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Fluid/gravity correspondence: A nonconformal realization in compactified D4 branes

Chao Wu,[1,*] Yidian Chen,[1,†] and Mei Huang[1,2,‡]

[1]Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
[2]Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China

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We develop the framework of boundary derivative expansion (BDE) formalism of fluid/gravity correspondence in a compactified D4-brane system, which is a nonconformal background used in top-down holographic QCD models. Such models contain the D4-D6 model and the Sakai-Sugimoto (SS) model, with the background of the compactified black D4 branes under the near-horizon limit. By using the dimensional reduction technique, we derive a 5D Einstein gravity minimally coupled with three scalar fields from the 10D D4-brane background. Following the BDE formalism of fluid/gravity correspondence in the conformal background, we directly derive all the first order transport coefficients for nonconformal gluonic matter. The results of the ratio of the bulk to shear viscosity and the sound speed agree with those obtained from the Green-Kubo method. This agreement guarantees the validity of the BDE formalism of fluid/gravity duality in the nonconformal D-brane background, which can be used to calculate the second order transport coefficients in nonconformal background.

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I. INTRODUCTION

The quantum chromodynamics (QCD) phase transition and properties of hot/dense quark matter at high temperature and baryon density are some of the most important topics of high energy nuclear physics. The Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) provide the opportunity to investigate properties of nuclear matter at high temperature and small baryon density. It is now believed that the system created at RHIC/LHC is a strongly coupled quark-gluon plasma (sQGP) and behaves like a nearly "perfect" fluid [1,2]. One crucial quantity is the shear viscosity over entropy density η/s, which is required to be very small to fit the elliptic flow at RHIC/LHC. The result from AdS/CFT correspondence gives the lower bound of $\eta/s = \frac{1}{4\pi}$ [3,4], which is very close to the value used to fit the elliptic flow v_2 [5–7].

The anti–de Sitter/conformal field theory (AdS/CFT) duality [8–10] is discovered through pioneering works on the near-horizon structure of black branes (or black holes) [11–17] and the scattering process of branes and bulk probe fields [18–21]. It is generalized to a nonconformal brane background in the near-horizon limit in Ref. [22], which is called the gauge/gravity duality nowadays. The gravity/gauge duality or AdS/CFT correspondence provides a revolutionary method to tackle the problem of strongly coupled gauge theories. It has been widely used to investigate QCD phenomenology, e.g., glueballs [23–25], hadron spectra [26–30], the deconfinement phase transition [27–29,31], and transport properties [32].

The shear viscosity in AdS/CFT was firstly calculated in Ref. [32] through relations between the Green-Kubo formula[1] and the absorption cross section of gravitons [18–21]. Studies on the near equilibrium QGP from AdS/CFT duality in [35] gives a recipe of extracting two-point real-time thermal correlators via classical bulk action. Following [35], the authors of [36,37] calculated the first order transport coefficients in near extremal D3 brane background and found that, in the long-distance and low-frequency limit, these correlators turn into hydrodynamical forms. Second order transport coefficients of this system were calculated in [38]. The framework that investigates transport properties of a fluid via its corresponding gravity is called the fluid/gravity correspondence, and the most notable feature in the above works [36–38] is the use of Green-Kubo formula; thus one may call it the Green-Kubo formalism of fluid/gravity correspondence.

While the Green-Kubo formalism becomes popular in extracting transport properties of liquidlike plasma,[2] another systematic and powerful formalism—the boundary derivative expansion (BDE) formalism [40,41]—has been developed. The most remarkable feature for this formalism is the use of boundary dependent boost parameters for the bulk metric in the (in-going) Eddington-Finkelstein coordinate. Expansions are implemented with respect to boundary derivatives of the boost parameters and all the

*wuchao@ihep.ac.cn
†chenyd@ihep.ac.cn
‡huangm@ihep.ac.cn

[1]This was first proposed by Kubo [33] in statistical mechanics and recast into field theory formalism by Hosaya *et al.* [34].

[2]The literature on this topic can be found in the references of [39].

dissipative terms of boundary fluid are metric perturbations (in the large r limit) solved from the Einstein equation. The first example of the BDE formalism of fluid/gravity correspondence is a duality of AdS_5 black hole in the bulk to finite temperature conformal $\mathcal{N} = 4$ SYM plasma in the boundary [40], where the transport coefficients were calculated to second order. The BDE formalism was applied in several other models with an AdS_5 black hole background: (1) the AdS_5-dilaton model [42], where the gravity side is an AdS_5 black hole plus a boundary dependent dilaton field, while in the boundary is a fluid with forcing terms; and (2) the charged AdS_5 black hole model [43,44], where the bulk is a charged AdS_5 black hole, and the fluid on the boundary has a chemical potential. The addition of the Chern-Simons term for the $U(1)$ gauge field causes the appearance of vorticity in the first order dissipative expansion of R-charge current.

The development of fluid/gravity correspondence interweaves with the studies on hydrodynamical modes on the world volume of the blackfold [45–48], which opens a window to extract the dynamical information on the world volume of black branes in flat spacetime. The most obvious difference of this kind of research from the fluid/gravity correspondence is the need for a Dirichlet boundary condition on a finite cutoff surface. In [49], the effective hydrodynamics on a $p + n + 2$ dimensional "rigid wall" located at $r = R$ in a $D = p + n + 3$ dimensional spacetime is studied, where p and $n + 1$ are the number of (spatial) dimensions of the brane and the sphere, respectively. Based on this, Emparan *et al.* [50] studied the effective hydrodynamics on the world volume of a black D3 brane to first order. Both the thermodynamical and the viscous quantities depend on the location of the cavity and the horizon; however, η/s of this model is still $\frac{1}{4\pi}$. Erdmenger *et al.* [51] investigated the effective hydrodynamics of rotating black D3 branes. The common feature of [50,51] is the use of dimensional reduction, which transforms the effects of transverse directions into massless fields on the longitudinal directions in which the branes lie. This prompts our focus on the world volume theory.

The Green-Kubo formalism and BDE formalism of fluid/gravity correspondence provide powerful tools for us to study the liquidlike QGP (see, e.g., [52,53] for a phenomenological review on heavy ion collisions for theorists). Generally speaking, QGP is a liquidlike plasma with small shear viscosity, and can be described by relativistic hydrodynamics quite well. Results from the lattice show that QGP exhibits nonconformal properties, especially around the critical temperature T_c, e.g., the shear viscosity over entropy density ratio has a minimum around T_c, and the bulk viscosity over entropy density shows a peak around T_c [54–57]. This behavior has been described in bottom-up holographic QCD models [58–61]. However, current studies using fluid/gravity duality from a top-down method are mostly on AdS background, whose dual fluid is of course conformal and thus may only reflect the properties of QGP at the conformal regime, i.e., above $2T_c$. However, when we are at the nonconformal zone around T_c, AdS gravity may no longer be proper for a nonconformal gauge theory.

