PHYSICAL REVIEW D **93,** 066007 (2016)

Scaling behavior of regularized bosonic strings

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(Received 1 November 2015; published 22 March 2016)

We implement a proper-time UV regularization of the Nambu-Goto string, introducing an independent metric tensor and the corresponding Lagrange multiplier, and treating them in the mean-field approximation justified for long strings and/or when the dimension of space-time is large. We compute the regularized determinant of the 2D Laplacian for the closed string winding around a compact dimension, obtaining in this way the effective action, whose minimization determines the energy of the string ground state in the mean-field approximation. We discuss the existence of two scaling limits when the cutoff is taken to infinity. One scaling limit reproduces the results obtained by the hypercubic regularization of the Nambu-Goto string as well as by the use of the dynamical triangulation regularization of the Polyakov string. The other scaling limit reproduces the results obtained by canonical quantization of the Nambu-Goto string.

DOI: 10.1103/PhysRevD.93.066007

I. INTRODUCTION

Recently there has been an increased interest in the spectrum of the large-N QCD string. It has been investigated both by numerical simulations [1–10] and by analytic studies [11–20]. The two major questions to be addressed are: what is the effective action of the QCD string at large distances and what is the spectrum of this string? Addressing the former question implies that we have to modify the Nambu-Goto action by adding operators which are less relevant in the long-string limit, while the latter question requires a consistent quantization of the string in $D = 4$ dimensions.

String theory is generically a nonlinear problem, since the Nambu-Goto action, representing the area of the string world sheet, is not a quadratic function of the fields. However, a gauge fixing makes the action quadratic with certain constraints imposed on physical states. This is the essence of the canonical quantization successfully applied to the relativistic string in the 1970's, which leads to consistent results in the critical dimension ($D = 26$ for the bosonic string) and on the mass shell.

A subtle feature of the quantized string theory is the emergence of ultraviolet divergences which have to be regularized and the theory has to be renormalized in order to remove these divergences. In quantum field theory the regularization is customarily done by cutting off momenta squared above a certain value Λ^2. In string theory such a cutoff has to be done for a certain choice of the world-sheet coordinates (or the choice of gauge) at $\Lambda^2\sqrt{g}$, where g is the determinant of the world-sheet metric tensor, to comply with diffeomorphism invariance. If g_{ab} is the induced metric, this may actually result in a complicated nonlinear problem.

The importance of such a dependence of the cutoff on the world-sheet metric is seen already in the very first computation by Brink and Nielsen [21] of the energy due to zero-point fluctuations of an open string with fixed ends separated by the distance L. The classical energy is $E_{\rm cl} = K_0 L$. The energy of zero-point fluctuations is given by the sum over the string oscillator modes:

$$E_0 = K_0 \frac{D-2}{2} \sum_{n=1}^{n_{\rm max}} \frac{n}{2E_{\rm cl}} = \frac{D-2}{2} \left(\frac{\pi n_{\rm max}^2}{2L} - \frac{\pi}{12L} \right). \qquad (1)$$

The universal (i.e. regularization independent) second term on the right-hand side comes as the difference between the actual sum of discrete modes and an integral approximation to the sum. Diffeomorphism invariance requires that the maximal number of modes, $n_{\rm max}$, is L-dependent: $n_{\rm max} = L\Lambda/\pi$. We thus obtain

$$E_0 = (D-2) \left(\frac{\Lambda^2 L}{4\pi} - \frac{\pi}{24L} \right). \qquad (2)$$

Therefore the divergence contributes only to the string tension, but not to the lowest mass, which is determined by the (universal) second term on the right-hand side, whose negative sign is associated with a tachyon. If we naively used $n_{\rm max} = {\rm const}$, it would result in a divergent (and positive) mass squared of the lowest state of the string.

The dependence of the cutoff on the world-sheet metric is crucial for the path-integral formulation of string theory [22], where the world-sheet metric g_{ab} and the target-space

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position X^μ, $\mu = 1, \ldots, D$, of the string world sheet are independent (for a careful discussion of this point we refer to the book by Polykov [23]). Owing to diffeomorphism invariance, the world-sheet metric can be diagonalized, $g_{ab} = e^{\varphi}\delta_{ab}$, by choosing the conformal gauge. While the classical action does not depend on φ, it emerges in the effective action after the path integration over X^μ (and ghosts) because of the divergences of these path integrals and the corresponding dependence of the cutoff on φ. However, the remaining path integral over φ decouples on the mass shell in the critical dimension, and then the results obtained in the 1970's using the operator formalism are reproduced. For $D \neq 26$ and/or off shell, the path integral over φ has to be taken into account and plays an important role for the consistency.

In the Polyakov formulation of string theory the path integral over the target-space string coordinates (and ghosts) is Gaussian and results in the determinant of the 2D Laplace-Beltrami operator with proper boundary conditions. For an open string with fixed ends these are the Dirichlet boundary conditions, for which the computation of the determinant was performed in [24,25]. For slowly varying fields φ, the effective action is determined by the conformal anomaly and given by the so-called Liouville action. Remarkably, the path integral over φ can be consistently treated [18] order by order in the inverse string length and/or in the limit of large D, where the WKB expansion about the saddle points applies. The result for the ground-state energy as a function of L coincides with the well-known Alvarez-Arvis spectrum [26,27]. It reveals a tachyonic singularity at distances $L \le L_0$ with $-K_0^2 L_0^2$ being the tachyon mass squared. For larger distances this quantity is well behaved.

The Alvarez-Arvis spectrum follows [27,28] from the canonical quantization of an open string with the Dirichlet boundary condition. Similarly, no effects of nonlinearities are seen in the computation [26] of the Nambu-Goto path integral at large D when one uses the zeta-function regularization. However, one may wonder whether this regularization always can be used since a powerlike divergence is missing by construction. It is thus not obvious to which extent the dependence of the cutoff on the metric, which was the origin of the nonlinearity, is correctly captured using the zeta-function regularization. For this reason we would like to repeat the computation using a regularization where the UV cutoff is given by a dimensional parameter, like the Pauli-Villars or proper-time regularization. Without such a dimensional parameter it is hard to follow how the regularization affects the renormalization of the string tension or the masses in the theory.

There exist two latticelike string theories[1] where the UV cutoff is explicitly a dimensional parameter, the lattice length $a \sim 1/\Lambda$. In the first lattice approach the starting point is the Nambu-Goto action, and the path integral over string configurations is regularized by considering surfaces embedded on a hypercubic lattice, living on the plaquettes of the lattice. We denote this theory the hypercubic lattice string theory (HLS) [30]. The second lattice theory is a regularization of the Polyakov string theory, often denoted dynamical triangulation (DT) [31], since summation over the intrinsic geometries of the world sheet in the path integral is performed by summing over a suitable class of equilateral triangulations, each with link length a. The target space variables X^μ then live on the vertices of the triangulations and in this way a target space triangulation is defined. Both theories are naturally defined in any Euclidean R^D target space, where $D \geq 2$, and both theories lead to the following picture: one can define a two-point function which falls off exponentially at large target space distances, thus defining a renormalized lowest, positive mass of the theories. However, once this two-point function is defined, the string tension of the theory does not scale [30,32], i.e. it goes to infinity when the cutoff $a \to 0$. Thus, these regularized theories seemingly had little to do with ordinary bosonic string theory.

However, so-called noncritical string theory, which can be viewed as an extension of string theory to $D < 2$, showed that the DT regularization indeed captured precisely what one would view as string theory in this region (it is not known how to extend HLS to $D < 2$). Since bosonic string theory ceases to be tachyonic for $D < 2$, the problem for the lattice versions of string theory seems to be the fact that by construction they have no tachyons. From their very construction, the logarithm of the two-point function in these theories is subadditive, leading to a mass larger than or equal zero (see for instance [29] for a discussion).

Clearly the continuum bosonic string theory manages to perform subtractions which result in a negative m^2 (and a finite, positive renormalized string tension), but it has never been fully understood how to reconcile the continuum calculations with the lattice calculations where it seems plain impossible to obtain a negative m^2. In order to avoid this conundrum we will try to put ourselves in a string theory situation where there is no tachyon in the continuum formulation, where $D > 2$ and where there is thus a chance that the lattice and the continuum formulations might agree, precisely as they agree in the case $D < 2$, as mentioned above.

Remarkably, such a comparison is possible in the large D limit if we consider a closed string which propagates a distance L and which is wrapped around one target space dimension compactified to a circle with circumference $\beta \ll L$. For β not too small there is no tachyon according the old calculation by Alvarez [26]. We will repeat the calculation, using the Nambu-Goto action and a Lagrange multiplier for the induced world-sheet metric. In the large D

[1]For an introduction see the book [29].

limit a saddle point calculation is reliable and we will perform the calculation using a proper time cutoff a which serves as the equivalent to the lattice cutoff mentioned above. We will find a remarkable situation: one can follow the philosophy of the lattice approach and first renormalize the two-point function. It leads naturally to a certain renormalization of the string tension. However, this renormalization is then incompatible with a finite effective action for an "extended" string where $L \gg \beta$, and where $\beta \gg a$. From the perspective of the effective action of the extended string it implies that the effective string tension goes to infinity when the cutoff $a \to 0$, i.e. precisely the situation encountered for the lattice regularizations. However, due to a scale invariance of the effective action, it is in the continuum possible to avoid this situation by a rescaling of the target space coordinates, but the price one pays is the introduction of the tachyon.

The rest of this article is organized as follows: In Sec. II we consider the particle, rather than the string, using the length of the world line as the action, the particle equivalent of the Nambu-Goto action for the string. The technique and many of the results are the same as for the string, just simpler. In Sec. III we consider a path-integral formulation of the Nambu-Goto string, introducing an independent metric tensor and the corresponding Lagrange multiplier. In Sec. IV we find a saddle-point solution to the path-integral formulation of the Nambu-Goto string which is justified by the mean-field approximation and becomes exact at large D. The dependence of the ground-state energy on L is computed in Sec. V. We also evaluate there the value of the area of typical surfaces which dominate the path integral. Our main results concerning two possible scaling behaviors are presented in Secs. V and VI. In Sec. VII we show how the same results can be obtained using the Polyakov string formulation. The spectrum of excited states is briefly discussed in Sec. VIII. Our main results are summarized in Sec. IX. In Appendix A we remind the reader of some results for path integral of the relativistic particle. In Appendix B we compute the induced metric for the string and find its unexpected coordinate dependence near the boundary.

II. SUM OVER PATHS FOR THE RELATIVISTIC PARTICLE

Before performing the calculation for the bosonic string it is instructive to consider a similar calculation for the relativistic particle, using the length of the world line as the action, the particle equivalent of the Nambu-Goto action we will use for the string.

We rewrite the action (the bare mass times the length of the path) as

$$S = m_0 \int d\omega \sqrt{\dot{x}^2} = m_0 \int d\omega \sqrt{h} + \frac{m_0}{2} \int d\omega \lambda (\dot{x}^2 - h), \tag{3}$$

where $\dot{x} = dx(\omega)/d\omega$ and where we have introduced an independent world-line metric h which is a tensor $[h(\omega) = h_{11}(\omega)]$, and a Lagrange multiplier $\lambda = \alpha/\sqrt{h}$ with α being a scalar.

The classical equations of motion are

$$\frac{1}{\sqrt{h}} \frac{d}{d\omega} \lambda \dot{x}^\mu = 0, \tag{4a}$$

$$h = \dot{x}^2, \tag{4b}$$

$$\lambda = \frac{1}{\sqrt{h}}. \tag{4c}$$

A generic solution is

$$x^1_{\rm cl}(\omega) = \int_0^\omega d\omega' \sqrt{h_{\rm cl}(\omega')}, \quad x^2_{\rm cl} = \cdots = x^D_{\rm cl} = 0. \tag{5}$$

We can choose the (static) gauge, where

$$x^1_{\rm cl} = \frac{\omega}{\omega_L} L, \qquad \sqrt{h_{\rm cl}} = \frac{1}{\lambda} = \frac{L}{\omega_L}, \qquad \omega \in [0, \omega_L] \tag{6}$$

and L is the distance in the target space between the end points of the path. Of course any change of parametrization $\omega \to \omega'(\omega)$ will also provide us with a classical solution $x'(\omega') = x(\omega)$ and $h'_{11}(\omega')(d\omega')^2 = h_{11}(\omega)(d\omega)^2$.

