{43}$$

where $\epsilon = +1$ or -1 and σ_0 is a constant defined in (15).

For $\epsilon = -1$, the physical range of the expansion parameter $\mathcal{H}_0$ is given by

$$0 \le \mathcal{H}_0^2\ell^2 \le 2, \tag{44}$$

corresponding to the energy density interval

$$-\frac{\sigma_0}{2}\frac{\mu\ell^4}{a_0^4} \le \rho_0 \le \frac{\sigma_0}{2}\left(1 - \frac{\mu\ell^4}{a_0^4}\right). \tag{45}$$

In this case, Eq. (43) agrees with the RSII Friedmann equation (17) at quadratic order in ρ and linear order in μ.

For $\epsilon = +1$, the physical range of $\mathcal{H}_0$ is given by

$$\infty > \mathcal{H}_0^2\ell^2 \ge 2, \tag{46}$$

corresponding to

$$-\infty < \rho_0 \le \frac{\sigma_0}{2}\left(1 - \frac{\mu\ell^4}{a_0^4}\right). \tag{47}$$

Note that the density ρ_0 is negative when $\mathcal{H}_0^2$ lies outside the interval

$$2 - 2\sqrt{1 - \mu\ell^4/a_0^4} \le \mathcal{H}_0^2\ell^2 \le 2 + 2\sqrt{1 - \mu\ell^4/a_0^4}. \tag{48}$$

The second Friedmann equation is obtained by combining the time derivative of (42) with the energy conservation

$$\dot\rho_0 + 3H_0(\rho_0 + p_0) = 0. \tag{49}$$

One finds

$$\dot H_0 - \frac{\kappa}{a_0^2} = -4\pi G_{\rm N}(\rho_0 + p_0) + \frac{\ell^2}{2}\left(\dot H_0 - \frac{\kappa}{a_0^2}\right)\mathcal{H}_0^2 - \frac{2\ell^2\mu}{a_0^4}, \tag{50}$$

which may also be written in the form

$$\frac{\ddot a_0}{a_0}\left(1 - \frac{\ell^2}{2}\mathcal{H}_0^2\right) + \mathcal{H}_0^2 = \frac{4\pi G_{\rm N}}{3}(\rho_0 - 3p_0). \tag{51}$$

Given $a_0(\tau)$, the Friedmann equations (42) and (50) on the boundary describe the equation of state $p_0 = p_0(\rho_0)$ in a parametric form.

Nota bene (N.B.): As in the RSII cosmology, in the low energy limit $\ell\mathcal{H}_0 \ll 1$, Eqs. (42) and (51) reduce to the standard Friedmann equations for a two-component fluid consisting of dark radiation and the fluid obeying the equation of state $p_0 = p_0(\rho_0)$.

Remarkably, Eq. (42) has been also derived in other contexts. For $\kappa = 1$ and constant ρ_0 with (B19), Eq. (42) coincides with the saddle point of the spatially closed minisuperspace partition function dominated by matter fields conformally coupled to gravity [34]. A variant of Eq. (42) has been derived by Lidsey [35] from the generalized uncertainty principle and the first law of thermodynamics applied to the apparent horizon entropy. The quartic term with $\kappa = 0$ in (42) has been derived quite recently as a quantum correction to the Friedmann equation using thermodynamic arguments at the apparent horizon [36].

It is worth addressing the holographic cosmology of de Sitter type, i.e., for a constant $\rho_0 = \Lambda/(8\pi G_{\rm N})$, with $\mu = 0$. A static representation of the de Sitter boundary spacetime has been recently discussed [37] in the context of AdS/CFT. Using the standard $\kappa = 1$, 0, and -1 representations of the de Sitter geometry

$$ds^2 = \begin{cases} d\tau^2 - h^{-2}\cosh^2 h\tau(d\chi^2 + \sin^2\chi d\Omega^2), & \kappa = 1, \\ d\tau^2 - e^{2h\tau}(d\chi^2 + \chi^2 d\Omega^2), & \kappa = 0, \\ d\tau^2 - h^{-2}\sinh^2 h\tau(d\chi^2 + \sinh^2\chi d\Omega^2), & \kappa = -1, \end{cases} \tag{52}$$

Eq. (43) yields

$$h^2 = \frac{2}{\ell^2}\left(1 + \epsilon\sqrt{1 - \frac{\Lambda}{4\pi G_{\rm N}\sigma_0}}\right). \tag{53}$$

[1] The reason for the disagreement is twofold: first, there is a difference by a factor of 2 because the regularization used in Ref. [13] was one sided whereas in Ref. [33] was two sided. Another factor of 2 disagreement is due to an unconventional definition of the stress tensor in Ref. [33].

By making use of (2) with $\gamma = 1$ and (B19), Eq. (43) may be expressed as

$$\mathcal{H}_0^2 = \frac{1}{32\pi b G_N}\left(1 + \epsilon\sqrt{1 - \frac{64\pi}{3} b G_N \Lambda}\right), \tag{54}$$

which coincides with the equation for $\mathcal{H}_0^2$ of Pelinson *et al.* [38] for the anomaly induced inflation. Equation (54) with $\epsilon = +1$ and $\Lambda = 0$ describes the Starobinski inflation model [39]. With $\epsilon = -1$ and $\Lambda \ll 1/G_N$, one recovers at linear order the standard de Sitter cosmology with the expansion rate $\mathcal{H}_0^2 = \Lambda/3$.

IV. HOLOGRAPHIC MAP

The bulk metric that approaches the metric (32) as we approach the boundary $z = 0$ is expressed in the form (20) where the functions $\mathcal{A}$ and $\mathcal{N}$ are derived in Ref. [13] and are expressed in terms of a_0 as

$$\mathcal{A}^2 = a_0^2\left[\left(1 - \frac{\mathcal{H}_0^2 z^2}{4}\right)^2 + \frac{1}{4}\frac{\mu z^4}{a_0^4}\right], \tag{55}$$

$$\mathcal{N} = \frac{\dot{\mathcal{A}}}{\dot{a}_0}. \tag{56}$$

The spacetime (20) may be regarded as a z foliation of the bulk with an FRW cosmology on each z slice. For a constant a_0, e.g., $a_0 = \ell$, one recovers the static AdS-Schwarzschild solution (6), in which case the metric at the boundary $z = 0$ $(r \to \infty)$ represents the static Einstein universe.

The Hubble expansion rate corresponding to a z-cosmology is defined as

$$H \equiv \frac{\dot{\mathcal{A}}}{\mathcal{N}\mathcal{A}} = H_0 \frac{a_0}{\mathcal{A}} \tag{57}$$

and similarly

$$\mathcal{H} \equiv H^2 + \frac{\kappa}{\mathcal{A}^2}\mathcal{H}_0\frac{a_0}{\mathcal{A}}, \tag{58}$$

where $\mathcal{H}_0$ is defined by (37).

It is of interest to express the cosmological scale $\mathcal{A} = \mathcal{A}(\tau, z)$, the lapse function $\mathcal{N} = \mathcal{N}(\tau, z)$, and the Hubble rate $H = H(\tau, z)$ at an arbitrary z slice in terms of $\mathcal{A}_{\rm br} = \mathcal{A}(\tau, z_{\rm br})$, $\mathcal{N}_{\rm br} = \mathcal{N}(\tau, z_{\rm br})$, and $H_{\rm br} = H(\tau, z_{\rm br})$ on another slice $z_{\rm br}$. To make a connection with the RSII cosmology, we can identify $(\ell/z_{\rm br})\mathcal{A}_{\rm br} = a(t(\tau, z_{\rm br}))$ and $(\ell/z_{\rm br})\mathcal{N}_{\rm br} = n(t(\tau, z_{\rm br}))$ (see Appendix A), where $a(t)$ and $n(t)$ are the functions that appear in the line element (11) induced on the RSII brane.

First, using (57), we can express (55) as an equation for a_0^2, $\mathcal{A}^2$, and H^2 and similarly as another equation for a_0^2, $\mathcal{A}_{\rm br}^2$, and $H_{\rm br}^2$. Eliminating a_0^2 from these two equations, we find

$$\mathcal{A} = \frac{\mathcal{A}_{\rm br}}{\sqrt{2}}\left[\left(1 + \frac{1}{2}\mathcal{H}_{\rm br}^2 z_{\rm br}^2\right)\left(1 + \frac{z^4}{z_{\rm br}^4}\right) - \mathcal{H}_{\rm br}^2 z^2 + \epsilon\sqrt{1 + \mathcal{H}_{\rm br}^2 z_{\rm br}^2 - \frac{\mu z_{\rm br}^4}{\mathcal{A}_{\rm br}^4}}\left(1 - \frac{z^4}{z_{\rm br}^4}\right)\right]^{1/2}, \tag{59}$$

where

$$\mathcal{H}_{\rm br}^2 = H_{\rm br}^2 + \frac{\kappa}{\mathcal{A}_{\rm br}^2} \tag{60}$$

and $\epsilon = +1$ or -1. Thus, the map is not unique due to the sign ambiguity in front of the square root. However, consistency with (55) in the limit $z_{\rm br} \to 0$ and continuity of the metric requires $\epsilon = +1$ in the region $z \geq z_{\rm br}$, whereas for $z < z_{\rm br}$ both signs are allowed. This remaining nonuniqueness is removed for the two-sided braneworld by the Z_2 symmetry $z \leftrightarrow z_{\rm br}^2/z$, in which case $\epsilon = -1$ is fixed for the branch $z < z_{\rm br}$. Then, the metric (20) becomes invariant under the transformation $z \to \bar{z} = z_{\rm br}^2/z$. For the benefit of a joint description of one-sided and two-sided versions, the expression (59) may be written as

$$\mathcal{A} = \frac{\mathcal{A}_{\rm br}}{\sqrt{2}}\left[\left(1 + \frac{1}{2}\mathcal{H}_{\rm br}^2 z_{\rm br}^2\right)\left(1 + \frac{z^4}{z_{\rm br}^4}\right) - \mathcal{H}_{\rm br}^2 z^2 + \mathcal{E}(z)\sqrt{1 + \mathcal{H}_{\rm br}^2 z_{\rm br}^2 - \frac{\mu z_{\rm br}^4}{\mathcal{A}_{\rm br}^4}}\left(1 - \frac{z^4}{z_{\rm br}^4}\right)\right]^{1/2}, \tag{61}$$

where we have introduced a two-valued step function,

$$\mathcal{E}(z) = \begin{cases} +1, & \text{for } z \geq z_{\rm br}, \\ -1, & \text{for } z < z_{\rm br}, \text{two-sided version}, \\ +1 \quad \text{or} \quad -1, & \text{for } z < z_{\rm br}, \text{one-sided version}. \end{cases} \tag{62}$$

