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$1/N$ perturbations in superstring bit models

Charles B. Thorn*

Institute for Fundamental Theory, Department of Physics, University of Florida, Gainesville, Florida 32611, USA

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We develop the $1/N$ expansion for stable string bit models, focusing on a model with bit creation operators carrying only transverse spinor indices $a = 1, \ldots, s$. At leading order ($N = \infty$), this model produces a (discretized) light cone string with a "transverse space" of s Grassmann worldsheet fields. Higher orders in the $1/N$ expansion are shown to be determined by the overlap of a single large closed chain (discretized string) with two smaller closed chains. In the models studied here, the overlap is not accompanied with operator insertions at the break/join point. Then, the requirement that the discretized overlap has a smooth continuum limit leads to the critical Grassmann "dimension" of $s = 24$. This "protostring," a Grassmann analog of the bosonic string, is unusual, because it has no large transverse dimensions. It is a string moving in one space dimension, and there are neither tachyons nor massless particles. The protostring, derived from our pure spinor string bit model, has 24 Grassmann dimensions, 16 of which could be bosonized to form 8 compactified bosonic dimensions, leaving 8 Grassmann dimensions—the worldsheet content of the superstring. If the transverse space of the protostring could be "decompactified," string bit models might provide an appealing and solid foundation for superstring theory.

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I. INTRODUCTION

When string theory is formulated in light cone coordinates ($x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$) [1], it is natural to propose that string is a composite of elementary entities called string bits [2,3]. Incorporating supersymmetry in string bit models leads to the proposal [4] that the superstring bit creation operator has the structure

$$(\bar{\phi}_{[a_1\cdots a_n]})^{\beta}_{\alpha}(\boldsymbol{x}), \qquad a_i = 1, \ldots, s,$$
$$n = 0, \ldots, s, \qquad \alpha, \beta = 1, \ldots, N, \tag{1}$$

where $\boldsymbol{x}$ denotes the transverse coordinates of the light cone and the square brackets in the subscript remind us that the a_i's are completely antisymmetric. The a_i's are spinor indices of the transverse rotation group, and α, β are color indices, which are introduced to formulate a dynamics that favors the formation of long (closed) chains of string bits. The bit number operator $M = \sum_n \mathrm{tr}(\bar{\phi}_{[a_1\cdots a_n]}\phi_{[a_1\cdots a_n]}/n!)$ is naturally identified with the "+" component of momentum $P^+ = (P^0 + P^1)/\sqrt{2} = mM$. Chains with $M \to \infty$ would then become continuous closed strings. It is central to the string bit hypothesis that string bits are not *a priori* confined in chains but that chain formation is a consequence of the dynamics. Such dynamics will arise generically in the 't Hooft $N \to \infty$ limit [5]. In this original formulation of string bit models, there is an aspect of 't Hooft's idea of holography [6] in that the fundamental string bits only move in the transverse space, while the strings behave as though moving in transverse space plus an extra spatial dimension x^-, the canonical conjugate of P^+.

Recently, we have noted that the transverse coordinates are extraneous, and this led to the proposal that the bits have no space to move at all [7]. This proposal is a rather more drastic form of holography in which all space dimensions, and not just the longitudinal one, emerge from the dynamics of string formation from string bits [8]. The idea is that, with suitable dynamics, some spin degrees of freedom carried by the string bit can, on long chains, fluctuate collectively as one-dimensional spin waves. Such spin waves are well known to act as a compactified bosonic coordinate. In these two papers, the string bit creation operator is then taken to be the simpler

$$(\bar{\phi}_{a_1\cdots a_n})^{\beta}_{\alpha}, \qquad a_i = 1, \ldots, s,$$
$$n = 0, \ldots, s, \qquad \alpha, \beta = 1, \ldots, N, \tag{2}$$

where, here and from now on, we suppress the square brackets around the spinor indices. In Ref. [8], a further set of "flavor" indices is appended to the ϕ's to serve as the seed for the transverse coordinates. However, we refrain from adding them here, because we hope that the seeds for transverse space can somehow be found by enlarging the value of s. Fluctuations in the a's produce,

*thorn@phys.ufl.edu

on long chains, the Grassmann worldsheet fields $\theta^a_{L,R}$ of the Green-Schwarz type [9]. If $s = 24$, eight of the θ's could take the role of the superstring Grassmann fields, but the remaining 16 could be bosonized into 8 (compactified) transverse coordinates.

In this article, our aim is to study, by perturbing in $1/N$, not only the precise manner in which string bit dynamics lead to the free superstring spectrum at $N = \infty$ but also how the $1/N$ corrections lead to the three string interaction vertex of string theory. We will start in Sec. II by setting up the systematic $1/N$ perturbation expansion in string bit models. We will see that the structure of this perturbation theory follows that of Mandelstam's interacting string diagrams [10]. Then, we proceed to apply this formalism to our pure spinor string bit model, which we also call the protostring bit model. In Sec. III, we obtain the exact zeroth order spectrum (at $N = \infty$). In Sec. IV, we discuss the calculation of the overlap that describes the cubic vertex. We formulate the overlap calculation for finite bit number chains and then discuss the continuum limit in which the bit numbers of each string tend to infinity at fixed ratio. These results are aided by numerical calculations using MATLAB. We compare our conclusions with those in the literature for various situations. In Appendix C, we present an analytic computation of the overlap in the continuum limit. Then, we use this result to determine how various operator insertions behave when inserted at the break/join point of the string to two string transition. Finally, in Sec. V, we put everything together to construct the total vertex, respecting the requirement that the physical amplitudes have a finite continuum limit. As seen in Ref. [2], this last requirement determines the critical dimension. We list and compare the ways in which this requirement is met for the bosonic string, for the IIB superstring [9,11], for the Ramond-Neveu-Schwarz (RNS) string [12], and finally for the new "protostring" which is the outcome of our pure spinor superstring bit model. We close with a discussion of the properties of the protostring, which has 24 Grassmann dimensions, has no transverse bosonic dimensions, and is expected to have a spectrum with no massless particles, i.e. that it possesses a mass gap. We include three Appendixes containing technical details to supplement the main text.

II. $1/N$ EXPANSION

The string bit model we focus on in this article takes as fundamental variables the creation operators of (2). Their (anti)commutation relations are given in Appendix A. We shall keep s a general positive integer, and we shall analyze the Hamiltonian H_S given in Ref. [8] and quoted in detail [see Eq. (A7)] in Appendix A, along with its action on color singlet states. To guide the reader's eye, we display here the $s = 1$ Hamiltonian:

$$H^{s=1} = \frac{2}{N}\mathrm{Tr}[(\bar{a}^2 - i\bar{b}^2)a^2 - (\bar{b}^2 - i\bar{a}^2)b^2 + (\bar{a}\,\bar{b} + \bar{b}\,\bar{a})ba + (\bar{a}\,\bar{b} - \bar{b}\,\bar{a})ab]. \tag{3}$$

In this special case, there is one bosonic bit $\phi = a$ and one fermionic bit $\phi_1 = b$. The Hamiltonian for general s is a good deal more complex.

For the analysis to follow, it will be convenient to introduce Grassmann coordinates θ^a, $a = 1, \ldots, s$ and define a super bit creation operator

$$\psi(\theta) = \sum_{k=0}^{s} \frac{1}{k!}\bar{\phi}_{c_1\cdots c_k}\theta^{c_1}\cdots\theta^{c_k}. \tag{4}$$

The $\bar{\phi}$ can be recovered from ψ recursively by taking multiple Grassmann derivatives, starting with s derivatives which singles out $\bar{\phi}_{a_1\cdots a_s}$, Then, single out $\phi_{a_1\cdots a_{s-1}}$ by applying $s-1$ derivatives on ψ minus the contribution of $\bar{\phi}_{a_1\cdots a_s}$, and so on. To work on the color singlet subspace of Fock space, we define an empty state $|0\rangle$ and the set of trace operators

$$T(\theta_1, \ldots, \theta_k) = \mathrm{Tr}\psi(\theta_1)\cdots\psi(\theta_k), \tag{5}$$

where the θ's are s-component Grassmann variables. Then, the color singlet subspace is spanned by states of the form

$$T(\theta_1, \ldots, \theta_K)T(\eta_1, \ldots, \eta_L)\cdots|0\rangle. \tag{6}$$

A. Action of H on multitrace states

In Appendix A, we present the Hamiltonian as the sum of five terms and give the action of each term on color singlet states.. We can summarize the action of $H = \sum_{i=1}^{5} H_i$ on multitrace states by defining

$$\bar{h}_{kl} = 2\left(s - 2\theta^a_k\frac{d}{d\theta^a_k}\right) + 2\theta^a_k\frac{d}{d\theta^a_l} + 2\theta^a_l\frac{d}{d\theta^a_k} - 2i\theta^a_k\theta^a_l - 2i\frac{d}{d\theta^a_k}\frac{d}{d\theta^a_l} \tag{7}$$

$$\bar{h} = \sum_{k=1}^{M}\bar{h}_{k,k+1}. \tag{8}$$

Then, there follows

$$HT(\theta_1\cdots\theta_M)|0\rangle = \bar{h}T(\theta_1\cdots\theta_M)|0\rangle + \frac{1}{N}\sum_{k,l\neq k,k+1}\bar{h}_{kl}T(\theta_l\cdots\theta_k) \times T(\theta_{k+1}\cdots\theta_{l-1})|0\rangle \tag{9}$$

$$HT(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)|0\rangle = (\bar{h}_\theta+\bar{h}_\eta)T(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)+\frac{1}{N}\text{Fission Terms}$$
$$+\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\bar{h}_{kl}T(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})|0\rangle$$
$$+\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\bar{h}_{lk}T(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)|0\rangle. \tag{10}$$