Splitting $x^\mu = x^\mu_{\rm cl} + x^\mu_{\rm q}$ where $x^\mu_{\rm cl}$ is given by (5) and then integrating over $x^\mu_{\rm q}$, we find the effective action

$$S_{\rm eff} = m_0 \int d\omega \sqrt{h} + \frac{m_0}{2} \int d\omega \lambda (\dot{x}^2_{\rm cl} - h) - d\Lambda \int d\omega \frac{\sqrt[4]{h}}{\sqrt{\lambda}} + \frac{d}{2} \log\left(\Lambda \int d\omega \frac{\sqrt[4]{h}}{\sqrt{\lambda}}\right). \tag{7}$$

More specifically the integration over $x^\mu_{\rm q}$ results in the determinant of the diffeomorphism invariant differential operator

$$\mathcal{O} = -\frac{1}{\sqrt{h}} \frac{d}{d\omega} \lambda \frac{d}{d\omega}. \tag{8}$$

We regularize this determinant by using a proper time cutoff

$$\operatorname{tr} \log \mathcal{O} = -\int_{a^2}^\infty \frac{d\tau}{\tau} \operatorname{tr} e^{-\tau \mathcal{O}}, \qquad a^2 \equiv \frac{1}{4\pi\Lambda^2}. \tag{9}$$

By an explicit calculation for constant h and λ we obtain

$$\operatorname{tr}\log\left(-\frac{1}{\sqrt{h}}\frac{\mathrm{d}}{\mathrm{d}\omega}\lambda\frac{\mathrm{d}}{\mathrm{d}\omega}\right)$$
$$=-\int_{a^2}^{\infty}\frac{\mathrm{d}\tau}{\tau}\sum_{n=1}^{\infty}\exp\left[-\frac{\tau}{\sqrt{h}}\lambda\left(\frac{\pi n}{\omega_L}\right)^2\right]$$
$$=-\frac{\omega_L}{\sqrt{\pi}a}\frac{\sqrt[4]{h}}{\sqrt{\lambda}}+\log\frac{\omega_L\sqrt[4]{h}}{a\sqrt{\lambda}}, \qquad (10)$$

which finally leads to (7).

In addition to the path integral over x^μ which resulted in the effective action (7), we have path integrals over the fields h and λ. As is well known,[2] the path integral over λ is saturated by a constant value of α owing to localization, after which the dependence on h enters only via the length $\tau=\int \mathrm{d}\omega\sqrt{h}$ of the path. The path integral over h [factorized over reparametrizations $f(\omega)$, $f'(\omega)\geq 0$, of the path] can then be substituted by an ordinary integral over τ:

$$\int\frac{\mathcal{D}h}{\mathcal{D}f}\cdots=\int\frac{\mathrm{d}\tau}{\sqrt{\tau}}\det{}^{1/2}\left(-\frac{1}{\sqrt{h}}\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{\sqrt{h}}\frac{\mathrm{d}}{\mathrm{d}\omega}\right)\cdots. \qquad (11)$$

This will change $D\to D-1$ in the linear divergence of the effective action in accordance with the fact that there are only $D-1$ independent degrees of freedom for the relativistic path.

In the rest of this section we shall ignore such a shift of D assuming that D is large. For the string the shift will be from D to $D-2$. We will use the notation d for the shifted value of D with the understanding that it makes no difference in the large D limit. For $m_0\sim d$ all the terms in the action (7) would be of order d, so the Jacobian displayed in Eq. (11) will be not essential. We can then compute the integral over τ by the saddle-point method. Equivalently, we can simply compute the path integrals over h and λ at large d by the saddle-point method, minimizing the effective action (7), without introducing the variable τ. This is exactly how we shall proceed in the next sections when we deal with the relativistic string.

Minimizing (7) with respect to h, we obtain the equation for α:

$$1-\alpha-\frac{d\Lambda}{2m_0\sqrt{\alpha}}+\frac{d}{4m_0\sqrt{\alpha}\int\mathrm{d}\omega\sqrt{h/\alpha}}=0. \qquad (12)$$

The solution is an ω-independent constant. Since we can always choose h to be constant in one dimension by change of parametrization, this shows that it is not inconsistent to choose both h and λ constant, as was done in the calculation (10).

Minimizing (7) with respect to λ, we obtain the equation for h:

[2] See, e.g. the book [33], Section 9.1.

$$\dot{x}_{\mathrm{cl}}^2-h+\frac{d\Lambda}{m_0\alpha^{3/2}}h-\frac{dh}{2m_0\alpha^{3/2}\int\mathrm{d}\omega\sqrt{h/\alpha}}=0, \qquad (13)$$

relating h and x_{cl}. Using Eq. (12), we write

$$h=\frac{\alpha}{(3\alpha-2)}\dot{x}_{\mathrm{cl}}^2=\frac{\alpha}{(3\alpha-2)}\frac{L^2}{\omega_L^2}. \qquad (14)$$

From (12) it follows that the "bare" mass m_0 has to diverge as Λ for $\Lambda\to\infty$.

At the minimum, we have the following leading large L behavior;

$$S_{\mathrm{eff}}=m_0(3\alpha-2)\int\mathrm{d}\omega\sqrt{h}=m_0\sqrt{\alpha(3\alpha-2)}L. \qquad (15)$$

Since our effective action is just the logarithm of the free particle propagator, we know that the leading L behavior is

$$S_{\mathrm{eff}}=m_{\mathrm{ph}}L+\mathcal{O}(\log L), \qquad (16)$$

where m_{ph} is the physical mass of the particle. This is obtained by choosing

$$\alpha=\frac{2}{3}+\frac{m_{\mathrm{ph}}^2}{2m_0^2}. \qquad (17)$$

Equation (12) then says

$$m_0=\sqrt{\frac{27}{8}}d\Lambda+\sqrt{\frac{3}{2}}\frac{m_{\mathrm{ph}}^2}{d\Lambda}, \qquad (18)$$

the scaling relation well known from treating the relativistic particle as a limit of a random walk process with average step length $a\sim 1/\Lambda$ ([29]). The value $\alpha=2/3$ is far away from the classical value $\alpha=1$. However, Eqs. (12) and (13) have semiclassical power expansions in $d\Lambda/m_0$ ($1/m_0\propto\hbar$), starting out with $\alpha=1$ and decreasing towards $\alpha=2/3$ with decreasing m_0. The radius of convergence of this expansion corresponds precisely to $\alpha=2/3$, as is shown in Appendix A, and the value $m_0=d\Lambda\sqrt{27/8}$ associated with $\alpha=2/3$ is thus the natural quantum point of the free particle. As we will see the situation will be similar for the string.

Classically the length of the particle path is $\int\mathrm{d}\omega\sqrt{h_{\mathrm{cl}}}=L$. However, the average path in the path integral is much longer, as is clear from (14), which shows that the average length of such a path is

$$\ell=\left\langle\int\mathrm{d}\omega\sqrt{\dot{x}^2}\right\rangle=\int\mathrm{d}\omega\sqrt{h}=\sqrt{\frac{\alpha}{3\alpha-2}}L. \qquad (19)$$

The reason for the divergence of ℓ when the cutoff $a\to 0$ is of course the quantum fluctuations of x_q. One can explicitly calculate (see Appendix A) that in the limit where $a\to 0$ we have

$$\langle \dot{x}_q^2 \rangle = \frac{d\Lambda \sqrt[4]{h}}{m_0 \lambda^{3/2}} = h. \tag{20}$$

Equation (19) shows that the Hausdorff dimension of such a path is two in the scaling limit and that the proper time cutoff a, even if introduced as a diffeomorphism invariant cutoff in parameter space $\omega \in [0, \omega_L]$, has a consistent interpretation as a length a in target space. Let us assume that a can be interpreted as a typical smallest length scale probed in target space. Then we can view the path of length ℓ as made of $n_\ell = \ell/a$ pieces or "building blocks." Similarly the classical path $x_{\rm cl}$ consists of $n_L = L/a$ building blocks and (19) reads in the scaling limit:

$$n_\ell = \sqrt{\frac{3}{8\pi}} \frac{d}{m_{\rm ph} L} n_L^2, \tag{21}$$

which tells us that the path of length ℓ with end points separated by a distance L in target space has Hausdorff dimension $d_H = 2$.

We remind the reader that there is nothing wrong with the result that the average length of a path appearing in the path integral diverges when the cutoff is removed. As is well known, even in ordinary quantum mechanics, such a path is not an observable. In the Heisenberg picture the operators $\hat{x}^\mu(t)$ do not commute at different times and attempts to measure $\hat{x}^\mu(t)$ at successive small time intervals Δt will precisely result in an average fractal path with $d_H = 2$ in the limit $\Delta t \to 0$.

The fact that the Hausdorff dimension of the paths is equal 2 is consistent with the interpretation of the proper time cutoff as a length scale also in target space. From the explicit expressions (8) and (9) it is clear that in the classical limit where $\lambda/\sqrt{h} = (\omega_L/L)^2$ oscillating modes with mode number $n > L/\pi a$ will be suppressed, telling us that we can probe distances down to a in target space by the fluctuating field x^μ. However, from (19) the corresponding mode cutoff in the quantum case is $n > \ell/\pi a \propto L/m_{\rm ph} a^2$. The fact that the path has length $\ell \gg L$ implies that we have to use a much larger frequency ω when expanding x^μ in modes in order to obtain the same resolution in target space.

It is possible to perform a different scaling. Suppose we insist that ℓ is finite. This is what one would do if we considered one-dimensional gravity and $x^\mu(\omega)$ were fields living in this one-dimensional world.[3] In such a world we expect the leading term in the effective action (15) to be proportional to the one-dimensional volume $\ell = \int d\omega \sqrt{h}$, i.e. one would write

$$S_{\rm eff} = \tilde{m}_{\rm ph} \ell, \qquad \tilde{m}_{\rm ph} = m_0 (3\alpha - 2), \tag{22}$$

or

$$\alpha = \frac{2}{3} + \frac{\tilde{m}_{\rm ph}}{3 m_0} \tag{23}$$

and

$$m_0 = \sqrt{\frac{27}{8}} d\Lambda + \sqrt{\frac{2}{3}} \tilde{m}_{\rm ph} \tag{24}$$

instead of the scaling (17) and (18). From the perspective of such a one-dimensional world a finite ℓ implies that our former target space L is as small as the cutoff $a \sim 1/\Lambda$. However, from the viewpoint of our one-dimensional world x^μ is just a field and in the split $x = x_{\rm cl} + x_q$, where integration over quantum x_q produces the different scaling of L and ℓ, we are free to perform a renormalization of the background field,

$$x_{\rm cl} = Z^{1/2} x_R, \qquad Z = (3\alpha - 2)/\alpha. \tag{25}$$

The field renormalization Z has a standard perturbative expansion,

$$Z = 1 - m_0^{-1} d\Lambda + \mathcal{O}(m_0^{-2}), \tag{26}$$

in terms of the coupling constant m_0^{-1}, which in perturbation theory is always assumed to be small even compared to the cutoff. By such a renormalization we obtain a new L_R in target space, $L = Z^{1/2} L_R$, which scales the same way as ℓ and the effective action is simply changed from $m_{\rm ph} L$ to $\tilde{m}_{\rm ph} L_R$ (and a complete calculation of effective action which also includes power corrections will lead to identical expressions, except for an allover, cutoff dependent normalization factor).