Furthermore, applying the definition (56) to $\mathcal{N}$ and $\mathcal{N}_{\rm br}$ combined with (61), we find

$$\frac{\mathcal{N}}{\mathcal{N}_{\rm br}} = \frac{\mathcal{A}}{\mathcal{A}_{\rm br}} + \left(\frac{z_{\rm br}^2\mathcal{A}_{\rm br}^2\mathcal{H}_{\rm br}\dot{\mathcal{H}}_{\rm br}}{4\mathcal{A}\dot{\mathcal{A}}_{\rm br}} + \frac{\mu z_{\rm br}^4}{2\mathcal{A}\mathcal{A}_{\rm br}^3}\right) \times \frac{\mathcal{E}(z)(1 - z^4/z_{\rm br}^4)}{\sqrt{1 + \mathcal{H}_{\rm br}^2 z_{\rm br}^2 - \mu z_{\rm br}^4/\mathcal{A}_{\rm br}^4}} + \frac{z_{\rm br}^2\mathcal{A}_{\rm br}^2\mathcal{H}_{\rm br}\dot{\mathcal{H}}_{\rm br}}{4\mathcal{A}\dot{\mathcal{A}}_{\rm br}}\left(1 - \frac{z^2}{z_{\rm br}^2}\right)^2. \tag{63}$$

The map between the holographic and RSII cosmologies is schematically illustrated in Fig. 1.

Note that the quantity $\mathcal{H}_{\rm br}$ in (61) and (63) is identical to that defined in (25) for the RSII cosmology. Besides, it is clear by construction that the functions $\mathcal{A}$ and $\mathcal{N}$ in (61) and (63) are, up to a sign, equal to those in the line element (20). The general expression (61) agrees with that of Brax and Peschanski [15] obtained for the two-sided model with $z_{\rm br} = \ell$ and $\kappa = 0$.

Next, we derive a relation between the Hubble rate H on an arbitrary z slice and the Hubble rate H_0 on the $z = 0$ boundary. Using (57) and (58), we find

$$\mathcal{H}^2 = 2\mathcal{H}_{\rm br}^2\left[\left(1+\frac{\mathcal{H}_{\rm br}^2 z_{\rm br}^2}{2}\right)\left(1+\frac{z^4}{z_{\rm br}^4}\right) - \mathcal{H}_{\rm br}^2 z^2 + \mathcal{E}(z)\sqrt{1+\mathcal{H}_{\rm br}^2 z_{\rm br}^2 - \frac{\mu z_{\rm br}^4}{\mathcal{A}_{\rm br}^4}\left(1-\frac{z^4}{z_{\rm br}^4}\right)}\right]^{-1}. \quad (64)$$

Evaluating this expression at $z = 0$, we find the relationship between $z = 0$ cosmology and the cosmology on the brane at $z = z_{\rm br}$,

$$\mathcal{H}_0^2 = 2\mathcal{H}_{\rm br}^2\left(1+\frac{\mathcal{H}_{\rm br}^2 z_{\rm br}^2}{2}+\mathcal{E}_0\sqrt{1+\mathcal{H}_{\rm br}^2 z_{\rm br}^2-\frac{\mu z_{\rm br}^4}{\mathcal{A}_{\rm br}^4}}\right)^{-1}, \quad (65)$$

where $\mathcal{E}_0 \equiv \mathcal{E}(0) = -1$ for the two-sided and $\mathcal{E}_0 = +1$ or -1 for the one-sided version of the RSII model. The inverse relation can be obtained either from (64) by taking the limit

$$\begin{array}{ccc} d\tau^2 - a_0^2 d\Omega_\kappa^2 & \xrightarrow{\tau\to\tilde{\tau}} & (1/\mathcal{N}^2)d\tilde{\tau}^2 - a_0^2 d\Omega_\kappa^2 \\ \downarrow z & & \downarrow z \\ \mathcal{N}^2 d\tau^2 - \mathcal{A}^2 d\Omega_\kappa^2 & \xrightarrow[\tau\to\tilde{\tau}]{} & d\tilde{\tau}^2 - \mathcal{A}^2 d\Omega_\kappa^2 \end{array}$$

FIG. 1. Mapping of the holographic cosmology on the $z = 0$ boundary into the braneworld cosmology on an arbitrary z slice. The times τ and $\tilde{\tau}$ are the holographic and RSII synchronous times, respectively.

$z_{\rm br} \to 0$ and replacing $z \to z_{\rm br}$ or by making use of (57) and (55). Either way, we find

$$\mathcal{H}_{\rm br}^2 = \mathcal{H}_0^2\left[1-\frac{\mathcal{H}_0^2 z_{\rm br}^2}{2}+\frac{1}{16}\left(\mathcal{H}_0^4+\frac{4\mu}{a_0^4}\right)z_{\rm br}^4\right]^{-1}. \quad (66)$$

The functional dependence of $\mathcal{H}_{\rm br}^2$ vs $\mathcal{H}_0^2$ is depicted in Fig. 2 for two values of the black-hole mass parameter: $\mu = 0$ (left panel) and $\mu\ell^4/a_0^4 = 1/2$ with $z_{\rm br}^2/\ell^4 = 2$ (right panel). The shaded areas in both panels denote the region defined by (48), i.e., the region in which $\rho_0 > 0$. The function assumes a maximal value

$$\mathcal{H}_{\rm br}^2|_{\rm max} = \frac{8\sqrt{4+\mu z_{\rm br}^4/a_0^4}}{z_{\rm br}^2(\sqrt{4+\mu z_{\rm br}^4/a_0^4}-2)^2} \quad (67)$$

at

$$\mathcal{H}_0^2|_{\rm max} = \frac{2}{z_{\rm br}^2}\sqrt{4+\frac{\mu z_{\rm br}^4}{a_0^4}}. \quad (68)$$

For $\mu = 0$, the maximum becomes a singularity at $\mathcal{H}_0^2|_{\rm max} = 4/z_{\rm br}^2$. The part of the domain where

$$\mathcal{H}_0^2 < \mathcal{H}_0^2|_{\rm max} \quad (69)$$

corresponds to the branch $\mathcal{E}(z) = +1$ for $z < z_{\rm br}$ of Eq. (61), and hence the condition (69) is met for the one-sided version only. The remaining part,

$$\mathcal{H}_0^2 \geq \mathcal{H}_0^2|_{\rm max}, \quad (70)$$

corresponds to the branch $\mathcal{E}(z) = -1$ for $z < z_{\rm br}$ and is relevant for both the one-sided and two-sided versions. From (66), in the limit $\mathcal{H}_0 \to \infty$, we find

$$\mathcal{H}_{\rm br} = \frac{4}{z_{\rm br}^2\mathcal{H}_0}. \quad (71)$$

However, it is important to note that the regime in which $\mathcal{H}_0$ does not satisfy Eq. (48) violates the weak energy condition

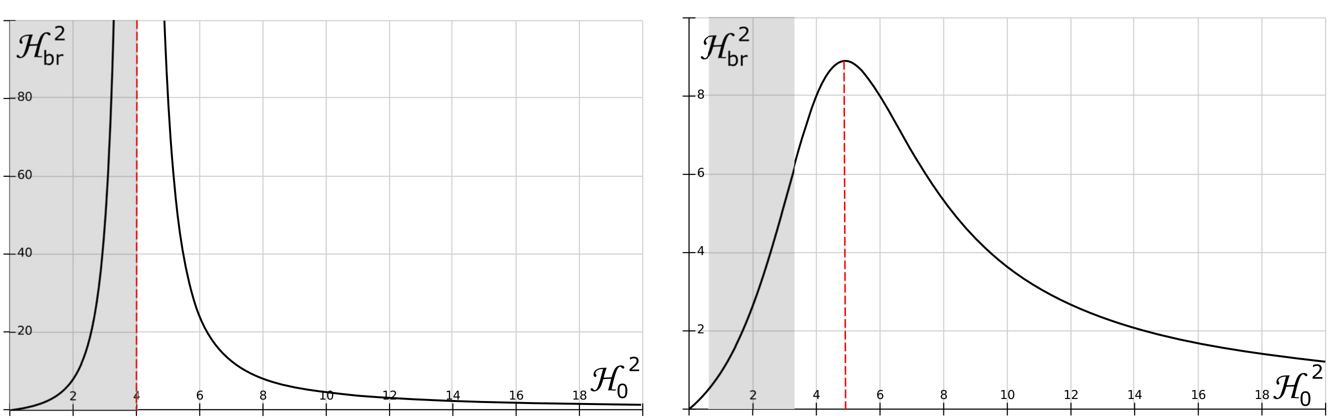

FIG. 2. $\mathcal{H}_{\rm br}^2$ as a function of $\mathcal{H}_0^2$ (both in units of $z_{\rm br}^{-2}$) defined by (66) for $\mu = 0$ (left panel) and $\mu\ell^4/a_0^4 = 2$ with $z_{\rm br}^2/\ell^2 = 2$ (right panel). The region left of the vertical dashed red line is relevant for the one-sided version only. The shaded area corresponds to the physical region $\rho_0 > 0$.

and a large $\mathcal{H}_0$ implies a large negative energy density. Thus, the large (negative) density limit on the holographic brane maps into the low-density limit on the RSII brane.