In the development of perturbation theory, we shall transfer the derivatives of $\bar{h}_{kl}$ to act on the coefficient amplitude multiplying each multitrace state, whence it will take the form

$$h_{kl} = -2\left(s-2\theta_k^a\frac{d}{d\theta_k^a}\right)-2\theta_k^a\frac{d}{d\theta_l^a}-2\theta_l^a\frac{d}{d\theta_k^a}-2i\theta_k^a\theta_l^a-2i\frac{d}{d\theta_k^a}\frac{d}{d\theta_l^a} \tag{11}$$

$$h=\sum_{k=1}^{M}h_{k,k+1}. \tag{12}$$

We shall also make use of Grassmann variables that satisfy a Clifford algebra:

$$S_k^a=\theta_k^a+\frac{d}{d\theta_k^a}, \qquad \tilde{S}_k^a=i\left(\theta_k^a-\frac{d}{d\theta_k^a}\right) \tag{13}$$

$$\{S_k^a,S_l^b\}=2\delta_{kl}\delta_{ab}, \qquad \{\tilde{S}_k^a,\tilde{S}_l^b\}=2\delta_{kl}\delta_{ab}, \qquad \{S_k^a,\tilde{S}_l^b\}=0. \tag{14}$$

Then, h_{kl} becomes

$$h_{kl}=-iS_k^aS_l^a+i\tilde{S}_k^a\tilde{S}_l^a-iS_k^a\tilde{S}_l^a+i\tilde{S}^a{}_kS_l^a+2iS_k^a\tilde{S}_k^a. \tag{15}$$

B. Systematic perturbation theory

We develop the $1/N$ expansion on Fock space, following the methods of Ref. [13]. At zeroth order, the first task is to solve the eigenvalue problem

$$h\psi_r(\theta_1,\ldots,\theta_{M_r})=E_r\psi_r(\theta_1,\ldots,\theta_{M_r}), \tag{16}$$

and then we change the single trace operators to energy basis

$$T_r=\int d^s\theta_1\cdots d^s\theta_{M_r}T(\theta_1,\ldots,\theta_{M_r})\psi_r(\theta_1,\ldots,\theta_{M_r}). \tag{17}$$

Because the T are cyclically symmetric, we may assume that the ψ_r satisfy the cyclic property

$$\psi_r(\theta_1,\ldots,\theta_{M_r})=(-)^{s(M_r-1)}\psi_r(\theta_2,\ldots,\theta_{M_r},\theta_1). \tag{18}$$

The potential minus sign is due to the fact that if s and M_r-1 are odd, the cyclic transform of the measure acquires a minus sign.

Define the conjugate to ψ_r, denoted $\bar{\psi}_r$, such that

$$\int d\theta_1\cdots d\theta_{M_r}\bar{\psi}_s(\theta_1,\ldots,\theta_{M_r})\psi_r(\theta_1,\ldots,\theta_{M_r})=\delta_{rs}, \tag{19}$$

and so the completeness relation is written

$$\sum_r \psi_r(\theta_1,\ldots,\theta_{M_r})\bar{\psi}_r(\phi_1,\ldots,\phi_{M_r}) = \delta(\theta-\phi), \tag{20}$$

where the delta function is understood to be symmetrized under cyclic permutations. In the energy basis, the action of H on a single trace state becomes

$$\begin{aligned} HT_r|0\rangle &= E_rT_r|0\rangle + \frac{1}{N}\int d\theta \sum_{l\neq k,k+1} \bar{h}_{kl}T(\theta_l\cdots\theta_k)T(\theta_{k+1}\cdots\theta_{l-1})|0\rangle\psi_r(\theta_1,\ldots,\theta_M) \\ &\equiv E_rT_r|0\rangle + \frac{1}{N}\sum_{s,t}T_sT_t|0\rangle V_{\rm str} \\ V_{\rm str} &= \sum_{l\neq k,k+1}\int d\theta\bar{\psi}_s(\theta_l\cdots\theta_k)\bar{\psi}_t(\theta_{k+1}\cdots\theta_{l-1})h_{kl}\psi_r(\theta_1,\ldots,\theta_M). \end{aligned} \tag{21}$$

We see that h_{kl} acts to the left on the eigenfunction $\bar{\psi}_s$, in which k, l label nearest neighbor pairs of θ's. We can normal order h_{kl} and get the normal ordering constant by calculating

$$\alpha_{kl} = \langle\psi_G|h_{kl}|\psi_G\rangle = \frac{1}{M_s}\langle\psi_G|h|\psi_G\rangle = \frac{E_G}{M_s}, \tag{22}$$

where the second equality follows because $\bar{\psi}_G$ is cyclically invariant. Thus, each term of $h=\sum_k h_{k,k+1}$ contributes an equal amount. In the continuum limit, $E_G \sim \alpha M_s + O(1/M_s)$, so that in this limit $\alpha_{kl}=\alpha$. Thus, we have

$$\langle\psi_s|h_{kl} = \langle\psi_s|(:h_{kl}: + \alpha). \tag{23}$$

The terms in the operator $:h_{kl}:$ are nominally of order M_s^{-1}, and so they nominally vanish in the continuum limit. However, it can be shown (see Appendix C) that the energy lowering components of S_k, S_l, $\tilde{S}_k$, or $\tilde{S}_l$, nominally of order $M_s^{-1/2}$, acting to the right give a Grassmann odd factor S of order 1 in the continuum limit (independently of which operator is chosen). In other words, the singularity at the joining point can produce a factor $M_s^{1/2}$ for each S_k. Thus, the terms in $:h_{kl}:$ with two such lowering operators can potentially contribute at order 1. Happily, the contribution is $S^2=0$ because S is Grassmann odd. Thus, in the continuous string limit of our model, the vertex is a pure overlap with no operator insertions at the joining point:

$$V_{\rm str} \sim \alpha \sum_{l\neq k,k+1}\int d\theta\bar{\psi}_s(\theta_l\cdots\theta_k)\bar{\psi}_t(\theta_{k+1}\cdots\theta_{l-1})\psi_r(\theta_1,\ldots,\theta_M). \tag{24}$$

The fission operation on any multitrace state acts on each trace factor just as shown above. On multitrace states, the Hamiltonian can also fuse any pair of traces into one as follows:

$$HT_sT_t|0\rangle \equiv (E_s+E_t)T_sT_t|0\rangle + \frac{1}{N}T_r|0\rangle W_{\rm rst} + \frac{1}{N}(T_uT_vT_tV_{uvs} + T_sT_uT_vV_{uvt})|0\rangle. \tag{25}$$

The second term on the right is the fusion term arising from

$$\begin{aligned} T_r|0\rangle W_{\rm rst} = \int d\theta d\phi \sum_{k=1}^{M_s}\sum_{l=1}^{M_t}[&\bar{h}_{kl}T(\theta_{k+1}\cdots\theta_k\phi_l\cdots\phi_{l-1}) \\ &+\bar{h}_{lk}T(\phi_{l+1}\cdots\phi_l\theta_k\cdots\theta_{k-1})]\psi_s(\theta_1\cdots\theta_{m_s})\psi_t(\phi_1\cdots\phi_{M_t}), \end{aligned} \tag{26}$$

from which we infer

$$W_{\rm rst} = \sum_{k=1}^{M_s}\sum_{l=1}^{M_t}\int d\theta d\phi[\bar{\psi}_r(\theta_{k+1}\cdots\theta_k\phi_l\cdots\phi_{l-1})h_{kl}\psi_s(\theta_1\cdots\theta_{M_s})\psi_t(\phi_1\cdots\phi_{M_t}) + \bar{\psi}_r(\phi_{l+1}\cdots\phi_l\theta_k\cdots\theta_{k-1})h_{lk}\psi_s(\theta_1\cdots\theta_{M_s})\psi_t(\phi_1\cdots\phi_{M_t})]. \tag{27}$$

Again, in the continuum limit, h_{kl} and h_{lk} can be replaced by α, in which case the two terms are equal, giving a net factor of 2.

The double sums in V and W have the simple interpretation of including all ways of splitting a chain in two or of joining two chains into one. In the first case, one picks two bits where the split takes case. In the second case, one must pick a bit on each chain where the two chains join. Since these events can happen with any pair of bits, one must sum over all choices. In the continuum limit, these double sums should go over to double integrals $\sum_{k,l} \to (1/m^2)\int d\sigma d\sigma'$, and since the factor α includes a factor $(1/m)$, the overlap should supply a factor of $1/M^3$ to get a finite continuum limit.