We will see that similar relations are valid for the Nambu-Goto string, but they will have more radical consequences in the string universe.

III. THE NAMBU-GOTO STRING

We now use the Nambu-Goto action and perform a calculation similar to the one for the particle. This was first done by Alvarez [26] at large D and extended by Pisarski and Alvarez [34] to the topology of a cylinder. As described in the Introduction the setup is the following: we have a closed string propagating a distance L and wrapped around a compactified dimension of circumference β. The action is

[3] Of course the gravity formulation would be even clearer if we had used to Brink-Howe-DiVecchia formulation, where we have an independent metric $h_{11}(\omega)$ and a free Gaussian field $x^\mu(\omega)$ coupled covariantly to $h_{11}(\omega)$, i.e. the particle equivalent of the Polyakov string formulation. However, the results will be the same as the ones we have already derived, so we will refrain from giving any details in the case of the particle. For the string we will consider the Polyakov formulation in addition to the Nambu-Goto formulation.

diffeomorphism invariant and the results should not depend on the chosen parametrization.

Introducing an auxiliary field λ^{ab} and independent metric field ρ_{ab}, we rewrite the Nambu-Goto action in the standard way as

$$K_0 \int \mathrm{d}^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int \mathrm{d}^2\omega \sqrt{\det \rho_{ab}} + \frac{K_0}{2} \int \mathrm{d}^2\omega \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho_{ab}). \tag{27}$$

Here λ^{ab} transforms under coordinate transformations as a tensor times the volume element and ρ_{ab} is a tensor. The path integration is performed independently over real values of X^μ and ρ_{ab} and over imaginary values of λ^{ab}.

The Euler-Lagrange equations, minimizing the right-hand side of Eq. (27) with respect to X^μ, λ^{ab} and ρ_{ab} are

$$\frac{1}{\sqrt{\det \rho}} \partial_a \lambda^{ab} \partial_b X^\mu = 0, \tag{28a}$$

$$\rho_{ab} = \partial_a X \cdot \partial_b X, \tag{28b}$$

$$\lambda^{ab} = \rho^{ab} \sqrt{\det \rho}, \qquad \det \rho \equiv \det \rho_{ab}. \tag{28c}$$

Choosing the world-sheet parametrization with ω_1 and ω_2 inside a $\omega_\beta \times \omega_L$ rectangle in the parameter space, we find from Eq. (28)

$$X^1_{\mathrm{cl}} = \frac{L}{\omega_L} \omega_1, \qquad X^2_{\mathrm{cl}} = \frac{\beta}{\omega_\beta} \omega_2, \qquad X^\perp_{\mathrm{cl}} = 0, \tag{29a}$$

$$[\rho_{ab}]_{\mathrm{cl}} = \mathrm{diag}\left(\frac{L^2}{\omega_L^2}, \frac{\beta^2}{\omega_\beta^2} \right), \tag{29b}$$

$$\lambda^{ab}_{\mathrm{cl}} = \mathrm{diag}\left(\frac{\beta \omega_L}{L \omega_\beta}, \frac{L \omega_\beta}{\beta \omega_L} \right). \tag{29c}$$

To analyze quantum fluctuations in the path-integral approach, it is convenient to split $X^\mu = X^\mu_{\mathrm{cl}} + X^\mu_{\mathrm{q}}$, where X^μ_{cl} is given by Eq. (29a), and perform the Gaussian path integral over X^μ_{q}. We may fix the gauge at this stage, e.g. by choosing $X^1_{\mathrm{q}} = X^2_{\mathrm{q}} = 0$, i.e. choosing the so-called static gauge, where fluctuations are transversal to the classical string world sheet.[4] The number of fluctuating X's then equals the number of dimensions transversal to the string world sheet: $d = D - 2$. We then obtain the effective action, governing the fields λ^{ab} and ρ_{ab},

$$S_{\mathrm{eff}} = K_0 \int \mathrm{d}^2\omega \sqrt{\det \rho} + \frac{K_0}{2} \int \mathrm{d}^2\omega \lambda^{ab} (\partial_a X_{\mathrm{cl}} \cdot \partial_b X_{\mathrm{cl}} - \rho_{ab}) + \frac{d}{2} \mathrm{tr} \log \left(-\frac{1}{\sqrt{\det \rho}} \partial_a \lambda^{ab} \partial_b \right), \tag{30}$$

where $d = D - 2$ is the number of fluctuating X's.

We use the proper-time regularization of the trace as in (9), now with

$$\mathcal{O} = -\frac{1}{\sqrt{\rho}} \partial_a \lambda^{ab} \partial_b, \qquad \sqrt{\rho} \equiv \sqrt{\det \rho}, \tag{31}$$

which reproduces the usual 2D Laplacian for λ^{ab} given by Eq. (28c).

Using the invariance of the measure in the path integral over X^μ, λ^{ab} and ρ_{ab}, we derive the following exact set of the quantum Schwinger-Dyson equations:

$$\left\langle F[\lambda, \rho] \frac{1}{\sqrt{\rho}} \partial_a \lambda^{ab} \partial_b X^\mu_{\mathrm{cl}} \right\rangle = 0, \tag{32a}$$

$$\langle \rho_{ab} F[\lambda, \rho] \rangle = \langle \partial_a X \cdot \partial_b X F[\lambda, \rho] \rangle + \left\langle \frac{1}{K_0} \frac{\delta F[\lambda, \rho]}{\delta \lambda^{ab}} \right\rangle, \tag{32b}$$

$$\left\langle \frac{\lambda^{ab}}{\sqrt{\rho}} F[\lambda, \rho] \right\rangle = \left\langle \rho^{ab} \left(1 - \frac{d}{2K_0 \sqrt{\rho}} \langle \omega | \mathrm{e}^{-a^2 \mathcal{O}} | \omega \rangle \right) F[\lambda, \rho] \right\rangle + \left\langle \frac{2}{K_0 \sqrt{\rho}} \frac{\delta F[\lambda, \rho]}{\delta \rho_{ab}} \right\rangle. \tag{32c}$$

Here $F[\lambda, \rho]$ is an arbitrary functional of λ^{ab} and ρ_{ab}.

In the mean-field approximation, which becomes exact at large D, we can disregard fluctuations of λ^{ab} and ρ_{ab} around the saddle-point values, i.e. simply substitute them by mean values. This is analogous to what happens in the N-component sigma model at large N, where we can disregard quantum fluctuations of the Lagrange multiplier.

Disregarding the quantum fluctuations means in the path-integral language that the path integrals over λ^{ab} and ρ_{ab} are given by saddle points. These saddle points can be alternatively found from the whole set of the Schwinger-Dyson equations (32), assuming factorization.[5] The Schwinger-Dyson equations are then reduced to three equations for the saddle-point values $\bar{\lambda}^{ab} \equiv \langle \lambda^{ab} \rangle$ and $\bar{\rho}_{ab} \equiv \langle \rho_{ab} \rangle$:

$$\frac{1}{\sqrt{\bar{\rho}}} \partial_a \bar{\lambda}^{ab} \partial_b X^\mu_{\mathrm{cl}} = 0, \tag{33a}$$

[4] Fixing a static gauge produces a ghost determinant, which is a determinant of an operator of multiplication by a function. At large D this determinant can be ignored, but may become essential to next orders of the $1/D$-expansion.

[5] See e.g. page 247 of the book [35].

$$\bar{\rho}_{ab} = \langle \partial_a X \cdot \partial_b X \rangle, \tag{33b}$$

$$\frac{\bar{\lambda}^{ab}}{\sqrt{\bar{\rho}}} = \bar{\rho}^{ab}\left(1 - \frac{d}{2K_0\sqrt{\bar{\rho}}}\langle\omega|\mathrm{e}^{-a^2\mathcal{O}}|\omega\rangle\right). \tag{33c}$$

Equations (33a) and (33b) look similar to the classical Eqs. (28a) and (28b), while Eq. (33c) contains an additional term compared to the classical Eq. (28c), due to the fact that operator $\mathcal{O}$ in Eq. (31) depends explicitly on ρ.

Using the known Seeley expansion for the cylinder, we write Eq. (33c) in the bulk (i.e. away from the boundary) as

$$\bar{\lambda}^{ab} = \bar{\rho}^{ab}\sqrt{\bar{\rho}}\left(1 - \frac{d\Lambda^2}{2K_0\sqrt{\det \bar{\lambda}^{ab}}}\right). \tag{34}$$

This equation possesses the solution

$$\bar{\lambda}^{ab} = C\bar{\rho}^{ab}\sqrt{\bar{\rho}}, \qquad C = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2K_0}}, \tag{35}$$

which generalizes the classical solution (28c). Note that C is fixed to be the ω-independent value between $1/2$ and 1 given by Eq. (35). This will play a crucial role in the following.

It is interesting to note that if we straightforwardly insert Eq. (35) into the action (27), it results in a Polyakov-like expression

$$S = \frac{CK_0}{2}\int \mathrm{d}^2\omega\sqrt{\rho}\rho^{ab}\partial_a X \cdot \partial_b X + K_0(1-C)\int \mathrm{d}^2\omega\sqrt{\rho} \tag{36}$$

with independent ρ_{ab} and X^μ. In the action (36) the coefficients of the quadratic in X^μ term and the volume term obey a certain relation. As we shall see below in Sec. VII, this is necessary for the consistency.

Let us now discuss how $\bar{\lambda}^{ab}$ depends on the world-sheet coordinate ω. If we choose the conformal gauge, where ρ_{ab} is proportional to δ_{ab}, we have from Eq. (35) $\bar{\lambda}^{ab} = C\delta^{ab}$, i.e. $\bar{\lambda}^{ab}$ is constant. This obviously satisfies Eq. (33a). For general coordinates we expect that $\bar{\lambda}^{ab}$ may depend only on ω_1 because of the cylinder geometry. We therefore obtain from Eq. (33a) the restriction

$$\partial_1\bar{\lambda}^{11} = 0, \qquad \partial_1\bar{\lambda}^{12} = 0, \tag{37}$$

so $\bar{\lambda}^{11}$ and $\bar{\lambda}^{12} = \bar{\lambda}^{21}$ have to be ω_1-independent. Because $\det\bar{\lambda}^{ab} = C^2$, we conclude that $\bar{\lambda}^{22}$ is also ω_1-independent. Thus $\bar{\lambda}^{ab}$ is constant.