A natural choice for a top-down holographic way to tackle the strongly coupled nonconformal plasma is to build models using the nonconformal D-branes. Such studies include, e.g., [62] for D1-brane and [63] for Dp-branes with $p \geq 2$, where the Green-Kubo method is used. There is another interesting work on this topic which can handle more cases, including $p = 0$, 1 and the fundamental strings in type II string theory (but $p = 5$ excluded): the Ref. [64], where the BDE formalism in Fefferman-Graham coordinates developed in [65] was used. Besides the Dp-brane backgrounds like in Refs. [63,64], one may also use the compactified D-brane backgrounds, e.g., the compactified D4 brane. The compactified D4 brane is the background of the D4-D6 model [29] and the Sakai-Sugimoto (SS) model [30], which are two nonconformal holographic QCD models from top-down. The background of these two models is the compactified D4 black branes under the near-horizon limit. There are some previous studies on the transport properties of this background. The sound speed and bulk to shear ratio were calculated in [66]; the shear to entropy density ratio was argued in this reference to be $1/4\pi$ by showing that the SS model background is in the class of [67]. Using the null horizon focusing equation, Eling and Oz [68] also calculated the ratio of the bulk viscosity to the shear viscosity.

Based on the above review of the relevant literatures, one can see that there is a lack of parallel formulation with [40]. This motivates us to develop the framework of BDE formalism of fluid/gravity duality for nonconformal gauge theory plasma. In this paper, by using the BDE formalism of fluid/gravity correspondence, we calculate the first order transport coefficients of the nonconformal QGP under the quenched limit on the boundary of both the D4-D6 and SS model's background. Our results are consistent with former studies by other methods. The previous results together with ours reveal that the plasma of the D4-D6 and SS model is nonconformal with a small bulk viscosity and saturates the KSS bound [3,4], and this agreement guarantees the validity of the BDE formalism of fluid/gravity correspondence for nonconformal D-brane backgrounds with more than one submanifold reduced. This work can been seen as a nonconformal counterpart that is parallel with the AdS_5 construction in fluid/gravity correspondence of Bhattacharyya *et al.* [40].

This paper is organized as follows: After the Introduction, we will firstly give the preliminaries from a 10D compactified black D4-brane background to a five-dimensional one in Sec. II in order to make a connection with the recipe of fluid/gravity correspondence. Then, in Sec. III, we will solve all the first order perturbative *Ansätze* and get the metric which perturbatively solves the Einstein equation to the first order. By making use of this solution, we calculate the boundary stress tensor for the QGP that corresponds to the bulk of the SS model in Sec. IV and analyze its transport properties. We give the discussion and outlook in Sec. V.

II. THE SETUP

In this section, following [66], we will show how to derive the action and classical background of the D4-D6 and SS model into 5D form through dimensional reduction as in [50,51]. The purpose of doing this is to make a connection with [40]; more details can be found in the Appendix.

The D4-brane action of type IIA theory in Einstein frame is given as

$$S=\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[R^{(10)}-\frac{1}{2}(^{10}\nabla\phi)^2-\frac{g_s^2}{2\cdot 4!}e^{\frac{\phi}{2}}F_4^2\right], \tag{1}$$

where $2\kappa_{10}^2=(2\pi)^7 g_s^2 l_s^8$ is the 10D gravitational coupling and $^{10}\nabla$ stands for 10D nabla. G is the determinant of the following diagonal 10D metric tensor:

$$ds^2=e^{2\alpha_1 A}g_{MN}dx^M dx^N + e^{2\alpha_2 A}(e^{2\beta_1 B}dy^2+e^{2\beta_2 B}\gamma_{ab}d\theta^a d\theta^b), \tag{2}$$

where g_{MN}, A and B only depend on x^M, the coordinates of the first 5 dimensions, and γ_{ab} with $a,b=1,2,3,4$ is the metric on the S^4. $\alpha_{1,2}$ and $\beta_{1,2}$ are four parameters whose value will be clear in the following context. The explicit forms of the dilaton and Ramond-Ramond (RR) field are given in (19). It should be noticed here that y is also a compact dimension and we will integrate out both y and the four-sphere to get a 5D effective theory.

From Eq. (2), we have $\sqrt{-G}=e^{5(\alpha_1+\alpha_2)A+(\beta_1+4\beta_2)B}\times\sqrt{-g}\sqrt{\gamma}$ with $\gamma=\det\gamma_{ab}$ the determinant of the metric on the unit four-sphere. During the reduction process, we have used the following relation:

$$S\sim\int d^{10}x\sqrt{-G}(R^{(10)}+\cdots) = \int d^5x\sqrt{-g}e^{(3\alpha_1+5\alpha_2)A+(\beta_1+4\beta_2)B}(R+\cdots). \tag{3}$$

To avoid the appearance of nonminimal coupling of the gravity with the scalar field in the reduced theory, one should set

$$\alpha_1=-\frac{5}{3},\qquad \alpha_2=1,\qquad \beta_1=4,\qquad \beta_2=-1, \tag{4}$$

so Eq. (2) becomes

$$ds^2=e^{-\frac{10}{3}A}g_{MN}dx^M dx^N+e^{2A+8B}dy^2+e^{2A-2B}d\Omega_4^2. \tag{5}$$

From Eq. (A7), the 10D Ricci scalar has the form of

$$R^{(10)}=e^{\frac{10}{3}A}\left[R+\frac{10}{3}\nabla^2A-\frac{40}{3}(\partial A)^2-20(\partial B)^2\right] + 12e^{-2A+2B}. \tag{6}$$

During the reduction process, we have

$$\sqrt{-G}=\sqrt{-g}\sqrt{\gamma}e^{-\frac{10}{3}A}, \tag{7}$$

$$\sqrt{-G}R^{(10)}=\sqrt{-g}\sqrt{\gamma}\left(R+\frac{10}{3}\nabla^2A-\frac{40}{3}(\partial A)^2 - 20(\partial B)^2+12e^{-\frac{16}{3}A+2B}\right), \tag{8}$$

$$\sqrt{-G}(^{10}\nabla\phi)^2=\sqrt{-g}\sqrt{\gamma}e^{-\frac{10}{3}A}G^{MN}\partial_M\phi\partial_N\phi = \sqrt{-g}\sqrt{\gamma}(\partial\phi)^2, \tag{9}$$

$$\sqrt{-G}\frac{g_s^2}{2\cdot 4!}e^{\frac{\phi}{2}}F_4^2=\sqrt{-g}\sqrt{\gamma}\frac{Q_4^2}{2}e^{\frac{\phi}{2}-\frac{34}{3}A+8B}. \tag{10}$$

