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Speed limit in internal space of domain walls via all-order effective action of moduli motion

Minoru Eto[1] and Koji Hashimoto[2]

[1]*Department of Physics, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan*

[2]*Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan*

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We find that motion in internal moduli spaces of generic domain walls has an upper bound for its velocity. Our finding is based on our generic formula for all-order effective actions of internal moduli parameter of domain wall solitons. It is known that the Nambu-Goldstone mode Z associated with spontaneous breaking of translation symmetry obeys a Nambu-Goto effective Lagrangian $\sqrt{1-(\partial_0 Z)^2}$ detecting the speed of light ($|\partial_0 Z| = 1$) in the target spacetime. Solitons can have internal moduli parameters as well, associated with a breaking of internal symmetries such as a phase rotation acting on a field. We obtain, for generic domain walls, an effective Lagrangian of the internal moduli ϵ to all orders in $(\partial\epsilon)$. The Lagrangian is given by a function of the Nambu-Goto Lagrangian: $L = g\left(\sqrt{1+(\partial_\mu\epsilon)^2}\right)$. This shows generically the existence of an upper bound on $\partial_0\epsilon$, i.e., a speed limit in the internal space. The speed limit exists even for solitons in some nonrelativistic field theories, where we find that ϵ is a type I Nambu-Goldstone mode that also obeys a nonlinear dispersion to reach the speed limit. This offers a possibility of detecting the speed limit in condensed matter experiments.

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I. INTRODUCTION

Solitons are used everywhere in physics, ranging from elementary particle physics where domain walls are used for brane-world scenarios, to condensed matter physics where walls dividing two phases can be observed at every scale. The dynamics of solitons is governed by low energy modes propagating on the solitons, which are often identified as Nambu-Goldstone massless modes. Thus, providing a full effective action of Nambu-Goldstone modes on solitons is quite important to characterize any physics in phases with broken symmetries.

Internal symmetries are indispensable in physics. When an internal symmetry is broken by a domain wall, an associated Nambu-Goldstone mode ϵ appears (see Fig. 1). The mode, called an internal moduli, is governed by some effective action. So far, little is known for an all-order expression of the effective action, since normally one employs so-called Manton's method [1,2] (the moduli approximation) to calculate the action order by order. In this article, we provide a generic form of the effective action of an internal moduli parameter ϵ of a generic domain wall to all orders in $\partial\epsilon$,

$$L_{\mathrm{dw}} = g\left(m\sqrt{1+(\partial_\mu\epsilon)^2}\right), \tag{1}$$

where g is a system-dependent function, $\mu = 0, 1, \ldots, d-1$ spans the domain wall world volume coordinates, and m is the mass of the original theory. Thus, our result paves the way to construct the effective actions of various solitons in generic models. For simplicity, we mainly focus on the domain wall in this article.

The argument $\sqrt{1+(\partial_\mu\epsilon)^2}$ of the effective Lagrangian is eventually of the form of Nambu-Goto action [3,4]. The action (1) provides generically an upper limit of the moduli motion $v \equiv \partial_0\epsilon$ in the internal space, since $\sqrt{1+(\partial_\mu\epsilon)^2} = \sqrt{1-v^2}$. Therefore, the internal space of domain walls has a speed limit: "an internal speed of light." Note that any internal space is nothing related to the real spacetime, so generically it would be anticipated that there

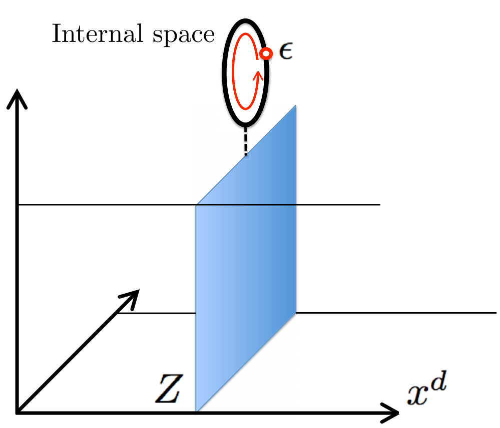

FIG. 1. A domain wall with an internal moduli parameter (shown as a point on a circle above the wall). The motion of the internal moduli ϵ probes the internal "space."

should not be any speed limit in the internal space. Indeed, in the standard breaking of vacuum symmetry, there appears no speed limit for $\partial_0\epsilon$; see Appendix A. Here, we find that for the breaking via solitons there exists the speed limit in the internal space. Note that, as is the speed of light, the upper bound is not a kind of cutoff. It is given far below the UV cutoff of the theory.

As for translational symmetry that is not internal, it is known that a Nambu-Goto action governs the Nambu-Goldstone mode Z (see Fig. 1). Any domain wall in a flat spacetime in any dimensions supported by a relativistic field theory has a Nambu-Goto effective action for Z in a static gauge [5,6], see also Appendix C,

$$L_{\rm dw} = -\mathcal{T}_{\rm dw}\sqrt{1+(\partial_\mu Z)^2}. \tag{2}$$

The value of Z shows the location of the domain wall. The action is valid to all orders in ∂Z, when higher $(\partial)^n Z$ for $n \geq 2$ is ignored. The Nambu-Goto action reflects the special relativity of the spacetime in which the domain wall lives. The effective action (2), with the wall velocity $\partial_0 Z$ in the transverse direction, is equal to an action of a relativistic particle, $-\sqrt{1-(\partial_0 Z)^2}$. So the structure of the effective action shows that the upper limit of the domain wall motion is the speed of light.

Note that the Nambu-Goto action (2) for Z was derived based on the original Lorentz invariance in $(d+1)$-dimensional spacetime. Thus, emergence of the speed of light in the Nambu-Goto action is quite natural since it is a direct consequence of the Lorentz symmetry. On the other hand, any internal moduli effective action cannot rely on the original Lorentz symmetry. Our result (1) shows that, nevertheless, the speed limit shows up in the internal space, which is a novel feature that we report here. We emphasize that the speed limit is nothing to do with any cutoff scales of the theories under consideration, as is the usual speed of light in spacetime.

Interestingly, our generic strategy can also be applied to nonrelativistic models that frequently appear as effective theories in condensed matter systems. Recently nonrelativistic Nambu-Goldstone modes such as magnons attracted much attention [7–11]. Because of our all-order effective action, the internal moduli has a type I dispersion for small velocity, while has a speed limit due to the nonlinear dispersion. The speed limit may be observed in experiments with symmetry-broken orders, such as magnetic domain walls.

II. GENERIC EFFECTIVE ACTIONS

A. Generic effective action in relativistic theory

We consider a generic Lagrangian of a relativistic complex scalar field $\phi(x^\mu, z)$ in $d+1$-dimensional spacetime ($\mu = 0, 1, \ldots, d-1$ and $x^d = z$). The Lagrangian is assumed to have only two derivatives at maximum, for simplicity;

$$\mathcal{L} = -F(|\phi|^2)(|\partial_\mu\phi|^2 + |\partial_z\phi|^2 + m^2|\phi|^2) - V(|\phi|^2), \tag{3}$$

with generic F and V. The field ϕ has a mass m (which is far below the cutoff of the theory). This system has a $U(1)$ global symmetry that rotates the phase of the field, $\phi \to e^{im\epsilon_0}\phi$ where ϵ_0 is a constant real parameter for the internal space S^1. We assume the existence of a static domain wall solution with an S^1 moduli parameter [12]

$$\phi = e^{im\epsilon_0}\phi_0(z; m), \tag{4}$$

which solves the equation of motion of (3),

$$\begin{aligned} &-\partial_\mu(F\partial^\mu\phi) - \partial_z(F\partial^z\phi) + m^2F'|\phi|^2\phi + m^2F\phi \\ &\quad + V'\phi + F'\phi|\partial_\mu\phi|^2 + F'\phi|\partial_z\phi|^2 = 0. \end{aligned} \tag{5}$$

In particular, the function $\phi_0(z; m)$ obeys

$$\begin{aligned} &-\partial_z(F\partial_z\phi_0) + m^2(F'|\phi_0|^2 + F)\phi_0 \\ &\quad + V'\phi_0 + F'\phi_0|\partial_z\phi_0|^2 = 0. \end{aligned} \tag{6}$$

Now, a motion of the internal moduli parameter ϵ can be encoded as $\epsilon = \epsilon_\mu x^\mu$ with a constant vector ϵ_μ, which amounts to ignoring $\partial\partial\epsilon$. With this spacetime-dependent internal moduli parameter ϵ, let us make an ansatz for a generic solution $\phi = e^{im\epsilon}\varphi_0(z)$ ($\epsilon = \epsilon_\mu x^\mu$). Plugging this into Eq. (5), we find that φ obeys the following equation:

$$\begin{aligned} &-\partial_z(F\partial_z\varphi_0) + m^2(1+(\epsilon_\mu)^2)(F'|\varphi|^2 + F)\varphi_0 \\ &\quad + V'\varphi_0 + F'\varphi_0|\partial_z\varphi_0|^2 = 0. \end{aligned} \tag{7}$$

Comparing this with (6), we can regard φ as a static solution in the model (3) with m being replaced with $m\sqrt{1+(\epsilon_\mu)^2}$. Namely, we get a generic solution

$$\varphi_0(z) = \phi_0\left(z; m\sqrt{1+(\epsilon_\mu)^2}\right). \tag{8}$$

Thus, a new exact solution with moduli motion is

$$\phi = e^{im\epsilon_\mu x^\mu}\phi_0\left(z; m\sqrt{1+(\epsilon_\mu)^2}\right). \tag{9}$$

Substituting the solution (9) to the action (3), we obtain the effective action of the internal moduli $\epsilon(x)$ of a generic domain wall,

$$\begin{aligned} S_{\rm dw} &\equiv \int d^dx dz \mathcal{L}|_{\phi = e^{im\epsilon_\mu x^\mu}\phi_0(z; m\sqrt{1+(\epsilon_\mu)^2})} \\ &= \int d^dx\, g\left(m\sqrt{1+(\partial_\mu\epsilon)^2}\right), \end{aligned} \tag{10}$$

where $g(m)$ is the on-shell action of the original static domain wall, $g(m) \equiv \int dz \mathcal{L}|_{\phi=\phi_0(z;m)}$. Interestingly, the effective action (10) is *not* a Nambu-Goto type $\sqrt{1+(\partial_\mu\epsilon)^2}$, but generically *a function of the Nambu-Goto*.

Some details for deriving the above result can be found in Appendix B.

B. Generic effective action in nonrelativistic theory

Condensed matter systems can be approximated by a complex scalar field as an order parameter, while its theory is nonrelativistic. Consider the following form [13] of the scalar field Lagrangian,

$$\mathcal{L}^{(\mathrm{NR})} = -F(|\phi|^2)\left(\frac{im_0}{2}(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}) + |\partial_i\phi|^2 + |\partial_z\phi|^2 + m^2|\phi|^2\right) - V(|\phi|^2), \tag{11}$$

with $i = 1, 2, \ldots, d-1$, which can be obtained from the relativistic Lagrangian (3) by inclusion of a chemical potential with a certain scaling. The static domain wall (4) remains as a solution, while an exact solution with moving moduli is given by

$$\phi = e^{im\epsilon}\phi_0\left(z; m\sqrt{1 - \frac{m_0}{m}\partial_0\epsilon + (\partial_i\epsilon)^2}\right). \tag{12}$$

Then we can repeat the same argument to arrive at the domain wall effective action

$$S^{(\mathrm{NR})}_{\mathrm{dw}} = \int d^d x g\left(m\sqrt{1 - \frac{m_0}{m}\partial_0\epsilon + (\partial_i\epsilon)^2}\right). \tag{13}$$

The effective Lagrangian is invariant under the charge conjugation $\epsilon \to -\epsilon$ with the time reversal, as in the original Lagrangian (11).

C. Speed limit in internal space

The Hamiltonian for the relativistic case is calculated as

$$H = -\frac{m(\partial_0\epsilon)^2}{\sqrt{1+(\partial_\mu\epsilon)^2}}g' - g, \tag{14}$$

which generically diverges at

$$(\partial_0\epsilon)^2 - (\partial_i\epsilon)^2 = 1. \tag{15}$$

Reaching the speed limit (15) expends infinite energy, as in the case of the speed of light in our spacetime. Therefore, the internal motion has the speed limit (15). The speed limit exists essentially owing to the higher order corrections in $\partial\epsilon$. In fact it cannot be seen in the usual moduli approximation to a finite order in $\partial\epsilon$. Notice that the normalization of the moduli parameter is given by $e^{im\epsilon}$. So the speed limit in the internal space is $1/m$. This scale is far below the cutoff scale of the original theory (3).