As an application, consider the energy eigenvalue problem in perturbation theory. Start by expanding the sought eigenstate in trace states,

$$|E\rangle = \sum_r T_r|0\rangle C_r^1 + \sum_{st} T_s T_t|0\rangle C_{st}^2 + \sum_{stu} T_s T_t T_u|0\rangle C_{stu}^3 + \ldots, \tag{28}$$

and require that $(H-E)|E\rangle = 0$:

$$0 = \sum_r (E_r - E)T_r|0\rangle C_r^1 + \sum_{st}(E_s + E_t - E)T_sT_t|0\rangle C_{st}^2 + \sum_{stu}(E_s+E_t+E_u-E)T_sT_tT_u|0\rangle C_{stu}^3 + \frac{1}{N}\sum_{\rm str} T_sT_t|0\rangle V_{\rm str}C_r^1 + \frac{1}{N}\sum_{\rm rst} T_r|0\rangle W_{\rm rst}C_{st}^2 + \ldots. \tag{29}$$

Then, equating coefficients of like terms, we have the sequence of equations

$$(E_r - E)C_r^1 + \frac{1}{N}\sum_{st} W_{\rm rst}C_{st}^2 = 0 \tag{30}$$

$$(E_s + E_t - E)C_{st}^2 + \frac{1}{N}\sum_r V_{\rm str}C_r^1 + C^3\text{Terms} = 0, \tag{31}$$

and so on. For example, we can choose the C_r^1 with common E_r to be nonzero at zeroth order and all other C's zero at zeroth order. Then, the C_3 terms in the second equation are of order $1/N^2$, so we obtain

$$C_{st}^2 = \frac{1}{E - E_s - E_t}\frac{1}{N}\sum_r V_{\rm str}C_r^1 + O(N^{-2}) \tag{32}$$

$$(E - E_r)C_r^1 = \frac{1}{N^2}\sum_{st} W_{\rm rst}\frac{1}{E - E_s - E_t}\sum_u V_{\rm stu}C_u^1 + O(N^{-3}). \tag{33}$$

In the $M \to \infty$ limit, when the energy eigenvalues become continuous, the first equation can be interpreted as the amplitude for a single string to decay into two strings. For any finite M, the second equation shows that the eigenvalues of the matrix

$$\Delta_{ru} = \frac{1}{N^2}\sum_{st} W_{\rm rst}\frac{1}{E_r - E_s - E_t}V_{\rm stu}, \qquad E_u = E_r \tag{34}$$

determine the lowest order energy shifts to the level E_r.

III. DIAGONALIZING h

In the preceding section, we have shown how the $1/N$ expansion of string bit models is determined by what we might call first quantized string calculations. The legacy of the underlying string bit models for these calculations is essentially the provision of a fundamental cutoff, namely the interpretation of a continuous P^+ by the discrete bit number. Finding the eigenvalues of h is straightforward, because h is bilinear in Clifford variables. Therefore, it can be solved by finding energy raising and lowering operators. This was done in Ref. [4], so we just give here a quick sketch of the procedure and results. Because h is the sum of terms h_a, each of which contain only the variables with component a, it suffices to work with just one component. In the following, we suppress the spinor index.

To begin, it is convenient to introduce Fourier transforms:

$$B_n = \frac{1}{\sqrt{M}}\sum_{k=1}^{M} S_k e^{-2\pi i k n/M}, \qquad \tilde{B}_n = \frac{1}{\sqrt{M}}\sum_{k=1}^{M} \tilde{S}_k e^{-2\pi i k n/M} \tag{35}$$

$$S_k = \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} B_n e^{2\pi i k n/M}, \qquad \tilde{S}_k = \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} \tilde{B}_n e^{2\pi i k n/M} \tag{36}$$

$$\{B_m, B_n\} = 2\delta_{m+n,M}, \qquad \{\tilde{B}_m, \tilde{B}_n\} = 2\delta_{m+n,M}, \qquad \{\tilde{B}_m, B_n\} = 0. \tag{37}$$

Then, we can express h in terms of these:

$$h = \sum_{n=1}^{M-1}\left[-iB_{M-n}B_n e^{2\pi i n/M} + i\tilde{B}_{M-n}\tilde{B}_n e^{2\pi i n/M} + 2iB_{M-n}\tilde{B}_n\left(1-\cos\frac{2\pi n}{M}\right)\right]. \tag{38}$$

We search for eigenoperators:

$$\begin{aligned}[h, B_n + \xi\tilde{B}_n] &= B_n\left(-4\sin\frac{2\pi n}{M} + 4i\xi\left(1-\cos\frac{2\pi n}{M}\right)\right) + \tilde{B}_n\left(4\xi\sin\frac{2\pi n}{M} - 4i\left(1-\cos\frac{2\pi n}{M}\right)\right)\\ &\equiv \Delta(B_n + \xi\tilde{B}_n).\end{aligned} \tag{39}$$

This implies

$$1 + \xi^2 + 2i\xi\cot\frac{\pi n}{M} = 0. \tag{40}$$

Solving the quadratic equation gives

$$\begin{aligned}\xi_\pm &= -i\cot\frac{\pi n}{M} \pm i\sqrt{1+\cot^2\frac{\pi n}{M}} = \begin{cases} i\tan\frac{\pi n}{2M} \\ -i\cot\frac{\pi n}{2M}\end{cases}\\ \Delta_\pm &= -4\sin\frac{2\pi n}{M} + 4i\xi_\pm\left(1-\cos\frac{2\pi n}{M}\right) = 8\sin\frac{\pi n}{M}\left(-\cos\frac{\pi n}{M} + i\xi_\pm\sin\frac{\pi n}{M}\right) = \mp 8\sin\frac{\pi n}{M}.\end{aligned} \tag{41}$$

We therefore define energy lowering operators,

$$F_n = B_n\cos\frac{\pi n}{2M} + i\tilde{B}_n\sin\frac{\pi n}{2M}, \tag{42}$$

and raising operators,

$$\bar{F}_n = B_n\sin\frac{\pi n}{2M} - i\tilde{B}_n\cos\frac{\pi n}{2M}, \tag{43}$$

which can be inverted,

$$B_n = F_n \cos\frac{\pi n}{2M} + \bar{F}_n \sin\frac{\pi n}{2M}, \qquad i\tilde{B} = F_n \sin\frac{\pi n}{2M} - \bar{F}_n \cos\frac{\pi n}{2M}. \tag{44}$$

We notice that

$$F_n^\dagger = B_{M-n} \cos\frac{\pi n}{2M} - i\tilde{B}_{M-n} \sin\frac{\pi n}{2M} = \bar{F}_{M-n} \tag{45}$$

$$\{F_n, \bar{F}_m\} = 2\sin\frac{\pi n}{2M}\cos\frac{\pi m}{2M}\delta_{n+m,M} + 2\cos\frac{\pi n}{2M}\sin\frac{\pi m}{2M}\delta_{n+m,M} = 2\delta_{n+m,M}. \tag{46}$$

Applying h to a state satisfying $F_n|G\rangle = 0$ for all n leads to a calculation of the ground energy. Remembering there is a contribution for each of the s components, we find

$$E_G = -4s\sum_{n=1}^{M-1} \sin\frac{n\pi}{M} \sim -\frac{8s}{\pi}M + \frac{2\pi s}{3M} + O(M^{-3}). \tag{47}$$

Since $M = P^+/m$ is conserved in all processes, it can be harmlessly subtracted, and we can identify the string tension by comparing the $1/M$ term to the string P_G^-:

$$P_G^- = \frac{T_0}{4m}\left(E_G + \frac{8s}{\pi}M\right) \sim \frac{\pi s T_0}{6P^+}(1 + O(M^{-2})). \tag{48}$$

At $N = \infty$, the lowest squared mass in this model is $s\pi T_0/3 > 0$; i.e., there is a mass gap.

IV. THREE CLOSED BIT CHAIN OVERLAP

As seen in Sec. II, the terms in the $1/N$ expansion are determined by the overlap integrals V_{rst} and W_{rst}. Let us focus on the second of these. We can calculate it in the raising and lowering operator formalism by expressing the ground state $|G\rangle$ of the large string in terms of raising and lowering operators of the two small strings, applied to the tensor product of the ground states of the small strings.

Divide the M spin variables into L ($k = 1, \ldots L$) and $K = M - L$ ($k = L+1, \ldots, M$) variables. Then, for each subset, we define modes

$$S_k = \frac{1}{\sqrt{L}}\sum_{n=0}^{L-1} B_n^{(1)} e^{2\pi ikn/L}, \qquad 1 \le k \le L \tag{49}$$

$$S_k = \frac{1}{\sqrt{K}}\sum_{n=0}^{K-1} B_n^{(2)} e^{2\pi i(k-L)n/(M-L)}, \qquad L+1 \le k \le M, \tag{50}$$

and likewise for $\tilde{S}_k$, putting a tilde on the corresponding B's. Then, introduce the vectors

$$v_m^k = \frac{1}{\sqrt{M}} e^{2\pi ikm/M}, \qquad k = 1, \ldots, M; \qquad m = 0, \ldots, M-1 \tag{51}$$

$$v_n^{(1)k} = \frac{1}{\sqrt{L}} e^{2\pi ikm/L}, \qquad k = 1, \ldots, L; \qquad n = 0, \ldots, L-1 \tag{52}$$

$$v_n^{(2)k} = \frac{1}{\sqrt{K}} e^{2\pi i(k-L)m/K}, \qquad k = L+1, \ldots, M; \qquad n = 0, \ldots, M-L-1. \tag{53}$$

Then, the B_n, $\tilde{B}_n$ are related to the $B_n^{(1)}$, $\tilde{B}_n^{(1)}$ and $B_n^{(2)}$, $\tilde{B}_n^{(2)}$ by

$$B_m = \sum_{n=0}^{L-1} B_n^{(1)} v_m^\dagger v_n^{(1)} + \sum_{n=0}^{M-L-1} B_n^{(2)} v_m^\dagger v_n^{(2)} \tag{54}$$

$$\tilde{B}_m = \sum_{n=0}^{L-1} \tilde{B}_n^{(1)} v_m^\dagger v_n^{(1)} + \sum_{n=0}^{M-L-1} \tilde{B}_n^{(2)} v_m^\dagger v_n^{(2)}. \quad (55)$$

Zero modes require special attention. First, we note that

$$B_0 = B_0^{(1)}\sqrt{\frac{L}{M}} + B_0^{(2)}\sqrt{\frac{K}{M}},$$
$$\tilde{B}_0 = \tilde{B}_0^{(1)}\sqrt{\frac{L}{M}} + \tilde{B}_0^{(2)}\sqrt{\frac{K}{M}}. \quad (56)$$