The relationship (66) simplifies at a particular point $z_{\rm br} = \sqrt{2}\ell$. Applying (42) at $z_{\rm br} = \sqrt{2}\ell$, we obtain

$$\mathcal{H}^2_{\sqrt{2}\ell} = \mathcal{H}_0^2\left(1 - \frac{8\pi G_{\rm N}\ell^2}{3}\rho_0\right)^{-1}, \tag{72}$$

where ρ_0 is the effective energy density of matter on the holographic brane, as defined in (41).

N.B.: Due to the Z_2 symmetry, the brane at $z = 0$ ($y = -\infty$) must be identical to the brane at $z = \infty$. In the RSII model, the second brane is pushed off to $z = \infty$, and hence the holographic brane at $z = 0$ is identical to the second brane of the RSII model.

Next, we analyze a few special cases in two scenarios: the holographic and the RSII cosmological scenario with the primary braneworld located at $z = 0$ and $z = z_{\rm br}$, respectively. In each of the two scenarios, we assume the presence of matter on the primary brane only and no matter in the bulk.

A. Holographic scenario

In the holographic scenario, the primary braneworld is at the AdS boundary at $z = 0$ evolving according to the Friedmann equations (42) and (50). The cosmology on the RSII brane at an arbitrary z slice emerges as a reflection of the boundary cosmology. We would like to express the cosmological parameters on the RSII brane at $z = z_{\rm br}$ in terms of the parameters on the holographic brane at $z = 0$. If the density ρ_0 and pressure p_0 on the holographic brane are known, the cosmological scale a_0 may be derived by integrating (42) and (51). On the other hand, given $a_0(\tau)$ on the boundary, Eqs. (42) and (50) define the equation of state $p_0 = p_0(\rho_0)$ in a parametric form. Using (55) and (A62), the scale a on the RSII brane can be expressed as

$$a^2 = \frac{\ell^2}{z_{\rm br}^2} a_0^2 Q^2(z_{\rm br}), \tag{73}$$

where

$$Q^2(z_{\rm br}) = \left(1 - \frac{\mathcal{H}_0^2 z_{\rm br}^2}{4}\right)^2 + \frac{1}{4}\frac{\mu z_{\rm br}^4}{a_0^4}, \tag{74}$$

Thus, given the RSII-brane position $z_{\rm br}$, the mapping from the holographic to RSII cosmology is unique.

Knowing a_0, we may calculate the effective density of matter on the RSII brane assuming the Friedmann equation (13) holds. Solving (26) for ρ, one finds

$$\rho = \sigma_0\sqrt{1 + \mathcal{H}_{\rm br}^2 z_{\rm br}^2 - \frac{\mu z_{\rm br}^4}{\mathcal{A}_{\rm br}^4}} - \sigma. \tag{75}$$

Then, by making use of (66) and (73), we obtain the effective density ρ on the RSII brane expressed in terms of a_0 and $\mathcal{H}_0$ on the holographic brane,

$$\rho = \sigma_0\sqrt{1 + \frac{\mathcal{H}_0^2 z_{\rm br}^2}{Q^2(z_{\rm br})} - \frac{\mu z_{\rm br}^4}{a_0^4 Q^4(z_{\rm br})}} - \sigma, \tag{76}$$

where the function $Q(x)$ is defined by (74). Using (43) and (66), we can also express ρ as an explicit function of a_0 and ρ_0,

$$\frac{\rho}{\sigma_0} = \left[1 + 2\frac{z_{\rm br}^2}{\ell^2}\left(1 + \epsilon\sqrt{1 - \frac{2\rho_0}{\sigma_0} - \frac{\mu\ell^4}{a_0^4}}\right)Q^{-2} - \frac{\mu z_{\rm br}^4}{a_0^4}Q^{-4}\right]^{1/2} - \frac{\sigma}{\sigma_0}, \tag{77}$$

where

$$Q^2 = \left[1 - \frac{z_{\rm br}^2}{2\ell^2}\left(1 + \epsilon\sqrt{1 - \frac{2\rho_0}{\sigma_0} - \frac{\mu\ell^4}{a_0^4}}\right)\right]^2 + \left(\frac{z_{\rm br}^2}{2\ell^2}\right)^2\frac{\mu\ell^4}{a_0^4}. \tag{78}$$

Equation (77) simplifies considerably when the brane is placed at $z_{\rm br} = \ell$. In this case, we find

$$\frac{\rho}{\sigma_0} = \left|\frac{1 + \rho_0/\sigma_0 - \epsilon\sqrt{1 - 2\rho_0/\sigma_0 - \mu\ell^4/a_0^4}}{1 - \rho_0/\sigma_0 - \epsilon\sqrt{1 - 2\rho_0/\sigma_0 - \mu\ell^4/a_0^4}}\right| - \frac{\sigma}{\sigma_0}. \tag{79}$$

Note that the function $\rho = \rho(\rho_0, a_0)$ is not uniquely defined, although the mapping $a_0 \to a$ is unique.

With no black hole in the bulk, i.e., for $\mu = 0$, the density ρ is a function of ρ_0 only. For $\mu \neq 0$, the density ρ depends on both ρ_0 and a_0. If the function $a_0 = a_0(\tau)$ is known or the equation of state $p = p(\rho_0)$ is specified, the scale a_0 will be an implicit function of ρ_0 through the Friedmann equations (42) and (51). However, as we have specified neither the equation of state nor the function $a_0 = a_0(\tau)$, we treat a_0 as a parameter and show the functional dependence $\rho = \rho(\rho_0)$ for various values of $\mu\ell^4/a_0^4$ and various $z_{\rm br}^2/\ell^2$ in Figs. 3 and 4 for $\epsilon = -1$ and $+1$, respectively.

It is of interest to analyze the expression (77) in the three limiting regimes: $|\rho_0/\sigma_0| \ll 1$, $\rho_0/\sigma_0 \to -\infty$, and $z_{\rm br} \ll \ell$. Consider first the regime of large negative ρ_0. Taking the limit $\rho_0/\sigma_0 \to -\infty$ of (77), we find

$$\frac{\rho}{\sigma_0} = \epsilon\frac{4\ell^2}{z_{\rm br}^2}\sqrt{-\frac{\sigma_0}{2\rho_0}}. \tag{80}$$