The speed limit sounds quite counterintuitive, since the phase rotation acting on a field, $\phi \to e^{im\epsilon}\phi$, can be arbitrarily fast in principle, as is mentioned in the introduction. However, the physical reason for the existence of the speed limit is the stability of the domain wall. Normally, the mass m is related to the energy levels of the nonzero modes on the domain wall; thus, putting energy in the internal zero mode on the domain wall more than m means exciting too many nonzero modes, leading to a demolition of the wall itself.

We notice that the effective Hamiltonian in the nonrelativistic case derived from (13) also diverges at

$$1 - (m_0/m)\partial_0\epsilon + (\partial_i\epsilon)^2 = 0. \tag{16}$$

Therefore, there again exists a speed limit. In particular, the speed limit is $\partial_0(m\epsilon) = m^2/m_0$ for $\partial_i\epsilon = 0$. Note that the speed limit exists only for a certain direction of motion, and there is no speed limit at $\partial_0(m\epsilon) = -m^2/m_0$. It is intriguing that even within a nonrelativistic theory the domain wall internal space can have a speed limit, which can be tested in experiments realizing the domain wall with an internal degree of freedom.

III. EXAMPLES

A. Generalized Nambu-Goto via domain wall

Only a special class of Lagrangians leads to a Nambu-Goto effective action. The condition for having a Nambu-Goto effective action in (10) is $g(m) \propto m$. This is equivalent to having $V(|\phi|^2) = 0$ in the original Lagrangian (3), due to a scaling symmetry in $\mathcal{L}$.

A particular example that leads to a Nambu-Goto effective action is a massive $\mathbb{C}P^1$ sigma model with the Fubini-Study metric,

$$F(|\phi|^2) = (1+|\phi|^2)^{-2}, \qquad V = 0. \tag{17}$$

The explicit domain wall solution is $\phi_0 = e^{mz}$, and the on-shell action is $g(m) = -m$. Thus, the effective action of the internal moduli ϵ is given by a Nambu-Goto [15],

$$S^{(\mathrm{massive}\,\mathbb{C}\mathrm{P}^1)}_{\mathrm{dw}} = -m\int d^d x\sqrt{1+(\partial_\mu\epsilon)^2}. \tag{18}$$

Figure 2 is a plot of the Hamiltonian of the Nambu-Goto system (18) for $\epsilon_\mu = \omega\delta_{0\mu}$. It diverges at the speed limit in the internal space, $\omega = \pm m$.

With a nontrivial V, we can derive various effective action of the internal space. As an interesting example, we introduce the following V in the massive $\mathbb{C}P^1$ model (see Appendix E for details):

$$V = 4\lambda|\phi|^2(1-|\phi|^2)^2(1+|\phi|^2)^{-4}. \tag{19}$$

By the redefinition $\phi = e^{i\Phi}\tan\frac{\Theta}{2}$, a part of the massive $\mathbb{C}P^1$ model can be recast to a sine-Gordon model with $m^2\sin^2\Theta$ potential term. The additional potential (19) gives

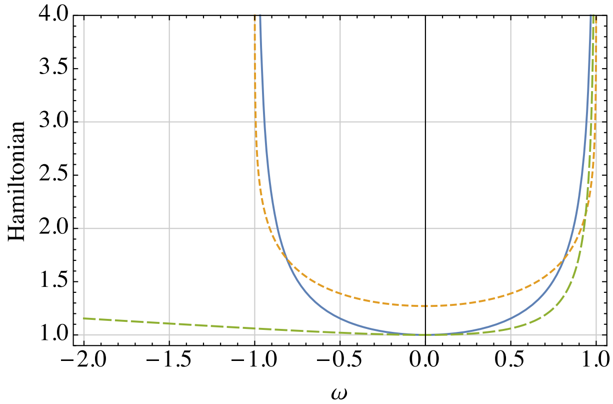

FIG. 2. A plot of the effective Hamiltonian of a domain wall, as a function of $\omega \equiv \partial_0\epsilon$ for $\partial_i\epsilon = 0$. The solid line: relativistic $\mathbb{C}P^1$ sigma model (giving a Nambu-Goto). The dashed line: relativistic modified $\mathbb{C}P^1$ sigma model ($\lambda = 1/2$). Long dashed line: nonrelativistic $\mathbb{C}P^1$ sigma model ($m_0 = 1$). The Hamiltonians ($m = 1$) diverge at the speed limits (15)–(16).

$\lambda \sin^2 2\Theta$. Thus, the massive $\mathbb{C}P^1$ model with (19) includes as a part the so-called double sine-Gordon model that has been investigated for soliton confinement phenomena. The static domain wall solution is

$$\phi_0(z; m, \lambda) = \left[\frac{\sqrt{4\lambda + m^2\cosh^2(\tilde{m}z)} + m^2\sinh^2(\tilde{m}z)}{\sqrt{4\lambda + m^2\cosh^2(\tilde{m}z)} - m^2\sinh^2(\tilde{m}z)}\right]^{\frac{1}{2}}$$

with $\tilde{m} \equiv \sqrt{4\lambda + m^2}$, and using this, we obtain the effective action for ϵ as

$$S_{\rm dw} = -\int d^dx \left(\sqrt{\tilde{\lambda}} + \frac{L_{\rm NG}^2}{4\sqrt{\lambda}} \tanh^{-1}\sqrt{\frac{\lambda}{\tilde{\lambda}}}\right), \tag{20}$$

where $\tilde{\lambda} \equiv \lambda + L_{\rm NG}^2/4$, and $L_{\rm NG} = -m\sqrt{1 + (\partial_\mu\epsilon)^2}$ is the Nambu-Goto Lagrangian. The Hamiltonian is given by

$$H = \sqrt{\tilde{\lambda}} + \frac{m^2(1 + (\partial_0\epsilon)^2 + (\partial_i\epsilon)^2)}{4\sqrt{\lambda}} \tanh^{-1}\sqrt{\frac{\lambda}{\tilde{\lambda}}}, \tag{21}$$

which diverges at the speed limit (15), as in Fig. 2. A conserved charge for the symmetry $\epsilon \to \epsilon + \delta$ is

$$Q = \frac{m^2}{\sqrt{4\lambda}} \tanh^{-1}\sqrt{\frac{4\lambda}{m^2(1 + (\partial_\mu\epsilon)^2) + 4\lambda}}\,\partial_0\epsilon. \tag{22}$$

This also diverges at the speed limit [16]. With $\epsilon = \epsilon_\mu x^\mu$, H and Q coincide with the tension and Noether charge calculated in the original nonlinear sigma model, which serves as a nontrivial consistency. See also Appendix E for other models.

B. Type I Nambu-Goldstone mode

We found the generic form (13) of the effective action of ϵ to all orders $\partial\epsilon$. In the nonrelativistic massive $\mathbb{C}P^1$ case with (17), expanding (13) in $\partial\epsilon$ up to quadratic order, we obtain

$$L^{(\rm NR)} = m\left[1 - \frac{1}{2}\frac{m_0}{m}\dot{\epsilon} - \frac{1}{8}\left(\frac{m_0}{m}\dot{\epsilon}\right)^2 + \frac{1}{2}(\partial_i\epsilon)^2\right]. \tag{23}$$

Note that one cannot obtain the $\dot{\epsilon}^2$ term by the usual order-by-order moduli approximation: we need the exact solution (12) to get (23). From (23), the dispersion relation is given by

$$\omega = \frac{2m}{m_0}|\mathbf{k}|. \tag{24}$$

Thus, the Nambu-Goldstone mode ϵ is type I (that is, linear and relativistic) [17]. Note that our speed limit $\omega(= \dot{\epsilon}) = m/m_0$ means the upper bound for the value of the internal speed ω.

C. Fattening and destroying the wall

As we anticipated, the motion of the internal moduli ϵ destroys the wall when it exceeds the speed limit. The effective Hamiltonian acquires an imaginary part, which signals the decay of the domain wall itself. The speed limit is given by the scalar mass m; turning on the moduli motion $\partial_0\epsilon$ reduces the mass to $m\sqrt{1 - (\partial_0\epsilon)^2}$. So the speed limit amounts effectively to the massless limit in the original theory.

The mass is inversely related to the width of the domain wall, so we can expect that the internal moduli motion will make the domain wall thicker, and finally decays smoothly by the fattening. In Fig. 3, we plot the energy density of the

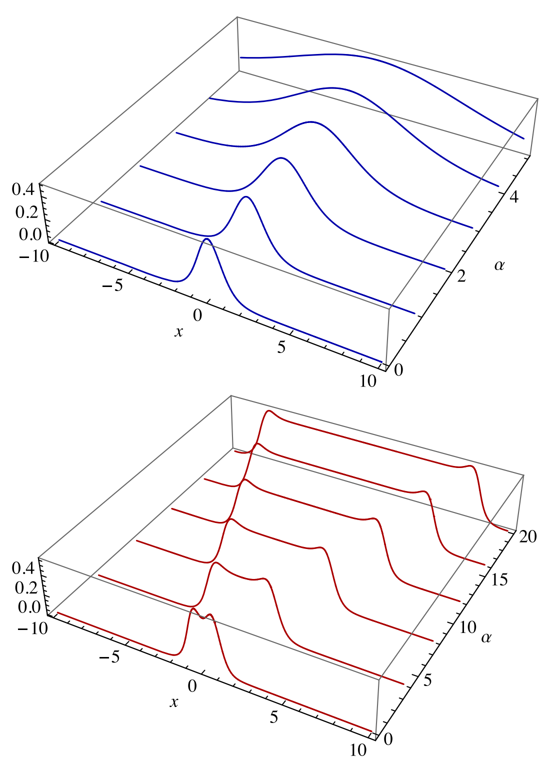

FIG. 3. Energy density profiles of the domain wall in the $\mathbb{C}P^1$ with (upper) / without (lower) V in (19), when we change the internal moduli $\partial_0\epsilon$. The parameter α is defined by $\partial_0\epsilon = 1 - \exp[-\alpha]$. $\alpha = 0$ corresponds to the original domain wall. We chose $m = 1$ and $\lambda = 1/2$ for this plot.

exact domain wall solutions in the massive $\mathbb{C}P^1$ sigma model with/without V (19), by changing $|\partial_0\epsilon|$. We can clearly see the fattening of the domain walls. In the model with V, at $\omega = 0$ we can see that a single domain wall consists of two constituent walls connecting $\Theta = 0 \to \pi/2$ and $\pi/2 \to \pi$, respectively. They are confined. As ω increases, a large repulsive force appears that deconfines the constituent domain walls.

IV. CONCLUSION

We show generically the existence of a speed limit in internal moduli space ϵ of domain walls. The speed dependence of the Hamiltonian is calculated to all orders in $\partial\epsilon$. The effective Lagrangian is generically a function of the Nambu-Goto Lagrangian, in contrast to the transverse moduli Z obeying a Nambu-Goto. Even for nonrelativistic field theory, we find the speed limit, which may be seen in experiments with symmetry-broken orders.

Our calculation can extend to other spices of solitons. In [18], D. Tong studied the internal S^1 moduli of a 't Hooft-Polyakov Bogomol'nyi-Prasad-Sommerfield (BPS) monopole and showed that it obeys a Nambu-Goto action; see Appendix G. Note that, as we demonstrated, the phase of the domain wall in the massive $\mathbb{C}P^1$ model also obeys the Nambu-Goto action. This is not a coincidence. Indeed, 't Hooft-Polyakov monopole in the Higgs phase is identified with a kink in the massive $\mathbb{C}P^1$ model [19]. Thus, the kink-monopole correspondence may be valid to all orders in $\partial\epsilon$ [20].

Furthermore, even in the absence of solitons, a vacuum itself can have an internal moduli, where our method can be applied to show the nonexistence of the speed limit; see Appendixes A and F.

The speed limit suggests emergence of an extra dimension of the spacetime. In fact, we can derive the effective action of the internal moduli (10) by using a generalized boost along a newly introduced extra dimension S^1; see Appendix D for the details.

When the world volume of the domain wall is $(1+2)$ dimensional, we can take a dual of the generalized Nambu-Goto action to obtain a general nonlinear electrodynamics. Those generalizations of the Born-Infeld action may possess interesting structure, including a possible relation to D-branes [21,22] in string theory and an open string metric [23,24].

The existence of the speed limit in internal space is encouraging for brane-world scenario and possible cosmological models using a speed limit of inflation rolling [25]. Various applications to particle physics, cosmology, and condensed matter physics are expected.

In the appendixes, we provide detailed calculations that are used in the main part of the article, and some additional calculations that further support the claim in the article. First, we present a calculation for the case of the vacuum breaking of the internal symmetry and derive its effective action, to find out that there is no speed limit in the internal space. Next, we show a calculation to obtain the generic effective actions of the internal moduli for the domain wall, (10) and (13) in detail. Then, we provide a brief review of the derivation of the Nambu-Goto action for the translational moduli (2). We take a route different from that in [5,6]. Then, we rederive our main effective action of the internal moduli (10) by using extra dimensions, as an instructive exercise. After that, we present detailed calculations of explicit examples of the domain wall solutions, as well as a new example for a composite domain wall. Two more examples follow, an example of a symmetry breaking at a vacuum via extra dimensions, and an example of a 't Hooft-Polyakov monopole.