Therefore the D4 brane action Eq. (1) is reduced to its 5D form and takes the form of

$$S=\frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{40}{3}(\partial A)^2 - 20(\partial B)^2-V(\phi,A,B)\right],$$
$$V(\phi,A,B)=\frac{Q_4^2}{2}e^{\frac{\phi}{2}-\frac{34}{3}A+8B}-12e^{-\frac{16}{3}A+2B}, \tag{11}$$

where κ_5 is the 5D surface gravity with the following definition:

$$\frac{1}{2\kappa_5^2}\equiv\frac{V_1\Omega_4}{2\kappa_{10}^2}, \tag{12}$$

with $V_1=\int dy$ the volume of the compact circle. The system turns into a 5D Einstein gravity minimally coupled with three scalars ϕ, A, B and $V(\phi,A,B)$ is the scalar potential. The equations of motion (EOMs) for this reduced system are

$$E_{MN}-T_{MN}=0, \tag{13}$$

$$\nabla^2\phi-\frac{Q_4^2}{4}e^{\frac{\phi}{2}-\frac{34}{3}A+8B}=0, \tag{14}$$

$$\nabla^2A+\frac{17Q_4^2}{80}e^{\frac{\phi}{2}-\frac{34}{3}A+8B}-\frac{12}{5}e^{-\frac{16}{3}A+2B}=0, \tag{15}$$

$$\nabla^2B-\frac{Q_4^2}{10}e^{\frac{\phi}{2}-\frac{34}{3}A+8B}+\frac{3}{5}e^{-\frac{16}{3}A+2B}=0, \tag{16}$$

where

$$E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R \tag{17}$$

is the Einstein tensor in the 5D spacetime, and

$$\begin{aligned} T_{MN} \equiv & \frac{1}{2}\left(\partial_M\phi\partial_N\phi - \frac{1}{2}g_{MN}(\partial\phi)^2\right) \\ & + \frac{40}{3}\left(\partial_M A\partial_N A - \frac{1}{2}g_{MN}(\partial A)^2\right) \\ & + 20\left(\partial_M B\partial_N B - \frac{1}{2}g_{MN}(\partial B)^2\right) - \frac{1}{2}g_{MN}V, \end{aligned} \tag{18}$$

which can be viewed as the energy-momentum tensor in the bulk.

The classical solution for black D4 brane in Einstein frame reads

$$\begin{aligned} ds^2 &= H_4^{-\frac{3}{8}}(-f(r)dt^2 + d\vec{x}^2) \\ &\quad + H_4^{\frac{5}{8}}\frac{dr^2}{f(r)} + H_4^{-\frac{3}{8}}dy^2 + H_4^{\frac{5}{8}}r^2 d\Omega_4^2, \\ e^{\phi} &= e^{\Phi-\Phi_0} = H_4^{-\frac{1}{4}}, \qquad F_4 = g_s^{-1}Q_4\epsilon_4, \\ H_4 &= 1 + \frac{r_{Q4}^3}{r^3}, \qquad f(r) = 1 - \frac{r_H^3}{r^3}, \end{aligned} \tag{19}$$

where $g_s = e^{\Phi_0}$ and $Q_4 = (2\pi l_s)^3 g_s N_c/\Omega_4$.[3] Note that we write one of the directions in which the D4 brane lies (denoted by y) separately from the other three directions (denoted by $\{\vec{x}\}$) in order to compare with (5). Under the near-horizon limit, the above metric becomes

$$\begin{aligned} ds^2 &= \left(\frac{r}{L}\right)^{\frac{9}{8}}(-f(r)dt^2 + d\vec{x}^2) \\ &\quad + \left(\frac{L}{r}\right)^{\frac{15}{8}}\frac{dr^2}{f(r)} + \left(\frac{r}{L}\right)^{\frac{9}{8}}dy^2 + L^{\frac{15}{8}}r^{\frac{1}{8}}d\Omega_4^2, \end{aligned} \tag{20}$$

$$e^{\phi} = \left(\frac{r}{L}\right)^{\frac{3}{4}}, \tag{21}$$

where $L^3 = Q_4/3 = \pi g_s N_c l_s^3$. The above metric differs from the D4-D6 model and the SS model for the interchange of t with y. Also, it is in the Einstein frame, not string frame. Comparing Eq. (5) with Eq. (20), we have

$$e^A = L^{\frac{51}{80}}r^{\frac{13}{80}}, \qquad e^B = L^{-\frac{3}{10}}r^{\frac{1}{10}}, \tag{22}$$

[3]The normalization condition for Q_4 here is $2\kappa^2\mu_4 N_c = \int_{S^4} F_4$, where $2\kappa^2 = 2\kappa_{10}^2 g_s^{-2}$ and $\mu_4 = ((2\pi)^4 l_s^5)^{-1}$ is the D4-brane charge.

and the reduced 5D metric is

$$ds^2 = Lr^{\frac{5}{3}}(-f(r)dt^2 + d\vec{x}^2) + \frac{L^4}{r^{\frac{4}{3}}f(r)}dr^2. \tag{23}$$

From its Ricci scalar $R \sim -\frac{5}{6r^{11/3}}(14r^3 + r_H^3)$, when $r \to 0$, R will become minus infinity so $r = 0$ is the curvature singularity and away from that point the above metric will always be regular; thus we will only focus on the regime of $r > 0$ from now on. At the boundary $r \to \infty$, $R \to 0$ so Eq. (23) is asymptotically flat, which is not obvious for the appearance of $r^{5/3}$ in the first four dimensions.

We turn to the ingoing Eddington-Finkelstein coordinate by making the transformation $dt = dv - \frac{L^{3/2}}{r^{3/2}f(r)}dr$; then the above metric becomes

$$ds^2 = Lr^{\frac{5}{3}}(-f(r)dv^2 + d\vec{x}^2) + 2L^{\frac{5}{2}}r^{\frac{1}{6}}dvdr. \tag{24}$$

$r = 0$ is still the curvature singularity of this 5D metric but everywhere away from that is regular. Since we have already lost track of the dimensions in the process of dimensional reduction [Eq. (2)], keeping L explicit will be insignificant, so from now on we set $L = 1$, which means $Q_4 = 3$. After a boost of coordinates $dv = -u_\mu dx^\mu$, $dx^i = P^i_\mu dx^\mu$, where $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$, we have

$$\begin{aligned} & ds^2 = r^{\frac{5}{3}}(-f(r)u_\mu u_\nu dx^\mu dx^\nu + P_{\mu\nu}dx^\mu dx^\nu) - 2r^{\frac{1}{6}}u_\mu dx^\mu dr, \\ & u^\mu = \gamma(1,\beta_i),\ \gamma = \frac{1}{\sqrt{1-\beta_i^2}}. \end{aligned} \tag{25}$$

In the above metric, u^μ is the four-speed of the relativistic fluid with the normalization $u_\mu u^\mu = -1$. $P_{\mu\nu}$ is the projection tensor of the boundary with $P_{\mu\nu}P^{\nu\rho} = P^\rho_\mu$, which projects any tensor to the plane orthogonal to u^μ. As one can check, Eq. (25) is the zeroth order solution of 5D EOM. The boundary of Eq. (25) is actually a fluid with constant temperature and velocity, which is of course in global equilibrium.