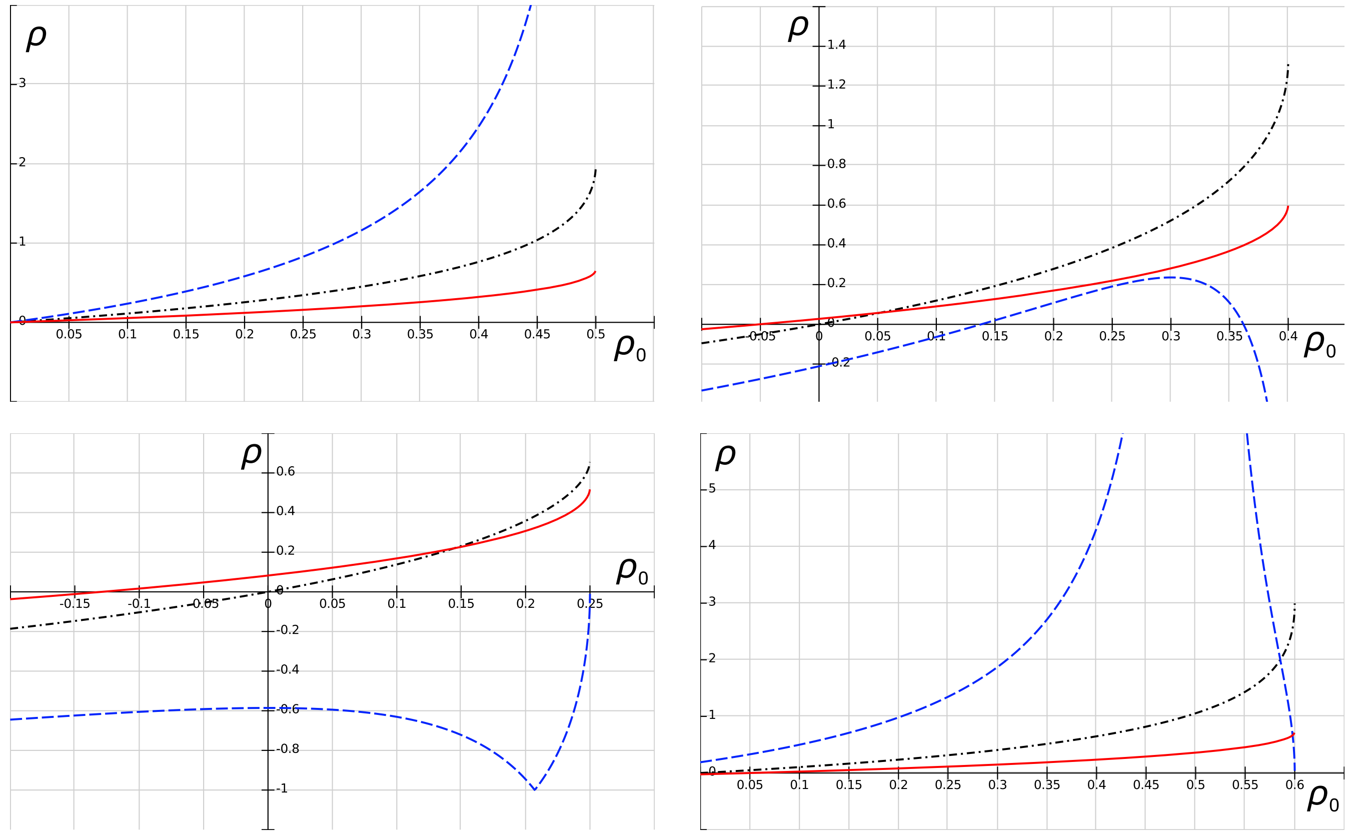

FIG. 3. The effective density on the RSII brane ρ as a function of the density on the holographic brane ρ_0 (both in units of σ_0) for $\epsilon = -1$, $\sigma = \sigma_0$, and $\mu\ell^4/a_0^4 = 0$ (top left), 0.2 (top right), 0.5 (bottom left), and -0.2 (bottom right). The full red, dash-dotted black, and dashed blue lines represent $z_{\rm br}^2/\ell^2 = 0.5$, 1, and 2, respectively.

This equation is equivalent to Eq. (71) and can be obtained directly from (71) by making use of the low-density limit of the RSII Friedmann equation (17) and the large $\mathcal{H}_0$ limit of the holographic Friedmann equation (42).

Next, consider the limit $z_{\rm br}/\ell \ll 1$. This case is important because, as discussed in Appendix B, in the limit $z_{\rm br} \to 0$, the RSII brane provides an infrared cutoff regularization of the on-shell bulk action. According to (68), in this limit, $\mathcal{H}_0|_{\rm max} \to \infty$, so the necessary condition (70) for the $\mathcal{E}(0) = -1$ cannot be met. Hence, the limit $z_{\rm br}/\ell \ll 1$ is relevant only for the $\mathcal{E}(0) = +1$ branch of the one-sided version. In this limit, $Q(z_{\rm br}) \to 1$, and the expression (77) reduces to

$$\rho = \sigma_0 - \sigma, \tag{81}$$

so, with the fine-tuning condition $\sigma = \sigma_0$, the effective density on the RSII brane vanishes as the brane position approaches the boundary at $z = 0$.

1. Low-density regime

The regime in which $|\rho_0/\sigma_0| \ll 1$ is relevant for the one-sided version only since in this case $\mathcal{H}_0 \ll 1$ and the necessary condition (70) for the two-sided version is not met unless $z_{\rm br} \gg \ell$. For $\epsilon = -1$, $\rho_0 \ll \sigma_0$, and $\mu\ell^4/a_0^4 \ll 1$, we find at linear order in μ and quadratic order in ρ_0

$$\frac{\rho}{\sigma_0} = 1 - \frac{\sigma}{\sigma_0} + \frac{z_{\rm br}^2}{\ell^2}\frac{\rho_0}{\sigma_0} + \frac{1}{2}\frac{z_{\rm br}^2}{\ell^2}\left(\frac{z_{\rm br}^2}{\ell^2} + 1\right)\frac{\rho_0^2}{\sigma_0^2} - \frac{1}{2}\frac{z_{\rm br}^2}{\ell^2}\left(\frac{z_{\rm br}^2}{\ell^2} - 1\right)\frac{\mu\ell^4}{a_0^4} + \dots . \tag{82}$$

Hence, at linear order, the effective energy density on the RSII brane equals the energy density on the holographic brane multiplied by a constant plus the cosmological constant term which can be eliminated by adopting the RSII fine-tuning condition $\sigma = \sigma_0$.

The effective pressure on the RSII brane can be easily derived by making use of the energy conservation equations (A51) on the RSII brane and (49) on the holographic brane. At linear order, one finds

$$p = -(\sigma_0 - \sigma) + \frac{z_{\rm br}^2}{\ell^2} p_0 + \dots . \tag{83}$$

Hence, at linear order, the effective fluid on the RSII brane satisfies the same equation of state as the fluid on the holographic brane. The cosmological constant term will

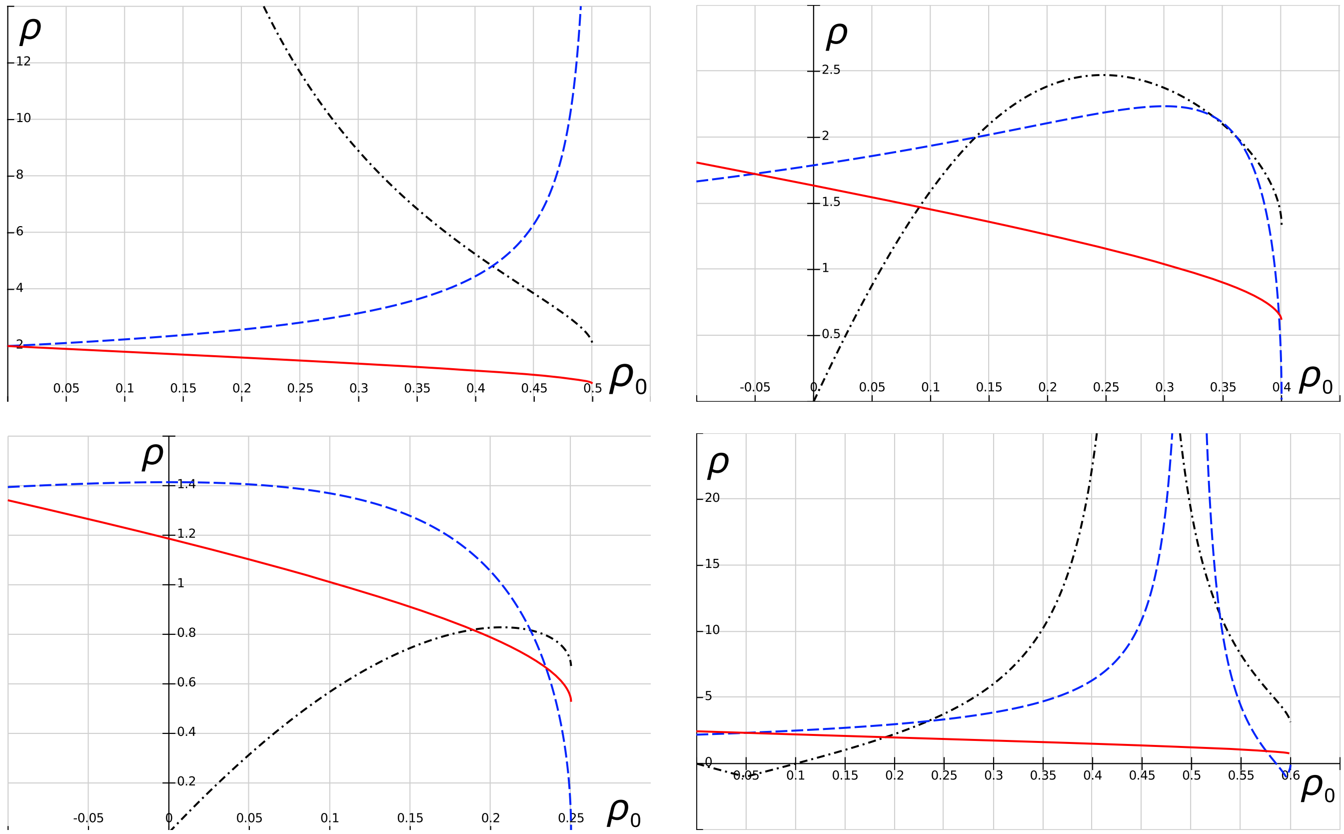

FIG. 4. Same as in Fig. 3 for $\epsilon = +1$.

vanish on both branes if the RSII fine-tuning condition is imposed, whereas the dark radiation contribution will be the same on the two branes only if $z_{\rm br} = \ell$. We recover the standard cosmology on both branes by choosing ℓ such that σ_0 is sufficiently large to suppress the quadratic and higher terms in (82).