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APPENDIX A: EFFECTIVE ACTION FOR A SYMMETRY BREAKING AT VACUUM

We consider a generic Lagrangian of a complex scalar field $\phi(x^\mu)$ $(\mu = 0, 1, 2, 3)$. The Lagrangian is, for simplicity,

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = -|\partial_\mu\phi|^2 - m^2|\phi|^2 - V(|\phi|^2). \tag{A1}$$

We consider a spontaneous symmetry breaking,

$$V = c|\phi|^2 + \lambda|\phi|^4, \tag{A2}$$

where $-\mu^2 = m^2 + c < 0$ to make sure that the symmetry breaking occurs, as the original Lagrangian looks like

$$\mathcal{L} = -|\partial_\mu\phi|^2 - (-\mu^2|\phi|^2 + \lambda|\phi|^4). \tag{A3}$$

We treat m and c independently in our analysis, although a physically important quantity is the combination μ.

Let us first consider a classical solution with a moving moduli,

$$\phi = v e^{im\epsilon_\mu x^\mu}, \tag{A4}$$

with a constant four vector ϵ_μ. Since rotation in the internal space excites massive modes, the amplitude v changes from the vacuum expectation value (VEV) as

$$v = \sqrt{\frac{\mu^2 - m^2\epsilon^2}{2\lambda}}, \qquad (\epsilon^2 = \epsilon_\mu\epsilon^\mu). \tag{A5}$$

Note that ω in $\epsilon_\mu = (\omega, 0, 0, 0)$ can take any value. Nonzero ω just shifts the VEV as $v = \sqrt{(\mu^2 + m^2\omega^2)/2\lambda}$. Namely, there is no limit for internal moduli in the homogeneous vacuum. The Hamiltonian for this configuration is

$$H = \frac{(\mu^2 - m^2\epsilon^2)((3\epsilon_0^2 + \epsilon_i^2)m^2 - \mu^2)}{4\lambda}. \tag{A6}$$

In order to obtain a low energy effective Lagrangian, we plug (A4) into the Lagrangian and replace $\epsilon_\mu \to \partial_\mu \epsilon(x^\mu)$. Eventually, we find the following effective Lagrangian:

$$L_{\rm eff} = \frac{1}{4\lambda}(\mu^2 - m^2\partial_\mu\epsilon\partial^\mu\epsilon)^2. \tag{A7}$$

The configuration (A4) still solves the equation of motion of this effective Lagrangian and the Hamiltonian is identical to Eq. (A6). The massive modes are correctly taken into account via the higher derivative corrections. Clearly, no speed limits for $\partial_0\epsilon$ appear in this effective Lagrangian.

APPENDIX B: SOME DETAILS FOR DERIVING THE GENERIC EFFECTIVE ACTION

In this subsection, we show details for deriving the effective action (10) and (13). Let us begin with the equations of motion of the relativistic Lagrangian (3),

$$\begin{aligned} &-\partial_\mu(F\partial^\mu\phi) - \partial_z(F\partial^z\phi) + m^2F'|\phi|^2\phi + m^2F\phi \\ &\quad + V'\phi + F'\phi|\partial_\mu\phi|^2 + F'\phi|\partial_z\phi|^2 = 0. \end{aligned} \tag{B1}$$

The domain wall solution perpendicular to the z axis,

$$\phi(z) = e^{im\epsilon_0}\phi_0(z; m), \tag{B2}$$

is a solution to the reduced equation

$$\begin{aligned} &-\partial_z(F\partial_z\phi_0) + m^2(F'|\phi_0|^2 + F)\phi_0 + V'\phi_0 \\ &\quad + F'\phi_0|\partial_z\phi_0|^2 = 0. \end{aligned} \tag{B3}$$

Let us make an ansatz for a generic solution

$$\phi = e^{im\epsilon}\varphi_0(z), \qquad (\epsilon = \epsilon_\mu x^\mu). \tag{B4}$$

Plugging this into Eq. (B1), we find that φ obeys the following equation:

$$\begin{aligned} &-\partial_z(F\partial_z\varphi_0) + m^2(1 + (\epsilon_\mu)^2)(F'|\varphi|^2 + F)\varphi_0 \\ &\quad + V'\varphi_0 + F'\varphi_0|\partial_z\varphi_0|^2 = 0. \end{aligned} \tag{B5}$$

Comparing this with (B3), we can regard φ as a static solution in the model (3) with m being replaced with $m\sqrt{1 + (\epsilon_\mu)^2}$. Namely, we get the generic solution

$$\varphi_0(z) = \phi_0(z; m\sqrt{1 + (\epsilon_\mu)^2}). \tag{B6}$$

Correspondingly, if we plug $\phi(x^\mu, z) = e^{im\epsilon}\varphi(x^\mu, z)$ into the Lagrangian (3), we find the Lagrangian for φ,

$$\begin{aligned} \tilde{\mathcal{L}} = &-F(|\varphi|^2)(|\partial_\mu\varphi|^2 + |\partial_z\varphi|^2 + m\epsilon_\mu j^\mu \\ &+ m^2(1 + (\epsilon_\mu)^2))|\varphi|^2) - V(|\varphi|^2), \end{aligned} \tag{B7}$$

with $j^\mu = i(\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi)$. Therefore, the on-shell Lagrangian for the generic solution (B4) is directly obtained as

$$\begin{aligned} \int dz\mathcal{L}(\phi = e^{im\epsilon}\varphi_0) &= \int dz\tilde{\mathcal{L}}(\varphi_0) \\ &= g(m\sqrt{1 + (\epsilon_\mu)^2}), \end{aligned} \tag{B8}$$

with $g(m) = \int dz\mathcal{L}(\phi = e^{im\epsilon_0}\phi_0(z; m))$. Finally, just replacing ϵ_μ by $\partial_\mu\epsilon$, we reach the final result (10).

We can repeat the same argument for the nonrelativistic model. The equations of motion for the nonrelativistic Lagrangian (11) are given as

$$\begin{aligned} &\frac{im_0}{2}\partial^0(F\phi) - \partial_I(F\partial_I\phi) + m^2(F'|\phi|^2 + F)\phi + V'\phi \\ &\quad + \frac{im_0}{2}\{F'(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi})\phi + F\partial_0\phi\} + F'|\partial_I\phi|^2\phi = 0, \end{aligned} \tag{B9}$$

with $x^I = x^1, x^2, \ldots, x^{d-1}, z$. Clearly, (B2) is a solution of this equation. Furthermore, the equation for φ_0 with the ansatz (B4) is obtained as

$$\begin{aligned} &-\partial_z(F\partial_z\varphi_0) + (m^2 - m_0\epsilon_0 + m^2(\epsilon_i)^2)(F'|\varphi_0|^2 + F)\varphi_0 \\ &\quad + V'\varphi_0 + F'\varphi_0|\partial_z\varphi_0|^2 = 0. \end{aligned} \tag{B10}$$

Comparing this with (B3), we find the generic solution in the nonrelativistic model is given by

$$\varphi_0(z) = \phi_0\left(z; m\sqrt{1 - \frac{m_0}{m}\epsilon_0 + (\epsilon_i)^2}\right). \tag{B11}$$

Correspondingly, if we plug $\phi(x^\mu, z) = e^{im\epsilon}\varphi(x^\mu, z)$ into the nonrelativistic Lagrangian (11), we find the Lagrangian for φ,

$$\begin{aligned} \tilde{\mathcal{L}}^{(\rm NR)} = &-F(|\varphi|^2)\left(\frac{im_0}{2}(\bar{\varphi}\partial_0\varphi - \varphi\partial_0\bar{\varphi}) + |\partial_i\varphi|^2 \right. \\ &\left. + |\partial_z\varphi|^2 + m\epsilon_i j^i + (m^2 - m_0\epsilon_0 + (\epsilon_i)^2)|\varphi|^2\right) \\ &- V(|\varphi|^2). \end{aligned} \tag{B12}$$

Therefore, the on-shell value of the original nonrelativistic Lagrangian for the generic solution is given by

$$\int dz\mathcal{L}^{(\mathrm{NR})}(\phi = e^{im\epsilon}\varphi_0) = \int dz\tilde{\mathcal{L}}^{(\mathrm{NR})}(\varphi_0)$$
$$= g\left(m\sqrt{1 - \frac{m_0}{m}\epsilon_0 + (\epsilon_i)^2}\right). \tag{B13}$$

APPENDIX C: DERIVING NAMBU-GOTO EFFECTIVE ACTION FOR TRANSLATIONAL MODULI

Let us start with a real scalar field ϕ in $d+1$ dimensions. The Lagrangian is given by $\mathcal{L}$ and thus the action is written as

$$S = \int d^{d+1}x\mathcal{L}[\phi]. \tag{C1}$$

Suppose we obtain a domain wall as a classical solution of this system,

$$\phi = \phi_0(z), \qquad (z = x^d). \tag{C2}$$

The domain wall world volume is perpendicular to the direction z, so it is flat along the remaining directions x^μ ($\mu = 0, 1, \ldots, d-1$). The obvious zero mode Z of the domain wall is the position of the domain wall in the z axis. Inclusion of the zero mode gives us a generic solution

$$\phi = \phi_0(z - Z) \tag{C3}$$

where Z is a constant parameter. Turning on Z does not cost any energy and this remains as a classical solution of the original system $\mathcal{L}$.

Now, we are interested in the low energy effective description of the domain wall. The zero mode Z can depend on the world volume coordinates x^μ, as $Z(x^\mu)$. Ignoring the higher derivatives such as $(\partial)^2 Z$, we should be able to obtain an effective action of the domain wall

$$S_{\mathrm{dw}} = \int dx^{d-1}L_{\mathrm{dw}}[\partial_\mu Z]. \tag{C4}$$

The domain wall Lagrangian L_{dw} is a functional of $\partial_\mu Z$ only, as the potential term $V(Z)$ should not appear because Z is a zero mode.

The easiest way to get L_{dw} is to make a Lorentz transformation. Consider the following Lorentz transformation,

$$\tilde{z} = \Lambda^d{}_d z + \Lambda^d{}_\nu x^\nu, \tag{C5}$$

$$\tilde{x}^\mu = \Lambda^\mu{}_d z + \Lambda^\mu{}_\nu x^\nu, \tag{C6}$$

where Λ is an $SO(1,d)$ transformation matrix whose determinant is 1. Then obviously we can generate a new classical solution

$$\phi = \phi_0(\tilde{z}). \tag{C7}$$

This solution is a tilted domain wall moving in a transverse direction with a constant velocity. Since

$$\tilde{z} = \Lambda^d{}_d(z + (\Lambda^d{}_d)^{-1}\Lambda^d{}_\mu x^\mu), \tag{C8}$$

it is possible to regard

$$Z(x^\mu) = -(\Lambda^d{}_d)^{-1}\Lambda^d{}_\mu x^\mu. \tag{C9}$$

So, once we obtain a domain wall effective action as a function of the Lorentz transformation matrix elements Λ, using (C9) we can regard it as a domain wall effective action for $Z(x^\mu)$.

Let us calculate the effective action of the domain wall. Substituting the transformed solution (C7) into the original action, we can obtain the effective action

$$S_{\mathrm{dw}} = \int d^d x \int dz\, \mathcal{L}|_{\phi=\phi_0(\tilde{z})}. \tag{C10}$$

To compute this integral, we use the following trick. The new domain wall solution (C7) depends only on $\tilde{z}$ while the spacetime coordinates x^μ should be kept as they are, so we consider a general coordinate transformation (note that it is not a Lorentz transformation),

$$\tilde{z} = \Lambda^d{}_d z + \Lambda^d{}_\nu x^\nu, \tag{C11}$$

$$\tilde{x}^\mu = x^\mu. \tag{C12}$$

The transformation for z is the same as the Lorentz transformation (C5) while x^μ is kept intact. For this general coordinate transformation, the Jacobian is found as

$$\frac{d\tilde{x}}{dx} = \begin{pmatrix} \delta^\mu{}_\nu & 0 \\ \Lambda^d{}_\nu & \Lambda^d{}_d \end{pmatrix}, \tag{C13}$$

so the metric is transformed as

$$g^{\tilde{d}\tilde{d}} = (\Lambda^d{}_d)^2 + \Lambda^d{}_\mu\Lambda^d{}_\nu\eta^{\mu\nu} = 1. \tag{C14}$$

The last equality is due to the $SO(1,d)$ nature of the Lorentz transformation Λ. Note that there appear nontrivial off-diagonal elements of the metric such as $g^{\tilde{d}\mu}$, although they are irrelevant in the following calculations.