As is shown in Sec. IV below, both $\bar{\rho}_{ab}$ and $\bar{\lambda}^{ab}$ are in fact diagonal as a consequence of the diagonal form (29a) of the classical solution and Eq. (33b). However, $\bar{\rho}_{11}$ and $\bar{\rho}_{22}$ are *not* constant and depend on ω_1 near the boundary in a nontrivial way in order that the boundary conditions are satisfied (as is discussed in Appendix B). Equation (35) then implies that $\bar{\rho}_{11}$ and $\bar{\rho}_{22}$ will have the same ω_1 dependence if $\bar{\lambda}^{11}$ and $\bar{\lambda}^{22}$ are constant since we have

$$\bar{\lambda}^{11}\bar{\rho}_{11} = \bar{\lambda}^{22}\bar{\rho}_{22}. \tag{38}$$

IV. SADDLE-POINT SOLUTION AT LARGE *d*

To compute $\bar{\rho}_{ab}$ from Eq. (33b), we note that

$$\langle \partial_a X_{\mathrm{q}} \cdot \partial_b X_{\mathrm{q}}\rangle = \frac{d}{K_0}\frac{\delta}{\delta\lambda^{ab}(\omega)}\operatorname{tr}\log\left[\frac{1}{\sqrt{\rho}}(-\partial_c\lambda^{cd}\partial_d)\right]. \tag{39}$$

As we have shown in the previous section, the saddle-point value $\bar{\lambda}^{ab}$ is ω-independent. We have therefore a weaker relation

$$\int \mathrm{d}^2\omega\langle \partial_a X_{\mathrm{q}} \cdot \partial_b X_{\mathrm{q}}\rangle = \frac{d}{K_0}\frac{\partial}{\partial\bar{\lambda}^{ab}}\operatorname{tr}\log\left[\frac{1}{\sqrt{\bar{\rho}}}(-\partial_c\bar{\lambda}^{cd}\partial_d)\right]. \tag{40}$$

Using the proper-time regularization (9), we write for the given world-sheet coordinates explicitly

$$\begin{aligned}\operatorname{tr}\log&\left[\frac{1}{\sqrt{\bar{\rho}}}(-\partial_a\bar{\lambda}^{ab}\partial_b)\right]\\ &= -\int_{a^2}^{\infty}\frac{\mathrm{d}\tau}{\tau}\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\exp\left\{-\frac{\tau}{\sqrt{\bar{\rho}}}\left[\bar{\lambda}^{11}\left(\frac{\pi n}{\omega_L}\right)^2\right.\right.\\ &\quad\left.\left.+\bar{\lambda}^{22}\left(\frac{2\pi m}{\omega_\beta}\right)^2 + (\bar{\lambda}^{12}+\bar{\lambda}^{21})\left(\frac{\pi n}{\omega_L}\right)\left(\frac{2\pi m}{\omega_\beta}\right)\right]\right\}.\end{aligned} \tag{41}$$

We have substituted here a constant value of $\sqrt{\bar{\rho}}$ because $\bar{\rho}_{ab}$, as is already pointed out (see Appendix B), depends on ω_1 only near the boundary, and the contribution from such a region will be suppressed in the closed string channel as β/L. Below we will present formulas which are valid also for ω_1-dependent $\bar{\rho}_{11}$ and $\bar{\rho}_{22}$, and where this phenomenon can be explicitly observed.

The right-hand side of Eq. (41) can be differentiated with respect to $\bar{\lambda}^{ab}$. Acting with $\partial/\partial\bar{\lambda}^{12}$ we find

$$\begin{aligned}\int &\mathrm{d}^2\omega\langle\partial_1 X_{\mathrm{q}}\cdot\partial_2 X_{\mathrm{q}}\rangle\\ &= \frac{d}{K_0}\sum_{m,n}\frac{(\frac{\pi n}{\omega_L})(\frac{2\pi m}{\omega_\beta})}{\bar{\lambda}^{11}(\frac{\pi n}{\omega_L})^2 + \bar{\lambda}^{22}(\frac{2\pi m}{\omega_\beta})^2 + 2\bar{\lambda}^{12}(\frac{\pi n}{\omega_L})(\frac{2\pi m}{\omega_\beta})}\\ &\quad\times\exp\left\{-\frac{a^2}{\sqrt{\bar{\rho}}}\left[\bar{\lambda}^{11}\left(\frac{\pi n}{\omega_L}\right)^2 + \bar{\lambda}^{22}\left(\frac{2\pi m}{\omega_\beta}\right)^2\right.\right.\\ &\quad\left.\left.+2\bar{\lambda}^{12}\left(\frac{\pi n}{\omega_L}\right)\left(\frac{2\pi m}{\omega_\beta}\right)\right]\right\},\end{aligned} \tag{42}$$

where we have substituted $\bar{\lambda}^{21} = \bar{\lambda}^{12}$. From Eq. (42) it follows that

$$\bar{\rho}_{12} = 0, \qquad \bar{\lambda}^{12} = 0 \tag{43}$$

is a solution. That it is the correct solution can be shown order by order of the semiclassical expansion in d/K_0, starting from the classical solution (29c) and using Eq. (35). The reason for (43) can be traced to the diagonal form (29a) of the classical solution. We thus conclude that $\bar{\rho}_{ab}$ and $\bar{\lambda}^{ab}$ are diagonal.

For diagonal and in general ω-dependent $\bar{\rho}_{ab}$ and constant $\bar{\lambda}^{ab}$ we have

$$\begin{aligned}\operatorname{tr}\log\left[\frac{1}{\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}(-\bar{\lambda}^{11}\partial_1^2 - \bar{\lambda}^{22}\partial_2^2)\right] &= -\frac{\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}{\sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}}\Lambda^2 + \frac{\beta\Lambda}{\sqrt{\bar{\lambda}^{22}}} \\ &\quad + 2\log\eta\left(\frac{\mathrm{i}}{2}\sqrt{\frac{\bar{\lambda}^{11}}{\bar{\lambda}^{22}}}\frac{\omega_\beta}{\omega_L}\right),\end{aligned} \tag{44}$$

where the quadratic and linear divergences are as they should be for the proper-time regularization. The finite term is given as usual [36] by the Dedekind eta function. Equation (44) coincides with the trace log of the 2D Laplacian for the cylinder, extracted from the general formula [25].

To avoid confusion, we point out that the boundary divergence in Eq. (44) (given by the second term on the right-hand side) is linked to the bulk divergence (given by the first term on the right-hand side). No contradiction with the open-closed string duality emerges in this case in contrast to Ref. [37], where it was argued that the boundary term is ruled out by open-closed string duality. In the so-called analytic regularization employed in [37] one effectively is using $\Lambda = 0$, and in that case the boundary term indeed vanishes.

We shall concentrate on the closed-string sector, where $L \gg \beta$ (i.e. a long cylinder), and in this case the second term on the right-hand side of Eq. (44) can be neglected. We then use the modular transformation of the η-function

$$\eta\left(\frac{\mathrm{i}\tau}{2}\right) = \sqrt{\frac{2}{\tau}}\eta\left(\frac{2\mathrm{i}}{\tau}\right) \tag{45}$$

and the asymptote

$$\eta\left(\frac{\mathrm{i}\tau}{2}\right) \to \mathrm{e}^{-\pi\tau/24} \tag{46}$$

to get[6]

$$\begin{aligned}\operatorname{tr}\log\left[\frac{1}{\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}(-\bar{\lambda}^{11}\partial_1^2 - \bar{\lambda}^{22}\partial_2^2)\right] &= -\frac{\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}{\sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}}\Lambda^2 - \frac{\pi}{3}\sqrt{\frac{\bar{\lambda}^{22}}{\bar{\lambda}^{11}}}\frac{\omega_L}{\omega_\beta}.\end{aligned} \tag{47}$$

Substituting the regularized trace log from Eq. (47) into Eq. (40), we finally obtain

$$\begin{aligned}\frac{1}{\omega_\beta\omega_L}\int d^2\omega\bar{\rho}_{11} &= \frac{L^2}{\omega_L^2} + \frac{\pi d}{6K_0}\sqrt{\frac{\bar{\lambda}^{22}}{(\bar{\lambda}^{11})^3}}\frac{1}{\omega_\beta^2} \\ &\quad + \frac{d\Lambda^2}{2K_0}\frac{\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}{\sqrt{(\bar{\lambda}^{11})^3\bar{\lambda}^{22}}},\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{1}{\omega_\beta\omega_L}\int d^2\omega\bar{\rho}_{22} &= \frac{\beta^2}{\omega_\beta^2} - \frac{\pi d}{6K_0\sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}}\frac{1}{\omega_\beta^2} \\ &\quad + \frac{d\Lambda^2}{2K_0}\frac{\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}{\sqrt{\bar{\lambda}^{11}(\bar{\lambda}^{22})^3}}.\end{aligned} \tag{49}$$

To solve these equations, we substitute

$$\bar{\lambda}^{11} = C\sqrt{\frac{\bar{\rho}_{22}}{\bar{\rho}_{11}}}, \qquad \bar{\lambda}^{22} = C\sqrt{\frac{\bar{\rho}_{11}}{\bar{\rho}_{22}}} \tag{50}$$

as it follows from Eq. (35) for diagonal $\bar{\rho}_{ab}$ and use the already mentioned fact that $\bar{\rho}_{11}$ and $\bar{\rho}_{22}$ have the same ω-dependence [see Eq. (38)]. We then find the following solution:

$$\begin{aligned}\frac{1}{\omega_\beta\omega_L}\int d^2\omega\bar{\rho}_{11} &= \frac{L^2}{\omega_L^2}\frac{(\beta^2 - \frac{\beta_0^2}{2C})}{(\beta^2 - \frac{\beta_0^2}{C})}\frac{C}{2C-1}, \\ \frac{1}{\omega_\beta\omega_L}\int d^2\omega\bar{\rho}_{22} &= \frac{1}{\omega_\beta^2}\left(\beta^2 - \frac{\beta_0^2}{2C}\right)\frac{C}{2C-1}\end{aligned} \tag{51}$$

and

$$\begin{aligned}\bar{\lambda}^{11} &= C\frac{\omega_L}{\omega_\beta L}\sqrt{\beta^2 - \beta_0^2/C}, \\ \bar{\lambda}^{22} &= C\frac{\omega_\beta L}{\omega_L}\frac{1}{\sqrt{\beta^2 - \beta_0^2/C}}\end{aligned} \tag{52}$$

with

$$\beta_0^2 = \frac{\pi d}{3K_0}. \tag{53}$$

It should be noted that the same solution can be obtained by a straightforward minimization of the effective action (30) with Eq. (47) inserted for the trace log and assuming that $\bar{\rho}_{ab}$ and $\bar{\lambda}^{ab}$ are diagonal and constant:

[6]In Eq. (47) the sign of the first term on the right-hand side is negative with the proper-time regularization, but it may be positive for other regularizations. For instance, cutting of the mode expansion at some maximal mode number leads to a positive term as shown in Ref. [38].

$$S_{\text{eff}} = \frac{K_0}{2}\left(\bar{\lambda}^{11}\frac{L^2\omega_\beta}{\omega_L} + \bar{\lambda}^{22}\frac{\beta^2\omega_L}{\omega_\beta} + 2\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} - \bar{\lambda}^{11}\int d^2\omega\bar{\rho}_{11} - \bar{\lambda}^{22}\int d^2\omega\bar{\rho}_{22}\right) - \frac{\pi d}{6}\sqrt{\frac{\bar{\lambda}^{22}}{\bar{\lambda}^{11}}}\frac{\omega_L}{\omega_\beta} - \frac{d\int d^2\omega\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}{2\sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}}\Lambda^2. \tag{54}$$

This simply repeats the original Alvarez computation except that we start from an arbitrary $\omega_\beta \times \omega_L$ rectangle in the parameter space and use the proper-time regularization rather than the zeta-function regularization. We reproduce the results [26], when $\omega_L = L$, $\omega_\beta = \beta$ and $\Lambda = 0$ as it is when using the zeta-function regularization. However, we emphasize once again that the more cumbersome approach we have used by solving Eq. (33) leads to the solution (51)–(52) without invoking the assumption that $\bar{\rho}_{ab}$ and $\bar{\lambda}^{ab}$ are diagonal and constant.

V. THE LATTICELIKE SCALING LIMIT

Substituting the solution (51)–(52) into Eq. (54), we obtain

$$S_{\text{eff}}^{\text{sp}} = K_0 CL\sqrt{\beta^2 - \beta_0^2/C} \tag{55}$$

for the saddle-point value of the effective action. Further, we find that the average area of a surface which appears in the path integral is

$$\mathcal{A} = \langle \text{Area} \rangle = \int d^2\omega \langle \sqrt{\det \rho_{ab}} \rangle = \int d^2\omega \sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} = L\frac{(\beta^2 - \beta_0^2/2C)}{\sqrt{\beta^2 - \beta_0^2/C}}\frac{C}{(2C-1)}. \tag{56}$$

Formulas (55) and (56) are our main results, valid for $L \gg \beta$ in the mean-field approximation. Let us now discuss the physical implications of these formulas.

First, formula (35) for the constant C (which plays the same role as α in our discussion of the random walk) shows that the bare string tension K_0 needs to be renormalized in order for C to remain real. Also, C is clearly constrained to take values between $1/2$ and 1. Second, all calculations are done with a proper time cutoff $a \sim 1/\Lambda$, which as in the random walk case can be thought of as the shortest distance one can measure in target space. Thus it is questionable if it makes sense to consider a $\beta < a$, i.e. it does probably not make sense to enter the regime where S_{eff} ceases to be real.