In order to mimic slight deviations from local equilibration and the anisotropy of the fluid, we promote the four parameters in Eq. (25) to be x^μ dependent: $r_H \to r_H(x)$, $u_\mu \to u_\mu(x)$, with the requirement that $|\frac{\partial u}{T}| \ll 1$, where T is the local temperature of the fluid. Then Eq. (25) becomes

$$\begin{aligned} ds^2 = r^{\frac{5}{3}}(&-f(r_H(x),r)u_\mu(x)u_\nu(x)dx^\mu dx^\nu \\ &+ P_{\mu\nu}(x)dx^\mu dx^\nu) - 2r^{\frac{1}{6}}u_\mu(x)dx^\mu dr, \end{aligned} \tag{26}$$

which is no longer the solution of 5D EOM, but we can make it the solution again by putting some perturbations in. Using the method of [40], we should firstly expand the fluid quantities of Eq. (26) at some special point, say, $x^\mu = 0$ in the local rest frame of the fluid, as

$$u_\mu = -\delta^0_\mu + x^\nu\partial_\nu\beta_j\delta^j_\mu, \qquad r_H(x^\mu) = r_H(0) + x^\mu\partial_\mu r_H. \tag{27}$$

$r_H(x^\mu = 0)$ is the location of the event horizon corresponding to $x^\mu = 0$ in the boundary; it relates with the local equilibrium temperature of the fluid at that point. In order to keep the formulations neat, we will just denote it as r_H in the following calculations but one should always remember that it is a local quantity at $x^\mu = 0$. Then we have

$$u_\mu dx^\mu = -dv + x^\mu \partial_\mu \beta_i dx^i, \qquad u_\mu u_\nu dx^\mu dx^\nu = dv^2 - 2x^\mu \partial_\mu \beta_i dx^i dv,$$
$$P_{\mu\nu} dx^\mu dx^\nu = d\vec{x}^2 - 2x^\mu \partial_\mu \beta_i dx^i dv, \qquad f(r_H(x), r) = f(r) - \frac{3r_H^2}{r^3} x^\mu \partial_\mu r_H. \tag{28}$$

Thus Eq. (26) becomes

$$ds^2 = \left(-r^{\frac{5}{3}} f + \frac{3r_H^2}{r^{\frac{4}{3}}} x^\mu \partial_\mu r_H\right) dv^2 - \frac{2r_H^3}{r^{\frac{4}{3}}} x^\mu \partial_\mu \beta_i dv dx^i + 2r^{\frac{1}{6}} dv dr + r^{\frac{5}{3}} d\vec{x}^2 - 2r^{\frac{1}{6}} x^\mu \partial_\mu \beta_i dx^i dr. \tag{29}$$

The above metric deviates the solution of the Einstein equation slightly by the first order boundary derivatives at x^μ; we will see that adding some perturbation terms will make it the solution again, and these perturbations are solved in the next section.

III. THE FIRST ORDER PERTURBATIONS

The $SO(3)$ symmetry in Eq. (23) separates the perturbations into tensors, vectors, and scalars of $SO(3)$, and we will make use of this advantage to solve these three kinds of perturbations one by one. Generally speaking, all the perturbation *Ansätze* will have the form

$$P(r) \times \begin{cases} \partial_i \beta_i, & \text{for the scalar part;} \\ \partial_v \beta_i, & \text{for the vector part;} \\ \sigma_{ij}, & \text{for the tensor part;} \end{cases} \tag{30}$$

where $P(r)$ is some function of r and can be solved through the Einstein equation with the boundary conditions as

(i) $P(r)$ is regular at $r = r_H$;
(ii) $\lim_{r\to\infty} \frac{P(r)}{r^n} \to 0$.

Here $n = 0$ or $n = 3$ depends on the nature of perturbation terms. We can see that the perturbations will always be of the form of Eq. (30) with the above boundary condition implemented.

A. The tensor part

We set the tensor part perturbation as

$$ds^2_{(1)T} = r^{\frac{5}{3}} \alpha_{ij}(r) dx^i dx^j. \tag{31}$$

The EOM that α_{ij} satisfies is

$$E_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{kl} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}. \tag{32}$$

It turns out that the differential equation for α_{ij} is

$$\frac{d}{dr}\left(r^4 f \frac{d\alpha_{ij}}{dr}\right) = -5r^{\frac{3}{2}}\sigma_{ij}, \tag{33}$$

where $\sigma_{ij} \equiv \partial_{(i}\beta_{j)} - \frac{1}{3}\delta_{ij}\partial_k\beta_k$ is the spatial part of shear stress tensor. The purpose for writing the EOM for the tensor part like this is due to the traceless of α_{ij}: the trace part of EOM should be removed from the diagonal components. The equation for the first order tensor perturbations takes similar form for different models, as can be seen, e.g., from [40,50,51]; the reason for this may be due to the universality of the shear viscosity in supergravity[4] [3,67]. Since the metric of the SS model is also in the class of [67], it is natural for Eq. (33) to take such a form. We write α_{ij} as $\alpha_{ij} = F(r)\sigma_{ij}$; then $F(r)$ can be solved from

$$F'' + \frac{4r^3 - r_H^3}{r^4 f} F' + \frac{5}{r^{\frac{5}{2}} f} = 0, \tag{34}$$

from which the result can be solved as

$$F(r) = C_2 + \frac{1}{3\sqrt{r_H}}\left[2\sqrt{3}\left(\arctan\frac{1-2\sqrt{r/r_H}}{\sqrt{3}} - \arctan\frac{1+2\sqrt{r/r_H}}{\sqrt{3}}\right) + \ln\frac{(\sqrt{r}+\sqrt{r_H})^2(r+\sqrt{rr_H}+r_H)}{(r-\sqrt{rr_H}+r_H)} + C_1 \ln\frac{r^3 - r_H^3}{r^3} - \ln(\sqrt{r}-\sqrt{r_H})^2\right]. \tag{35}$$

[4]The tensor part perturbation corresponds to the shear viscous term.

Regularity at $r = r_H$ requires $C_1 = 2$ and the normalizability at $r \to \infty$ requires $C_2 = 2\pi/\sqrt{3r_H}$; thus

$$F(r) = \frac{1}{3\sqrt{r_H}}\left[2\sqrt{3}\left(\arctan\frac{1-2\sqrt{r/r_H}}{\sqrt{3}} - \arctan\frac{1+2\sqrt{r/r_H}}{\sqrt{3}} + \pi\right) + \ln\frac{(\sqrt{r}+\sqrt{r_H})^4(r+\sqrt{rr_H}+r_H)^2(r^2+rr_H+r_H^2)}{r^6}\right]. \quad (36)$$

It is regular in the whole regime of $r > 0$ and vanishes to 0 asymptotically.

B. The vector part

For the vector part, we set the perturbation *Ansatz* as

$$ds^2_{(1)V} = -\frac{2r_H^3}{r^{\frac{4}{3}}} w_i dx^i dv. \quad (37)$$

The constraint equation for the vector perturbation is

$$g^{rv}(E_{vi} - T_{vi}) + g^{rr}(E_{ri} - T_{ri}) = 0, \quad (38)$$

which gives

$$\partial_i r_H + 2r_H \partial_v \beta_i = 0. \quad (39)$$

The dynamical equation is

$$E_{vi} - T_{vi} = 0. \quad (40)$$

It turns out that $w_i(r)$ satisfies

$$w_i'' - \frac{2}{r}w_i' - \frac{5r^{\frac{1}{2}}}{2r_H^3}\partial_v\beta_i = 0, \quad (41)$$

from which the solution is given as

$$w_i(r) = -\frac{2r^{\frac{5}{2}}}{r_H^3}\partial_v\beta_i + \frac{1}{3}r^3 C_{1i} + C_{2i}. \quad (42)$$