When we substitute the new solution (C7) into the Lagrangian, we may use the new Lagrangian transformed by the general coordinate transformation (C11)–(C12), since the Lagrangian is a scalar quantity under the general coordinate transformation. The solution depends only on $\tilde{z}$, so all derivatives ∂_μ acting on the solution vanish. The only terms that are relevant are $g^{\tilde{d}\tilde{d}}\partial_{\tilde{d}}\phi\partial_{\tilde{d}}\phi^*$ and its higher derivative analogues. Since $g^{\tilde{d}\tilde{d}} = 1$, this concludes that the Lagrangian with the new solution is equal to the

Lagrangian with the original solution, via a simple replacement z by $\tilde{z}$. Then the domain wall effective action (C10) can be evaluated as

$$\int d^d x \int dz\, \mathcal{L}|_{\phi=\phi_0(\tilde{z})} = \int d^d x \int d\tilde{z} \sqrt{-\det \tilde{g}}\, [\mathcal{L}|_{\phi=\phi_0(z)}]_{z \text{ replaced by } \tilde{z}}. \quad \text{(C15)}$$

The general coordinate transformation (C11)–(C12) gives

$$\det \tilde{g} = \frac{1}{(\Lambda^d{}_d)^2}, \quad \text{(C16)}$$

so we obtain

$$S_{\rm dw} = \frac{1}{\Lambda^d{}_d} \int d^d x \int d\tilde{z} [\mathcal{L}|_{\phi=\phi_0(z)}]_{z \text{ replaced by } \tilde{z}} = \frac{1}{\Lambda^d{}_d} \int d^d x \int dz \mathcal{L}|_{\phi=\phi_0(z)}. \quad \text{(C17)}$$

We define the on-shell value of the Lagrangian with the classical solution (C2) integrated over z as a tension $\mathcal{T}$ of the domain wall,

$$\mathcal{T}_{\rm dw} \equiv -\int dz \mathcal{L}|_{\phi=\phi_0(z)}. \quad \text{(C18)}$$

Then we find the domain wall effective Lagrangian

$$L_{\rm dw} = -\mathcal{T}_{\rm dw} \frac{1}{\Lambda^d{}_d}. \quad \text{(C19)}$$

Using the $SO(1,d)$ relation and the relation (C9), we find

$$\frac{1}{(\Lambda^d{}_d)^2} = 1 + (\Lambda^d{}_d)^{-2} \Lambda^d{}_\mu \Lambda^d{}_\nu \eta^{\mu\nu} = 1 + (\partial_\mu Z)^2, \quad \text{(C20)}$$

so we finally obtain the domain wall effective Lagrangian

$$L_{\rm dw} = -\mathcal{T}_{\rm dw} \sqrt{1 + (\partial_\mu Z)^2}. \quad \text{(C21)}$$

This is nothing but a Nambu-Goto Lagrangian. Hence, we conclude that the effective action of the translational zero mode $Z(x^\mu)$ of any domain wall is a Nambu-Goto action, up to the first derivative of the zero mode $Z(x^\mu)$.

APPENDIX D: EXTRA DIMENSIONS AND DERIVING EFFECTIVE ACTION FOR INTERNAL MODULI

Previously we derived the effective action of the internal moduli of the domain wall, using the equations of motion and explicit solutions. Here, we utilize an embedding to a spacetime with an extra dimension to derive the same expressions for the effective action of the internal moduli.

First, we study what form of the Lagrangian can give a domain wall with an internal moduli space S^1. Next, we embed the scalar system into a higher dimensional spacetime where the internal phase rotation is related to the extra dimensional coordinate. We obtain generic effective action of the domain wall. Then, we study what condition should be met for the action to be Nambu-Goto type. Finally, we study the speed limit in the internal space.

1. System with a domain wall with S^1 internal moduli

We consider a generic Lagrangian of a complex scalar field $\phi(x^\mu, z)$ in a flat $(d+1)$-dimensional spacetime $(\mu = 0, \ldots, d-1)$. The Lagrangian is assumed to have only two derivatives at maximum, for simplicity;

$$S = \int d^{d+1} x \mathcal{L},$$
$$\mathcal{L} = -F(|\phi|^2)(|\partial_\mu \phi|^2 + |\partial_z \phi|^2 + m^2 |\phi|^2) - V(|\phi|^2). \quad \text{(D1)}$$

The mass term is intentionally separated from the potential functional $V(|\phi|^2)$. This system has a $U(1)$ global symmetry that rotates the phase of the field,

$$\phi \to e^{im\epsilon}\phi, \qquad \epsilon \in \mathbb{R}. \quad \text{(D2)}$$

We assume the existence of a domain wall solution

$$\phi = \phi_0(z) \quad \text{(D3)}$$

where ϕ_0 is a real function that interpolates two vacua that have the same energy. We need an S^1 moduli for the domain wall, so the domain wall needs to be a solution even if we rotate it as

$$\phi = e^{im\epsilon}\phi_0(z). \quad \text{(D4)}$$

This, in particular, means that the vacua, $\phi(z = -\infty)$ and $\phi(z = \infty)$, have to be a fixed point of the $U(1)$ symmetry. Otherwise the $U(1)$ rotation changes the vacuum and so the moduli become non-normalizable, which means there is no sense in discussing effective action of the moduli parameters.

Generically the fixed points of the phase rotation are only $\phi = 0$ and $\phi = \infty$ $(1/\phi = 0)$, so this condition of having the S^1 moduli constrains the Lagrangian as follows: the total potential

$$V_{\rm tot} \equiv m^2 |\phi|^2 F(|\phi|^2) + V(|\phi|^2) \quad \text{(D5)}$$

has two minima at $\phi = 0$ and $\phi = \infty$. In the following, we treat this system. Several examples are given later.

2. Generalized Lorentz boost creating new solutions

Whether the internal moduli parameter can be regarded as another spatial coordinate or not is an important question. As we have seen in the previous section, the Nambu-Goto action can be obtained by the Lorentz transformation of the soliton solution. So, here, we consider a Lorentz transformation including the internal moduli direction. We are careful about what situation we perform the Lorentz transformation in, below.

First, we introduce the internal coordinate. Since the S^1 moduli is associated with the phase $U(1)$ symmetry acting on the field ϕ, it is natural to upgrade the phase factor of the domain wall solution to the additional spatial coordinate α. The new internal space needs to be compact, so we define a new complex scalar field $\Phi(x^\mu, z, \alpha)$ that lives in the $(1+d+1)$-dimensional spacetime spanned by $x^M \equiv (x^\mu, z, \alpha)$, as

$$\Phi(x^\mu, z, \alpha) = e^{im\alpha} \sum_{n=-\infty}^{\infty} e^{in\alpha/R} \phi^{(n)}(x^\mu, z). \qquad \text{(D6)}$$

The form is written in a Fourier expansion where n is the Fourier mode number. In addition, we have introduced an overall factor $e^{im\alpha}$ by the following reason. Let us prepare the following $(d+2)$-dimensional action

$$S = \frac{1}{2\pi R} \int_0^{2\pi R} d\alpha \int d^d x dz \mathcal{L}_{d+2}, \qquad \text{(D7)}$$

$$\mathcal{L}_{d+2} = -F(|\phi|^2)|\partial_M \phi|^2 - V(|\phi|^2). \qquad \text{(D8)}$$

Then, substituting the expansion (D6), the factor $e^{im\alpha}$ provides exactly the mass term in the $(d+1)$-dimensional action (3). In fact, when we have only the $n=0$ mode in (D6), then substituting it to (D7) exactly reproduces (3). The factor $e^{im\alpha}$ in (D6) manifests a twisted periodicity condition

$$\Phi(x^\mu, z, \alpha + 2\pi R) = e^{2\pi i m R} \Phi(x^\mu, z, \alpha), \qquad \text{(D9)}$$

which is called Scherk-Schwarz compactification [26].

Now, we have the domain wall solution $\phi = \phi_0(z)$ of the original $\mathcal{L}$; then

$$\Phi = \Phi_0(\alpha, z) \equiv e^{im\alpha} \phi_0(z) \qquad \text{(D10)}$$

is a solution of the equation of motion of $\mathcal{L}_{d+2}$. This solution just has the $n=0$ component of the expansion (D6).

Since it is a solution of $\mathcal{L}_{d+2}$, and since $\mathcal{L}_{d+2}$ appears to be Lorentz invariant in $(d+2)$ dimensions, we can make use of the $(d+2)$-dimensional Lorentz transformation to create a new classical solution. In particular, we are interested in the internal coordinate α; let us make a transformation in the subspacetime spanned by α and x^μ. As in the previous section, the Lorentz transformation is

$$\tilde{\alpha} = \Lambda^\alpha{}_\alpha \alpha + \Lambda^\alpha{}_\mu x^\mu, \qquad \text{(D11)}$$

$$\tilde{x}^\mu = \Lambda^\mu{}_\alpha \alpha + \Lambda^\mu{}_\nu x^\nu. \qquad \text{(D12)}$$

We enact this transformation to obtain a new solution,

$$\begin{aligned} \Phi &= \Phi_0(\tilde{\alpha}, z) \\ &= e^{im\tilde{\alpha}} \phi_0(z) \\ &= e^{im\Lambda^\alpha{}_\alpha \alpha} e^{im\Lambda^\alpha{}_\mu x^\mu} \phi_0(z). \end{aligned} \qquad \text{(D13)}$$

Note that this is a solution of $\mathcal{L}_{d+2}$ while it is not a solution of the original $(d+1)$-dimensional $\mathcal{L}$. The reason is that to make the reduction to the $d+1$ dimensions we need the relation (D10) where the dependence on the extra dimension is $e^{im\alpha}$, while the new solution (D13) has a different phase factor $e^{im\Lambda^\alpha{}_\alpha \alpha}$. In fact, a new $(d+1)$-dimensional solution that can be read off from (D13) as

$$\phi = e^{im\Lambda^\alpha{}_\mu x^\mu} \phi_0(z) \qquad \text{(D14)}$$

is not a solution of $\mathcal{L}$ but a solution of

$$\tilde{\mathcal{L}} = \mathcal{L}|_{m \to m\Lambda^\alpha{}_\alpha}. \qquad \text{(D15)}$$

The reason is obvious. The α dependence in the new solution (D13) provides, together with the α derivatives in $\mathcal{L}_{d+2}$, a new mass term that is $m^2(\Lambda^\alpha{}_\alpha)^2$ instead of the original m^2. In other words, we define a new solution

$$\phi = e^{i(m/\Lambda^\alpha{}_\alpha)\Lambda^\alpha{}_\mu x^\mu} \phi_0(z; m/\Lambda^\alpha{}_\alpha), \qquad \text{(D16)}$$

where we have replaced m by $m/\Lambda^\alpha{}_\alpha$; then this (D16) is a solution of the original Lagrangian $\mathcal{L}$. In this way, we can create new solutions by a generalized Lorentz transformation in the space including the internal direction. Taking $\Lambda^\alpha{}_\alpha = 1/\sqrt{1+(\epsilon_\mu)^2}$ and $\Lambda^\alpha{}_\mu = \epsilon_\mu/\sqrt{1+(\epsilon_\nu)^2}$, we get the solution (9).

3. General effective action for the internal S^1 moduli

We are ready for evaluating the effective action. Let us first calculate it as an effective action of a $(d+2)$-dimensional solution. (That is, we here calculate first the effective action without the above replacement of m. Later we incorporate the effect of the replacement.) The on-shell action is

$$S = \frac{1}{2\pi R} \int_0^{2\pi R} d\alpha \int d^d x dz \mathcal{L}_{d+2}|_{\Phi = \Phi_0(\tilde{\alpha}, z)}, \qquad \text{(D17)}$$

and to evaluate this explicitly we perform a general coordinate transformation

$$\tilde{\alpha} = \Lambda^\alpha{}_\alpha \alpha + \Lambda^\alpha{}_\mu x^\mu, \qquad \tilde{x}^\mu = x^\mu \qquad \text{(D18)}$$

as in the previous section. Then, using $g^{\tilde{\alpha}\tilde{\alpha}} = 1$ and $\sqrt{-\det \tilde{g}} = 1/\Lambda^\alpha{}_\alpha$, we find

$$
\begin{aligned}
S &= \frac{1}{2\pi R}\int_{\Lambda^{\alpha}{}_{\mu}x^{\mu}}^{2\pi R\Lambda^{\alpha}{}_{\alpha}+\Lambda^{\alpha}{}_{\mu}x^{\mu}} \frac{d\tilde{\alpha}}{\Lambda^{\alpha}{}_{\alpha}} \\
&\quad \times \int d^d x dz [\mathcal{L}_{d+2}|_{\Phi=\Phi_0(\alpha,z)}]_{\alpha\,\text{replaced by}\,\tilde{\alpha}} \\
&= \frac{1}{2\pi R\Lambda^{\alpha}{}_{\alpha}} 2\pi R\Lambda^{\alpha}{}_{\alpha}\int d^d x dz \mathcal{L}_{d+2}|_{\Phi=\Phi_0(\alpha,z)} \\
&= \int d^d x dz \mathcal{L}_{d+2}|_{\Phi=\Phi_0(\alpha,z)} \\
&= \int d^d x g(m). \qquad \text{(D19)}
\end{aligned}
$$

In the last equality we have defined the on-shell action for the original domain wall with the $(d+1)$-dimensional Lagrangian,

$$
g(m) \equiv \int dz \mathcal{L}|_{\phi=\phi_0(z)}. \qquad \text{(D20)}
$$

(D19) means that the on-shell action is independent of the Lorentz transformation parameters appearing in the solution.