For $\epsilon = +1$ and $z_{\rm br}^2/\ell^2 = 1$, ρ diverges in the limit $\rho_0 \to 0$. For an arbitrary $z_{\rm br}^2/\ell^2$, we find at linear order

$$\frac{\rho}{\sigma_0} = \frac{z_{\rm br}^2/\ell^2 + 1}{z_{\rm br}^2/\ell^2 - 1} - \frac{\sigma}{\sigma_0} + \frac{z_{\rm br}^2/\ell^2}{(z_{\rm br}^2/\ell^2 - 1)^2}\frac{\rho_0}{\sigma_0} - \frac{z_{\rm br}^2/\ell^2}{2(z_{\rm br}^2/\ell^2 - 1)^3}\frac{\mu\ell^4}{a_0^4} + \dots . \tag{84}$$

In this case, the effective energy density on the RSII brane at linear order differs from the energy density on the holographic brane by a multiplicative constant. Besides, for $\sigma = \sigma_0$, the effective cosmological constant does not vanish and is equal to

$$\Lambda_{\rm br} = \frac{6}{\ell^2}\frac{z_{\rm br}^2/\ell^2 + 1}{z_{\rm br}^2/\ell^2 - 1} - \frac{6}{\ell^2}. \tag{85}$$

This scenario offers a few interesting possibilities. Suppose the energy density ρ_0 on the holographic brane describes matter with the equation of state satisfying $3p_0 + \rho_0 > 0$, as for, e.g., cold dark matter. According to (84) and (85), we have an asymptotically de Sitter universe on the RSII brane. The location of the brane is crucial. For $z_{\rm br}$ of the order of ℓ (excluding $z_{\rm br} = \ell$), we could have the standard ΛCDM cosmology on the RSII brane if we included a cosmological constant term in ρ_0 and fine tuned it to cancel $\Lambda_{\rm br}$ up to a small phenomenologically acceptable contribution. In principle, this could work even without the RSII fine-tuning condition $\sigma = \sigma_0$. For small $\ell/z_{\rm br}$, if we impose the RSII fine-tuning condition, both the constant and linear terms will be suppressed by a factor $\ell^2/z_{\rm br}^2$. So we can choose the ratio $\ell/z_{\rm br}$ such that the effective cosmological constant $\Lambda_{\rm br}$ fits the observed value today,

$$\Lambda = 3\Omega_\Lambda H_0^2, \tag{86}$$

where H_0 is today's Hubble constant of the order of 2.5×10^{-40} GeV. Expanding (85) for small $\ell/z_{\rm br}$ and equating $\Lambda_{\rm br}$ with Λ, we find

$$\frac{\ell^2}{z_{\rm br}^2} = \frac{\sqrt{\Omega_\Lambda}}{2} H_0 \ell \lesssim 10^{-28}, \tag{87}$$

where the numerical estimate of the right-hand side is obtained for $\Omega_\Lambda \simeq 0.7$ and the Newtonian potential constraint (4) at small distances with $\ell \lesssim 10^{12}$ GeV^{-1}.

B. RSII cosmological scenario

In the RSII scenario, the primary braneworld is the RSII brane at $z = z_{\rm br}$ with cosmology determined by Eqs. (17) and (18). Observers at the boundary brane at $z = 0$ experience the emergent cosmology which is a reflection of the RSII cosmology. We would like to express the cosmological scale and effective energy density on the holographic brane at $z = 0$ in terms of cosmological scale and energy density on the RSII brane. For simplicity, in the following, we assume the RSII fine-tuning condition $\sigma = \sigma_0$. If the density ρ and pressure p on the RSII brane are known, the cosmological scale a may be derived by integrating (17) and (18). On the other hand, given $a(\tau)$ on the RSII brane, Eqs. (17) and (18) define the equation of state $p = p(\rho)$ in a parametric form. From (59), we find the scale a_0 on the holographic brane expressed in terms of a,

$$a_0^2 = \frac{a^2}{2}\frac{z_{\rm br}^2}{\ell^2}\left(1 + \frac{\mathcal{H}_{\rm RSII}^2\ell^2}{2} + \mathcal{E}_0\sqrt{1 + \mathcal{H}_{\rm RSII}^2\ell^2 - \frac{\mu\ell^4}{a^4}}\right), \tag{88}$$

where, as before, $\mathcal{E}_0 \equiv \mathcal{E}(0) = -1$ for the two-sided and $\mathcal{E}_0 = +1$ or -1 for the one-sided version of the RSII model. Thus, the mapping $a \to a_0$ is unique only for the two-sided model.

Knowing a, we may calculate the effective density of matter on the holographic brane assuming the Friedmann equation (42) holds. As before, this can be done for an arbitrary $z_{\rm br}$.

Using (13), (65), and (88), we can express the Hubble rate $\mathcal{H}_0$ and the scale a_0 in terms of ρ and a:

$$\mathcal{H}_0^2 = \frac{4}{z_{\rm br}^2}\frac{(\rho/\sigma_0 + 1)^2 - 1 + \mu\ell^4/a^4}{(\rho/\sigma_0 + 1 + \mathcal{E}_0)^2 + \mu\ell^4/a^4}, \tag{89}$$

$$a_0^2 = \frac{a^2}{4}\frac{\ell^2}{z_{\rm br}^2}\left[\left(\frac{\rho}{\sigma_0} + 1 + \mathcal{E}_0\right)^2 + \frac{\mu\ell^4}{a^4}\right]. \tag{90}$$

Next, using (43) to replace $\mathcal{H}_0^2$ in (89), substituting the expression (90) for a_0, and solving for ρ_0, we find

$$\frac{\rho_0}{\sigma_0} = 4\frac{\ell^2}{z_{\rm br}^2}\frac{(\rho/\sigma_0 + 1)^2 - 1 + (\ell/z_{\rm br})^2(\rho/\sigma_0 + 1)(\mathcal{E}_0 - \rho/\sigma_0 - 1) + (1 - \ell^2/z_{\rm br}^2)\mu\ell^4/a^4}{(\rho/\sigma_0 + 1 + \mathcal{E}_0)^2 + \mu\ell^4/a^4}. \tag{91}$$

To simplify the analysis, consider $z_{\rm br} = \ell$. For the two-sided RSII model along with the $\mathcal{E}_0 = -1$ branch of the one-sided model, we have

$$\frac{\rho_0}{\sigma_0} = -4\frac{\rho/\sigma_0 + 2}{(\rho/\sigma_0^2)^2 + \mu\ell^4/a^4}. \tag{92}$$

Thus, the two-sided model with positive energy density ρ and positive μ maps into a holographic cosmology with negative effective energy density ρ_0. For $\mu = 0$, the density ρ_0 diverges for small ρ as $1/\rho$. The one-sided model maps into two branches: the $\mathcal{E}_0 = -1$ branch identical with the two-sided map and the $\mathcal{E}_0 = +1$ branch, in which case we find

$$\frac{\rho_0}{\sigma_0} = \frac{4\rho/\sigma_0}{(\rho/\sigma_0^2 + 2)^2 + \mu\ell^4/a^4}, \tag{93}$$

yielding smooth positive ρ_0. Note that the inverse function $\rho = \rho(\rho_0)$ of (91) for $\mu = 0$ and $z_{\rm br} = \ell$ coincides with the function defined by (79) for $\mu = 0$ if we set $\mathcal{E}_0 = +1$ for $\rho_0 > 0$ and $\mathcal{E}_0 = -1$ for $\rho_0 < 0$.

V. SUMMARY AND CONCLUSIONS

We have explicitly constructed the holographic mapping between two cosmological braneworld scenarios: holographic and RSII braneworld. In the holographic scenario, the primary braneworld is at the boundary of AdS_5 with emergent cosmology at the RSII braneworld. In the RSII scenario, the primary braneworld is located at an arbitrary nonzero $z = z_{\rm br}$, and the cosmology at the $z = 0$ boundary is emergent. In both scenarios, we have established a holographic map between these two braneworld cosmologies.

We have assumed the presence of matter on the primary braneworld only. The emergent cosmology is governed by the Friedman equations with effective energy density and pressure. We have obtained functional relations between cosmological scales a_0 and a, Hubble rates H_0 and H, and effective energy densities ρ_0 and ρ in the two scenarios. We have analyzed two versions of the RSII models: the so called one-sided and two-sided versions. We have demonstrated that the map between the cosmological scales is unique for the two-sided RSII model, whereas in the one-sided model, the mapping from the holographic to the RSII cosmology is a two-valued function.

In particular, we have studied the low-density regime, i.e., the regime in which $\rho \simeq \rho_0 \ll 1/(G_{\rm N}\ell^2)$. The low-density regime can be made simultaneous only in the one-sided RSII model since the necessary condition (69) for the two-sided version is not met if both ρ_0 and ρ are small. The low-density regime on the two-sided RSII brane corresponds to the high negative energy density limit on the holographic brane.

The analysis presented here is open to speculations. For example, it is conceivable that our Universe is a one-sided RSII braneworld, the cosmology of which is emergent from the primary holographic cosmology. If ρ_0 on the holographic brane describes matter with the equation of state satisfying $3p_0 + \rho_0 > 0$, as for, e.g., cold dark matter, in the one-sided model, we will, according to (84) and (85), have an asymptotically de Sitter universe on the RSII brane. With the AdS curvature ℓ satisfying the Newtonian potential constraint, if we choose an appropriate brane location so that $\Lambda_{\rm br}$ fits the observed value today, we could produce the standard ΛCDM cosmology on the RSII brane. Unfortunately, in this scenario, we have to push the brane as far as $10^{28}\ell$ away from the boundary which seems rather unnatural. Another way to recover the standard cosmology is to involve a negative holographic brane tension in addition to ρ_0 and fine tune it to cancel $\Lambda_{\rm br}$ up to a small phenomenologically acceptable contribution.

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APPENDIX A: SECOND RANDALL-SUNDRUM MODEL

Our curvature conventions are as follows: $R^a{}_{bcd} = \partial_c\Gamma^a_{db} - \partial_d\Gamma^a_{cb} + \Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de}$ and $R_{ab} = R^s{}_{asb}$, so that Einstein's equations are $R_{ab} - \frac{1}{2}RG_{ab} = +8\pi G T_{ab}$. The dynamics of a 3-brane in a $4+1$-dimensional bulk is described by the total action as the sum of the bulk and brane actions

$$S = S_{\rm bulk} + S_{\rm br}. \tag{A1}$$

The bulk action is given by

$$S_{\rm bulk} = \frac{1}{8\pi G_5}\int d^5x\sqrt{G}\left[-\frac{R^{(5)}}{2} - \Lambda_5\right] + S_{\rm GH}[h], \tag{A2}$$

where Λ_5 is the bulk cosmological constant related to the AdS curvature radius as $\Lambda_5 = -6/\ell^2$. The Gibbons-Hawking boundary term is given by an integral over the brane hypersurface Σ:

$$S_{\rm GH}[h] = \frac{1}{8\pi G_5}\int_\Sigma d^4x\sqrt{-h}K[h]. \tag{A3}$$

The quantity K is the trace of the extrinsic curvature tensor K_{ab} defined as

$$K_{ab} = h^c_a h^d_b n_{d;c}, \tag{A4}$$

where n^a is a unit vector normal to the brane pointing toward increasing z, h_{ab} is the induced metric

$$h_{ab} = G_{ab} + n_a n_b, \tag{A5}$$

and $h \equiv \det h_{ab}$ is its determinant. Observers reside on the positive tension brane with action

$$S_{\rm br}[h] = \int d^4x\sqrt{-h}(-\sigma + \mathcal{L}^{\rm matt}[h]), \tag{A6}$$

where they see the metric $h_{\mu\nu}$.