It is then convenient to define the relative zero mode operators

$$b_0 = B_0^{(1)}\sqrt{\frac{K}{M}} - B_0^{(2)}\sqrt{\frac{L}{M}},$$
$$\tilde{b}_0 = \tilde{B}_0^{(1)}\sqrt{\frac{K}{M}} - \tilde{B}_0^{(2)}\sqrt{\frac{L}{M}}, \quad (57)$$

and it is easy to confirm the Clifford algebra

$$\{B_0, B_0\} = \{\tilde{B}_0, \tilde{B}_0\} = \{b_0, b_0\} = \{\tilde{b}_0, \tilde{b}_0\} = 2$$
$$\{B_0, b_0\} = \{\tilde{B}_0, \tilde{b}_0\} = \{B_0, \tilde{b}_0\} = \{b_0, \tilde{B}_0\}$$
$$= \{B_0, \tilde{B}_0\} = \{b_0, \tilde{b}_0\} = 0. \quad (58)$$

We can now rewrite the overlap conditions as

$$B_m = b_0 v_m^\dagger w_0 + \sum_{n=1}^{L-1} B_n^{(1)} v_m^\dagger v_n^{(1)} + \sum_{n=1}^{M-L-1} B_n^{(2)} v_m^\dagger v_n^{(2)} \quad (59)$$

$$\tilde{B}_m = \tilde{b}_0 v_m^\dagger w_0 + \sum_{n=1}^{L-1} \tilde{B}_n^{(1)} v_m^\dagger v_n^{(1)} + \sum_{n=1}^{M-L-1} \tilde{B}_n^{(2)} v_m^\dagger v_n^{(2)} \quad (60)$$

$$w_0 \equiv v_0^{(1)}\sqrt{\frac{K}{M}} - v_0^{(2)}\sqrt{\frac{L}{M}}. \quad (61)$$

Finally, in order to calculate the three chain vertex, we relate the energy lowering operators for the large chain F_m for $m \neq 0$ to the raising and lowering operators for the smaller chains. The zero mode operators commute with h, but it is convenient to define $f_0 = (b_0 + i\tilde{b}_0)/2$ and $\bar{f}_0 = f_0^\dagger = (b_0 - i\tilde{b}_0)/2$ which satisfy

$$f_0^2 = \bar{f}_0^2 = 0, \qquad \{f_0, \bar{f}_0\} = 1. \quad (62)$$

Let f_n be the $M-1$ operators f_0, $F_n^{(1)}/\sqrt{2}$, $F_n^{(2)}/\sqrt{2}$, so that $\{f_m, f_n^\dagger\} = \delta_{mn}$, and we can write

$$F_m = \sqrt{2}\sum_{n=0}^{M-2}(f_n C_{mn} + f_n^\dagger S_{mn}), \quad (63)$$

where the matrices C, S are given in Appendix B.

Then, we seek the ground state of the large chain in the form

$$|G\rangle = \exp\left\{\frac{1}{2}\sum_{k,l} D_{kl} f_k^\dagger f_l^\dagger\right\}|0\rangle[\det(I + DD^\dagger)]^{-1/4}, \quad (64)$$

where $f_k|0\rangle = 0$ for $k = 0, \ldots, M-2$. $F_m|G\rangle = 0$ is equivalent to

$$C_{mn}D_{nl} + S_{ml} = 0. \quad (65)$$

From $CD = -S$, we compute

$$C(I + DD^\dagger)C^\dagger = CC^\dagger + SS^\dagger$$
$$\det C \det(I + DD^\dagger) \det C^\dagger = \det(CC^\dagger + SS^\dagger)$$
$$\det(I + DD^\dagger) = \frac{\det(CC^\dagger + SS^\dagger)}{\det(CC^\dagger)}. \quad (66)$$

Using MATLAB to study these determinants numerically, we find that $\det(CC^\dagger + SS^\dagger) = 1$, and we also confirm the behavior

$$\det CC^\dagger \sim \frac{0.9290}{[KLM]^{1/6}}\left(\frac{L}{M}\right)^{[M/K-L/M]/3-2/3} \times \left(\frac{K}{M}\right)^{[M/L-K/M]/3-2/3}. \quad (67)$$

Here, $K = M - L$ is the number of bits in one of the smaller strings. It is interesting to compare this determinant for the Grassmann overlap to the corresponding one for a single bosonic string coordinate,

$$\det XX^\dagger = \frac{2.1528}{[KLM]^{1/6}}\left(\frac{L}{M}\right)^{-[M/K-L/M]/3} \times \left(\frac{K}{M}\right)^{-[M/L-K/M]/3}, \quad (68)$$

which was also calculated numerically with MATLAB. This bosonic determinant, apart from the numerical factor, can be understood based on the conformal mapping properties of the worldsheet [10,14]. There is an intimate relation between the Grassmann and bosonic overlaps reflected in the fact that the superstring overlap involves the product of the two,

$$\det CC^\dagger \det XX^\dagger = \frac{2.0000}{[KLM]^{1/3}}\left(\frac{KL}{M^2}\right)^{-2/3}$$
$$= 2.0000\frac{M}{KL}, \quad (69)$$

in which the combination is greatly simplified. This simplification is the content of the Green-Schwarz statement that the bosonic and spinor worldsheet determinants cancel each other; it is associated with the dependence of an off-shell vertex on the interaction time $e^{-ia\Delta P^-}$. The measure contribution to ΔP^- is

$$\frac{s-d}{6}\Delta\frac{1}{2P^+}\to 0 \tag{70}$$

for $d\to s$, which is the supersymmetry requirement. It is important to appreciate that the cancellation is actually incomplete, and moreover the part left over is essential to account for the eventual Poincaré invariance.

Of course, the string bit model studied in this article produces no bosonic coordinates but only Grassmann ones. As such, the requirement that the vertex has a finite continuum limit, i.e., that it scales as M^{-3} at large M with L/M, K/M fixed, determines $s=24$. We call this interesting string model the protostring.

V. PROTOSTRING THEORY

To summarize our work, we have found that the Grassmann overlap scales as $M^{-s/8}$ if there are s Grassmann worldsheet fields. The scaling of the bosonic overlap is $M^{-d/8}$ for d transverse worldsheet scalars. If one combines these, one gets $M^{-(s+d)/8}$. With no operator insertions at the break/join point, the smooth continuum limit would require $s+d=24$. We should stress that the smooth continuum limit requirement will not necessarily ensure Poincaré invariance based on the Lorentz group $SO(d+1,1)$. The bosonic string has no Grassmann worldsheet fields, so the critical dimension should be $d=24$, which in this case is enough for Poincaré invariance. The superstring has $s=d=8$ which does not give a smooth continuum limit. But we also know that the superstring requires an operator insertion proportional to $\Delta X^i\Delta X^j$ at the joining point. This insertion produces an additional factor M^{-1}, which combined with the $s=d=8$ overlaps ensures a smooth continuum limit. Poincaré supersymmetry requires a further insertion of an eighth order Grassmann polynomial $P_{ij}(S)$ which, as shown in Appendix C, has no effect on the overall scaling behavior.

The bosonic string, although inherently unstable, has played an important role in the formal string literature, because of the economy and simplicity of its interactions—reflected in the absence of operator insertions.. We now see that there are several other possibilities that do not require insertions, namely a transverse worldsheet system with $d\le 24$ bosonic and $24-d$ Green-Schwarz fermionic worldsheet fields. The simple string bit model, with $s=24$, analyzed in this paper, produces the $d=0$ model on this list. This is a pure Grassmann analog of the bosonic string and as such should be of particular interest in string theory. The superstring and the RNS string both require operator insertions at the vertex break/join point. In order to compare the various possibilities, we need to know the scaling laws of various insertions. And for these, we need to know the overlap for the excited states. The matrix D is well known in the continuum limit, and with that knowledge, one can obtain the needed scaling laws. In Appendix C, we discuss these issues for insertions of S variables, with the conclusion that they scale as M^0. To compare to the other major possibilities, we have prepared tables of the various scaling laws.

TABLE I. Enhancement of scaling laws for operator insertions on overlaps of the bosonic (ΔX), Green-Schwarz (S), and RNS (Γ) types.

Insertion	Enhancement	Net
ΔX	$M^{1/2}$	$M^{-1/2}$
S	$M^{1/2}$	M^0
Γ	$M^{1/4}$	$M^{-1/4}$

We first note the nominal scaling rules for insertions. For bosonic variables, the insertion $\Delta X=X_{k+1}-X_k\sim m\frac{\partial X}{\partial\sigma}$ nominally scales as $\Delta X\sim M^{-1}$. Similarly, S_k, $\Gamma_k\sim\sqrt{m}(S(\sigma),\Gamma(\sigma))$ nominally scales as S_k, $\Gamma_k\sim M^{-1/2}$. Here, Γ^k are the RNS fermionic worldsheet field. However, the fission/fusion singularity enhances these expectations, as illustrated in Table I. Note that the enhancement is different in the RNS and Green-Schwarz overlaps. The various overlap scaling laws are compared in Table II. Finally, the various string models with total vertices and the critical dimension, determined by requiring the total vertex to scale as M^{-3}, are displayed in Table III. As we have discussed, the protostring is a Grassmann analog of the bosonic string. However, there are striking differences. For one, the bosonic string has a tachyonic ground state, whereas the lowest mass squared of the protostring is positive. Accordingly, the protostring is stable. Another interesting feature of the protostring is that its worldsheet degrees of freedom match those of the superstring: 16 of the Grassmann worldsheet fields can be bosonized into 8 compactified bosonic worldsheet fields. These, together with the remaining eight Grassmann fields, match the worldsheet fields of the superstring. However, since the compactification radius is fixed, it is not obvious how to achieve large spatial dimensions. There is the hope that some deformation of the protostring, which enables large

TABLE II. Summary of overlap scaling laws along with insertion rules.