However, we have to remember the fact that the new solution (D14) is not a solution of $\mathcal{L}$ but a solution of $\tilde{\mathcal{L}}$ defined in (D15) that is given by the replacement $m \to m\Lambda^{\alpha}{}_{\alpha}$. So, as mentioned in (D16), to have a solution of the original $(d+1)$-dimensional Lagrangian, we need to make a redefinition of m as $m \to m/\Lambda^{\alpha}{}_{\alpha}$. Therefore, the correct effective action of the domain wall described by the solution (D16) is given by (D19) with the replacement,

$$
S_{\mathrm{dw}} = \int d^d x g(m/\Lambda^{\alpha}{}_{\alpha}). \qquad \text{(D21)}
$$

We reinterpret the factor $1/\Lambda^{\alpha}{}_{\alpha}$ as a function of the moduli. The procedure is already given in the previous section. Comparing the constant moduli parameter ϵ in the solution (D4) and the new solution (D16), we can regard the dynamical moduli parameter to have a configuration in the new solution as

$$
\epsilon(x^{\mu}) = \Lambda^{\alpha}{}_{\mu}x^{\mu}/\Lambda^{\alpha}{}_{\alpha}. \qquad \text{(D22)}
$$

This is reminiscent of (C9) for the translational zero mode Z in the previous section. So, similarly, we have

$$
\frac{1}{(\Lambda^{\alpha}{}_{\alpha})^2} = 1 + (\Lambda^{\alpha}{}_{\alpha})^{-2}\Lambda^{\alpha}{}_{\mu}\Lambda^{\alpha}{}_{\nu}\eta^{\mu\nu} = 1 + (\partial_{\mu}\epsilon)^2. \qquad \text{(D23)}
$$

Substituting this expression to the domain wall effective action (D21), we obtain the final form of the generic effective action for the internal moduli $\epsilon(x^{\mu})$ of the domain wall as

$$
S_{\mathrm{dw}} = \int d^d x g(m\sqrt{1+(\partial_{\mu}\epsilon)^2}). \qquad \text{(D24)}
$$

This precisely reproduces (10).

The generic action is a function of the Nambu-Goto action. In the next subsection, we learn a condition of having the Nambu-Goto form.

4. The condition for having the Nambu-Goto

Our final expression for the effective action of the internal moduli is (D24). Obviously, the condition that this action becomes a Nambu-Goto action is to have a linear $g(m) = Am$ where A is a constant.

We find below that a sufficient condition to have a Nambu-Goto action for the effective action, in other words, to have a linear $g(m)$, is to start with

$$
V = 0 \qquad \text{(D25)}
$$

in the original $(d+1)$-dimensional Lagrangian. When $V=0$, the total action is

$$
S = -\int d^{d+1}x F(|\phi|^2)(|\partial_z\phi|^2 + m^2|\phi|^2). \qquad \text{(D26)}
$$

Here, we put $\partial_{\mu}\phi = 0$, which is satisfied for the solution. The m dependence can be absorbed into the rescaling $z_{\mathrm{new}} = mz$, such that

$$
S = -m\int d^d x \int dz_{\mathrm{new}} F(|\phi|^2)(|\partial_z^{\mathrm{new}}\phi|^2 + |\phi|^2). \qquad \text{(D27)}
$$

In this expression the derivative $\partial_z^{\mathrm{new}}$ is with respect to z_{new}. Now the m dependence appears only as an overall factor, so the equation of motion is independent of m. Then the domain wall solution is written as $\phi_0 = f(z_{\mathrm{new}})$, which is independent of m. Recovering the m dependence, we obtain a solution $\phi_0 = f(mz)$. Substituting this into the action, we obtain an on-shell action

$$
S_{\mathrm{dw}} = -m\int d^d x \left[\int dz F(|\phi|^2)(|\partial_z\phi|^2 + |\phi|^2)|_{\phi=f(z)}\right] \qquad \text{(D28)}
$$

where the last factor written with "[ ]" is independent of m. Denoting it as $-A$, we obtain a linear dependence,

$$
g(m) = Am. \qquad \text{(D29)}
$$

So, in summary, for a $(d+1)$-dimensional system whose action is given by

$$
S = -\int d^{d+1}x F(|\phi|^2)(|\partial_{\mu}\phi|^2 + |\partial_z\phi|^2 + m^2|\phi|^2), \qquad \text{(D30)}
$$

the internal moduli appearing in the solution as

$$
\phi = e^{im\epsilon(x^{\mu})}f(mz) \qquad \text{(D31)}
$$

has an effective action of the Nambu-Goto form,

$$
S_{\mathrm{dw}} = \int d^d x Am\sqrt{1+(\partial_{\mu}\epsilon)^2}, \qquad \text{(D32)}
$$

where the overall coefficient A is given by

$$A \equiv -\int dz F(|\phi|^2)(|\partial_z\phi|^2 + |\phi|^2)|_{\phi=f(z)}. \quad \text{(D33)}$$

The Nambu-Goto action in $1+2$ dimensions is equivalent to the Born-Infeld action. So we complete the derivation of the Born-Infeld action as an effective action of an internal moduli of a domain wall, for a class of $1+3$-dimensional complex scalar field theories whose Lagrangian is of the form (D30).

5. The speed limit in internal space

The Nambu-Goto action (D32) shows the existence of the speed limit. As we parametrized the internal space as (D31), this shows that the speed limit in the internal space is m. So, the mass term in the original action has quite an important property: it serves as the speed limit in the internal space.

Looking at the generic effective action (D24) that we obtained, it exhibits interesting structure: it is a function of a Nambu-Goto Lagrangian. Since the Nambu-Goto Lagrangian indicates the speed limit, the generic action may have the speed limit in the internal space. The critical speed is indeed the mass, m.

One may notice that the separation between the mass term and the potential term V is arbitrary in our calculation. Indeed, one can split the original mass term as $m^2|\phi|^2 = (m_1)^2|\phi|^2 + (m_2)^2|\phi|^2$ where $m = \sqrt{(m_1)^2 + (m_2)^2}$, and regard the $(m_2)^2$ term as a potential term, while regarding the $(m_1)^2$ term as a mass term. Following the same procedure as the previous subsection, we obtain an effective action

$$S_{\rm dw} = \int d^dx h(m_1\sqrt{1+(\partial_\mu\eta)^2}, m_2) \quad \text{(D34)}$$

where

$$h(m_1, m_2) \equiv \int dz \mathcal{L}|_{\phi=\phi_0(z)}. \quad \text{(D35)}$$

Here, the internal moduli field η is defined as

$$\phi = e^{im_1\eta(x^\mu)} f(mz). \quad \text{(D36)}$$

Notice the difference from (D31): the definitions of the moduli fields are related as

$$m_1\eta(x^\mu) = m\epsilon(x^\mu). \quad \text{(D37)}$$

Now, obviously the splitting of the mass term should not change the resultant effective action, so (D34) should be equal to (D24). Indeed, if one notices the equality

$$h(m_1, m_2) = g(m)|_{m=\sqrt{(m_1)^2+(m_2)^2}}, \quad \text{(D38)}$$

it is easy to show the equivalence of (D34) and (D24), through the relation (D37).

APPENDIX E: EXAMPLES OF EFFECTIVE ACTIONS ON DOMAIN WALLS

1. Massive $\mathbb{C}P^1$ sigma model

An example satisfying this condition $V=0$ and also the condition of having two minima in (D5) is a massive $\mathbb{C}P^1$ sigma model (17),

$$F(|\phi|^2) = \frac{1}{(1+|\phi|^2)^2}, \qquad V = 0. \quad \text{(E1)}$$

The vacua are located at $\phi = 0$ and $\phi = \infty$ as mentioned above, and an explicit domain wall solution is

$$\phi_0(z) = e^{mz}. \quad \text{(E2)}$$

Since the system is with $V=0$, the solution is of the form $f(mz)$ as explained in the previous section. The on-shell action is given by

$$S_{\rm dw} = -m\int d^dx\left[\int dz \frac{|\partial_z\phi|^2 + |\phi|^2}{(1+|\phi|^2)^2}\bigg|_{\phi=e^z}\right] = \int d^dx(-m). \quad \text{(E3)}$$

Thus, the effective action is of the Nambu-Goto type given in Eq. (D32) with $A=-1$.

Let us verify if the Nambu-Goto action correctly describes dynamics of the domain wall. For that purpose, it is simple to see a time-dependent solution, namely, the so-called Q-kink domain wall [21]. The solution is obtained through a standard Bogomol'nyi technique as

$$\begin{aligned} M &= m\int dz \frac{|\dot\phi - i\phi\sin\alpha|^2 + |\phi' - \phi\cos\alpha|^2 - i(\dot\phi\phi^* - \phi\dot\phi^*)\sin\alpha + (|\phi|^2)'\cos\alpha}{(1+|\phi|^2)^2} \\ &\geq mT\cos\alpha + mQ\sin\alpha, \end{aligned} \quad \text{(E4)}$$

with the Noether charge $Q = \int dz \frac{-i(\dot\phi\phi^* - \phi\dot\phi^*)}{(1+|\phi|^2)^2}$ and the topological charge $T = \int dz \frac{(|\phi|^2)'}{(1+|\phi|^2)^2}$. The energy bound from below is the most stringent when $\tan\alpha = Q/T$. Expressing $\sin\alpha = \omega/m$ and $\cos\alpha = \sqrt{1-(\omega/m)^2}$, we get the BPS solution and mass formula of the Q-kink domain wall

$$\phi = e^{i\tilde{\omega}x^0 + \sqrt{1-\tilde{\omega}^2}z}, \quad M = m\sqrt{T^2+Q^2} = \frac{m}{\sqrt{1-\tilde{\omega}^2}}, \quad \tilde{\omega} = \frac{\omega}{m}. \tag{E5}$$

We can derive the same mass formula from the effective Lagrangian of the Nambu-Goto type. The Lagrangian is

$$L_{\rm NG} = -m\sqrt{1+(\partial_\mu \epsilon)^2}. \tag{E6}$$

A conjugate momentum is given by $\pi_\epsilon = \frac{\delta L_{\rm NG}}{\delta \partial_0 \epsilon} = -m\partial^0\epsilon/\sqrt{1+(\partial_\mu\epsilon)^2}$. Then, the Hamiltonian takes the form

$$H_{\rm NG} = (\partial_0\epsilon)\pi_\epsilon - L_{\rm NG} = m\frac{1+(\partial_i\epsilon)^2}{\sqrt{1+(\partial_\mu\epsilon)^2}}, \quad (i=1,2). \tag{E7}$$

Reading x^0 dependence of $\epsilon(x^\mu)$ from the Q-kink domain wall solution (E5), we find $\epsilon(x^\mu) = \tilde{\omega}x^0$. Plugging this into $H_{\rm NG}$, we find

$$H_{\rm NG} = \frac{m}{\sqrt{1-\tilde{\omega}^2}}. \tag{E8}$$

This is precisely the same as the BPS mass formula given in Eq. (E5). Thus, we confirm the Nambu-Goto action works very well as the effective action.

2. An additional potential to massive $\mathbb{C}P^1$ sigma model: A model

In the previous subsection, we have seen the simplest example in which the effective action of the domain wall corresponds to the usual Nambu-Goto action. In this section, we see an example that the effective theory is not of the simple Nambu-Goto type.

We again consider the massive $\mathbb{C}P^1$ model. But we introduce an additional higher order interaction term (19)

$$\mathcal{L} = \frac{|\partial_M\phi|^2 - m^2|\phi|^2}{(1+|\phi|^2)^2} - V, \tag{E9}$$

$$V = \frac{4\lambda|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^4}, \tag{E10}$$

where we assume $\lambda \geq 0$. The total scalar potential is positive definite, so that $\phi = 0$ and $\phi = \infty$ remain as the global vacua with zero vacuum energy. In addition, a new local minimum appears at $|\phi| = 1$ due to the additional term in the potential. Thus, a domain wall interpolating the vacua $\phi = 0, \infty$ still exists but it is deformed compared to the one in the massive $\mathbb{C}P^1$ sigma model in the previous subsection. An advantage of the particular choice of the potential (E10) is that we have an analytic solution of the domain wall with which we can analytically obtain an effective action of the deformed domain wall.