At first glance it seems impossible to obtain a finite S_{eff} by renormalizing K_0 in (55), since K_0 is of order Λ^2. However, let us try to imitate as closely as possible the calculation of the two-point function of the string by choosing, for a fixed cutoff a or Λ, β as small as possible without entering into the tachyonic regime of S_{eff}. Thus we choose

$$\beta_{\min}^2 = 2\beta_0^2\frac{K_0}{2d\Lambda^2} = \frac{\pi}{3}\frac{1}{\Lambda^2} = \frac{1}{3}(2\pi a)^2. \tag{57}$$

This choice ensures that $\beta_{\min}^2 > \beta_0^2/C$ for all values of $K_0 > 2d\Lambda^2$ and that $\beta_0^2/C \to \beta_{\min}^2$ for $K_0 \to 2d\Lambda^2$. With this choice we have

$$S_{\text{eff}} = \sqrt{\frac{\pi}{3}}\frac{K_0 CL}{\Lambda}\sqrt{2C-1}. \tag{58}$$

Only if $\sqrt{2C-1} \sim 1/\Lambda$ can we obtain a finite limit for $\Lambda \to \infty$. Thus we are forced to renormalize K_0 as follows:

$$K_0 = 2d\Lambda^2 + f\frac{M_{\text{ph}}^4}{\Lambda^2}, \qquad f = \frac{18d}{\pi^2}. \tag{59}$$

With this renormalization we find

$$S_{\text{eff}} = dM_{\text{ph}}L. \tag{60}$$

Since the partition function in this case has the interpretation as a kind of two-point function for a string propagating a distance L, we have the following leading L behavior of the two-point function:

$$Z(L) \sim e^{-S_{\text{eff}}} = e^{-dM_{\text{ph}}L + \mathcal{O}(\log L)}, \tag{61}$$

where the mass is a tunable parameter. We note that the situation is very similar to the situation for the free particle. In that case we had the classical value $\alpha = 1$ and a semiclassical expansion in $1/m_0$ which interpolated between $\alpha = 1$ and the quantum value $\alpha = 2/3$. Here we have the classical value $C = 1$ and a semiclassical expansion in $1/K_0$, which interpolates between $C = 1$ and the quantum value $C = 1/2$.

In the scaling limit (59) we can calculate the average area $\langle \text{Area} \rangle = \mathcal{A}$ of a surface using (56):

$$\mathcal{A} \propto \frac{L}{M_{\text{ph}}^3 a^2}. \tag{62}$$

It diverges. If we view the surface as made from $n_{\mathcal{A}}$ building blocks of size a^2, we find

$$n_{\mathcal{A}} \propto \frac{1}{(M_{\text{ph}}L)^3}n_L^4, \qquad n_L = \frac{L}{a}, \tag{63}$$

telling us that the Hausdorff dimension of the surface is $d_H = 4$ since $n_L = L/a$ is a typical linear extent of the surface measured in units of the cutoff a.

Let us finally turn to the situation where $L \gg \beta \gg a$. In this case we have a real extended minimal surface of area $A_{\min} = L \times \beta$, around which the string fluctuates. In this case we find from (56) that

$$\mathcal{A} \propto \frac{A_{\rm min}}{M_{\rm ph}^2 a^2}. \tag{64}$$

Again this can be written in terms of building blocks as

$$n_{\mathcal{A}} \propto \frac{1}{M_{\rm ph}^2 A_{\rm min}} n_{A_{\rm min}}^2, \qquad n_{A_{\rm min}} = \frac{A_{\rm min}}{a^2}, \tag{65}$$

showing that the Hausdorff dimension of the surface is still 4 for this kind of surfaces.

Let us now discuss what we define as the physical string tension. With the given boundary conditions the string extends over the minimal area $A_{\rm min}$ and we write the partition function as

$$Z(K_0, L, \beta) = {\rm e}^{-S_{\rm eff}(K_0, L, \beta)} = {\rm e}^{-K_{\rm ph} A_{\rm min} + \mathcal{O}(L,\beta)}. \tag{66}$$

This is precisely the way one would define the physical (renormalized) string tension in a gauge theory, with L, β being the side lengths of a Wilson loop and $L, \beta \gg a$, where a is the lattice link length. This is also the way the physical string tension is defined in lattice string theories like HLS and DT. Let us rewrite (59) as

$$K_0 = 2d\Lambda^2 + \frac{\tilde{K}_{\rm ph}^2}{2d\Lambda^2}, \tag{67}$$

very similar to the relation between the bare mass m_0 and the renormalized mass $m_{\rm ph}$. From the explicit form of $S_{\rm eff}$ given in (56) we have

$$K_{\rm ph} = K_0 C = d\Lambda^2 + \frac{1}{2}\tilde{K}_{\rm ph} + \mathcal{O}(1/\Lambda^2). \tag{68}$$

Thus the physical string tension as defined above diverges as the cutoff Λ is taken to infinity. However, the first correction is finite and behaves as we would have liked $K_{\rm ph}$ to behave, namely as $\tilde{K}_{\rm ph} \propto dM_{\rm ph}^2$.

We have encountered a situation identical to the one met in HLS and DT: it is possible by renormalizing the coupling constant to define a two-point function with a positive, finite mass. The Hausdorff dimension of the ensemble of surfaces is $d_H = 4$, but then the effective string tension defined as in (66) will be infinite. In addition the relation (68) is *precisely* the relation one finds in the lattice string theories. To make things clear, let us rephrase our scaling relations in dimensionless units like it is done in the lattice theories. Denote

$$K_0 a^2 = \mu, \qquad d/2\pi = \mu_c,$$
$$K_{\rm ph} a^2 = \mathcal{K}, \qquad M_{\rm ph} a^2 = \mathcal{M}. \tag{69}$$

Then the renormalization we have encountered [Eqs. (59) and (68)] can be rewritten as

$$\mathcal{M}(\mu) = c_1(\mu - \mu_c)^{1/4},$$
$$\mathcal{K}(\mu) = \mathcal{K}(\mu_c) + c_2(\mu - \mu_c)^{1/2},$$
$$\mathcal{K}(\mu_c) = \mu_c/2 > 0. \tag{70}$$

These are the scaling relations obtained in lattice string theory and we have now reproduced them by a standard continuum mean-field calculation.

VI. SCALING TO THE STANDARD STRING THEORY LIMIT

The scaling limit of the previous section was essentially particlelike, because the string tension has remained infinite. Remarkably, it is possible to have yet another scaling behavior which is stringlike and where the string tension is finite.

We have made a decomposition $X^\mu = X_{\rm cl}^\mu + X_{\rm q}^\mu$, where the parameters L and β refer to the "background" field $X_{\rm cl}^\mu$. In standard quantum field theory we usually have to perform a renormalization of the background field to obtain a finite effective action. It is possible to do the same here by scaling

$$X_{\rm cl}^\mu = Z^{1/2} X_R^\mu, \qquad Z = (2C-1)/C. \tag{71}$$

Notice that the field renormalization Z has a standard perturbative expansion

$$Z = 1 - \frac{d\Lambda^2}{2K_0} + \mathcal{O}(K_0^{-2}) \tag{72}$$

in terms of the coupling constant K_0^{-1}, which in perturbation theory is always assumed to be small, even compared to the cutoff.

However, in the limit $C \to 1/2$ it has dramatic effects since, working with renormalized lengths L_R and β_R defined as in (71):

$$L_R = \sqrt{\frac{C}{2C-1}} L, \qquad \beta_R = \sqrt{\frac{C}{2C-1}} \beta, \tag{73}$$

we now obtain for the effective action

$$S_{\rm eff} = K_R L_R \sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}, \qquad K_R = K_0(2C-1) \equiv \tilde{K}_{\rm ph}. \tag{74}$$

The renormalized coupling constant K_R indeed makes $S_{\rm eff}$ finite and is identical to the $\tilde{K}_{\rm ph}$ defined in (67). If we view L_R and β_R as representing physical distances, (74) tells us that we indeed have a renormalized, finite string tension $\tilde{K}_{\rm ph}$ in the scaling limit. In fact (74) is identical to the continuum string theory formula.

The "price" we pay for this rescaling of lengths is that we have introduced a tachyon in the theory. Before rescaling we argued that the negative term under the square root was of the order of the cutoff a^2 and there was thus no compelling reason to view it as responsible for a tachyon. However, now it has become finite and in fact it is precisely (minus) the closed bosonic string tachyon mass squared:

$$M^2_{\rm tachyon} = \frac{\pi d}{3\tilde{K}_{\rm ph}}. \tag{75}$$

Looking at (74) there is no compelling reason why β_R could not be smaller than $M_{\rm tachyon}$. However, let us write (71) in the following form:

$$\frac{\beta}{2\pi a} = \sqrt{\frac{1}{\pi d}}\sqrt{K_R}\beta_R. \tag{76}$$

Thus, insisting that $\beta/2\pi a > 1$, since a plays the role of a cutoff distance in target space, implies that $\beta_R^2 > \pi d/K_R$, i.e. we are outside the tachyon region of (74). Being deep into the tachyonic region, i.e. having $\beta_R \ll M_{\rm tachyon}/K_R$ means that originally $\beta \ll a$, clearly a situation which is strange starting out for instance in a hypercubic lattice theory with lattice spacing a.

The background field renormalization we have performed in the string case is very similar to the one we made for the particle, and we can give it the same interpretation: the background field renormalization is such that the average area $\mathcal{A}$ is finite, as one would define it to be if we considered a theory of two-dimensional gravity coupled to some matter fields. In fact, if we insert the scaling (71) for X^μ and (74) for K_0 in the expression (56) for $\mathcal{A}$, we obtain

$$\mathcal{A} = L_R \frac{(\beta_R^2 - \frac{\pi d}{6K_R})}{\sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}}, \tag{77}$$

which is cutoff independent and thus finite when the cutoff is removed. The area is simply the minimal area for $\beta_R \gg M_{\rm tachyon}/K_R$ and diverges when $\beta_R \to M_{\rm tachyon}/K_R$.

One may wonder if it is possible to have a continuum theory when $L, \beta \sim a$, i.e. of the order of the cutoff. Of course it is not in general. But in the scaling limit (67), where $(2C-1) \to \tilde{K}_{\rm ph}/2d\Lambda^2$, the actual cutoff is $\sim a/\sqrt[4]{\bar\rho} \propto a\sqrt{2C-1}$, which is much smaller than a. After the renormalization (73) the cutoff becomes $\sim a$ in the units, where the "physical" distances L_R and β_R are finite, that is still much smaller than the distances.

This phenomenon can be explicitly seen within the mode expansion, quite similarly to what is discussed in Sec. II for the relativistic particle. The exponent of the cutting factor [like in Eq. (42)] at the saddle point is

$$\begin{aligned}
&\sum_{m,n} \frac{a^2}{\sqrt{\bar\rho}}\left[\bar\lambda^{11}\left(\frac{\pi n}{\omega_L}\right)^2 + \bar\lambda^{22}\left(\frac{2\pi m}{\omega_\beta}\right)^2\right] \\
&\quad = \sum_{m,n} a^2(2C-1)\left[\left(\frac{\pi n}{L}\right)^2 + \left(\frac{2\pi m}{\sqrt{\beta^2-\beta_0^2/C}}\right)^2\right] \\
&\quad = \sum_{m,n} a^2 C\left[\left(\frac{\pi n}{L_R}\right)^2 + \left(\frac{2\pi m}{\sqrt{\beta_R^2 - \pi d/3K_R}}\right)^2\right].
\end{aligned} \tag{78}$$

So the modes are cut off at $n_{\rm max} \sim a^{-1}L_R$, $m_{\rm max} \sim a^{-1}\sqrt{\beta_R^2 - \pi d/3K_R}$. These numbers are as large as usual ($\sim a^{-1}$) also in the scaling limit described in this section.