Overlap	Scaling	Insertion	Scaling
V_X	$M^{-d/8}$	ΔX	$M^{-1/2}$
V_S	$M^{-s/8}$	S	M^0
V_Γ	$M^{-d/16}$	Γ	$M^{-1/4}$

TABLE III. Total vertices and critical dimensions for bosonic, Green-Schwarz, RNS, and protostring overlaps.

Type	Total vertex	Critical dimension
Bosonic string	V_X	$d = 24$
Boseprotostring	$V_X V_S$	$d + s = 24,\ d, s > 0$
IIB superstring	$\Delta X^i \Delta X^j \mathcal{P}_{ij}(S) V_X V_S$	$d = s = 8$
RNS string	$(\Gamma \cdot \Delta X)^2 V_\Gamma V_X$	$d = 8$
Protostring	V_S	$s = 24$

transverse dimensions, can be found to produce the actual superstring. Taken as it is given here, the protostring moves in one space dimension (there are no large transverse dimensions). The Lorentz group $SO(1,1)$ in two spacetime dimensions has just a single generator which scales the $\pm$ components oppositely, which is probably ensured by the scaling laws for the continuum limit.

What about the boseprotostring models ($s + d = 24$, $s, d > 0$)? As long as $d < s$, they possess the stability enjoyed by the protostring. When $d > s$, they possess tachyons which render them unstable as is the bosonic string. The case $d = s = 12$ is the marginal case. While the stable ones are not produced by the string bit models analyzed in this paper, more elaborate string bit models have been devised [8], which are likely to serve as their foundation. If the d bosonic coordinates are compactified, the Lorentz group is the same for these stable boseprotostring models as for the protostring and just as likely to be respected. If the d coordinates are not compact, invariance under the Lorentz group $SO(d+1,1)$ will not be ensured. But they would still be interesting and well-defined quantum systems. The worldsheet degrees of freedom of these models vary with d. Counting each Grassmann dimension as half a bosonic dimension, the total number of effective bosonic coordinates is $d + (24-d)/2 = 12 + d/2$. We have already noted the match of number of degrees of freedom between the protostring ($d = 0$) and superstring. We can note a similar coincidence between the $d = 12$ case and the heterotic superstring. The worldsheet system of the latter has $12/2 + 24/2 = 18$ effective bosonic dimensions. Recall that, with regard to stability, the $d = 12$ case is marginal, with massless particles. All the models with $d > 12$ are unstable.

The protostring's foundation as a simple stable string bit model recommends the latter as a solid starting point for defining string theory more generally. While perhaps not as tantalizing, the boseprotostring models offer their own open questions to investigate further. Exploring string bit models and the string models they produce remains a promising direction for future research.

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APPENDIX A: HAMILTONIAN AND ITS ACTION ON COLOR SINGLETS

The string bit creation and annihilation operators satisfy the (anti)commutation relations

$$[(\phi_{a_1\cdots a_m})_\alpha{}^\beta, (\bar{\phi}_{b_1\cdots b_n})_\gamma{}^\delta]_\pm \equiv (\phi_{a_1\cdots a_m})_\alpha{}^\beta (\bar{\phi}_{b_1\cdots b_n})_\gamma{}^\delta - (-)^{mn} (\bar{\phi}_{b_1\cdots b_n})_\gamma{}^\delta (\phi_{a_1\cdots a_m})_\alpha{}^\beta = \delta_{mn}\delta_\alpha{}^\delta \delta_\gamma{}^\beta \sum_P (-)^P \delta_{a_1 b_{P_1}} \cdots \delta_{a_n b_{P_n}}, \tag{A1}$$

which incorporate the fact that $\bar{\phi}$ creates a boson if n is even and a fermion if n is odd. The sum over P is over all permutations of $1, 2, \ldots, n$.

The Hamiltonian analyzed in this paper is the one called H_S in Ref. [8]. We quote

$$H_S = H_1 + H_2 + H_3 + H_4 + H_5, \tag{A2}$$

where the H_i are

$$H_1 = \frac{2}{N} \sum_{n=0}^{s} \sum_{k=0}^{s} \frac{s-2n}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1\cdots a_n} \bar{\phi}_{b_1\cdots b_k} \phi_{b_1\cdots b_k} \phi_{a_1\cdots a_n} \tag{A3}$$

$$H_2 = \frac{2}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1\cdots a_n} \bar{\phi}_{bb_1\cdots b_k} \phi_{b_1\cdots b_k} \phi_{ba_1\cdots a_n} \tag{A4}$$

$$H_3 = \frac{2}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{ba_1\cdots a_n} \bar{\phi}_{b_1\cdots b_k} \phi_{bb_1\cdots b_k} \phi_{a_1\cdots a_n} \tag{A5}$$

$$H_4 = \frac{2i}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1\cdots a_n} \bar{\phi}_{b_1\cdots b_k} \phi_{bb_1\cdots b_k} \phi_{ba_1\cdots a_n} \tag{A6}$$

$$H_5 = -\frac{2i}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{ba_1\cdots a_n} \bar{\phi}_{bb_1\cdots b_k} \phi_{b_1\cdots b_k} \phi_{a_1\cdots a_n}. \tag{A7}$$

H_S commutes with the supersymmetry operators

$$Q^a = \sum_{n=0}^{s-1} \frac{(-)^n}{n!} \mathrm{Tr}[e^{i\pi/4} \bar{\phi}_{a_1\cdots a_n} \phi_{aa_1\cdots a_n} + e^{-i\pi/4} \bar{\phi}_{aa_1\cdots a_n} \phi_{a_1\cdots a_n}]$$
$$\{Q^a, Q^b\} = 2M\delta_{ab}, \tag{A8}$$

which will guarantee equal numbers of bosonic and fermionic eigenstates at each energy level.

Using the commutation relations (A1), it is straightforward to obtain the action of the H_i on single trace states:

$$H_1 T(\theta_1, \ldots, \theta_M)|0\rangle = 2\sum_{k=1}^{M} \left(s - 2\theta_k^a \frac{d}{d\theta_k^a}\right) T(\theta_1, \ldots, \theta_M)|0\rangle + \frac{2}{N} \sum_{k=1}^{M} \left(s - 2\theta_k^a \frac{d}{d\theta_k^a}\right) \sum_{l\neq k,k+1} T(\theta_l \cdots \theta_k) T(\theta_{k+1} \cdots \theta_{l-1})|0\rangle \tag{A9}$$

$$H_2 T(\theta_1, \ldots, \theta_M)|0\rangle = 2\sum_{k=1}^{M} \theta_k^a \frac{d}{d\theta_{k+1}^a} T(\theta_1, \ldots, \theta_M)|0\rangle + \frac{2}{N} \sum_{k=1}^{M} \sum_{l\neq k,k+1} \theta_k^a \frac{d}{d\theta_l^a} T(\theta_l \cdots \theta_k) T(\theta_{k+1} \cdots \theta_{l-1})|0\rangle \tag{A10}$$

$$H_3 T(\theta_1, \ldots, \theta_M)|0\rangle = 2\sum_{k=1}^{M} \theta_{k+1}^a \frac{d}{d\theta_k^a} T(\theta_1, \ldots, \theta_M)|0\rangle + \frac{2}{N} \sum_{k=1}^{M} \sum_{l\neq k,k+1} \theta_l^a \frac{d}{d\theta_k^a} T(\theta_l \cdots \theta_k) T(\theta_{k+1} \cdots \theta_{l-1})|0\rangle \tag{A11}$$

$$H_4 T(\theta_1, \ldots, \theta_M)|0\rangle = -2i\sum_{k=1}^{M} \theta_k^a \theta_{k+1}^a T(\theta_1, \ldots, \theta_M)|0\rangle - \frac{2i}{N} \sum_{k=1}^{M} \sum_{l\neq k,k+1} \theta_k^a \theta_l^a T(\theta_l \cdots \theta_k) T(\theta_{k+1} \cdots \theta_{l-1})|0\rangle \tag{A12}$$

$$H_5 T(\theta_1, \ldots, \theta_M)|0\rangle = -2i\sum_{k=1}^{M} \frac{d}{d\theta_k^a} \frac{d}{d\theta_{k+1}^a} T(\theta_1, \ldots, \theta_M)|0\rangle - \frac{2i}{N} \sum_{k=1}^{M} \sum_{l\neq k,k+1} \frac{d}{d\theta_k^a} \frac{d}{d\theta_l^a} T(\theta_l \cdots \theta_k) T(\theta_{k+1} \cdots \theta_{l-1})|0\rangle. \tag{A13}$$

We note that the differential operators are applied to the nearest neighbors on the same trace when they involve two distinct Grassmann variables.

The action of the H_i on multitrace states takes two forms. When both annihilation operators contract on the same trace, the action can be read off from the preceding formulas. When they act on different traces, the action is to fuse them into a single trace as follows,

$$H_1 T(\theta_1 \cdots \theta_K) T(\eta_1 \cdots \eta_L)|0\rangle_{\text{Fusion}} = +\frac{2}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} \left(s - 2\theta_k^a \frac{d}{d\theta_k^a}\right) T(\theta_{k+1} \cdots \theta_k \eta_l \cdots \eta_{l-1})|0\rangle + \frac{2}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} \left(s - 2\eta_l^a \frac{d}{d\eta_l^a}\right) T(\theta_k \cdots \theta_{k-1} \eta_{l+1} \cdots \eta_l)|0\rangle \tag{A14}$$

$$H_2 T(\theta_1 \cdots \theta_K) T(\eta_1 \cdots \eta_L)|0\rangle_{\text{Fusion}} = +\frac{2}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} \theta_k^a \frac{d}{d\eta_l} T(\theta_{k+1} \cdots \theta_k \eta_l \cdots \eta_{l-1})|0\rangle + \frac{2}{N} \sum_{k=1}^{K} \sum_{l=1}^{L} \eta_l^a \frac{d}{d\theta_k} T(\theta_k \cdots \theta_{k-1} \eta_{l+1} \cdots \eta_l)|0\rangle, \tag{A15}$$

with similar transcriptions for the other H_i. In each case, the differential operators have the same structure as the fission terms, but the states on the right are a suitable pair of single trace states. And when there are two distinct Grassmann operators, they act on the nearest neighbors on the large trace.