The basic equations are the bulk field equations outside the brane,

$$R^{(5)}_{ab} - \frac{1}{2}R^{(5)}G_{ab} = \Lambda_5 G_{ab}, \tag{A7}$$

and junction conditions [40]

$$[[K^\mu_\nu - \delta^\mu_\nu K^\alpha_\alpha]] = 8\pi G_5(\sigma\delta^\mu_\nu + T^\mu_\nu), \tag{A8}$$

where the energy-momentum tensor $T^\mu_\nu = {\rm diag}(\rho, -p, -p, -p)$ describes matter on the brane and $[[f]]$ denotes the discontinuity of a function $f(z)$ across the brane, i.e.,

$$[[f(z)]] = \lim_{\epsilon\to 0}(f(z_{\rm br} + \epsilon) - f(z_{\rm br} - \epsilon)). \tag{A9}$$

To derive the RSII model solution, it is convenient to use Gaussian normal coordinates $x_a = (x_\mu, y)$ with the fifth coordinate y related to the Fefferman-Graham coordinate z by $z = \ell e^{y/\ell}$. Then, in the two-sided version with the Z_2 symmetry $y - y_{\rm br} \leftrightarrow y_{\rm br} - y$, one identifies the region $-\infty < y \le y_{\rm br}$ with $y_{\rm br} \le y < \infty$. We start with a simple ansatz for the line element

$$ds^2_{(5)} = \psi^2(y) g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2, \tag{A10}$$

where the warp factor ψ^2 is a function of y. We assume that $\psi^2 \to 0$ as $y \to \infty$ and

$$\psi^2(y_{\rm br}) = 1. \tag{A11}$$

Then, the total action (A2) may be brought to the form [41,42]

$$S[g] = \frac{1}{8\pi G_5}\int d^4x\sqrt{-g}\int dy\left[-\frac{R}{2}\psi^2 - 4(\psi^3\psi')' + 6\psi^2(\psi')^2 - \Lambda^{(5)}\psi^4\right] + S_{\rm GH}[g] + S_{\rm br}[g], \tag{A12}$$

where R is the four-dimensional Ricci scalar associated with the metric $g_{\mu\nu}$ and the prime $'$ denotes a derivative with respect to y. The extrinsic curvature is easily calculated using the definition (A4) and the unit normal vector $n^\mu = (0,0,0,0,1)$. We find the nonvanishing components

$$K_{\mu\nu} = n_{\mu;\nu} = -\Gamma^a_{\mu\nu} n_a = \psi\psi' g_{\mu\nu}. \tag{A13}$$

The fifth coordinate in (A12)) may be integrated out if $\psi \to 0$ sufficiently fast as we approach $y = \infty$.

The functional form of ψ is found by solving the Einstein equations (A7) outside the brane. Using the components of the Ricci tensor

$$R^{(5)}_{55} = -4\frac{\psi''}{\psi}, \qquad R^{(5)}_{5\mu} = 0, \tag{A14}$$

$$R^{(5)}_{\mu\nu} = R_{\mu\nu} + (3\psi'^2 + \psi\psi'')g_{\mu\nu}, \tag{A15}$$

and the Ricci scalar

$$R^{(5)} = \frac{R}{\psi^2} + 12\frac{\psi'^2}{\psi^2} + 8\frac{\psi''}{\psi}, \tag{A16}$$

we find the 55 and $\mu\nu$ components of the Einstein equations, respectively, as

$$6\frac{\psi'^2}{\psi^2} + \Lambda^{(5)} + \frac{R}{2\psi^2} = 0 \tag{A17}$$

and

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = (3\psi'^2 + 3\psi\psi'' + \Lambda^{(5)}\psi^2)g_{\mu\nu}. \tag{A18}$$

The unique solution to (A17) and (A18) which satisfies the condition (A11) and vanishes at $y = \infty$ is

$$\psi = e^{-(y-y_{\rm br})/\ell}, \tag{A19}$$

where $\ell = \sqrt{-6/\Lambda^{(5)}}$. With this solution, the metric (A10) is AdS_5 in normal coordinates because the constant factor $e^{y_{\rm br}/\ell}$ may be removed by a coordinate translation $y \to \tilde{y} = y - y_{\rm br}$. Equation (A18) reduces to the four-dimensional Einstein equation in empty space,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{A20}$$

This equation should follow from the variation of the action (A12) with $\mathcal{L}_{\rm matt} = 0$ after integrating out the fifth coordinate. For this to happen it is necessary that the last three terms in square brackets are canceled by the boundary term and the brane action without matter. Using (A13), one finds that the integral of the second term in square brackets is precisely canceled by the Gibbons-Hawking term. Then, the cancellations of the remaining terms will take place if

$$\frac{\gamma}{8\pi G_5}\int_{y_{\rm br}}^\infty dy[6\psi^2(\psi')^2 - \Lambda^{(5)}\psi^4] = \sigma, \tag{A21}$$

where

$$\gamma = \begin{cases} 1, & \text{one-sided RSII,} \\ 2, & \text{two-sided RSII.} \end{cases} \tag{A22}$$

This equation yields the RSII fine-tuning condition

$$\sigma = \sigma_0 \equiv \frac{3\gamma}{8\pi G_5 \ell}. \tag{A23}$$

In this way, after integrating out the fifth dimension, the total effective four-dimensional action assumes the form of the standard Einstein-Hilbert action without cosmological constant,

$$S = \frac{1}{8\pi G_{\rm N}}\int d^4x\sqrt{-g}\left(-\frac{R}{2}\right), \tag{A24}$$

where $G_{\rm N}$ is the Newton constant defined by

$$\frac{1}{G_{\rm N}} = \frac{\gamma}{G_5}\int_{y_{\rm br}}^\infty dy\psi^2 = \frac{\gamma\ell}{2G_5}. \tag{A25}$$

Then, the constant σ_0 in (A23) is given by

$$\sigma_0 = \frac{3}{4\pi G_{\rm N}\ell^2}, \tag{A26}$$

so the RSII fine-tuning condition does not depend on the sidedness γ if σ_0 is expressed in terms of the four-dimensional Newton constant.

It is worth noting that the fine-tuning condition (A23) and (A25) follows directly from the junction conditions (A8) and the metric (A10) with (A19). For a static brane at $y = y_{\rm br}$, we find

$$K_{\mu\nu}|_{y=y_{\rm br}+\epsilon} = -\frac{1}{\ell}g_{\mu\nu}. \tag{A27}$$

For the one-sided version, we can set

$$K_{\mu\nu}|_{y=y_{\rm br}-\epsilon} = 0, \tag{A28}$$

whereas the two-sided version or the Z_2 symmetry implies

$$K_{\mu\nu}|_{y=y_{\rm br}-\epsilon} = -K_{\mu\nu}|_{y=y_{\rm br}+\epsilon}. \tag{A29}$$

Then, from (A8), we find

$$\frac{3\gamma}{\ell}\delta^\mu_\nu = 8\pi G_5\sigma\delta^\mu_\nu, \tag{A30}$$

yielding (A23).

Equation (A20) admits any Ricci flat metric as its solution. The trivial solution $g_{\mu\nu} = \eta_{\mu\nu}$ gives the original RSII model [5] with an empty Minkowski brane located at an arbitrary $y = y_{\rm br}$ in the AdS/Z_2 bulk. More general solutions with a black hole on the brane are first considered in Ref. [43] and discussed in more detail in Ref. [44].

The RSII model can be extended to include a brane with spherical or hyperbolic geometry embedded in the AdS-Schwarzschild geometry [45,46]. In this case, it is convenient to represent the bulk metric in Schwarzschild coordinates (6). It is worth mentioning that the solution (6) is closely related to the D3-brane solution of ten-dimensional supergravity corresponding to a stack of $N_{\rm D}$ coincident D3-branes. If we identify the AdS curvature radius with $\ell = \ell_s (4\pi g_s N_{\rm D})^{1/4}$, where g_s is the string coupling constant, and $\ell_s = \sqrt{\alpha'}$ is the fundamental string length, a near-horizon nonextremal D3-brane metric is given by [47]

$$ds^2_{(10)} = ds^2_{\rm ASch} - \ell^2 d\Omega_5^2. \tag{A31}$$

The coordinates r and z are related by

$$\frac{r^2}{\ell^2} = \frac{\ell^2}{z^2} - \frac{\kappa}{2} + \frac{\kappa^2 + 4\mu}{16}\frac{z^2}{\ell^2}. \tag{A32}$$

The brane is placed at $z_{\rm br} < z_{\rm h}$ corresponding to the $r_{\rm br} > r_{\rm h}$. The location of the horizon $r_{\rm h}$ is the positive solution to the equation $f(r) = 0$ yielding

$$r_{\rm h}^2 = \frac{\ell^2}{2}\left(\sqrt{\kappa^2 + 4\mu} - \kappa\right), \qquad z_{\rm h}^2 = \frac{4\ell^2}{\sqrt{\kappa^2 + 4\mu}}. \tag{A33}$$

The fifth coordinate is cut at the horizon [45], so the bulk in the one-sided version is the section of spacetime defined by $z_{\rm br} \le z < z_{\rm h}$. In the two-sided version, one identifies the region $z_{\rm br} \le z < z_{\rm h}$ with $z_{\rm br}^2/z_{\rm h} < z \le z_{\rm br}$ with a fixed point at $z = z_{\rm br}$. Note that the RSII braneworld may be arbitrarily close to the AdS boundary since $z_{\rm br}$ can be chosen arbitrarily small but not zero.