It is easy to see that the above general solution is regular at r_H. The other boundary condition for the vector part perturbation is

$$\lim_{r\to\infty}\frac{w_i}{r^3} \to 0, \quad (43)$$

which means C_{1i} must be 0. The appearance of C_{2i} will cause the $(0i)$ components of the boundary stress tensor to go out of the Landau frame. So if one likes to express the boundary stress tensor in the Landau frame, C_{2i} should be set to 0. Thus the final result for the vector perturbation of first order is

$$ds^2_{(1)V} = 4r^{\frac{7}{6}}\partial_v\beta_i dv dx^i. \quad (44)$$

C. The scalar part

The scalar part, similar to other works on the effective hydrodynamics of black branes, e.g., [50,51], is the most complicated part. We set the scalar part perturbation as

$$ds^2_{(1)S} = \frac{k(r)}{r^{\frac{4}{3}}}dv^2 + r^{\frac{5}{3}}h(r)\delta_{ij}dx^i dx^j + 2r^{\frac{1}{6}}j(r)dvdr. \quad (45)$$

In our case, the gauge condition $\mathrm{tr}[g_{(0)}^{-1}g_{(1)}] = 0$ [40,43] cannot be used here for solving the scalar part perturbation, since this will cause inconsistencies when solving the EOMs and make the surface stress tensor unrenormalizable. Other gauge conditions like $h(r) = 1$ [44] cannot be used either, since the spatial trace part of the metric is nontrivial in the nonconformal case here. Thus we need to keep all three unknowns. However, the labor cost to solve all of them gives us a bonus that a bulk viscous term will appear in the surface stress tensor, which does not appear in the conformal models with AdS gravity like in Refs. [40,43,44]. We have two constraint equations for the scalar sector:

$$g^{rr}(E_{rv} - T_{rv}) + g^{rv}(E_{vv} - T_{vv}) = 0, \quad (46)$$

$$g^{rr}(E_{rr} - T_{rr}) + g^{rv}(E_{rv} - T_{rv}) = 0, \quad (47)$$

which separately give

$$\partial_v r_H = -\frac{2}{5}r_H\partial_i\beta_i \quad (48)$$

and

$$3(5r^3 - 2r_H^3)h' - 30r^2 j - 5k' + 10r^{\frac{3}{2}}\partial_i\beta_i = 0. \quad (49)$$

We also have a total number of seven dynamical equations for scalar perturbations. Four of them,

$$E_{rr} - T_{rr} = 0, \quad (50)$$

$$E_{rv} - T_{rv} = 0, \quad (51)$$

$$E_{vv} - T_{vv} = 0, \quad (52)$$

$$\sum_{i=1}^{3}(E_{ii} - T_{ii}) = 0, \quad (53)$$

come from the Einstein equation Eq. (13), and three of them,

$$\nabla^2\phi - \frac{9}{4}e^{\frac{\phi}{2}-\frac{34}{3}A+8B} = 0, \tag{54}$$

$$\nabla^2 A + \frac{153}{80}e^{\frac{\phi}{2}-\frac{34}{3}A+8B} - \frac{12}{5}e^{-\frac{16}{3}A+2B} = 0, \tag{55}$$

$$\nabla^2 B - \frac{9}{10}e^{\frac{\phi}{2}-\frac{34}{3}A+8B} + \frac{3}{5}e^{-\frac{16}{3}A+2B} = 0, \tag{56}$$

are from the three scalar field equations in the 5D bulk, namely, Eqs. (14)–(16). This looks horrible at first glance, but fortunately, not all of them give a useful message. It turns out that Eq. (51) and Eq. (52) come out of linear compositions of specific constraints with the Einstein equation of the scalar sector, so they are not independent equations, and Eqs. (54)–(56) give the same differential equation for the three unknown scalar perturbations. So we only need to solve Eqs. (49), (50), (53), and (54) to nail down Eq. (45), among which the last three equations are

$$0 = 6rh'' + 9h' - 10j', \tag{57}$$

$$0 = 12r^4 f h'' + 12(4r^3 - r_H^3)h' - 6rk'' - 3k' - 6(5r^3 - 2r_H^3)j' - 90r^2 j + 20r^{\frac{3}{2}}\partial_i\beta_i, \tag{58}$$

$$0 = 2r^3 f j' + 12r^2 j + 2k' - 3r^3 f h' - 2r^{\frac{3}{2}}\partial_i\beta_i. \tag{59}$$

We will choose Eqs. (49), (57), (59) to solve the three unknown scalar perturbations that we set in Eq. (45). From Eq. (49) we have

$$6r^2 j + k' = \frac{3}{5}(5r^3 - 2r_H^3)h' + 2r^{\frac{3}{2}}\partial_i\beta_i, \tag{60}$$

and after putting it into Eq. (59), we get

$$-10r^3 f j' = (15r^3 + 3r_H^3)h' + 10r^{\frac{3}{2}}\partial_i\beta_i. \tag{61}$$

Then, putting the above equation into Eq. (57) one can finally get the equation for h:

$$\frac{d}{dr}\left(r^4 f\frac{dh}{dr}\right) + \frac{5}{3}r^{\frac{3}{2}}\partial_i\beta_i = 0. \tag{62}$$

Without losing generality, we set $h = F_h(r)\partial_i\beta_i$, and $F_h(r)$ satisfies

$$\frac{d}{dr}\left(r^4 f\frac{dF_h}{dr}\right) = -\frac{5}{3}r^{\frac{3}{2}}. \tag{63}$$

If one compares the above equation with Eq. (33), one can get $F_h = F/3$ without solving it; thus

$$h = \frac{1}{3}F(r)\partial_i\beta_i. \tag{64}$$

Inserting h into Eq. (57) one has the equation for $j = F_j(r)\partial_i\beta_i$ as

$$10F_j' = 2rF'' + 3F'. \tag{65}$$

This is a first order differential equation, and the solution can be obtained by direct integration; the result is

$$F_j(r) = -\frac{2}{5}\frac{r^{\frac{5}{2}} - r_H^{\frac{5}{2}}}{r^3 - r_H^3} + C_j + \frac{1}{30\sqrt{r_H}}\left[2\sqrt{3}\left(\arctan\frac{1-2\sqrt{r/r_H}}{\sqrt{3}} - \arctan\frac{1+2\sqrt{r/r_H}}{\sqrt{3}}\right) + \ln\frac{(\sqrt{r}+\sqrt{r_H})^4(r+\sqrt{rr_H}+r_H)^2(r^2+rr_H+r_H^2)}{r^6}\right]. \tag{66}$$

Since the above expression is already regular at $r = r_H$, the remaining boundary condition for j is

$$\lim_{r\to\infty} F_j \to 0. \tag{67}$$

Thus $C_j = \frac{\sqrt{3}\pi}{15\sqrt{r_H}}$, so we have finally

$$F_j(r) = -\frac{2}{5}\frac{r^{\frac{5}{2}} - r_H^{\frac{5}{2}}}{r^3 - r_H^3} + \frac{1}{10}F. \tag{68}$$