In order to illustrate the situation better, let us rewrite the Lagrangian in terms of a spherical coordinate

$$\phi = e^{i\Phi}\tan\frac{\Theta}{2}. \tag{E11}$$

Then, the Lagrangian is written in the following form:

$$\mathcal{L} = \frac{1}{4}[(\partial_M\Theta)^2 + \sin^2\Theta(\partial_M\Phi)^2 - m^2\sin^2\Theta - \lambda\sin^2 2\Theta]. \tag{E12}$$

The scalar potential in this Lagrangian is identical to the so-called double sine-Gordon potential; see Fig. 4. The double sine-Gordon model has been studied for a long time, and a domain wall solution is known to be

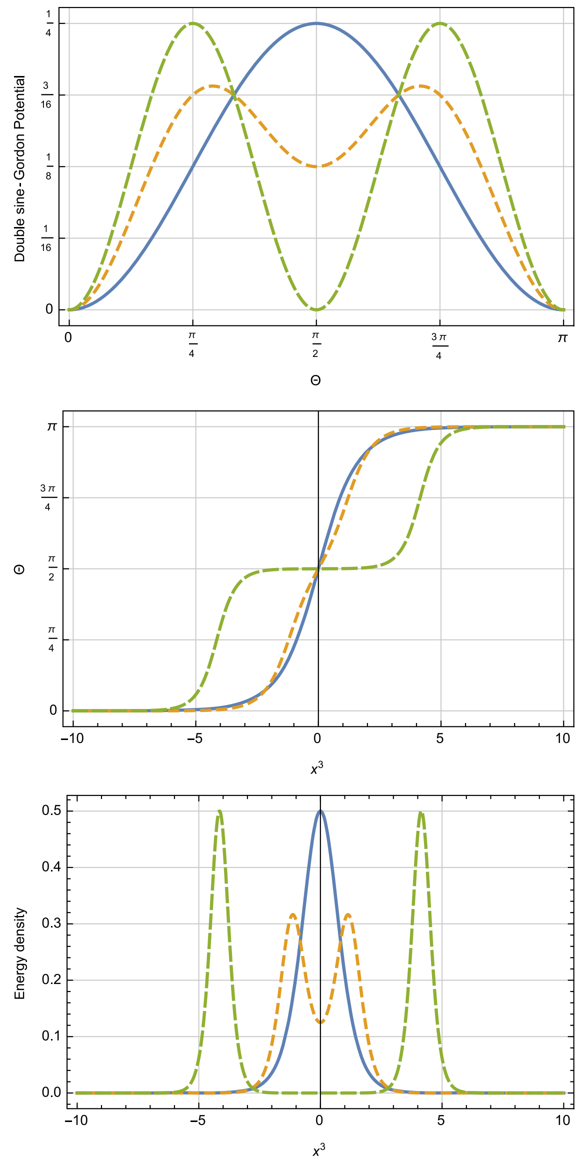

FIG. 4. The upper, middle, and lower panels show the double sine-Gordon potential, the domain wall solutions, and energy densities, respectively. Blue (solid), yellow (dashed), and green (long-dashed) lines correspond to $(m,\lambda) = (1,0), (1/2,1/2), (10^{-3},1)$.

$$\Theta = \arccos\left[\mp\frac{\sinh\left(m\sqrt{1+\gamma}(z-Z)\right)}{\sqrt{\cosh^2(m\sqrt{1+\gamma}(z-Z))+\gamma}}\right],$$
$$\Phi = \epsilon, \qquad \gamma = \frac{4\lambda}{m^2}, \tag{E13}$$

where Z and ϵ are constants. For the upper (lower) sign, Θ goes to 0 (π) as $z \to -\infty$ and to π (0) as $z \to +\infty$.

Now, we can easily translate the above domain wall solution in terms of the original $\mathbb{C}P^1$ field ϕ. It is of a slightly complicated form

$$\phi_0 = e^{i\epsilon}\left[\frac{\sqrt{\gamma+\cosh^2(m\sqrt{1+\gamma}(z-Z))} \pm \sinh\left(m\sqrt{1+\gamma}(z-Z)\right)}{\sqrt{\gamma+\cosh^2(m\sqrt{1+\gamma}(z-Z))} \mp \sinh\left(m\sqrt{1+\gamma}(z-Z)\right)}\right]^{\frac{1}{2}}. \tag{E14}$$

For the upper (lower) sign, ϕ goes to 0 (∞) as $z \to -\infty$ and to ∞ (0) as $z \to +\infty$. Let us obtain the mass of the domain wall. It can simply be done by making use of the standard Bogomol'nyi technique as

$$\begin{aligned} M &= \int dz \frac{1}{(1+|\phi|^2)^2}\left\{\left|\phi' \mp e^{i\epsilon} m\sqrt{|\phi|^2 + \frac{\gamma|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}}\right|^2 \right. \\ &\quad \left. \pm (e^{-i\epsilon}\phi' + e^{i\epsilon}\bar{\phi}')m\sqrt{|\phi|^2 + \frac{\gamma|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}}\right\} \\ &\geq \pm m \int dz \frac{(e^{-i\epsilon}\phi' + e^{i\epsilon}\bar{\phi}')}{(1+|\phi|^2)^2}\sqrt{|\phi|^2 + \frac{\gamma|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}}, \end{aligned} \tag{E15}$$

where ϵ is an arbitrary real constant. The Bogomol'nyi bound is saturated when the following first order differential equation is satisfied:

$$\phi' = \pm m e^{i\epsilon}\sqrt{|\phi|^2 + \frac{\gamma|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}}. \tag{E16}$$

This is indeed solved by ϕ given in Eq. (E14). Thus, the mass is given by

$$\begin{aligned} M(m,\gamma) &= m\int_0^\infty df \frac{2}{(1+f^2)^2}\sqrt{f^2 + \frac{\gamma f^2(1-f^2)^2}{(1+f^2)^2}} \\ &= \frac{1}{2}\left[\sqrt{m^2+4\lambda} + \frac{m^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{m^2}}\right]. \end{aligned} \tag{E17}$$

This is, of course, identical to the mass formula of the double sine-Gordon domain wall known in the literature.

We now extend the above solution to have a Q charge, which is a new solution existing only in the deformed massive $\mathbb{C}P^1$ model. In order to get the solution, the Bogomol'nyi technique is again useful,

$$\begin{aligned} M &= \int \frac{dz}{(1+|\phi|^2)^2}\left\{|\dot{\phi} - i\omega\phi|^2 + \left|\phi' \mp e^{i\epsilon(t)}\tilde{m}\sqrt{|\phi|^2 + \frac{\tilde{\gamma}|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}}\right|^2 \right. \\ &\quad \left. \pm \tilde{m}(e^{-i\epsilon(t)}\phi' + e^{i\epsilon(t)}\bar{\phi}')\sqrt{|\phi|^2 + \frac{\tilde{\gamma}|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}} - i\omega(\dot{\phi}\,\bar{\phi} - \phi\dot{\bar{\phi}})\right\} \\ &\geq \int dz\left\{\pm\tilde{m}\frac{(e^{-i\epsilon(t)}\phi' + e^{i\epsilon(t)}\bar{\phi}')}{(1+|\phi|^2)^2}\sqrt{|\phi|^2 + \frac{\tilde{\gamma}|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}} - \frac{i\omega(\dot{\phi}\,\bar{\phi} - \phi\dot{\bar{\phi}})}{(1+|\phi|^2)^2}\right\}, \end{aligned} \tag{E18}$$

where we have introduced

$$\tilde{m}^2 = m^2 - \omega^2, \qquad \tilde{\gamma} = \frac{4\lambda}{m^2-\omega^2}. \tag{E19}$$

The energy bound is saturated when the first order differential equations are satisfied,

$$\dot{\phi} = i\omega\phi, \qquad \phi' \pm e^{i\epsilon(t)}\tilde{m}\sqrt{|\phi|^2 + \frac{\tilde{\gamma}|\phi|^2(1-|\phi|^2)^2}{(1+|\phi|^2)^2}} = 0. \tag{E20}$$

This is solved by

$$\phi_Q = e^{i\omega t + i\epsilon_0}\left[\frac{\sqrt{\tilde{\gamma} + \cosh^2(\tilde{m}\sqrt{1+\tilde{\gamma}}(z-Z))} \pm \sinh(\tilde{m}\sqrt{1+\tilde{\gamma}}(z-Z))}{\sqrt{\tilde{\gamma} + \cosh^2(\tilde{m}\sqrt{1+\tilde{\gamma}}(z-Z))} \mp \sinh(\tilde{m}\sqrt{1+\tilde{\gamma}}(z-Z))}\right]^{\frac{1}{2}}. \tag{E21}$$

The mass of the Q-double sine-Gordon domain wall is given by

$$M_Q(m,\lambda,\omega) = \frac{1}{2}\left[\sqrt{m^2-\omega^2+4\lambda} + \frac{m^2+\omega^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{m^2-\omega^2+4\lambda}}\right]. \tag{E22}$$

Note that this corresponds to the mass of the usual $\mathbb{C}P^1$ Q domain wall in the limit $\lambda \to 0$ as

$$M_Q(m,0,\omega) = \sqrt{m^2-\omega^2} + \frac{\omega^2}{\sqrt{m^2-\omega^2}} = \frac{m^2}{\sqrt{m^2-\omega^2}}. \tag{E23}$$

The Q-charge density is given by

$$\begin{aligned} Q &= \int dx^3 \frac{i(\phi\dot{\phi}^* - \dot{\phi}\phi^*)}{(1+|\phi|^2)^2} \\ &= \left[\frac{\omega}{4\sqrt{\lambda}}\tanh^{-1}\left(\frac{2\sqrt{\lambda}}{\sqrt{4\lambda+m^2-\omega^2}}\tanh(x\sqrt{4\lambda+m^2-\omega^2})\right)\right]_{x^3=-\infty}^{x^3=+\infty} \\ &= \frac{\omega}{2\sqrt{\lambda}}\tanh^{-1}\frac{2\sqrt{\lambda}}{\sqrt{4\lambda+m^2-\omega^2}}. \end{aligned} \tag{E24}$$

Let us next compare the values of Lagrangian for the static and Q domain wall solutions. Substituting ϕ_0 in Eq. (E14) and ϕ_Q in Eq. (E21) into the Lagrangian (E9) and integrating it over z, we get

$$L[\phi_0] = -\frac{1}{2}\left[\sqrt{m^2+4\lambda} + \frac{m^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{m^2+4\lambda}}\right], \tag{E25}$$

$$L[\phi_Q] = -\frac{1}{2}\left[\sqrt{m^2-\omega^2+4\lambda} + \frac{m^2-\omega^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{m^2-\omega^2+4\lambda}}\right]. \tag{E26}$$

The latter can be obtained by just replacing m by $\sqrt{m^2-\omega^2}$ in the former. Having $\Lambda^\alpha{}_\alpha = \frac{1}{\sqrt{1-\frac{\omega^2}{m^2}}}$ as a Lorentz boost toward the $(d+2)$th direction, this replacement can be understood as exchanging $m \to m/\Lambda^\alpha{}_\alpha$. Thus, we have verified that the deformed $\mathbb{C}P^1$ model is in the category to which our prescription can apply.