APPENDIX B: FORMULAS FOR OVERLAP CALCULATIONS

The following matrix elements are needed in (55):

$$v_m^\dagger v_n^{(1)} = \frac{1}{\sqrt{ML}} \sum_{k=1}^{L} e^{2i\pi k(n/L - m/M)} = -\frac{1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{-2i\pi(n/L - m/M)}} \tag{B1}$$

$$\begin{aligned} v_m^\dagger v_n^{(2)} &= \frac{1}{\sqrt{MK}} \sum_{k=L+1}^{M} e^{2i\pi[(k-L)(n/K - m/M) - Lm/M]} \\ &= \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{-2i\pi(n/K - m/M)}} \end{aligned} \tag{B2}$$

$$\begin{aligned} v_m^\dagger w_0 &= \frac{-1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi m/M}} \sqrt{\frac{K}{M}} - \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi m/M)}} \sqrt{\frac{L}{M}} \\ &= -\frac{1}{\sqrt{LK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi m/M}}. \end{aligned} \tag{B3}$$

Then,

$$\begin{aligned} F_m = &\sum_{n=1}^{L-1} \left[F_n^{(1)} \cos\left(\frac{n\pi}{2L} - \frac{m\pi}{2M}\right) v_m^\dagger v_n^{(1)} + F_n^{(1)\dagger} \cos\left(\frac{n\pi}{2L} + \frac{m\pi}{2M}\right) v_m^\dagger v_{L-n}^{(1)} \right] \\ &+ \sum_{n=1}^{K-1} \left[F_n^{(2)} \cos\left(\frac{n\pi}{2K} - \frac{m\pi}{2M}\right) v_m^\dagger v_n^{(2)} + F_n^{(2)\dagger} \cos\left(\frac{n\pi}{2K} + \frac{m\pi}{2M}\right) v_m^\dagger v_{K-n}^{(2)} \right] \\ &+ \left(f_0 \cos\left(\frac{m\pi}{2M} - \frac{\pi}{4}\right) + f_0^\dagger \cos\left(\frac{m\pi}{2M} + \frac{\pi}{4}\right) \right) v_m^\dagger w_0 \sqrt{2}. \end{aligned} \tag{B4}$$

Then, the C, S matrices needed in (63) are given by

$$\begin{aligned} C_{m0} &= -\frac{1}{\sqrt{LK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi m/M}} \cos\left(\frac{m\pi}{2M} - \frac{\pi}{4}\right) \\ C_{mn1} &= -\frac{1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{-2i\pi(n/L - m/M)}} \cos\left(\frac{n\pi}{2L} - \frac{m\pi}{2M}\right) \\ C_{mn2} &= \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{-2i\pi(n/K - m/M)}} \cos\left(\frac{n\pi}{2K} - \frac{m\pi}{2M}\right) \end{aligned} \tag{B5}$$

$$\begin{aligned} S_{m0} &= -\frac{1}{\sqrt{LK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi m/M}} \cos\left(\frac{m\pi}{2M} + \frac{\pi}{4}\right) \\ S_{mn1} &= -\frac{1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi(n/L + m/M)}} \cos\left(\frac{n\pi}{2L} + \frac{m\pi}{2M}\right) \\ S_{mn2} &= \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2i\pi(n/K + m/M)}} \cos\left(\frac{n\pi}{2K} + \frac{m\pi}{2M}\right). \end{aligned} \tag{B6}$$

APPENDIX C: CONSTRUCTING $|G\rangle$ IN THE CONTINUUM LIMIT

The equations determining the matrix D can be analyzed in the continuum limit in which $L, M \to \infty$ with $x \equiv L/M$ fixed. Then, $K/M \equiv (M-L)/M = 1-x$. For this purpose, we consider this limit on the matrices C, S. This limit must be taken in eight separate cases corresponding to left and right moving waves on each of the three closed strings. It is convenient to remove some common factors of C, S using lowercase letters for the reduced matrices:

$$C_{mn} \equiv \frac{1 - e^{-2\pi imL/M}}{2\pi i} c_{mn},$$
$$S_{mn} \equiv \frac{1 - e^{-2\pi imL/M}}{2\pi i} s_{mn}. \tag{C1}$$

Then:

(1) Holding m, $n1$, $n2$ fixed,

$$c_{m0} \to \frac{1}{m\sqrt{2x(1-x)}}$$
$$c_{mn1} \to \frac{-1}{n/\sqrt{x} - m\sqrt{x}},$$
$$c_{mn2} \to \frac{1}{n/\sqrt{1-x} - m\sqrt{1-x}} \tag{C2}$$

$$s_{m0} \to \frac{1}{m\sqrt{2x(1-x)}}$$
$$s_{mn1} \to \frac{1}{n/\sqrt{x} + m\sqrt{x}},$$
$$s_{mn2} \to \frac{-1}{n/\sqrt{1-x} + m\sqrt{1-x}}. \tag{C3}$$

(2) Holding $m' = M - m, n1, n2$ fixed,

$$c_{m0} \to \frac{-1}{m'\sqrt{2x(1-x)}}, \qquad c_{mn1} \to 0,$$
$$c_{mn2} \to 0 \tag{C4}$$

$$s_{m0} \to \frac{1}{m'\sqrt{2x(1-x)}}, \qquad s_{mn1} \to 0,$$
$$s_{mn2} \to 0. \tag{C5}$$

(3) Holding m, $n'1 \equiv L - n1, n2$ fixed,

$$c_{m0} = \frac{1}{m\sqrt{2x(1-x)}}, \qquad c_{mn1} = 0,$$
$$c_{mn2} = \frac{1}{n/\sqrt{1-x} - m\sqrt{1-x}} \tag{C6}$$

$$s_{m0} = \frac{1}{m\sqrt{2x(1-x)}}, \qquad s_{mn1} = 0,$$
$$s_{mn2} = \frac{-1}{n/\sqrt{1-x} + m\sqrt{1-x}}. \tag{C7}$$

(4) Holding m', $n'1$, $n2$ fixed,

$$c_{m0} = \frac{-1}{m'\sqrt{2x(1-x)}},$$
$$c_{mn1} = \frac{1}{(n'/\sqrt{x} - m'\sqrt{x})}, \qquad c_{mn2} = 0 \tag{C8}$$

$$s_{m0} = \frac{1}{m'\sqrt{2x(1-x)}},$$
$$s_{mn1} = \frac{1}{(n'/\sqrt{x} + m'\sqrt{x})}, \qquad s_{mn2} = 0. \tag{C9}$$

(5) Holding m, $n1$, $n'2 = K - n2$ fixed,

$$c_{m0} = \frac{1}{m\sqrt{2x(1-x)}},$$
$$c_{mn1} = \frac{-1}{n/\sqrt{x} - m\sqrt{x}}, \qquad c_{mn2} = 0 \tag{C10}$$

$$s_{m0} = \frac{1}{m\sqrt{2x(1-x)}},$$
$$s_{mn1} = \frac{1}{n/\sqrt{x} + m\sqrt{x}}, \qquad s_{mn2} = 0. \tag{C11}$$

(6) Holding m', $n1$, $n'2$ fixed,

$$c_{m0} = \frac{-1}{m'\sqrt{2x(1-x)}}, \qquad c_{mn1} = 0,$$
$$c_{mn2} = \frac{-1}{(n'/\sqrt{1-x} - m'\sqrt{1-x})} \tag{C12}$$

$$s_{m0} = \frac{1}{m'\sqrt{2x(1-x)}}, \qquad s_{mn1} = 0,$$
$$s_{mn2} = \frac{1}{(n'/\sqrt{1-x} + m'\sqrt{1-x})}. \tag{C13}$$

(7) Holding m, $n'1$, $n'2$ fixed,

$$c_{m0} = \frac{1}{m\sqrt{2x(1-x)}}, \qquad c_{mn1} = 0,$$
$$c_{mn2} = 0 \tag{C14}$$

$$s_{m0} = \frac{1}{m\sqrt{2x(1-x)}}, \qquad s_{mn1} = 0,$$
$$s_{mn2} = 0. \tag{C15}$$

(8) Holding m', $n'1$, $n'2$ fixed,

$$c_{m0} = \frac{-1}{m'\sqrt{2x(1-x)}}$$
$$c_{mn1} = \frac{1}{(n'/\sqrt{x} - m'\sqrt{x})},$$
$$c_{mn2} = \frac{-1}{(n'/\sqrt{1-x} - m'\sqrt{1-x})} \tag{C16}$$

$$s_{m0} = \frac{1}{m'\sqrt{2x(1-x)}}$$
$$s_{mn1} = \frac{1}{(n'/\sqrt{x} + m'\sqrt{x})},$$
$$s_{mn2} = \frac{-1}{(n'/\sqrt{1-x} + m'\sqrt{1-x})}. \tag{C17}$$

The equation $CD + S = 0$ then breaks up into the series of equations

$$C_{m0}D_{0,l} + C_{mn1}D_{n1,l} + C_{mn2}D_{n2,l} + S_{ml} = 0 \tag{C18}$$

$$C_{mn1}D_{n1,0} + C_{mn2}D_{n2,0} + S_{m0} = 0 \tag{C19}$$

$$C_{m'0}D_{0,l'} + C_{m'n1'}D_{n1',l'} + C_{m'n2'}D_{n2',l'} + S_{m',l'} = 0 \tag{C20}$$

$$C_{m'n1'}D_{n1',0} + C_{m'n2'}D_{n2',0} + S_{m'0} = 0 \tag{C21}$$

$$C_{m0}D_{0,l'} + C_{mn1}D_{n1,l'} + C_{mn2}D_{n2,l'} = 0 \tag{C22}$$

$$C_{m'0}D_{0,l} + C_{m'n1'}D_{n1',l} + C_{m'n2'}D_{n2',l} = 0, \tag{C23}$$

where unprimed indices refer to the continuum limit holding m, $n1$, or $n2$, fixed and primed indices indicate holding $M-m$, $L-n1$, or $M-L-n2$ fixed. In these formulas, l is allowed to refer to either of the smaller strings, so unprimed it is held fixed, and primed $L-l$ or $M-L-l$ as appropriate is held fixed. In addition to these equations, the matrix D is required to be antisymmetric $D^T = -D$.