We set $k = F_k(r)\partial_i\beta_i$ likewise and substitute h and j into Eq. (49), and we can have

$$F_k(r) = -\frac{2\sqrt{3}\pi}{15\sqrt{r_H}}r^3 + \frac{4}{5}r^{\frac{5}{2}} + C_k - \frac{1}{15\sqrt{r_H}}(r^3 + 2r_H^3)\left[2\sqrt{3}\left(\arctan\frac{1-2\sqrt{r/r_H}}{\sqrt{3}} - \arctan\frac{1+2\sqrt{r/r_H}}{\sqrt{3}}\right) + \ln\frac{(\sqrt{r}+\sqrt{r_H})^4(r+\sqrt{rr_H}+r_H)^2(r^2+rr_H+r_H^2)}{r^6}\right]. \tag{69}$$

The integral constant C_k is fixed by the requirement that the final boundary stress tensor is in the Landau frame, which gives $C_k = -\frac{4\sqrt{3}\pi}{15}r_H^{\frac{5}{2}}$. So we finally have

$$k = \left(\frac{4}{5}r^{\frac{5}{2}} - \frac{1}{5}(r^3 + 2r_H^3)F\right)\partial_i\beta_i. \tag{70}$$

In order to make a consistent check, one may put h, j, and k into Eq. (58), it comes out that the three first order scalar perturbations that we have solved out satisfy Eq. (58) just right. So the scalar perturbations that we need to make Eq. (29) the solution of the Einstein equation again turn out to be

$$ds^2_{(1)S} = \left(\frac{F_k}{r^{\frac{4}{3}}}dv^2 + r^{\frac{5}{3}}F_h\delta_{ij}dx^i dx^j + 2r^{\frac{1}{6}}F_j dv dr\right)\partial_k\beta_k. \tag{71}$$

D. Global form of the full metric containing first order perturbations

Putting all the stuff of the zeroth and first order together, we get

$$ds^2 = \left(-r^{5/3}f + \frac{3r_H^2 x^\mu\partial_\mu r_H}{r^{4/3}} + \frac{F_k\partial_i\beta_i}{r^{4/3}}\right)dv^2 + \left(4r^{7/6}\partial_v\beta_i - \frac{2r_H^3 x^\mu\partial_\mu\beta_i}{r^{4/3}}\right)dx^i dv + 2r^{1/6}(1 + F_j\partial_i\beta_i)dvdr + r^{5/3}\left(\delta_{ij} + \frac{1}{3}F\delta_{ij}\partial_k\beta_k + F\sigma_{ij}\right)dx^i dx^j - 2r^{\frac{1}{6}}x^\mu\partial_\mu\beta_i dx^i dr. \tag{72}$$

The above is just the full solution of the first order at the vicinity of $x^\mu = 0$ in some special frame, whose covariant form can be constructed as

$$ds^2 = -r^{\frac{5}{3}}\left(f(r_H(x), r) - \frac{F_k(r_H(x), r)}{r^3}\partial_\rho u^\rho\right)u_\mu u_\nu dx^\mu dx^\nu - 2r^{\frac{7}{6}}(u_\mu a_\nu + u_\nu a_\mu)dx^\mu dx^\nu + r^{\frac{5}{3}}F(r_H(x), r)\sigma_{\mu\nu}dx^\mu dx^\nu + r^{\frac{5}{3}}\left(1 + \frac{1}{3}F(r_H(x), r)\partial_\rho u^\rho\right)P_{\mu\nu}dx^\mu dx^\nu - 2r^{\frac{1}{6}}(1 + F_j(r_H(x), r)\partial_\rho u^\rho)u_\mu dx^\mu dr, \tag{73}$$

where $\sigma_{\mu\nu} = P_\mu^\rho P_\nu^\sigma\partial_{(\rho}u_{\sigma)} - \frac{1}{3}P_{\mu\nu}\partial_\rho u^\rho$ is the 4D covariant shear viscous tensor and $a_\mu = u^\nu\partial_\nu u_\mu$ is the four-acceleration related with u_μ.

IV. THE BOUNDARY STRESS TENSOR AND TRANSPORT PROPERTIES

A. Derivation of boundary stress tensor

The system of this model is a five-dimensional Einstein gravity coupled with three scalar fields; its total action can be written as

$$S = S_{\text{bulk}} + S_{\text{GH}} + S_{c.t.}, \tag{74}$$

where S_{bulk} is the bulk action (11) and S_{GH} is the corresponding Gibbons-Hawking action,

$$S_{\text{GH}} = -\frac{1}{\kappa_5^2}\int d^4x\sqrt{-h}K, \tag{75}$$

where h_{MN} is the boundary metric tensor at a hyperplane with constant large r. K is the trace of the external curvature. The most crucial part in the total action is the bulk counterterm $S_{c.t.}$. Since the bulk metric is not AdS, the results of the counterterm for AdS spacetime [69] cannot be directly used here, but fortunately there are also works on the renormalization of nonconformal branes [70]. Here we adapt the counterterm used in [71], which they borrow from the much earlier work [28] on the renormalization of the black D4 brane; in the Einstein frame it has the form of

$$S_{c.t.} = \frac{1}{\kappa_{10}^2} \int d^9 x \sqrt{-H} \frac{5}{2} e^{-\frac{1}{12}\phi}. \tag{76}$$

Here H is the determinant of the boundary metric of Eq. (5),

$$ds^2 = e^{-\frac{10}{3}A} h_{\mu\nu} dx^\mu dx^\nu + e^{2A+8B} dy^2 + e^{2A-2B} d\Omega_4^2. \tag{77}$$

Note $x^M = \{x^\mu, r\}$. After the dimensional reduction on the above metric, Eq. (76) becomes

$$S_{c.t.} = \frac{1}{\kappa_5^2} \int d^4 x \sqrt{-h} \frac{5}{2} e^{-\frac{5}{3}A - \frac{1}{12}\phi}. \tag{78}$$

This counterterm contributes to the surface stress tensor as

$$\frac{2}{\sqrt{-h}} \frac{\delta S_{c.t.}}{\delta h^{\mu\nu}} = \frac{1}{\kappa_5^2} \left(-\frac{5}{2} e^{-\frac{5}{3}A - \frac{1}{12}\phi} h_{\mu\nu} \right). \tag{79}$$

Using Eq. (22) (and remember that we have set $L = 1$), one has the surface stress tensor with contribution from the counterterm as

$$T_{\mu\nu}^{\text{surf}} = \frac{1}{\kappa_5^2} \left(K_{\mu\nu} - h_{\mu\nu} K - \frac{5}{2} r^{-\frac{1}{3}} h_{\mu\nu} \right). \tag{80}$$

In the standard technic for $3+1$ decomposition of general relativity, h_{MN} is defined as

$$h_{MN} = g_{MN} - n_M n_N, \tag{81}$$

where $n_M = N \nabla_M r$ is the unit normal vector for a hyperplane at constant large r in the 5D bulk, of which the metric can be written as

$$ds^2 = (N^2 + N_M N^M) dr^2 + 2 N_M dr dx^M + h_{MN} dx^M dx^N. \tag{82}$$

$N = (g^{MN} \nabla_M r \nabla_N r)^{-\frac{1}{2}}$ is called the lapse function and N^M is the shift vector. The index of n_M and N^M goes up and down with h_{MN}. The external curvature K_{MN} is related with h_{MN} by

$$\begin{aligned} K_{MN} &= -\frac{1}{2} \mathcal{L}_n h_{MN} \\ &= -\frac{1}{2} (n^P \partial_P h_{MN} + \partial_M n^P h_{PN} + \partial_N n^P h_{PM}), \end{aligned} \tag{83}$$

in which $\mathcal{L}_n$ is the Lie derivative along the unit normal n^M.