Now, the effective action of the domain wall can be obtained by just replacing

$$m \to L_{\rm NG} = m\sqrt{1+(\partial_\mu\epsilon)^2} \tag{E27}$$

in the Lagrangian $L[\phi_0]$ given in Eq. (E25),

$$L_{\rm eff} = -\frac{1}{2}\left[\sqrt{L_{\rm NG}^2+4\lambda} + \frac{L_{\rm NG}^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{L_{\rm NG}^2+4\lambda}}\right]. \tag{E28}$$

This is very different from the standard Nambu-Goto Lagrangian ($\lambda \to 0$ limit is the Nambu-Goto Lagrangian). Conjugate momentum is given by

$$\Pi_\epsilon = \frac{m^2}{\sqrt{4\lambda}}\tanh^{-1}\sqrt{\frac{4\lambda}{m^2(1+(\partial_\mu\epsilon)^2)+4\lambda}}\partial_0\epsilon. \tag{E29}$$

Then, the Hamiltonian is

$$\begin{aligned} H_{\rm eff} = \frac{1}{2}\Bigg[&\sqrt{m^2(1+(\partial_\mu\epsilon)^2)+4\lambda} + \frac{m^2(1+(\partial_0\epsilon)^2+(\partial_i\epsilon)^2)}{\sqrt{4\lambda}} \\ &\times\tanh^{-1}\sqrt{\frac{4\lambda}{m^2(1+(\partial_\mu\epsilon)^2)+4\lambda}}\Bigg]. \end{aligned} \tag{E30}$$

Finally, we consider the solution in the effective theory

$$\epsilon(x^\mu) = \frac{\omega}{m} t + \epsilon_0. \tag{E31}$$

The corresponding energy density is

$$H_{\rm eff} = \frac{1}{2}\left[\sqrt{m^2 - \omega^2 + 4\lambda} + \frac{m^2 + \omega^2}{\sqrt{4\lambda}} \tanh^{-1}\sqrt{\frac{4\lambda}{m^2 - \omega^2 + 4\lambda}}\right]. \tag{E32}$$

This is nothing but $M_Q(m, \lambda, \omega)$ given in Eq. (E22). Furthermore, the Noether charge of the shift symmetry $\epsilon \to \epsilon + \delta$ for the solution (E31) is given by

$$Q_{\rm eff}\Big|_{\epsilon_Q} = \frac{\partial L}{\partial(m\dot{\epsilon})}\Big|_{\epsilon_Q} = \frac{1}{m}\Pi_\epsilon\Big|_{\epsilon_Q} = \frac{\omega}{2\sqrt{\lambda}} \tanh^{-1}\sqrt{\frac{4\lambda}{m^2 - \omega^2 + 4\lambda}}. \tag{E33}$$

This Noether charge in the effective theory is exactly the same as that in the original theory; see Eq. (E24). Thus, the effective Lagrangian (E28) correctly reproduces not only the mass but also the conserved charge.

3. An additional potential to massive $\mathbb{C}P^1$ sigma model: B model

Let us next consider a different scalar potential from Eq. (E10) for the $\mathbb{C}P^1$ model

$$V = \frac{\eta|\phi|^4}{(1 + |\phi|^2)^2}, \qquad (\eta > 0). \tag{E34}$$

This model is not explained in the main part of the paper, but we consider this model to give further support to the main result of the paper. This potential lifts the vacuum at the south pole $|\phi| = \infty$ while the point $\phi = 0$ is left as the unique vacuum. As shown in Fig. 5, the vacuum $|\phi| = \infty$ remains as a local minimum for $m^2 > \eta$ while it becomes a global maximum for $m^2 \leq \eta$. In terms of the spherical coordinate (E11), we find the Lagrangian in the following expression:

$$\mathcal{L} = \frac{1}{4}\left[(\partial_M \Theta)^2 + \sin^2\Theta(\partial_M \Phi)^2 - m^2 \sin^2\Theta - 4\eta \sin^2\frac{\Theta}{2}\right]. \tag{E35}$$

This is quite similar to the model (E12). A difference is the period of the scalar potential in the Θ direction. A static domain wall solution of the equations of motion is given by

$$\phi_0 = e^{i\epsilon} \frac{2m^2 e^{m(z-Z)}}{m^2 - \eta e^{2m(z-Z)}}. \tag{E36}$$

One should change coordinate from ϕ to $\phi' = 1/\phi$ in the vicinity of the point $\phi_0 \to \infty$. This solution interpolates the north pole ($\phi = 0$) at $z \to -\infty$, and passing through the south pole ($|\phi| = \infty$), it reaches back to the north pole. Since the field trajectory goes across the potential hill twice, the configuration is again a bound state of two domain walls at a finite distance. This solution might be unstable against generating tachyonic fluctuations because the S^1 trajectory surrounding the S^2 target space is contractible. In the following, we do not worry about the tachyonic modes at all. Instead, we concentrate on the zero modes and massive modes around the background solution given in Eq. (E36), since our main interest of this paper is put in the effective theory of the zero modes.

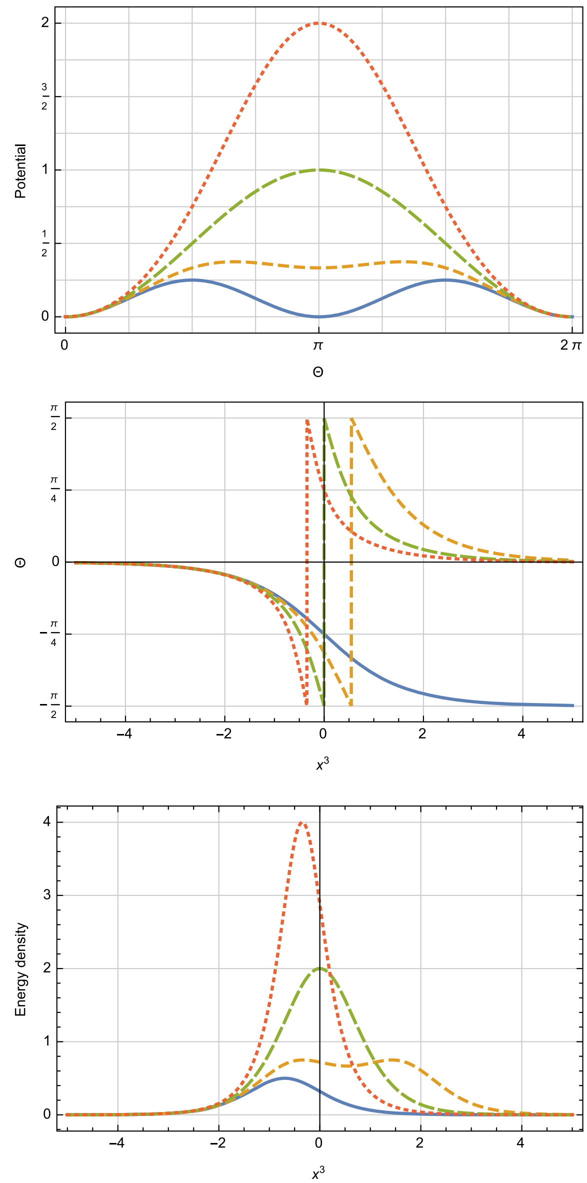

FIG. 5. The upper, middle, and lower panels show the double sine-Gordon potential B, the domain wall solutions, and energy densities, respectively. Blue (solid), yellow (dashed), green (long-dashed), and orange (short-dashed) lines correspond to $(m, \eta) = (1, 0), (1, 1/3), (1, 1)$, and (1,2).

Let us start with giving a Q-extension of the static domain wall solution in Eq. (E36),

$$\phi_Q = e^{i\omega t+\epsilon_0}\frac{2(m^2-\omega^2)e^{\sqrt{m^2-\omega^2}(z-Z)}}{m^2-\omega^2-\eta e^{2\sqrt{m^2-\omega^2}(z-Z)}}. \tag{E37}$$

We assume $\omega^2 < m^2$ because $\omega^2 = m^2$ corresponds to the vacuum solution, and $\omega^2 > m^2$ does not solve the equation of motion. The tension of this solution is given in a slightly complicated form as

$$M_Q^{m^2-\eta-\omega^2>0} = \frac{2(m^2-\eta)\sqrt{m^2-\omega^2}}{m^2-\eta-\omega^2} + \frac{\eta(m^2-\eta-2\omega^2)}{2(m^2-\eta-\omega^2)^{3/2}}\log\left(\frac{2m^2-\eta-2\omega^2+2\sqrt{(m^2-\omega^2)(m^2-\eta-\omega^2)}}{2m^2-\eta-2\omega^2-2\sqrt{(m^2-\omega^2)(m^2-\eta-\omega^2)}}\right), \tag{E38}$$

$$M_Q^{m^2-\eta-\omega^2=0} = \frac{4m^2}{\sqrt{m^2-\omega^2}}, \tag{E39}$$

$$M_Q^{m^2-\eta-\omega^2<0} = \frac{4(\eta-m^2)\sqrt{(m^2-\omega^2)(\eta-m^2+\omega^2)}+\pi\eta(\eta-m^2+2\omega^2)}{2(\eta-m^2+\omega^2)^{3/2}} + \frac{\eta(\eta-m^2+2\omega^2)}{(\eta-m^2+\omega^2)^{3/2}}\cot^{-1}\left(\frac{2\sqrt{(m^2-\omega^2)(\eta-m^2+\omega^2)}}{\eta-2m^2+2\omega^2}\right). \tag{E40}$$

As is done in the previous subsection, our first nontrivial check is comparing the value of integration of the Lagrangian over the transverse coordinate z for the static solution ϕ_0 and ϕ_Q,

$$L[\phi_0]^{m^2-\eta>0} = -2m + \frac{\eta}{2\sqrt{m^2-\eta}}\log\left(\frac{m(m-\sqrt{m^2-\eta})+m^2-\eta}{m(m+\sqrt{m^2-\eta})+m^2-\eta}\right), \tag{E41}$$

$$L[\phi_0]^{m^2-\eta=0} = -4m, \tag{E42}$$

$$L[\phi_0]^{m^2-\eta<0} = -2m - \frac{\eta}{\sqrt{\eta-m^2}}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{2m^2-\eta}{2m\sqrt{\eta-m^2}}\right)\right). \tag{E43}$$

As is expected, we find that $L[\phi_Q]$ is obtained by just replacing m by $\sqrt{m^2-\omega^2}$ in $L[\phi_0]$. Now, we can construct a low energy effective theory by replacing m with $L_{\rm NG} = m\sqrt{1+(\partial_\mu\epsilon)^2}$. Then, the Hamiltonian in the effective theory is obtained in a standard way. For ϵ dependent only on x^0 it is expressed as

$$H_{\rm eff}^{m^2-\eta-\dot\epsilon^2>0} = \frac{2(m^2-\eta)m\sqrt{1-\dot\epsilon^2}}{m^2(1-\dot\epsilon^2)-\eta} + \frac{\eta(m^2(1-2\dot\epsilon^2)-\eta)}{2(m^2(1-\dot\epsilon^2)-\eta)^{3/2}}\log\left(\frac{2m^2(1-\dot\epsilon^2)-\eta+2m\sqrt{(1-\dot\epsilon^2)(m^2(1-\dot\epsilon^2)-\eta)}}{2m^2(1-\dot\epsilon^2)-\eta-2m\sqrt{(1-\dot\epsilon^2)(m^2(1-\dot\epsilon^2)-\eta)}}\right), \tag{E44}$$

$$H_{\rm eff}^{m^2-\eta-\dot\epsilon^2=0} = \frac{4m}{\sqrt{1-\dot\epsilon^2}}, \tag{E45}$$

$$H_{\rm eff}^{m^2-\eta-\dot\epsilon^2<0} = \frac{4(\eta-m^2)m\sqrt{(1-\dot\epsilon^2)(\eta-m^2(1-\dot\epsilon^2))}+\pi\eta(\eta-m^2(1-2\dot\epsilon^2))}{2(\eta-m^2(1-\dot\epsilon^2))^{3/2}} + \frac{\eta(\eta-m^2(1-2\dot\epsilon^2))}{(\eta-m^2(1-\dot\epsilon^2))^{3/2}}\cot^{-1}\left(\frac{2m\sqrt{(1-\dot\epsilon^2)(\eta-m^2(1-\dot\epsilon^2))}}{\eta-2m^2(1-\dot\epsilon^2)}\right). \tag{E46}$$

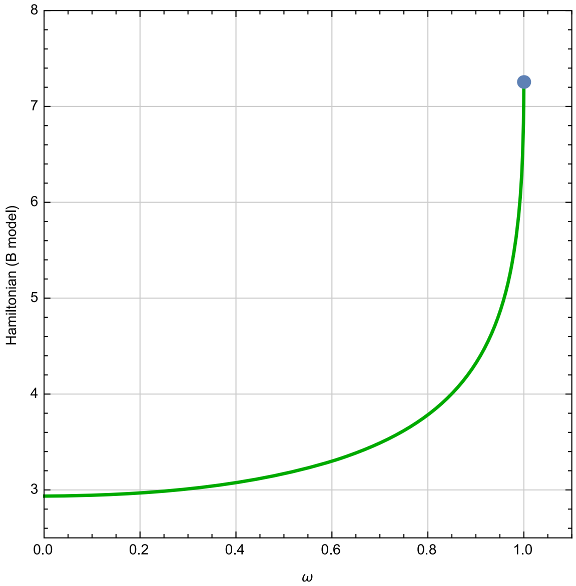

FIG. 6. The on-shell Hamiltonian for $m = 1$ and $\eta = 1/3$. $H = 2 - \frac{\log(49-20\sqrt{6})}{2\sqrt{6}}$ at $\omega = 0$ and $H = 4\pi/\sqrt{3}$ at $\omega = 1$.

By putting a solution $\epsilon = \omega t/m$ in the effective theory, we exactly reproduce the mass formula (E40) in the original nonlinear sigma model.