VII. POLYAKOV VERSUS NAMBU-GOTO FORMULATIONS

It is natural to ask if it is possible to reproduce the above results using the Polyakov formulation of string theory.

Let us rewrite the Nambu-Goto action as

$$S = (1-\alpha)K_0 \int {\rm d}^2\omega\sqrt{g} + \frac{\alpha K_0}{2}\int {\rm d}^2\omega\sqrt{g}g^{ab}\partial_a X\cdot\partial_b X, \tag{79}$$

where α is a constant and where g_{ab} is the induced metric,

$$g_{ab} = \partial_a X\cdot\partial_b X. \tag{80}$$

Let us now consider X^μ and g_{ab} as independent in (79). We then have the Polyakov formulation of string theory. Integrating over quantum fluctuations of X^μ, we arrive at the following effective action for g_{ab}:

$$\begin{aligned}
S_{\rm eff} &= (1-\alpha)K_0\int {\rm d}^2\omega\sqrt{g} \\
&\quad + \frac{\alpha K_0}{2}\int {\rm d}^2\omega\sqrt{g}g^{ab}\partial_a X_{\rm cl}\cdot\partial_b X_{\rm cl} \\
&\quad + \frac{d}{2}{\rm tr}\log\left(-\frac{\alpha}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b\right).
\end{aligned} \tag{81}$$

The invariance of the measure in the path integral over g_{ab} under a shift results in the Schwinger-Dyson equation,

$$\left\langle g^{ab}\left(1-\alpha - \frac{d}{2K_0\sqrt{g}}\langle\omega|{\rm e}^{a^2\alpha\Delta}|\omega\rangle\right)\right\rangle = 0. \tag{82}$$

Using the Seeley expansion (in the bulk)

$$\frac{1}{\sqrt{g}}\langle\omega|{\rm e}^{a^2\Delta}|\omega\rangle = \frac{1}{2\pi a^2} + \frac{1}{24\pi}R \tag{83}$$

we find this equation is consistent if α satisfies

$$K_0(1-\alpha) - \frac{d\Lambda^2}{2\alpha} = m^2, \tag{84}$$

where m^2 is finite, i.e. if the quadratic divergence cancels. Solving Eq. (84) for $m^2 \ll K_0 \sim \Lambda^2$, we find

$$\alpha = C = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2K_0}} \tag{85}$$

which is already familiar from the analysis where we used the Nambu-Goto action.

For the value (85) of α, the Λ^2 term in the action (81) vanishes, so the action looks like the one obtainable using the zeta-function regularization where $\Lambda = 0$ and $C = 1$. The equation for the Liouville field φ (which appears in the conformal gauge $\rho_{ab} = \mathrm{e}^{\varphi}\delta_{ab}$) is then the standard Liouville equation. In the stringlike scaling limit the constant m^2 in (84) is multiplied by $(2C-1)$ and the Liouville equation becomes a free field equation.

Thus the action (81) is consistent for $\alpha = 1$ only for analytic regularizations with $\Lambda = 0$. Otherwise, we have to add the nonvanishing first term. The Nambu-Goto formulation remarkably leads to the consistent action, as was shown in Eq. (36) above.

A few comments regarding the Polyakov formulation are in order. In the conformal gauge there appears the usual ghost determinant, which can be neglected at large d. Nevertheless, reparametrizations of the boundary remain essential and for the case of the cylinder the path integral over the reparametrizations (or, equivalently, over the boundary value of φ) reduces to an integration over the modular parameter ω_β/ω_L. The latter integral can be calculated at large d again by the saddle-point method which implies a minimization with respect to ω_β/ω_L. This is in contrast to the Nambu-Goto formulation, where ω_β/ω_L was arbitrary.

The fact that $\sqrt{g}$ enters the action (81) linearly allows us to compute the ground state energy. Fixing the conformal gauge, we find at the saddle point with respect to φ for our cylinder

$$S_{\mathrm{eff}} = \frac{K_0 C}{2}\left(L^2 \frac{\omega_\beta}{\omega_L} + \beta^2 \frac{\omega_L}{\omega_\beta}\right) - \frac{\pi d}{6}\frac{\omega_L}{\omega_\beta}. \tag{86}$$

Notice that the bulk value of g_{ab} does not enter Eq. (86).

Minimizing (86) with respect to ω_L/ω_β, we get

$$E_0 = K_0 C \sqrt{\beta^2 - \frac{\pi d}{3CK_0}} \tag{87}$$

that is the same as Eq. (55) for the Nambu-Goto formulation. For $C = 1$ we reproduce the results [39] obtained for the zeta-function regularization.

The mean area can be found by differentiating the partition function with respect to K_0:

$$-K_0 \frac{\partial}{\partial K_0} \log Z = K_{\mathrm{ph}} \langle \mathrm{Area} \rangle, \tag{88}$$

where the string tension $K_{\mathrm{ph}} = CK_0$ from Eq. (87), as in Eq. (68). Differentiating we find

$$\langle \mathrm{Area} \rangle = L \frac{(\beta^2 - \beta_0^2/2C)}{\sqrt{\beta^2 - \beta_0^2/C}} \frac{C}{(2C-1)}, \tag{89}$$

which is the same as Eq. (56) for the Nambu-Goto formulation.

However, it is not so clear how to link g_{ab} to the induced metric. They are only related by the boundary condition, stating they are the same at the boundary [25,40].

VIII. EXITED STATES

Masses of exited states can be extracted from the next terms in the expansion of the η-function. Using Eq. (45) with $\tau = \sqrt{\bar{\lambda}^{11}/\bar{\lambda}^{22}}\omega_\beta/\omega_L = L^{-1}\sqrt{\beta^2 - \beta_0^2/C}$, we expand it in the closed-string sector as

$$\begin{aligned}\eta\left(\frac{2\mathrm{i}}{\tau}\right)^{-d} &= \mathrm{e}^{d\pi/6\tau} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{-4\pi n/\tau})^{-d} \\ &= \mathrm{e}^{d\pi/6\tau} \sum_{N=0}^{\infty} d_N \mathrm{e}^{-4\pi N/\tau},\end{aligned} \tag{90}$$

where d_N are the level occupation numbers. Repeating the above computation, we obtain for the spectrum at level N:

$$E_N = K_{\mathrm{R}} \sqrt{\beta_{\mathrm{R}}^2 + \frac{2\pi}{K_{\mathrm{R}}}\left(4N - \frac{d}{6}\right)}. \tag{91}$$

For $N \sim d$ this results in a linear Regge trajectory with the "renormalized" Regge slope $1/8\pi K_{\mathrm{R}} = \alpha'/4$. As usual it is 4 times smaller for the closed string than for an open string.

Equation (91) is again the usual formula for the spectrum of excited string states, as it follows from the zeta-function regularization.

IX. DISCUSSION

Using a mean-field continuum calculation, which we expect to be reliable in the large-d limit, we have obtained the same result for the bosonic string as was originally obtained in lattice string theories (HLS or DT). In these theories it was impossible to define a finite physical string tension in the limit where the lattice cutoff a was taken to zero. Our mean-field calculation allows us to trace in detail how this nonscaling arises, and it also allows us to understand how one in the continuum theory can perform an alternative scaling which reproduces some of the standard continuum results of bosonic string theory, like formula (74). Rather surprisingly this scaling implies that the distances one considers in target space are comparable or even much smaller than the cutoff a one starts out imposing. If one had started out with a lattice string theory like HLS where the path integral is performed over surfaces embedded on a hypercubic lattice with link lengths a, it clearly makes no sense to consider target space distances less than a. While one in these theories *can* define a scaling limit for a two-point function, this scaling limit always considers distances much larger than a: when $a \to 0$ the correlation length stays finite in target space, i.e. it involves infinite many lattice spacings.

Working in a continuum formalism, nothing prevents us from making an additional rescaling like (71) of the target space, but from the point of the regularized theory we will, as shown explicitly by e.g. formula (76), always be at cutoff scale a for fixed rescaled distances β_R, L_R and fixed string tension K_R. In terms of the original variables L, β the continuum string limit describes a Lilliputian world, which is a world where the average area $\mathcal{A}$ remains finite [as shown in formula (77)] when the cutoff is removed. Having a finite $\mathcal{A}$ is natural from a two-dimensional world-sheet point of view, so our Lilliputians are like two-dimensional beings, while standard lattice scaling is an enterprise only for Gulliver. In the case of the particle this shift between the worlds of Gulliver and the Lilliputians is more or less an academic exercise in the sense that the resulting propagator was the same up to a cutoff dependent factor not depending on x^μ. However, in the string case the Lilliputian world is the one of standard continuum string theory, while the Gulliver world is one where strings are degenerated into so-called branched polymers due to the nonscaling of the string tension, as described long ago in the framework of the HST or DT regularization.

From a standard field theory perspective it seems a little contrived to insist that ℓ is finite, as one would naturally do in a one-dimensional quantum gravity theory, since the average length of a world line goes to infinite in the path integral when removing the cutoff. Nevertheless, as mentioned, this change of perspective has no consequences in the case of the particle, contrary to the case of strings. In ordinary continuum string calculations such a rescaling of "distances" X^μ which makes the average area $\mathcal{A}$ finite, is usually not mentioned explicitly. However, it is there. Using the conformal invariance of the world-sheet field theory involves a renormalization of the vertex operators $e^{ip_\mu \hat{X}^\mu(\omega)}$, i.e. effectively an adjustment of scales dictated by the fields $X^\mu(\omega)$.

It should also be mentioned that a finite $\mathcal{A}$ is more or less the starting point in noncritical string theory, which can be viewed as two-dimensional quantum gravity coupled to matter fields with central charge $c < 1$. In these theories the finite $\mathcal{A}$ is obtained by a renormalization of the two-dimensional gravitational cosmological term, i.e. the first term on the right-hand side of Eq. (79). Some of our calculations can formally be extended to the region $c < 1$, since this region, again formally, corresponds to $d < 0$. As is seen from our formulas everything is different if $d < 0$ and one obtains completely different scaling. Such a different scaling could well be consistent with the scaling obtained by the DT lattice theory which for $c < 1$ provides a regularization of two-dimensional quantum gravity coupled to matter, and where the scaling, contrary to the situation for $d > 0$, agrees with continuum noncritical string calculations. Our mean-field results might be reliable in the $d \to -\infty$ limit, but we have not investigated it in detail.

Our results are based on the mean-field approximation and reproduce in the stringlike scaling limit the spectrum obtained by the canonical quantization. It would be interesting to pursue our approach beyond the mean-field approximation, accounting for fluctuations of ρ_{ab} and λ^{ab} to next orders in $1/d$, to check whether or not the spectrum changes.

Finally let us emphasize again that the present paper has shown that it is important to treat the cutoff carefully if one wants to understand how continuum bosonic string theory is related to nonperturbative lattice formulations of string theory. Our treatment does not provide a cure for the bosonic tachyon problem since the corresponding scaling limit corresponds to "strings" with infinite string tension and not to the ordinary (tachyonic) continuum string. However, the explicit role of the cutoff may be important when one considers QCD strings where one for sure has a cutoff of the order Λ_{QCD}.

ACKNOWLEDGMENTS

We are indebted to Poul Olesen, Peter Orland and Arkady Tseytlin for valuable discussions. The authors acknowledge support by the ERC-Advance Grant No. 291092, "Exploring the Quantum Universe" (EQU). Y. M. thanks the Theoretical Particle Physics and Cosmology group at the Niels Bohr Institute for the hospitality.