1. Solving the continuum equations

We can solve these equations using a method due to J. Goldstone, who solved the analogous equations for the three bosonic open string vertex [15]. The final results can also be found in Ref. [16], which employs a different method. Since the matrices C, S involve reciprocals of linear combinations of integers, one guesses a function with poles at appropriate points. Goldstone's choice was

$$g(z) = \frac{\Gamma(1+zx)\Gamma(1+z(1-x))}{z\Gamma(1+z)} \frac{e^{z\xi}}{\sqrt{x(1-x)}} \tag{C24}$$

$$\xi \equiv -x\ln(x) - (1-x)\ln(1-x). \tag{C25}$$

The function $g(z)$ has poles at 0, $-n/x$, and $-n/(1-x)$ for n positive integers,

$$g(z) \sim \frac{1}{ng(n/x)} \frac{1}{z+n/x},$$
$$g(z) \sim \frac{1}{ng(n/(1-x))} \frac{1}{z+n/(1-x)}, \tag{C26}$$

respectively. At large z, g behaves as $\sqrt{2\pi}z^{-1/2}$. Since $g(z)$ has zeroes at $-1, -2, -3, \ldots$, $g(z)/(z+m)$ has the same poles as g as long as m is a positive integer. Then, we can expand

$$\begin{aligned}\frac{g(z)}{z+m} &= \sum_{n=1}^{\infty} \frac{1}{ng(n/x)} \frac{1}{z+n/x} \frac{1}{m-n/x} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{ng(n/(1-x))} \frac{1}{z+n/(1-x)} \\ &\quad \times \frac{1}{m-n/(1-x)} + \frac{1}{mz\sqrt{x(1-x)}} \\ &= \sum_{n=1}^{\infty} \frac{1}{ng(n/x)} \frac{1}{z+n/x} \sqrt{x} c_{mn1} \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{ng(n/(1-x))} \frac{1}{z+n/(1-x)} \sqrt{1-x} c_{mn2} \\ &\quad + \frac{\sqrt{2}}{z} c_{m0}.\end{aligned} \tag{C27}$$

We can recognize this as the first of our equations to solve if we put $z = l/x$ or $z = l/(1-x)$. Then, the left side becomes either $g(l/x)\sqrt{x}s_{ml1}$ or $g(l/(1-x))\sqrt{1-x}s_{ml2}$:

$$\begin{aligned}s_{ml1} &= \sum_{n=1}^{\infty} \frac{1}{ng(n/x)g(l/x)} \frac{1}{l/x+n/x} c_{mn1} \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{ng(n/(1-x))g(l/x)} \frac{1}{l/x+n/(1-x)} \\ &\quad \times \sqrt{\frac{1-x}{x}} c_{mn2} + \frac{\sqrt{2x}}{g(l/z)l} c_{m0}\end{aligned} \tag{C28}$$

$$\begin{aligned}-s_{ml2} &= \sum_{n=1}^{\infty} \frac{1}{ng(n/x)g(l/(1-x)} \frac{1}{l/(1-x)+n/x} \\ &\quad \times \sqrt{\frac{x}{1-x}} c_{mn1} - \sum_{n=1}^{\infty} \frac{1}{ng(n/(1-x))g(l/(1-x)} \\ &\quad \times \frac{1}{l/(1-x)+n/(1-x)} c_{mn2} \\ &\quad + \frac{\sqrt{2(1-x)}}{g(l/(1-x))l} c_{m0}.\end{aligned} \tag{C29}$$

Unfortunately, the inferred D_{nl} would not be antisymmetric. We can fix this by noticing the identity obtained by expanding $zg(z)/(z+m)$,

$$\frac{zg(z)}{z+m} = \sum_{n=1}^{\infty} \frac{-n/x}{ng(n/x)} \frac{1}{z+n/x} \sqrt{x} c_{mn1} - \sum_{n=1}^{\infty} \frac{-n/(1-x)}{ng(n/(1-x))} \frac{1}{z+n/(1-x)} \sqrt{1-x} c_{mn2}. \tag{C30}$$

Putting $z = l/x$ and $z = l/(1-x)$ gives

$$s_{ml1} = \sum_{n=1}^{\infty} \frac{-1}{lg(l/x)g(n/x)} \frac{1}{l/x+n/x} c_{mn1} - \sum_{n=1}^{\infty} \frac{-1}{lg(l/x)g(n/(1-x))} \frac{1}{l/x+n/(1-x)} \sqrt{\frac{x}{1-x}} c_{mn2} \tag{C31}$$

$$\begin{aligned} -s_{ml2} = &\sum_{n=1}^{\infty} \frac{-1}{lg(l/(1-x))g(n/x)} \frac{1}{l/(1-x)+n/x} \sqrt{\frac{1-x}{x}} c_{mn1} \\ &- \sum_{n=1}^{\infty} \frac{-1}{lg(l/(1-x))g(n/(1-x))} \frac{1}{z+n/(1-x)} c_{mn2}. \end{aligned} \tag{C32}$$

Taking the average of the two expressions for s_{ml} gives an antisymmetric solution to the first equation,

$$\begin{aligned} s_{ml1} = &\frac{1}{2} \sum_{n=1}^{\infty} \frac{l-n}{lng(n/x)g(l/x)} \frac{1}{l/x+n/x} c_{mn1} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{l\sqrt{(1-x)/x} - n\sqrt{x/(1-x)}}{lng(n/(1-x))g(l/x)} \\ &\times \frac{1}{l/x+n/(1-x)} c_{mn2} + \frac{\sqrt{x/2}}{g(l/z)l} c_{m0} \end{aligned} \tag{C33}$$

$$\begin{aligned} -s_{ml2} = &\frac{1}{2} \sum_{n=1}^{\infty} \frac{l\sqrt{x/(1-x)} - n\sqrt{(1-x)/x}}{ng(n/x)g(l/(1-x)} \frac{1}{l/(1-x)+n/x} c_{mn1} \\ &- \frac{1}{2} \sum_{n=1}^{\infty} \frac{l-n}{lng(n/(1-x))g(l/(1-x)} \frac{1}{l/(1-x)+n/(1-x)} c_{mn2} + \frac{\sqrt{(1-x)/2}}{g(l/(1-x))l} c_{m0}. \end{aligned} \tag{C34}$$

From these equations, we can read off some of the matrix elements of D:

$$D_{n1,l1} = -\frac{1}{2} \frac{l-n}{lng(n/x)g(l/x)} \frac{1}{l/x+n/x} \tag{C35}$$

$$D_{n2,l1} = \frac{1}{2} \frac{l\sqrt{(1-x)/x} - n\sqrt{x/(1-x)}}{lng(n/(1-x))g(l/x)} \frac{1}{l/x+n/(1-x)} \tag{C36}$$

$$D_{0,l1} = -\frac{\sqrt{x/2}}{g(l/x)l} \tag{C37}$$

$$D_{n1,l2} = \frac{1}{2} \frac{l\sqrt{x/(1-x)} - n\sqrt{(1-x)/x}}{ng(n/x)g(l/(1-x)} \frac{1}{l/(1-x)+n/x} \tag{C38}$$

$$D_{n2,l2} = -\frac{1}{2} \frac{l-n}{lng(n/(1-x))g(l/(1-x)} \frac{1}{l/(1-x)+n/(1-x)} \tag{C39}$$

$$D_{0,l2} = \frac{\sqrt{(1-x)/2}}{g(l/(1-x))l}. \tag{C40}$$

Next let us consider Eq. (C19). We examine (C27) and (C30) near $z = 0$. First, put $z = 0$ in (C30),

$$\frac{1}{m\sqrt{x(1-x)}} = \sqrt{2}s_{m0} = \sum_{n=1}^{\infty}\frac{-1}{ng(n/x)}\sqrt{x}c_{mn1} - \sum_{n=1}^{\infty}\frac{-1}{ng(n/(1-x))}\sqrt{1-x}c_{mn2}, \tag{C41}$$

from which we confirm that $D_{n1,0} = -D_{0,n1}$ and $D_{n2,0} = -D_{0,n2}$. As it turns out, we do not need (C27) here.