B. Transport properties of QGP in D4 holographic QCD model

The surface stress tensor that we obtain is

$$T_{\mu\nu}^{\text{surf}} = \frac{1}{2\kappa_5^2} \left(\frac{1}{2} r_H^3 P_{\mu\nu} + \frac{5}{2} r_H^3 u_\mu u_\nu - 2 r_H^{\frac{5}{2}} \sigma_{\mu\nu} - \frac{4}{15} r_H^{\frac{5}{2}} \partial_\rho u^\rho P_{\mu\nu} \right). \tag{84}$$

Comparing with the result in relativistic hydrodynamics, we get

$$T_{\mu\nu}^{\text{hydro}} = p P_{\mu\nu} + \varepsilon u_\mu u_\nu - 2\eta \sigma_{\mu\nu} - \zeta \partial_\rho u^\rho P_{\mu\nu}, \tag{85}$$

where p, ε, η, and ζ are the momentum density, the energy density, shear viscosity, and bulk viscosity, respectively. We can get the respective hydrodynamical quantities for our system as

$$\begin{aligned} p &= \frac{1}{2\kappa_5^2} \frac{1}{2} r_H^3, \qquad \varepsilon = \frac{1}{2\kappa_5^2} \frac{5}{2} r_H^3, \\ \eta &= \frac{1}{2\kappa_5^2} r_H^{\frac{5}{2}}, \qquad \zeta = \frac{1}{2\kappa_5^2} \frac{4}{15} r_H^{\frac{5}{2}}. \end{aligned} \tag{86}$$

From Eq. (23) we can get the temperature for the 5D spacetime as

$$T = \frac{3 r_H^{\frac{1}{2}}}{4\pi}. \tag{87}$$

As one can easily see from the above two expressions, both the thermodynamic and the transport coefficients only depend on temperature. This is due to the setup of this model. One can also get the entropy density as

$$s = \frac{\varepsilon + p}{T} = \frac{1}{2\kappa_5^2} 4\pi r_H^{\frac{5}{2}}. \tag{88}$$

So the ratios of shear and bulk viscosity to entropy density are

$$\frac{\eta}{s} = \frac{1}{4\pi}, \qquad \frac{\zeta}{s} = \frac{1}{15\pi}. \tag{89}$$

Here we meet the renowned $1/4\pi$ again and this suggests that both the bulk of the D4-D6 model and SS model belong to the class in [67], as [66] has pointed out. The bulk to shear ratio is

$$\frac{\zeta}{\eta} = \frac{4}{15}, \tag{90}$$

which is also the same as in [66] and [68]. It is interesting to compare our result Eq. (90) with the results of Refs. [63,64] in which the bulk to shear viscosity ratio are both $1/10$. This is understandable since the case we considered here is the compactified near-horizon, nonextremal D4 brane in which the relativistic fluid resides only on $1+3$ dimensions out of the $1+4$ dimensional D4 brane's world volume. The spacetime here comes from dimensional reduction on $S^1 \times S^4$. But in Refs. [63,64], for a D4 brane, the submanifold that is reduced is the S^4 and the relevant hydrodynamics is $1+4$ dimensional. Another consistency with [66] is the sound speed that can be obtained via thermodynamic quantities:

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \frac{1}{5}. \tag{91}$$

As a self-consistent check, we calculate the dispersion relations by using the constituent relation Eq. (84) as in [40]. If considering the temperature $r_H(x)$ and 3-velocity $\beta_i(x)$ has fluctuations as

$$r_H(x) = r_H + \delta r_H e^{-i\omega v + i\vec{k}\cdot\vec{x}}, \qquad \beta_i(x) = \delta\beta_i e^{-i\omega v + i\vec{k}\cdot\vec{x}}, \tag{92}$$

one can get the relations of the fluctuations by putting the above equations into the EOM of boundary fluid, i.e., the conservation equation for $T^{\rm surf}_{\mu\nu}$:

$$\partial^\mu T^{\rm surf}_{\mu\nu} = 0. \tag{93}$$

Treating δr_H and $\delta\beta_i$ as first order quantities, one can get the linear equation for the fluctuations

$$\frac{5}{2}\omega\delta r_H - r_H k_i \delta\beta_i = 0, \tag{94}$$

$$\frac{3i}{2}k_i\delta r_H + (r_H^{1/2}\vec{k}^2 - 3ir_H\omega)\delta\beta_i + \frac{3}{5}r_H^{1/2}k_i k_j \delta\beta_j = 0. \tag{95}$$

In order to make the above equations have a nontrivial solution, the determinant of coefficients should be 0, which gives

$$\begin{aligned} \omega &= -\frac{i}{3r_H^{1/2}}\vec{k}^2, \qquad \text{shear mode} \\ \omega &= \pm\frac{1}{\sqrt{5}}|\vec{k}| - i\frac{4}{15r_H^{1/2}}\vec{k}^2 + \mathcal{O}(|\vec{k}|^3), \qquad \text{sound mode.} \end{aligned} \tag{96}$$

Comparing with the following results in hydrodynamics,

$$\begin{aligned} \omega &= -i\frac{\eta}{\varepsilon+p}\vec{k}^2, \qquad \text{shear mode} \\ \omega &= c_s|\vec{k}| - i\frac{\zeta+\frac{4}{3}\eta}{2(\varepsilon+p)}\vec{k}^2, \qquad \text{sound mode,} \end{aligned} \tag{97}$$

one can read the following relations:

$$\frac{\eta}{\varepsilon+p} = \frac{1}{3r_H^{1/2}}, \qquad c_s^2 = \frac{1}{5}, \qquad \frac{\zeta+\frac{4}{3}\eta}{2(\varepsilon+p)} = \frac{4}{15r_H^{1/2}}. \tag{98}$$

Comparing with the results in Eq. (86), we can find perfect consistency.

V. DISCUSSIONS AND OUTLOOK

We develop the BDE formalism of fluid/gravity correspondence in a compactified black D4-brane background and investigate the transport properties of its gauge-side dual gluonic matter. Compactified D4 branes are the background of the D4-D6 and the SS models, which are the two nonconformal top-down holographic QCD models. The SS model is a holographic model whose dual field theory lives on the world-volume of the flavor D8-branes; it is convenient to extract hadronic properties such as the meson and baryon spectrums from the SS model, since the Dirac-Born-Infeld (DBI) action of D8 branes describes some meson effective theory like the chiral perturbation theory (χ-PT). However, in the SS model, people focus more on the flavor sector, and may ignore the bulk sector. Our current work focuses on the bulk sector, i.e., the compactified black D4-brane background whose asymptotic region is not an AdS spacetime. Therefore, we choose the compactified black D4-brane background to describe nonconformal gluonic matter.