APPENDIX A: MORE ON RELATIVISTIC PATHS

Let us explicitly check that the solution (17) to Eq. (12) is the one which sums up the semiclassical expansion in $1/m_0$. The proper exact solution to the cubic Eq. (12) at large L is

$$\alpha(r) = \frac{2}{3} + \frac{(1+i\sqrt{3})(2-27r^2+i3\sqrt{3}r\sqrt{4-27r^2})^{1/3}}{2^{1/3}6} + \frac{(1-i\sqrt{3})(2-27r^2-i3\sqrt{3}r\sqrt{4-27r^2})^{1/3}}{2^{1/3}6}, \tag{A1}$$

where $r = d\Lambda/2m_0$. It has the required series expansion

$$\alpha = 1 - r - \frac{r^2}{2} - \frac{5r^3}{8} - r^4 - \frac{231r^5}{128} - \frac{7r^6}{2} + \mathcal{O}(r^8), \tag{A2}$$

monotonically decreases with r and indeed for $r = \sqrt{\frac{2}{27}} - \delta$,

$$\alpha = \frac{2}{3} + 2\sqrt{\frac{2}{3}}\delta + \mathcal{O}(\delta^2). \tag{A3}$$

To explicitly compute the induced metric in the static gauge, we use the mode expansion

$$x_q = \sqrt{2}\sum_{n=1}^{\infty} a_n \sin\frac{\pi n\omega}{\omega_L}. \tag{A4}$$

We then obtain

$$\begin{aligned}\langle \dot{x}_q^2\rangle &= \frac{2}{\omega_L}\sum_{n=1}^{\infty}\left(\frac{\pi n}{\omega_L}\right)^2 \langle a_n^2\rangle \cos^2\frac{\pi n}{\omega_L}\omega \mathrm{e}^{-a^2\lambda(\frac{\pi n}{\omega_L})^2 h^{-1/2}} \\ &= \frac{2}{\omega_L}\frac{d}{m_0\lambda}\sum_{n=1}^{\infty}\cos^2\frac{\pi n}{\omega_L}\omega \mathrm{e}^{-a^2\lambda(\frac{\pi n}{\omega_L})^2 h^{-1/2}}. \end{aligned}\tag{A5}$$

Replacing the sum by an integral, we find

$$\langle \dot{x}_q^2\rangle = \frac{2d}{m_0\lambda}\int_0^\infty \mathrm{d}x \cos^2 x \mathrm{e}^{-a^2\lambda x^2 h^{-1/2}} = \frac{d\Lambda\sqrt[4]{h}}{m_0\lambda^{3/2}}. \tag{A6}$$

This simply reproduces Eq. (13) without the last term, which comes from the difference between the sum in Eq. (A5) and the integral in Eq. (A6).

We have seen that nothing unexpected happens with the induced metric in the case of paths. In particular, we can make it constant by choosing the proper-time gauge (6). This is in contrast to the case of surfaces, where the dependence of the induced metric on ω is present to fulfill the boundary condition for the component of the metric tensor along the boundary, as is demonstrated in Appendix B.

To see how the typical paths that dominate the path integral look, let us compute the averaged transverse displacement squared $\langle x_\perp^2\rangle$. Proceeding as in Eq. (A5), we find

$$\langle x_\perp^2\rangle = \frac{2}{\omega_L}\sum_{n=1}^{\infty}\langle a_n^2\rangle \sin^2\frac{\pi n}{\omega_L}\omega = \frac{2}{\omega_L}\frac{d}{m_0\lambda}\sum_{n=1}^{\infty}\frac{\sin^2\frac{\pi n}{\omega_L}\omega}{(\frac{\pi n}{\omega_L})^2} \tag{A7}$$

which is convergent. The sum is computable using Joncquiére's relation for dilogarithms which gives

$$\begin{aligned}\langle x_\perp^2\rangle &= \frac{d\omega}{m_0\lambda}\left(1-\frac{\omega}{\omega_L}\right) \\ &= \frac{dL}{m_0\sqrt{\alpha(3\alpha-2)}}\frac{\omega}{\omega_L}\left(1-\frac{\omega}{\omega_L}\right).\end{aligned}\tag{A8}$$

In the scaling regime (18) it tends to a finite value

$$\langle x_\perp^2\rangle = \frac{dL}{m_{\mathrm{ph}}}\frac{\omega}{\omega_L}\left(1-\frac{\omega}{\omega_L}\right) \tag{A9}$$

with the transverse displacement growing like $\sqrt{L}$, as it should for the Brownian motion in target space.

In the other scaling regime (24), the right-hand side of Eq. (A8) vanishes as $m_0^{-1} \sim \Lambda^{-1}$. However, if we renormalize the transverse coordinate in the same way as in Eq. (25), the transverse displacement would also be finite,

$$\langle x_{R\perp}^2\rangle = \frac{dL_R}{\tilde{m}_{\mathrm{ph}}}\frac{\omega}{\omega_L}\left(1-\frac{\omega}{\omega_L}\right), \tag{A10}$$

and coinciding with (A9).

We can also compute the correlator at noncoinciding "times" ω_1 and ω_2. The 1d Dirichlet Green function can be computed through the mode expansion quite similarly to Eqs. (A7)–(A10). We obtain

$$\begin{aligned}&\langle x_{\mathrm{q}}^\mu(\omega_1)x_{\mathrm{q}}^\nu(\omega_2)\rangle \\ &= \frac{2}{\omega_L}\frac{\delta^{\mu\nu}}{m_0\lambda}\sum_{n=1}^{\infty}\frac{\sin\frac{\pi n}{\omega_L}\omega_1 \sin\frac{\pi n}{\omega_L}\omega_2}{(\frac{\pi n}{\omega_L})^2} \\ &= \frac{\delta^{\mu\nu}L}{m_0\sqrt{\alpha(3\alpha-2)}}\left(\frac{\omega_1+\omega_2}{2\omega_L} - \frac{|\omega_1-\omega_2|}{2\omega_L} - \frac{\omega_1\omega_2}{\omega_L^2}\right),\end{aligned}\tag{A11}$$

which reproduces Eq. (A8) for $\omega_1 = \omega_2$. It vanishes if $\omega_i = 0$ (i.e. at the boundary), as the Dirichlet Green function should. The first term in the brackets makes it positive.

In Eq. (A11) the coefficient $\frac{L}{m_0\sqrt{\alpha(3\alpha-2)}}$ equals either $\frac{L}{m_{\mathrm{ph}}}$ in the scaling limit (18) or $\frac{L_R}{\tilde{m}_{\mathrm{ph}}}$ in the scaling limit (24), if we renormalize x_{q}^μ. So the continuum formulas are identical in both cases.

APPENDIX B: INDUCED METRIC IN THE WORLD-SHEET COORDINATES

Let us compute the induced metric $\langle \partial_a X\cdot\partial_b X\rangle$ in the string case to verify its coordinate dependence.

Using the mode expansion

$$X_q = 2\sum_{m,n\geq 0}\left(a_{mn}\cos\frac{2\pi m\omega_2}{\omega_\beta} + b_{mn}\sin\frac{2\pi m\omega_2}{\omega_\beta}\right)\sin\frac{\pi n\omega_1}{\omega_L}, \tag{B1}$$

we have explicitly for the quantum part of the induced metric

$$\begin{aligned}\langle \partial_1 X_q \cdot \partial_1 X_q \rangle &= \frac{2}{\omega_\beta \omega_L} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\frac{\pi n}{\omega_L}\right)^2 \left[(2-\delta_{m0})\langle a_{mn}^2\rangle \cos^2 \frac{2\pi m}{\omega_\beta}\omega_2 + 2\langle b_{mn}^2\rangle \sin^2 \frac{2\pi m}{\omega_\beta}\omega_2\right] \cos^2 \frac{\pi n}{\omega_L}\omega_1 \\ &= \frac{2}{\omega_\beta \omega_L} \frac{d}{K_0} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \frac{(\frac{\pi n}{\omega_L})^2}{\bar{\lambda}^{22}(\frac{2\pi m}{\omega_\beta})^2 + \bar{\lambda}^{11}(\frac{\pi n}{\omega_L})^2} \cos^2 \frac{\pi n}{\omega_L}\omega_1 . \end{aligned} \tag{B2}$$

The sum over m is convergent and easily done, while the sum over n can be substituted by an integral as $L \to \infty$ (the closed string channel). For the divergent part we find

$$\frac{d}{\pi K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} \int_0^\infty \mathrm{d}x \frac{x^2}{y^2+x^2} \cos^2\left(\frac{x\omega_1}{\sqrt{\bar{\lambda}^{11}}}\right) \mathrm{e}^{-\varepsilon x^2} = \frac{d}{K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} \frac{1}{8\pi\varepsilon} \left[1 - \frac{\varepsilon \lambda^{11}}{\omega_1^2} + \mathrm{e}^{-\frac{\omega_1^2}{\varepsilon \bar{\lambda}^{11}}} \left(2 + \frac{\varepsilon \bar{\lambda}^{11}}{\omega_1^2}\right)\right], \tag{B3}$$

where

$$\varepsilon = \frac{a^2}{\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}} = \frac{1}{4\pi\Lambda^2 \sqrt{\bar{\rho}_{11}\bar{\rho}_{22}}}. \tag{B4}$$

For the finite part we get

$$\begin{aligned} &\frac{d}{K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} \frac{1}{\pi} \int_0^\infty \mathrm{d}x x \left[\coth\left(\frac{\omega_\beta x}{2\sqrt{\bar{\lambda}^{22}}}\right) - 1\right] \cos^2\left(\frac{x\omega_1}{\sqrt{\bar{\lambda}^{11}}}\right) \\ &= \frac{\pi d}{6K_0\omega_\beta^2} \frac{\sqrt{\bar{\lambda}^{22}}}{(\bar{\lambda}^{11})^{3/2}} + \frac{d}{K_0 \sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}} \frac{1}{\pi} \left[\frac{1}{8\omega_1^2} - \frac{\pi^2 \bar{\lambda}^{22}}{2\omega_\beta^2 \lambda^{11}} \frac{1}{\sinh^2(\frac{2\pi\omega_1}{\omega_\beta}\sqrt{\frac{\bar{\lambda}^{22}}{\bar{\lambda}^{11}}})}\right]. \end{aligned} \tag{B5}$$

The first term on the right-hand side is familiar from the integrated version of Sec. IV. The second term makes the induced metric to be ω_1-dependent. At $\omega_1 = 0$ it is equal to the first term.

Adding (B3) and (B5), we finally obtain

$$\langle \partial_1 X_q \cdot \partial_1 X_q \rangle = \frac{d}{K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} \frac{1}{8\pi\varepsilon} \left[1 + \mathrm{e}^{-\frac{\omega_1^2}{\varepsilon \bar{\lambda}^{11}}} \left(2 + \frac{\varepsilon \bar{\lambda}^{11}}{\omega_1^2}\right)\right] + \frac{\pi d}{6K_0\omega_\beta^2} \frac{\sqrt{\bar{\lambda}^{22}}}{(\bar{\lambda}^{11})^{3/2}} - \frac{\pi d}{2K_0\omega_\beta^2} \frac{\sqrt{\bar{\lambda}^{22}}}{(\bar{\lambda}^{11})^{3/2}} \frac{1}{\sinh^2(\frac{2\pi\omega_1}{\omega_\beta}\sqrt{\frac{\bar{\lambda}^{22}}{\bar{\lambda}^{11}}})}. \tag{B6}$$

We see from Eq. (B2) that the mean value equals

$$\frac{1}{\omega_\beta \omega_L} \int \mathrm{d}^2\omega \langle \partial_1 X_q \cdot \partial_1 X_q \rangle = \frac{d}{8\pi\varepsilon K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} + \frac{\pi d}{6K_0\omega_\beta^2} \frac{\sqrt{\bar{\lambda}^{22}}}{(\bar{\lambda}^{11})^{3/2}} \tag{B7}$$

which coincides with the right-hand side of Eq. (B6) far away from the boundary. The fact that the induced metric is ω_1-dependent near the boundary does not affect the mean value, because its contribution to the mean value is $\mathcal{O}(1/L)$. This ω_1-dependence of the induced metric near the boundary is specific to the cylinder (and disk) topology. It would be missing for a torus.