The analysis of Eqs. (C20) and (C21) parallels that of (C18) and (C19). Indeed, inspection of the relevant equations shows that $D_{n',l'} = -D_{nl}$. It then remains to analyze the first two equations (C22) and (C23), which do not involve the s's. We therefore examine the difference between Eqs. (C28), (C29) and (C31), (C32):

$$0 = \sum_{n=1}^{\infty}\frac{x}{lng(n/x)g(l/x)}c_{mn1} - \sum_{n=1}^{\infty}\frac{\sqrt{x(1-x)}}{lng(n/(1-x))g(l/x)}c_{mn2} + \frac{\sqrt{2x}}{g(l/z)l}c_{m0} \tag{C42}$$

$$0 = \sum_{n=1}^{\infty}\frac{\sqrt{x(1-x)}}{lng(n/x)g(l/(1-x)}c_{mn1} - \sum_{n=1}^{\infty}\frac{1-x}{lng(n/(1-x))g(l/(1-x)}c_{mn2} + \frac{\sqrt{2(1-x)}}{g(l/(1-x))l}c_{m0}. \tag{C43}$$

The c's in (C23) are the negatives of those in (C22), but since they are homogeneous in c, the two equations are actually identical in form. The coefficient of c_{m0} in (C42) is seen to be $-2D_{0,l1} = 2D_{0,l1'}$ and in (C43) to be $+2D_{0,l2} = -2D_{0,l2'}$. We thus determine from (C22)

$$D_{n1,l1'} = \frac{x}{2lng(n/x)g(l/x)}, \qquad D_{n2,l1'} = -\frac{\sqrt{x(1-x)}}{2lng(n/(1-x))g(l/x)} \tag{C44}$$

$$D_{n2,l2'} = \frac{1-x}{2lng(n/(1-x))g(l/(1-x)}, \qquad D_{n1,l2'} = -\frac{\sqrt{x(1-x)}}{2lng(n/x)g(l/(1-x)}, \tag{C45}$$

and from (C23)

$$D_{n1',l1} = -\frac{x}{2lng(n/x)g(l/x)}, \qquad D_{n2',l1} = \frac{\sqrt{x(1-x)}}{2lng(n/(1-x))g(l/x)} \tag{C46}$$

$$D_{n2',l2} = -\frac{1-x}{2lng(n/(1-x))g(l/(1-x)}, \qquad D_{n1',l2} = \frac{\sqrt{x(1-x)}}{2lng(n/x)g(l/(1-x)}, \tag{C47}$$

and we see that the solution respects the antisymmetry of the matrix D. We note that the matrix elements coupling left and right moving spin waves factorize, unlike those coupling left to left and right to right.

2. Operator insertions

Superstring vertices typically require insertions at the join/break point. In terms of the string bit model, these points could be $k = 1$, $k = L$, $k = L + 1$, or $k = M = L + K$. In the first two cases, we can write

$$S_L = \frac{1}{\sqrt{L}}\sum_{n=0}^{L-1}B_n^{(1)}, \qquad S_1 = \frac{1}{\sqrt{L}}\sum_{n=0}^{L-1}B_n^{(1)}e^{2i\pi n/L}, \tag{C48}$$

and in the last two cases,

$$S_{L+1} = \frac{1}{\sqrt{K}}\sum_{n=0}^{K-1}B_n^{(2)}e^{2\pi in/K}, \qquad S_M = \frac{1}{\sqrt{K}}\sum_{n=0}^{K-1}B_n^{(1)}, \tag{C49}$$

with similar expressions for $\tilde{S}$. The presence of the factor $L^{-1/2}$ or $K^{-1/2}$ in all these expressions means that in the continuum limit the sums over n must diverge like $L^{1/2}$ or $K^{1/2}$ if a finite contribution is to occur. We use

$$B_n = F_n\cos\frac{\pi n}{2M} + \bar{F}_n\sin\frac{\pi n}{2M} \tag{C50}$$

$$i\tilde{B}_n = F_n\sin\frac{\pi n}{2M} - \bar{F}_n\cos\frac{\pi n}{2M}, \tag{C51}$$

and the divergence must come from the action of the lowering operators on $|G\rangle$. So the possibilities are

$$S_L|G\rangle \sim \frac{1}{\sqrt{L}}\sum_{n=1}^{L-1}\cos\frac{\pi n}{2L}D_{n1,l}f_l^\dagger|G\rangle \tag{C52}$$

$$i\tilde{S}_L|G\rangle \sim \frac{1}{\sqrt{L}}\sum_{n=1}^{L-1}\sin\frac{\pi n}{2L}D_{n1,l}f_l^\dagger|G\rangle = \frac{1}{\sqrt{L}}\sum_{n=1}^{L}\cos\frac{\pi n'}{2L}D_{n1',l}f_l^\dagger|G\rangle. \tag{C53}$$

In the case of S_L, the trig function suppresses the modes near $n = L$, whereas for $\tilde{S}_L$ the modes near $n = 0$ (or n' near L) are suppressed. As $L \to \infty$, the sum over n diverges as $L^{1/2}$, so these insertions are finite and nonzero in the continuum limit. This divergence can be seen by considering modes in the range $1 \ll n,\ n' \ll L$,

$$\begin{aligned}
&D_{n1,l1} \sim \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)}, \qquad D_{n2,l1} \sim -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)}, \qquad D_{n1,0} \sim \frac{1}{\sqrt{4\pi n}}\\
&D_{n1,l2} \sim -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{(1-x)}}{lg(l/(1-x)}, \qquad D_{n2,l2} \sim \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{1-x}}{lg(l/(1-x)}\\
&D_{n2,0} \sim -\frac{1}{\sqrt{4\pi n}}.
\end{aligned} \tag{C54}$$

The elements $D_{n',l'}$ behave as the negatives of these. The mixed elements behave as

$$D_{n1,l1'} = \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)}, \qquad D_{n2,l1'} = -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)} \tag{C55}$$

$$D_{n2,l2'} = \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{1-x}}{lg(l/(1-x)}, \qquad D_{n1,l2'} = -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{1-x}}{lg(l/(1-x)}, \tag{C56}$$

and from (C23),

$$D_{n1',l1} = -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)}, \qquad D_{n2',l1} = \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{x}}{lg(l/x)} \tag{C57}$$

$$D_{n2',l2} = -\frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{1-x}}{lg(l/(1-x)}, \qquad D_{n1',l2} = \frac{1}{2}\frac{1}{\sqrt{2\pi n}}\frac{\sqrt{1-x}}{lg(l/(1-x)}. \tag{C58}$$

For the insertion on string 1 at $k = L$ or $k = 1$, we need

$$D_{n1,l}f_l^\dagger = \frac{1}{\sqrt{8\pi n}}\left[\sqrt{2}f_0^\dagger + \sum_{l=1}^{\infty}\frac{\sqrt{x}}{lg(l/x)}(f_{l1}^\dagger + f_{l1'}^\dagger) - \sum_{l=1}^{\infty}\frac{\sqrt{1-x}}{lg(l/(1-x))}(f_{l2}^\dagger + f_{l2'}^\dagger)\right] \equiv \frac{1}{\sqrt{8\pi n}}S, \tag{C59}$$

and

$$D_{n1',l}f_l^\dagger = -\frac{1}{\sqrt{8\pi n}}\left[\sqrt{2}f_0^\dagger + \sum_{l=1}^{\infty}\frac{\sqrt{x}}{lg(l/x)}(f_{l1}^\dagger + f_{l1'}^\dagger) - \sum_{l=1}^{\infty}\frac{\sqrt{1-x}}{lg(l/(1-x))}(f_{l2}^\dagger + f_{l2'}^\dagger)\right] = -\frac{1}{\sqrt{8\pi n}}S. \tag{C60}$$

The insertion of $i\tilde{S}_L$ gives the negative of the insertion of S_L because the roles of $n1$ and $n1'$ are switched. Inspection shows that all possible insertions, S_L, $i\tilde{S}_L$, S_{L+1}, $i\tilde{S}_{L+1}$, S_1, $i\tilde{S}_1$, S_M, $i\tilde{S}_M$, involve the same operator S in the limit $M, L, M-l \to \infty$, and in fact yield the same factor up to a sign. In this limit, the surviving part of the sum over n involves

$$\begin{aligned}
\frac{1}{\sqrt{8\pi L}}\sum_{n=n_0}^{L/2}\frac{1}{\sqrt{n}}\left(\cos\frac{n\pi}{2L} - \sin\frac{n\pi}{2L}\right) &\sim \frac{1}{\sqrt{8\pi}}\int_{n_0/L}^{1}\frac{dx}{\sqrt{x}}\left(\cos\frac{x\pi}{2} - \sin\frac{x\pi}{2}\right)\\
&\to \frac{1}{\sqrt{8\pi}}\int_0^1\frac{dx}{\sqrt{x}}\left(\cos\frac{x\pi}{2} - \sin\frac{x\pi}{2}\right).
\end{aligned} \tag{C61}$$

When the insertion is placed on the other string, $M - L = K$ replaces L, but as long as both L and K are large, the same result ensues.

If we consider inserting more than one factor of S at the join/break point, we find no new operator structure in the continuum limit unless S belongs to a different worldsheet field than the first one. For instance, $S_k^2 = 1$ identically. If we apply say $S_k S_j$ to the overlap, with k, j on the same small string, we pass this operator through the exponential factors. If k, j are at the join/break point, the commutator terms approach one or two factors of S in the continuum limit. If only one factor is produced, it will multiply a creation term which vanishes in the continuum limit. If two factors are produced, $S^2 = 0$ because S is an anticommuting variable, and thus the only surviving contribution in the continuum limit is the result of applying $S_k S_j$ to the ground state of the string. The terms involving creation operators vanish in the continuum limit, so only a c-number arising from normal ordering $S_k S_j$ can survive.

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