The junction conditions for the brane placed at $r_{\rm br}$ yield two independent equations:

$$\frac{f^{1/2}(r_{\rm br})}{r_{\rm br}} = \frac{1}{\ell}\left(1 + \kappa\frac{\ell^2}{r_{\rm br}^2} - \mu\frac{\ell^4}{r_{\rm br}^4}\right)^{1/2} = \frac{8\pi G_5}{3\gamma}\sigma, \tag{A34}$$

$$\begin{aligned}\frac{1}{2f^{1/2}(r_{\rm br})}\frac{df}{dr}\bigg|_{r=r_{\rm br}} &= \frac{1}{f^{1/2}(r_{\rm br})}\left(\frac{r_{\rm br}}{\ell^2} + \mu\frac{\ell^2}{r_{\rm br}^3}\right)\\ &= \frac{8\pi G_5}{3\gamma}\sigma.\end{aligned} \tag{A35}$$

Solving these equations for $r_{\rm br}^2$ and σ, we obtain

$$r_{\rm br}^2 = \frac{2\mu\ell^2}{\kappa}, \tag{A36}$$

$$\sigma = \sigma_0\left(1 + \frac{\kappa^2}{4\mu}\right)^{1/2}. \tag{A37}$$

Clearly, for $\kappa = 0$, we must have $\mu = 0$, in which case we recover the standard RSII with a flat brane at an arbitrary $r_{\rm br} = \ell^2/z_{\rm br}$, and Eq. (A37) reduces to the fine-tuning condition (A23). In contrast, in the case of $\kappa^2 = 1$, the brane location is fixed by (A36) with the requirement that μ is positive or negative for positive or negative κ, respectively.

Next, we give a simple derivation of the RSII braneworld cosmology following Soda [48]. We start from the bulk line element in Schwarzschild coordinates (6) and allow the brane to move in the bulk along the fifth dimension r. In other words, the brane hypersurface Σ is time dependent and may be defined by

$$r - a(t) = 0, \tag{A38}$$

where $a = a(t)$ is an arbitrary function. The normal to Σ is then given by

$$n_\mu \propto \partial_\mu(r - a(t)) = (-\partial_t a, 0, 0, 0, 1), \tag{A39}$$

and using the normalization $g^{\mu\nu}n_\mu n_\nu = -1$, one finds the nonvanishing components

$$n_t = -\frac{f^{1/2}\partial_t a}{(f^2 - (\partial_t a)^2)^{1/2}}, \tag{A40}$$

$$n_r = \frac{f^{1/2}}{(f^2 - (\partial_t a)^2)^{1/2}}, \tag{A41}$$

where the function f is given by (7) with r replaced by a, i.e.,

$$f(a) = \frac{a^2}{\ell^2} + \kappa - \mu\frac{\ell^2}{a^2}. \tag{A42}$$

Using this, from (A40), (A41), and (A5), we find the induced line element on the brane,

$$ds^2_{\rm ind} = n^2(t)dt^2 - a(t)^2 d\Omega_\kappa^2, \tag{A43}$$

where

$$n^2 = f - \frac{(\partial_t a)^2}{f}. \tag{A44}$$

Assuming that either the relation (A28) or (A29) holds for the dynamical brane, the junction conditions (A8) may be written in the form

$$K_{\mu\nu}|_{r=a-\epsilon} = \frac{8\pi G_5}{3\gamma}[3T_{\mu\nu} - (\sigma + T)g_{\mu\nu}]. \tag{A45}$$

Then, the $\chi\chi$ component gives

$$\frac{f^{3/2}}{(f^2 - (\partial_t a)^2)^{1/2}} = \frac{8\pi G_5}{3\gamma}(\sigma + \rho)a. \tag{A46}$$

It turns out that the tt component gives the time derivative of the above equation and hence imposes no additional constraint. Using (A44), Eq. (A46) may be cast into the form

$$\frac{(\partial_t a)^2}{n^2 a^2} + \frac{f}{a^2} = \frac{1}{\ell^2 \sigma_0^2}(\sigma + \rho)^2. \tag{A47}$$

The first term on the left-hand side of (A47) is the square of the Hubble expansion rate for the metric (A43) on the brane,

$$H_{\rm RSII}^2 = \frac{(\partial_t a)^2}{n^2 a^2}. \tag{A48}$$

Substituting for f the expression (A42) into (A47), we obtain the effective Friedmann equation:

$$H_{\rm RSII}^2 + \frac{\kappa}{a^2} = \frac{(\sigma + \rho)^2}{\ell^2 \sigma_0^2} - \frac{1}{\ell^2} + \frac{\mu \ell^2}{a^4}. \tag{A49}$$

Employing the RSII fine-tuning condition $\sigma = \sigma_0$ and (2), Eq. (A49) may be expressed in the form

$$H_{\rm RSII}^2 + \frac{\kappa}{a^2} = \frac{8\pi G_{\rm N}}{3}\rho + \frac{\rho^2}{\ell^2 \sigma_0^2} + \frac{\mu \ell^2}{a^4}, \tag{A50}$$

which differs from the standard Friedmann equation by the last two terms on the right-hand side. Clearly, both versions of the RSII model yield identical brane cosmologies.

The second Friedmann equation is obtained by combining the time derivative of (A50) with respect to the synchronous time $\tilde{t} = \int n dt$ with the energy conservation

$$\frac{d\rho}{d\tilde{t}} + 3H_{\rm RSII}(\rho + p) = 0. \tag{A51}$$

One finds

$$\frac{dH_{\rm RSII}}{d\tilde{t}} - \frac{\kappa}{a^2} = -4\pi G_{\rm N}(\rho + p) - \frac{3\rho}{\ell^2 \sigma_0^2}(\rho + p) - 2\frac{\mu \ell^2}{a^4}, \tag{A52}$$

which may also be written in the form

$$\frac{1}{a}\frac{d^2 a}{d\tilde{t}^2} + H_{\rm RSII}^2 + \frac{\kappa}{a^2} = \frac{4\pi G_{\rm N}}{3}(\rho - 3p) - \frac{\rho}{\ell^2 \sigma_0^2}(\rho + 3p). \tag{A53}$$

Next, we derive explicit expressions for the coordinate transformation (19) for the brane position at $z = z_{\rm br}$. Using the total differentials

$$dt = \dot{t} d\tau + t' dz, \qquad dr = \dot{r} d\tau + r' dz, \tag{A54}$$

where the prime $'$ denotes the derivative with respect to z, the line element (6) transforms into

$$ds^2 = \left(f\dot{t}^2 - \frac{1}{f}\dot{r}^2\right)d\tau^2 - \left(\frac{1}{f}r'^2 - f t'^2\right)dz^2 + 2\left(f\dot{t}t' - \frac{1}{f}\dot{r}r'\right)dt dz - r^2 d\Omega_\kappa^2. \tag{A55}$$

The function f defined in (A42) has the argument $r = r(\tau, z)$. Comparing (A55) with (20), we find

$$\frac{\ell^2}{z^2}\mathcal{N}^2 = f\dot{t}^2 - \frac{\ell^2}{z^2}\frac{\dot{\mathcal{A}}^2}{f}, \tag{A56}$$

and requiring that the off-diagonal component of the metric vanishes and that the zz component equals $-\ell^2/z^2$, we obtain

$$t' = \frac{\dot{a}}{\dot{t}}\frac{r'}{f^2} \tag{A57}$$

and

$$\mathcal{N}(\tau, z) = \pm\frac{f\dot{t}}{r'} = \pm\frac{l}{z}\frac{\dot{\mathcal{A}}}{f t'}. \tag{A58}$$

Next, we specify $z = z_{\rm br}$. With the help of (22), (23), (A57), and (A58), we find the explicit expressions for r', t',

$$r'(\tau, z_{\rm br}) = -\frac{\ell}{z_{\rm br}}\frac{f}{n} = -\frac{\ell}{z_{\rm br}}\frac{f^{3/2}}{(f^2 - (\partial_t a)^2)^{1/2}}, \tag{A59}$$

$$t'(\tau, z_{\rm br}) = \frac{r'}{f^2}\frac{\dot{a}}{\dot{t}} = -\frac{\ell}{z_{\rm br}}\frac{f^{-1/2}}{(f^2 - (\partial_t a)^2)^{1/2}}\partial_t a, \tag{A60}$$

where the argument of f is $a(t(\tau, z_{\rm br}))$, whereas

$$\dot{r}(\tau, z_{\rm br}) = \dot{t}\partial_t a, \tag{A61}$$

and $\dot{t}(\tau, z_{\rm br})$ remains an arbitrary function of τ. However, the induced metric at $z = z_{\rm br}$ will have the form (A43) with t replaced by τ, if we identify

$$\frac{\ell^2}{z_{\rm br}}\mathcal{A}^2(\tau,z_{\rm br})=a^2(\tau),\quad \frac{\ell^2}{z_{\rm br}}\mathcal{N}^2(\tau,z_{\rm br})=n^2(\tau). \tag{A62}$$

Then, from (23) and (A62), it follows $|\dot{t}(\tau,z_{\rm br})|=1$, yielding

$$t(\tau,z_{\rm br})=\pm\tau+\text{const}. \tag{A63}$$

Imposing that t and τ increase simultaneously [14], we have

$$\dot{t}(\tau,z_{\rm br})=1\quad \dot{r}=\partial_t a. \tag{A64}$$

The sign in (A59) is fixed from the relation between r' and the fifth component of the unit normal to the brane in (t,r) coordinates, i.e.,

$$r'(\tau,z_{\rm br})=\frac{\ell}{z_{\rm br}}n^r. \tag{A65}$$

This equation follows from the transformation of $n^a=(0,0,0,0,-\ell/z_{\rm br})$ in (τ,z) to $n^a=(n^t,0,0,0,n^r)$ in (t,r) coordinates. Thus, with the minus sign in (A59), we maintain consistency with Eqs. (A40) and (A41) and the convention that n^a points toward increasing z (decreasing r).