Exactly at the boundary we have from Eq. (B6) the twice larger value than the mean value (B7),

$$\langle \partial_1 X_q \cdot \partial_1 X_q \rangle|_B = \frac{d}{4\pi\varepsilon K_0 \sqrt{(\bar{\lambda}^{11})^3 \bar{\lambda}^{22}}} + \frac{\pi d}{3K_0\omega_\beta^2} \frac{\sqrt{\bar{\lambda}^{22}}}{(\bar{\lambda}^{11})^{3/2}}. \tag{B8}$$

As we shall momentarily see, this guarantees for the gauge condition (38) to be satisfied in the bulk.

Analogously, we find

$$\begin{aligned}\langle\partial_2 X_q\cdot\partial_2 X_q\rangle &= \frac{1}{\omega_\beta\omega_L}\frac{2d}{K_0}\sum_{m,n}\frac{(\frac{2\pi m}{\omega_\beta})^2}{\bar\lambda^{22}(\frac{2\pi m}{\omega_\beta})^2+\bar\lambda^{11}(\frac{\pi n}{\omega_L})^2}\sin^2\frac{\pi n}{\omega_L}\omega_1\\ &= \frac{\bar\lambda^{11}}{\bar\lambda^{22}}\langle\partial_1 X_q\cdot\partial_1 X_q\rangle-\frac{1}{\omega_\beta\omega_L}\frac{2d}{K_0\bar\lambda^{22}}\sum_{m,n}\left[\frac{\bar\lambda^{11}(\frac{\pi n}{\omega_L})^2}{\bar\lambda^{22}(\frac{2\pi m}{\omega_\beta})^2+\bar\lambda^{11}(\frac{\pi n}{\omega_L})^2}-\sin^2\frac{\pi n}{\omega_L}\omega_1\right]\\ &= \frac{\bar\lambda^{11}}{\bar\lambda^{22}}\langle\partial_1 X_q\cdot\partial_1 X_q\rangle-\frac{\bar\lambda^{11}}{\bar\lambda^{22}}\langle\partial_1 X_q\cdot\partial_1 X_q\rangle|_B+\frac{d}{K_0\sqrt{\bar\lambda^{11}(\bar\lambda^{22})^3}}\frac{1}{4\pi\varepsilon}(1-\mathrm{e}^{-\omega_1^2/\varepsilon\bar\lambda^{11}}).\end{aligned}\tag{B9}$$

We see that at the boundary $\langle\partial_2 X_q\cdot\partial_2 X_q\rangle|_B=0$ because of the boundary condition $X_q|_B=0$ and because the derivative is along the boundary.

Using Eq. (B8) and adding the classical parts, we rewrite Eq. (B9) as the following relation between components of the whole induced metric:

$$\bar\lambda^{11}\langle\partial_1 X\cdot\partial_1 X\rangle=\bar\lambda^{22}\langle\partial_2 X\cdot\partial_2 X\rangle+\frac{d}{4\pi\varepsilon K_0\sqrt{\bar\lambda^{11}\bar\lambda^{22}}}\mathrm{e}^{-\omega_1^2/\varepsilon\bar\lambda^{11}}.\tag{B10}$$

We see that $\langle\partial_2 X\cdot\partial_2 X\rangle$ equals $\langle\partial_1 X\cdot\partial_1 X\rangle$ everywhere outside the ε-vicinity of the boundary, where a more careful analysis of the second term on the right-hand side is required.

Using the mode expansion, we can also compute the transversal size of the string. Proceeding as above, we get

$$\begin{aligned}\langle X_\mathrm{q}^2\rangle &= \frac{1}{\omega_\beta\omega_L}\frac{2d}{K_0}\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{1}{\bar\lambda^{22}(\frac{2\pi m}{\omega_\beta})^2+\bar\lambda^{11}(\frac{\pi n}{\omega_L})^2}\\ &\quad\times\sin^2\frac{\pi n}{\omega_L}\omega_1=\frac{d}{\pi K_0\sqrt{\bar\lambda^{11}\bar\lambda^{22}}}\sum_{n=1}^{\infty}\frac{1}{n}\\ &\quad\times\coth\left(\frac{\pi n}{\omega_L}\frac{\omega_\beta}{2}\sqrt{\frac{\bar\lambda^{11}}{\bar\lambda^{22}}}\right)\sin^2\left(\frac{\pi n}{\omega_L}\omega_1\right).\end{aligned}\tag{B11}$$

The divergent part of the sum in Eq. (B11) can be replaced by an integral which has a logarithmic domain for $\varepsilon\ll\omega_\beta^2/\bar\lambda^{22}$. The (logarithmically) divergent part is

$$\sum_{n=1}^{\infty}\frac{1}{2n}\coth\left(\frac{\pi n}{\omega_L}\frac{\omega_\beta}{2}\sqrt{\frac{\bar\lambda^{11}}{\bar\lambda^{22}}}\right)\mathrm{e}^{-\epsilon\lambda^{11}(\frac{\pi n}{\omega_L})^2}=\frac{1}{4}\log\frac{\omega_\beta^2}{\varepsilon\bar\lambda^{22}}.\tag{B12}$$

The finite part for $\beta\ll L$ is

$$\sum_{n=1}^{\infty}\frac{1}{n}\coth\left(\frac{\pi n}{\omega_L}\frac{\omega_\beta}{2}\sqrt{\frac{\bar\lambda^{11}}{\bar\lambda^{22}}}\right)\left[\sin^2\left(\frac{\pi n}{\omega_L}\omega_1\right)-\frac{1}{2}\right]=\pi\frac{\omega_1\sqrt{\bar\lambda^{22}}}{\omega_\beta\sqrt{\bar\lambda^{11}}}\left(1-\frac{\omega_1}{\omega_L}\right).\tag{B13}$$

Finally, we obtain for large L

$$\langle X_\mathrm{q}^2\rangle=\frac{d}{K_0C}\left\{\frac{1}{4\pi}\log\left[\frac{\omega_\beta\omega_L\sqrt{\beta^2-\beta_0^2/C}}{\varepsilon CL}\right]+\frac{\omega_1}{\omega_L}\left(1-\frac{\omega_1}{\omega_L}\right)\frac{L}{\sqrt{\beta^2-\beta_0^2/C}}\right\}.\tag{B14}$$

If we perform the above renormalization (71) of the length scale $X_\mathrm{q}^2\to(2C-1)[X_\mathrm{q}^2]_\mathrm{R}/C$, then K_0 in the denominator becomes K_R:

$$\langle[X_\mathrm{q}^2]_\mathrm{R}\rangle=\frac{d}{K_\mathrm{R}}\left\{\frac{1}{4\pi}\log\left[\frac{1}{a^2}\left(\beta_\mathrm{R}^2-\frac{\pi d}{6K_\mathrm{R}}\right)\right]+\frac{\omega_1}{\omega_L}\left(1-\frac{\omega_1}{\omega_L}\right)\frac{L_\mathrm{R}}{\sqrt{\beta_\mathrm{R}^2-\pi d/3K_\mathrm{R})}}\right\}.\tag{B15}$$

The first term on the right-hand side of Eq. (B15) is familiar from the open-string case. It has the logarithmic divergence which cannot be renormalized, so it always diverges. It is the same for the zeta-function regularization, where the log is replaced by $\zeta(1)=\infty$. The appearance of the second term on the right-hand side of Eq. (B15) is specific to a cylinder. It comes from the modes with $m=0$ (the zero mode) and is missing for an open string. It looks pretty much like the one in Eq. (A10) for the random paths, if we identify $\tilde m_R$ with the mass of the lowest string state which propagates the distance L_R.

It is not hard to compute a correlator analogous to (A11) in the string case. Setting $\omega_2'=\omega_2$, we find for $\beta\ll L$

$$\begin{aligned}\langle X^\mu_{\rm q}(\omega_1,\omega_2)X^\nu_{\rm q}(\omega_1',\omega_2)\rangle &= \frac{2\delta^{\mu\nu}}{\omega_\beta\omega_L K_0}\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{\pi n}{\omega_L}\omega_1\sin\frac{\pi n}{\omega_L}\omega_1'}{\bar\lambda^{22}(\frac{2\pi m}{\omega_\beta})^2+\bar\lambda^{11}(\frac{\pi n}{\omega_L})^2}\\ &=\frac{d}{2\pi K_0 C}\sum_{n=1}^{\infty}\frac{1}{n}\coth\left(\frac{\pi n}{\omega_L}\frac{\omega_\beta}{2}\right)\left[\cos\frac{\pi n}{\omega_L}(\omega_1-\omega_1')\right.\\ &\quad\left.-\cos\frac{\pi n}{\omega_L}(\omega_1+\omega_1')\right], \qquad \text{(B16)}\end{aligned}$$

where we set

$$\omega_\beta=\frac{\omega_L}{L}\sqrt{\beta^2-\beta_0^2/C} \qquad \text{(B17)}$$

for simplicity of the formulas.

Equation (B16) represents the 2D (nonregularized) Dirichlet Green function for a cylinder at $\omega_2'=\omega_2$. If $|\omega_1-\omega_1'|\ll\omega_\beta$, (B16) behaves as

$$\text{(B16)}\overset{|\omega_1-\omega_1'|\ll\omega_\beta}{\longrightarrow}-\frac{\delta^{\mu\nu}}{K_0 C}\frac{1}{2\pi}\log\frac{|\omega_1-\omega_1'|}{\omega_\beta}, \qquad \text{(B18)}$$

i.e. as the ordinary Green function. If $|\omega_1-\omega_1'|\gg\omega_\beta$, (B16) behaves as

$$\text{(B16)}\overset{|\omega_1-\omega_1'|\gg\omega_\beta}{\longrightarrow}\frac{\delta^{\mu\nu}}{K_0 C}\left(\frac{\omega_1+\omega_1'}{2\omega_\beta}-\frac{|\omega_1-\omega_1'|}{2\omega_\beta}-\frac{\omega_1\omega_1'}{\omega_\beta\omega_L}\right) \qquad \text{(B19)}$$

so only the zero mode (i.e. the $m=0$ modes) remains at large L with an exponential accuracy and the result is quite analogous to Eq. (A11) in the particle case. In the open-string case this zero mode was absent and the contribution of nonzero modes coincides with the open-string result [26].

We can also compute $\langle\partial_1 X_{\rm q}(\omega_1,\omega_2)\cdot\partial_1 X_{\rm q}(\omega_1',\omega_2)\rangle$ for $\omega_\beta\gg|\omega_1-\omega_1'|\gg\sqrt{\varepsilon}=a/\sqrt[4]{\bar\rho}$ by differentiating (B16) with respect to ω_1 and ω_1'. We then find

$$\langle\partial_1 X_{\rm q}(\omega_1,\omega_2)\cdot\partial_1 X_{\rm q}(\omega_1',\omega_2)\rangle\overset{\omega_1'\bar\omega_1}{\longrightarrow}\frac{d}{K_0 C}\frac{\pi}{2\omega_\beta^2}\left(-\frac{1}{\pi^2(\omega_1-\omega_1')^2}+\frac{1}{3}-\frac{1}{\sinh\frac{2\pi\omega_1}{\omega_\beta}}\right). \qquad \text{(B20)}$$

In particular, we recover this way the ω_1-dependent term in the final part of the induced metric displayed in Eq. (B6). The constant term is also reproduced but this could be a coincidence because it is in general regularization dependent.

Remarkably, the log divergence, contaminating Eqs. (B14) and (B15), is missing in the correlator (B16). It comes back if $|\omega_1-\omega_1'|\lesssim\sqrt{\varepsilon}=a/\sqrt[4]{\bar\rho}$. This could be most probably interpreted as an effect of spikes, i.e. very long thin pieces of surfaces of negligible area, of longitudinal size of the cutoff at the world sheet.

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