A peculiar feature of the B model is that the mass does not diverge at $\omega/m = 1$. Namely, there is no speed limit in the internal moduli. An intuitive explanation is the following. As $\omega \to m$, the effective mass becomes small. In the B model, the potential of the small period $(m^2 - \omega^2)\sin^2\Theta$ vanishes. This means that the two constituent domain walls are further confined into one large domain wall; see Fig. 5. Therefore, no flattering and destroying the domain wall occurs in this model; see Fig. 6.

APPENDIX F: VACUUM SYMMETRY BREAKING AND EXTRA DIMENSION

Analysis performed for (A1) can be done by using the extra dimension technique that we introduced earlier.

A classical solution is

$$\phi_0 = e^{im\epsilon}\xi, \qquad \epsilon \in \mathbb{R}. \tag{F1}$$

Here, ξ is the Higgs vet and is constant, given explicitly as

$$\xi = \sqrt{\frac{-c - m^2}{2\lambda}}. \tag{F2}$$

The vacuum depends on the parameter ϵ, which can later become a massless Nambu-Goldstone mode.

It is straightforward to obtain the on-shell action, but here we follow the procedures using the extra dimensions since it is instructive. First, we upgrade the system to a $(d+2)$-dimensional system,

$$\Phi(x^\mu, z, \alpha) = e^{im\alpha} \sum_{n=-\infty}^{\infty} e^{in\alpha/R} \phi^{(n)}(x^\mu). \tag{F3}$$

The action is

$$S = \frac{1}{2\pi R} \int_0^{2\pi R} d\alpha \int d^d x dz \mathcal{L}_{d+2}, \tag{F4}$$

$$\mathcal{L}_{d+2} = -|\partial_M \phi|^2 - V(|\phi|^2). \tag{F5}$$

The Sherk-Schwarz boundary condition is

$$\Phi(x^\mu, \alpha + 2\pi R) = e^{2\pi i m R} \Phi(x^\mu, \alpha). \tag{F6}$$

Let us consider a boosted solution in $(d+2)$ dimensions. The original solution is

$$\Phi = \Phi_0(\alpha) \equiv e^{im\alpha}\xi \tag{F7}$$

and it is boosted by a Lorentz transformation

$$\tilde{\alpha} = \Lambda^\alpha{}_\alpha \alpha + \Lambda^\alpha{}_\mu x^\mu, \qquad \tilde{x}^\mu = \Lambda^\mu{}_\alpha \alpha + \Lambda^\mu{}_\nu x^\nu. \tag{F8}$$

The result is

$$\begin{aligned} \Phi &= \Phi_0(\tilde{\alpha}) \\ &= e^{im\tilde{\alpha}}\xi \\ &= e^{im\Lambda^\alpha{}_\alpha \alpha} e^{im\Lambda^\alpha{}_\mu x^\mu}\xi. \end{aligned} \tag{F9}$$

With this new solution, we can calculate the effective action as before. With the general coordinate transformation

$$\tilde{\alpha} = \Lambda^\alpha{}_\alpha \alpha + \Lambda^\alpha{}_\mu x^\mu, \qquad \tilde{x}^\mu = x^\mu, \tag{F10}$$

the substitution of the boosted solution to the action gives

$$\begin{aligned} S &= \frac{1}{2\pi R} \int_{\Lambda^\alpha{}_\mu x^\mu}^{2\pi R \Lambda^\alpha{}_\alpha + \Lambda^\alpha{}_\mu x^\mu} \frac{d\tilde{\alpha}}{\Lambda^\alpha{}_\alpha} \int d^4x [\mathcal{L}_{d+2}|_{\Phi=\Phi_0(\alpha)}]_{\alpha \text{ replaced by } \tilde{\alpha}} \\ &= \int d^4x \mathcal{L}_{d+2}|_{\Phi=\Phi_0(\alpha)} \\ &= \int d^4x [-m^2\xi^2 - V(\xi^2)] \\ &= \int d^4x \frac{(c+m^2)^2}{4\lambda}. \end{aligned} \tag{F11}$$

Replacing m by $m/\Lambda^\alpha{}_\alpha = m\sqrt{1 + (\partial_\mu \epsilon)^2}$, we obtain the effective action for the internal moduli field $\epsilon(x^\mu)$ as

$$\begin{aligned} S &= \int d^4x \frac{(c + m^2 + m^2(\partial_\mu\epsilon)^2)^2}{4\lambda} \\ &= \int d^4x \left[\frac{\mu^4}{4\lambda} - \frac{m^2\mu^2}{2\lambda}(\partial_\mu\epsilon)^2 + \frac{m^4}{4\lambda}((\partial_\mu\epsilon)^2)^2\right]. \end{aligned} \tag{F12}$$

So, if we rescale the moduli field $\epsilon(x^\mu)$ as

$$N(x^\mu) \equiv \frac{m\mu}{\sqrt{\lambda}}\epsilon(x^\mu), \tag{F13}$$

the effective action is

$$S = \text{const.} + \int d^4x \left[-\frac{1}{2}(\partial_\mu N)^2 + \frac{\lambda}{4\mu^4}((\partial_\mu N)^2)^2\right]. \tag{F14}$$

The normalization of the Nambu-Goldstone field is encoded in the original field as

$$\phi = \frac{\sqrt{\mu^2 - \frac{\lambda}{\mu^2}(\partial_\mu N)^2}}{\sqrt{2\lambda}} \exp[i(\sqrt{\lambda}/\mu)N(x^\mu)]. \tag{F15}$$

This relation seems ill defined when the inside of the square root becomes negative. However, since the original scalar field is complex, there is no problem. The effective action itself does not provide any reality constraint; thus, there is no speed limit.

APPENDIX G: EFFECTIVE ACTION FOR 't HOOFT-POLYAKOV MONOPOLE

Let us turn to the case of the popular $SU(2)$ 't Hooft-Polyakov monopole, following [18]. We find that the effective action for the internal moduli parameter $\epsilon(t)$ for the $SU(2)$ 't Hooft-Polyakov monopole is given by a Nambu-Goto action, that is, an action for a relativistic particle whose position is given by $\epsilon(t)$. Therefore, we have a speed limit in the internal space.

We start with the $SU(2)$ Yang-Mills-Higgs theory

$$S = -\frac{1}{g^2}\int d^4x \left[\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi^a)^2\right] \tag{G1}$$

where we use the normalization of the $SU(2)$ generators as $\mathrm{tr}[T^aT^b] = \delta_{ab}/2$ $(a, b = 1, 2, 3)$, and the component expansion is $A_\mu = A^a_\mu T^a$ and $\phi = \phi^a T^a$. The field strength and the covariant derivative are defined as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{G2}$$

$$D_\mu\phi \equiv \partial_\mu\phi - i[A_\mu, \phi]. \tag{G3}$$

The BPS equation for a monopole is given by

$$B_i = D_i\phi, \qquad E_i = 0, \tag{G4}$$

with $B_i \equiv (1/2)\epsilon_{ink}F_{jk}$ and $E_i \equiv F_{i0}$. The monopole solution is given as

$$A^a_i = \epsilon_{aij}\hat{\mathbf{r}}_j(1-K(r))/r, \quad A^a_0 = \hat{\mathbf{r}}_a J(r)/r, \quad \phi^a = \hat{\mathbf{r}}_a H(r)/r, \tag{G5}$$

with

$$K(r) \equiv Cr/\sinh(Cr), \quad J(r) = 0, \quad H(r) = Cr\coth(Cr) - 1. \tag{G6}$$

Here, C is a constant parameter, and provides the Higgs VEV at spatial infinity, $\phi^a = C\hat{\mathbf{r}}_a$.

The 't Hooft-Polyakov monopole has $\mathbb{R}^3 \times S^1$ moduli space, and the latter S^1 is the internal moduli parameter. It is generated by an unbroken global part of the local symmetry generated by

$$U = \exp[-2i\epsilon\phi^a T^a/C] \tag{G7}$$

so that the periodicity for the constant moduli parameter ϵ is $0 \le \epsilon < 2\pi$. It is easy to see that the transformation leaves the scalar field solution intact, while it changes the gauge field solution by

$$\delta A_i = \epsilon\frac{2}{C}D_i\phi. \tag{G8}$$

This is the internal zero mode that we are interested in.

Now, we upgrade this constant internal moduli parameter ϵ to a one-dimensional field $\epsilon(t)$. Since the monopole is a pointlike object, its worldline is one dimensional, so the moduli can depend only on time t.

It is important to note that, once we consider a time-dependent $\epsilon(t)$, it amounts to an electric field. In fact, the transformation (G7) generates also an electric field,

$$\delta A_0 = (\partial_0\epsilon)\frac{2}{C}\phi. \tag{G9}$$

So the internal motion provides the electric field, and turns the monopole into a dyon. The famous Julia-Zee dyon solution is given by

$$K(r) \equiv C'r/\sinh(C'r), \tag{G10}$$

$$J(r) = \tanh\gamma H(r), \tag{G11}$$

$$H(r) = \cosh\gamma[C'r\coth(C'r) - 1]. \tag{G12}$$

Here, C' is related to the previous C as

$$C'\cosh\gamma = C \tag{G13}$$

such that the asymptotic value of the Higgs field ϕ is the same as that of the original monopole solution. The BPS equation for the dyon is

$$(\cosh\gamma)B_i = D_i\phi, \qquad (\coth\gamma)E_i = D_i\phi. \tag{G14}$$

Comparing (G11) in this Julia-Zee dyon solution with (G9), we find a relation

$$(\partial_0\epsilon)\frac{2}{C} = \tanh\gamma. \tag{G15}$$

We calculate the effective action of the zero mode $\epsilon(t)$. It is sufficient to calculate the on-shell action of the dyon.

Substituting the dyon solution to the original action, we obtain

$$S = -\frac{4\pi C'}{g^2}\int dt = -\frac{4\pi C}{g^2}\int dt \frac{1}{\cosh\gamma}. \tag{G16}$$

Using the relation (G15) between γ and the moduli field $\epsilon(t)$, we obtain the effective action for the internal moduli as

$$S = -\frac{8\pi}{g^2}\int dt\sqrt{(C/2)^2 - (\partial_0\epsilon)^2}. \tag{G17}$$

This is an action of a relativistic particle, in other words, a one-dimensional Nambu-Goto action. The speed of light is given by $C/2$.

As a consistency check, let us calculate the Hamiltonian and compare it with the dyon mass. The Hamiltonian calculated from the moduli effective action (G17) is

$$H = \frac{8\pi(C/2)^2}{g^2\sqrt{(C/2)^2 - (\partial_0\epsilon)^2}}. \tag{G18}$$

Substituting the relation (G15), this Hamiltonian is written with γ as

$$H = \frac{4\pi C}{g^2}\cosh\gamma = \frac{4\pi C'}{g^2}\cosh^2\gamma, \tag{G19}$$

which is exactly equal to the Julia-Zee dyon mass. So, we conclude that the moduli effective action (G17) describes correctly the dynamics of the moduli.

It is intriguing to note that the speed limit $C/2$ in the internal space turns out to be equal to the mass of the W-bosons. For the case of the $\mathbb{C}P^1$ domain walls the speed limit is given by the mass of the original scalar field, and we find a universal feature here: the internal speed limit is given by the mass of the original massive field, per second.

It might be interesting to consider the BPS 't Hooft-Polyakov monopole in the Higgs phase in $\mathcal{N} = 2$ supersymmetric QCD with $N_F = N_C = 2$ [19]. There, the VEV C of the adjoint field in an $SU(2)$ vector multiplet is determined by a fundamental quark mass matrix $M = m\sigma_3$ as $\langle\phi\rangle = M$. Namely, we have $C = 2m$. Since the monopole in the Higgs phase is pierced by a squeezed magnetic flux, a vortex string, the moduli space is $\mathbb{R}\times S^1$. Assuming the effective action (G17) is valid even in the Higgs phase, the speed limit in the internal S^1 space can be read as $\partial_0\epsilon = m$. In [19], it was found that the monopole can be identified with a kink inside the vortex string. Namely, the kink is a topological soliton in the $1+1$-dimensional massive $\mathbb{C}P^1$ model that is the low energy effective action of the internal moduli of a non-Abelian vortex [27–30]. The moduli space of the kink in $1+1$ dimensions is $\mathbb{R}^1\times S^1$ as we explained in the main body of the paper. More concretely, the massive $\mathbb{C}P^1$ sigma model for the monopole is given by $F = \frac{4\pi}{g^2}(1+|\phi|^2)^{-2}$ in Eq. (17). This leads to the Nambu-Goto action (18) with the factor m multiplied by $4\pi/g^2$. It gives the speed limit m and it is identical to the Nambu-Goto action (G17) with $C = 2m$.

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