APPENDIX B: RSII/CFT CONNECTION

Here, we demonstrate a connection between The RSII model and AdS/CFT correspondence. Our derivation follows Hawking *et al.* [49] (see also Ref. [50]). We start from the bulk action (A2) and regularize the action by placing the RSII brane near the AdS boundary, i.e., at $z=\epsilon\ell$, $\epsilon\ll 1$ so that the induced metric is $h_{\mu\nu}=1/\epsilon^2(g^{(0)}_{\mu\nu}+\epsilon^2\ell^2 g^{(2)}_{\mu\nu}+\ldots)$. The bulk splits in two regions, $0\le z\le\epsilon\ell$ and $\epsilon\ell\le z<\infty$, so the bulk action will consist of two pieces. We can either discard the region $0\le z\le\epsilon\ell$ (one-sided regularization) or invoke the Z_2 symmetry and identify two regions (two-sided regularization). Then, the regularized bulk action may be written as

$$S^{\rm reg}_{\rm bulk}=\gamma S_0, \tag{B1}$$

where

$$S_0=\frac{1}{8\pi G_5}\int_{z\ge\epsilon\ell}d^5x\sqrt{G}\left[-\frac{R^{(5)}}{2}-\Lambda_{(5)}\right]+S_{\rm GH}[h] \tag{B2}$$

and, as before, $\gamma=1$ for the one-sided and $\gamma=2$ for the two-sided regularization. Next, we renormalize the action by adding counterterms to S_0 [32,49],

$$S^{\rm ren}_0[G]=S_0[G]+S_1[h]+S_2[h]+S_3[h], \tag{B3}$$

such that the renormalized on-shell action is finite in the limit $\epsilon\to 0$,

$$S^{\rm ren}_0[g^{(0)}]=\lim_{\epsilon\to 0}S^{\rm ren}_0[h]. \tag{B4}$$

The counterterms are [32]

$$S_1[h]=-\frac{6}{16\pi G_5\ell}\int d^4x\sqrt{-h}, \tag{B5}$$

$$S_2[h]=-\frac{\ell}{16\pi G_5}\int d^4x\sqrt{-h}\left(-\frac{R[h]}{2}\right), \tag{B6}$$

$$S_3[h]=-\frac{\ell^3}{16\pi G_5}\int d^4x\sqrt{-h}\frac{\log\epsilon}{4}\times\left(R^{\mu\nu}[h]R_{\mu\nu}[h]-\frac{1}{3}R^2[h]\right). \tag{B7}$$

The last term is scheme dependent, and its integrand is proportional to the holographic conformal anomaly [51]. Now, we demand that the variation with respect to $h_{\mu\nu}$ of the total RSII action (A10), which is the sum of the regularized bulk action (B1) and the brane action (A6), vanishes; i.e., we require

$$\delta(S^{\rm reg}_{\rm bulk}[h]+S_{\rm br}[h])=0. \tag{B8}$$

By making use of (B5), this may be written as

$$\delta\left[\gamma S^{\rm ren}_0-\gamma S_3-\left(\sigma-\frac{3\gamma\ell}{8\pi G_5}\right)\int d^4x\sqrt{-h}+\int d^4x\sqrt{-h}\mathcal{L}_{\rm matt}-\frac{\gamma\ell}{16\pi G_5}\int d^4x\sqrt{-h}\frac{R[h]}{2}\right]=0. \tag{B9}$$

The third term gives the contribution to the cosmological constant and may be eliminated by imposing the RSII fine-tuning condition (A23). The variation of the scheme dependent S_3 may be combined with the first term so that

$$\delta(S^{\rm ren}_0-S_3)=\frac{1}{2}\int d^4x\sqrt{-h}\langle T^{\rm CFT}_{\mu\nu}\rangle\delta h^{\mu\nu}, \tag{B10}$$

where

$$\langle T^{\rm CFT}_{\mu\nu}\rangle=\frac{2}{\sqrt{-h}}\frac{\partial S^{(\rm ren)}_{\rm bulk}}{\partial h^{\mu\nu}}-\frac{2}{\sqrt{-h}}\frac{\partial S_3}{\partial h^{\mu\nu}}. \tag{B11}$$

The net effect of δS_3 is that it cancels the $\Box R$ term in the conformal anomaly [33] so the trace of the CFT stress tensor simply reads

$$\langle T^{\rm CFT}{}^{\mu}_{\mu}\rangle=-\frac{\ell^3}{64\pi G_5}\left(R^{\mu\nu}R_{\mu\nu}-\frac{1}{3}R^2\right). \tag{B12}$$

The variation equation (B9) yields four-dimensional Einstein's equations on the boundary,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N (\gamma \langle T^{\rm CFT}_{\mu\nu} \rangle + T^{\rm matt}_{\mu\nu}), \qquad (B13)$$

where we have employed the relation (A25) to express G_5 in terms of Newton's constant G_N. The quantity $T^{\rm matt}_{\mu\nu}$ is the energy-momentum tensor associated with the matter Lagrangian $\mathcal{L}^{\rm matt}$. Thus, the dynamics of the boundary universe is governed by the energy-momentum tensor $T^{\rm CFT}_{\mu\nu}$ of the CFT on the boundary in addition to the matter energy-momentum tensor $T^{\rm matt}_{\mu\nu}$. Obviously, the sidedness factor γ in front of $T^{\rm CFT}_{\mu\nu}$ shows that the required number of copies of CFT is either one or two depending on whether the braneworld is sitting at the cutoff boundary of a single patch of AdS_5 or in between two patches of AdS_5. Equation (B13) with (34) and $\gamma = 1$ coincides with Einstein's equations in Ref. [12] derived in a different way.

From (34), with the help of (29), we obtain the vacuum expectation value of the trace of the CFT energy-momentum tensor:

$$\langle T^{\rm CFT}{}^{\mu}_{\mu} \rangle = g^{(0)\mu\nu} \langle T^{\rm CFT}_{\mu\nu} \rangle = \frac{\ell^3}{16\pi G_5} [({\rm Tr} g^{(2)})^2 - {\rm Tr}(g^{(2)})^2]. \qquad (B14)$$

Furthermore, using (28), we can express the trace in the form (B12) which may be conveniently rearranged as

$$\langle T^{\rm CFT}{}^{\mu}_{\mu} \rangle = \frac{\ell^3}{128\pi G_5} (G_{\rm GB} - C^2), \qquad (B15)$$

where

$$G_{\rm GB} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \qquad (B16)$$

is the Gauss-Bonnet invariant and

$$C^2 \equiv C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2 R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \qquad (B17)$$

is the square of the Weyl tensor $C_{\mu\nu\rho\sigma}$.

The trace of the CFT energy-momentum tensor obtained in this way may be compared with the standard conformal anomaly calculated in field theory. The general result is [52]

$$\langle T^{\rm CFT}{}^{\mu}_{\mu} \rangle = b G_{\rm GB} - c C^2 + b' \Box R. \qquad (B18)$$

This expression will match (B15) if we ignore the $\Box R$ term, assume $b = c$, and identify

$$\frac{\ell^3}{G_5} = 128\pi c. \qquad (B19)$$

For a theory with n_s scalar bosons, n_f Weyl fermions, and n_v vector bosons, the standard calculations give [52,53]

$$b = \frac{n_s + (11/2) n_f + 62 n_v}{360(4\pi)^2}, \qquad (B20)$$

$$c = \frac{n_s + 3 n_f + 12 n_v}{120(4\pi)^2}. \qquad (B21)$$

Hence, in general, we have $b \neq c$. However, in the $\mathcal{N} = 4$ U(N) super-Yang-Mills theory, $n_s = 6N^2$, $n_f = 4N^2$, and $n_v = N^2$, in which case the equality $b = c$ holds and the conformal anomaly is correctly reproduced by the holographic expression (B15). In this case, Eq. (B19) reads [51]

$$\frac{\ell^3}{G_5} = \frac{2N^2}{\pi}. \qquad (B22)$$

It is worth mentioning that the coefficient c appears in the lowest order quantum correction to the Newtonian potential. The calculations based on one-loop corrections to the graviton propagator [10] yield the result

$$\Phi(r) = \frac{G_N M}{r} \left(1 + \gamma \frac{128\pi c G_N}{3r^2} \right), \qquad (B23)$$

which can be compared with (3). Here, γ is the number of copies of CFT coupled to gravity. Applying Eq. (B19), one finds the coefficient of the $1/r^2$ term equal to $\gamma l^3 G_N / 3G_5$ which agrees with (3) if one uses the RSII relation (2). Hence, as mentioned in Sec. I, the two-sided RSII model requires two copies of CFT coupled to gravity.

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