The strategy is to use the dimensional reduction technique on the compact structure of the SS model background in the Einstein frame, and one can get a 5D effective Einstein gravity minimally coupled with three scalars with exponential potentials. Following the standard BDE formalism of fluid/gravity correspondence, we derive the constituent relation and read the thermodynamical and hydrodynamical quantities such as the energy and momentum density and the shear and bulk viscosities. It is found that the ratio of bulk to shear viscosity and sound speed from our results are consistent with the previous studies on the transport properties of the SS model [66,68], which shows the validity of the BDE formalism of fluid/gravity correspondence in a nonconformal background. The calculation of second order transport coefficients are technically direct based on this work. What is more, this work offers us a nonconformal prototype in fluid/gravity duality that is in parallel with the AdS_5 construction of Bharttacharyya *et al.*, which provides us with the opportunity to study nonconformal systems with the 5D metric (23) given in this work.

As further applications, the most straightforward project is to calculate the second order transport coefficients of nonconformal gluonic matter by using the BDE formalism. The second order transport coefficients have been calculated in conformal systems with [43,44] or without [40,42] chemical potentials using the BDE formalism of fluid/gravity duality, and they have also been calculated via Green-Kubo formulas in both conformal [38] and nonconformal [72] systems. For the nonconformal Dp-brane backgrounds, the form of the second order viscous tensor has been predicted in [64].

One may also study the effective fluid on a cavity with finite $r = R$ using the compactified black D4-brane solution (not under the near-horizon limit) as in [50,51]. The most different point of the effective recipe from the present work is the dilaton; A and B should be boundary dependent since they both relate with H_4, and thus r_{Q4}. This is like the case in [51] where scalars also relate with the harmonic functions, but unlike that in [50], in which the scalar is only r dependent and the cutoff surface can be chosen as isodilatonic. Thus we should let those three scalars all have first order perturbations just like in [51], which may make us solve six equations in all for the scalar part perturbation.

Another interesting attempt in the future is to investigate nonconformal fluid with an axial chemical potential μ_5 via the SS model with smeared D0 charge on the D4-brane world volume [73]; this model can extract the axial chemical potential and the axial charge diffusion constant besides the hydrodynamical quantities in the present work. What is more, all the hydrodynamical quantities should be dependent on both temperature and axial chemical potential. We can also use the method developed in this work to investigate the newly found anomalous effects [74] (such as the chiral magnetic effect, chiral separation effect, and so on) analytically.

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APPENDIX: DIMENSIONAL REDUCTION FROM 10D TO 5D

We will do dimensional reduction on the following 10D diagonal background:

$$ds^2 = e^{2\alpha_1 A} g_{MN} dx^M dx^N + e^{2\alpha_2 A}(e^{2\beta_1 B} dy^2 + e^{2\beta_2 B}\gamma_{ab} d\theta^a d\theta^b). \tag{A1}$$

The nontrivial Christoffel symbols of this metric are (the ones with tildes are ten-dimensional components)

$$\begin{aligned}
&\tilde{\Gamma}^M_{NP} = \Gamma^M_{NP} + \alpha_1(\delta^M_N \partial_P A + \delta^M_P \partial_N A - g_{NP}\nabla^M A);\\
&\tilde{\Gamma}^M_{yy} = -(\alpha_2 \nabla^M A + \beta_1 \nabla^M B) e^{(-2\alpha_1+2\alpha_2)A+2\beta_1 B};\\
&\tilde{\Gamma}^y_{My} = \alpha_2 \partial_M A + \beta_1 \partial_M B;\\
&\tilde{\Gamma}^M_{ab} = -(\alpha_2 \nabla^M A + \beta_2 \nabla^M B) e^{(-2\alpha_1+2\alpha_2)A+2\beta_2 B}\gamma_{ab};\\
&\tilde{\Gamma}^a_{Mb} = (\alpha_2 \partial_M A + \beta_2 \partial_M B)\delta^a_b;\\
&\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc}.
\end{aligned} \tag{A2}$$

From the above results, we also have

$$\begin{aligned}
\tilde{\Gamma}^N_{MN} &= \Gamma^N_{MN} + 5\alpha_1 \partial_M A,\\
\tilde{\Gamma}^P_{MP} + \tilde{\Gamma}^y_{My} + \tilde{\Gamma}^a_{Ma} &= \Gamma^P_{MP} + (5\alpha_1 + 5\alpha_2)\partial_M A + (\beta_1 + 4\beta_2)\partial_M B,
\end{aligned} \tag{A3}$$

which can make our computation more convenient. The components of the Ricci tensors are

$$\begin{aligned}
\tilde{R}_{MN} = {}& R_{MN} - (3\alpha_1 + 5\alpha_2)\nabla_M \nabla_N A - (\beta_1 + 4\beta_2)\nabla_M \nabla_N B - \alpha_1 g_{MN} \nabla_P \nabla^P A\\
&+ (3\alpha_1^2 + 10\alpha_1\alpha_2 - 5\alpha_2^2)\partial_M A \partial_N A - (3\alpha_1^2 + 5\alpha_1\alpha_2) g_{MN}(\partial A)^2\\
&+ (\alpha_1\beta_1 - \alpha_2\beta_1 + 4\alpha_1\beta_2 - 4\alpha_2\beta_2)(\partial_M A \partial_N B + \partial_N A \partial_M B)\\
&- (\alpha_1\beta_1 + 4\alpha_1\beta_2) g_{MN} \partial_P A \partial^P B - (\beta_1^2 + 4\beta_2^2)\partial_M B \partial_N B;
\end{aligned} \tag{A4}$$

$$\tilde{R}_{yy} = -[\alpha_2\nabla^2 A + \beta_1\nabla^2 B + (3\alpha_1\alpha_2 + 5\alpha_2^2)(\partial A)^2 + (3\alpha_1\beta_1 + 6\alpha_2\beta_1 + 4\alpha_2\beta_2)\partial A\cdot\partial B + (\beta_1^2 + 4\beta_1\beta_2)(\partial B)^2]e^{(-2\alpha_1+2\alpha_2)A+2\beta_1 B}; \tag{A5}$$

$$\tilde{R}_{ab} = -[\alpha_2\nabla^2 A + \beta_2\nabla^2 B + (3\alpha_1\alpha_2 + 5\alpha_2^2)(\partial A)^2 + (3\alpha_1\beta_2 + \alpha_2\beta_1 + 9\alpha_2\beta_2)\partial A\cdot\partial B + (\beta_1\beta_2 + 4\beta_2^2)(\partial B)^2]e^{(-2\alpha_1+2\alpha_2)A+2\beta_2 B}\gamma_{ab} + 3\gamma_{ab}. \tag{A6}$$

Again, the components with tildes are ten-dimensional ones. Then we have the Ricci scalar,

$$R^{(10)} = [R - (8\alpha_1 + 10\alpha_2)\nabla^2 A - (2\beta_1 + 8\beta_2)\nabla^2 B - (12\alpha_1^2 + 30\alpha_1\alpha_2 + 30\alpha_2^2)(\partial A)^2 - 6(\alpha_1 + 2\alpha_2)(\beta_1 + 4\beta_2)\partial A\cdot\partial B - (2\beta_1^2 + 8\beta_1\beta_2 + 20\beta_2^2)(\partial B)^2]e^{-2\alpha_1 A} + 12e^{-2\alpha_2 A - 2\beta_2 B}. \tag{A7}$$

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