

**Loosening up the Skyrme model**

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We consider the Skyrme model with the addition of extra scalar potentials that decrease the classical binding energies of the Skyrmions to about the 3% level—without altering the pion mass—if we insist on keeping platonic symmetries that are usually possessed by Skyrmions. A side effect of the potentials under consideration is the smaller size of the 1-Skyrmion resulting in a smaller moment of inertia and in turn a larger spin contribution to the energy upon semiclassical quantization. After taking into account the quantum contributions we find total binding energies at the 6% level.

DOI: [10.1103/PhysRevD.93.065048](https://doi.org/10.1103/PhysRevD.93.065048)**I. INTRODUCTION**

The Skyrme model was introduced by Skyrme as a model for baryons in a low-energy effective field theory of pions [1,2]. It first caught serious attention when it was shown that its soliton—the Skyrmion—is the baryon in the large- $N_c$  limit of QCD [3,4]. It took, however, a while before the higher-charged Skyrmion solutions—corresponding to baryons with  $B > 2$ —were found. The breakthrough came with the introduction of the rational maps, where the Skyrmion is split into a radial component and a sphere which is mapped to a Riemann sphere that is wrapped  $B$  times [5,6]. The rational maps are believed to describe the minimizers of the energy functional of the Skyrmions to quite high precision for vanishing pion mass and for small  $B \leq 7$ . For small baryon numbers the pion mass does not have a big impact on the Skyrmion solutions. However, when the pion mass is turned on—at approximately the value of the physical pion mass—the Skyrmions prefer to order themselves as a crystal of alpha particles [7] as opposed to the fullerenes described by the rational maps. The Skyrmions do capture many phenomenological features of nuclear physics and moreover it gives a geometrical interpretation of the physics behind. Nevertheless, a long-standing problem of the Skyrme model—which has been evident from the different calibration attempts [8,9]—is that the binding energies naturally come out too large, about one order of magnitude too large. More precisely, the recalibration of the Skyrme model in Ref. [9] ameliorates the problem of the large binding energies by using a higher-charged Skyrmion ( $B = 6$ ) as input (as opposed to the calibration using the proton and delta resonance [8]).

The problem of too large binding energies has been the motivation for improving the Skyrme model and gave rise to three recent directions to do so. One attempt is to make a model with an infinite tower of mesons, which is truly BPS (Bogomol'nyi-Prasad-Sommerfield) in the limit where all

the mesons are included [10,11]. This model is derived from the self-dual Yang-Mills theory in five dimensions. The second line of research is based on a modified Lagrangian that is composed of only a sixth-order derivative term (as opposed to the standard kinetic term and the fourth-order Skyrme term) as well as a potential; this theory is called the BPS Skyrme model [12,13] and as opposed to the normal Skyrme model (that does not have solutions saturating its bound), its BPS bound on the energy can be saturated for solutions with arbitrary large baryon numbers. The third and last attempt to ameliorate the large binding energy was made from the observation that the pure Skyrme term (fourth order) as well as a unique potential to the fourth power saturates an energy bound [14] and thus is BPS for a single baryon ( $B = 1$ ). This model is called the lightly bound Skyrme model [15]. Its higher-charged Skyrmions, however, do not saturate said bound [14], but they do in fact lie so close to the bound that the model indeed gives rise to very small classical binding energies—of the order of experimental data.

Although the lightly bound Skyrme model is a promising attempt at producing viable binding energies for possibly all nuclei, it has a drastic difference with the normal Skyrme model, namely the shapes of the Skyrmions [15]. Its higher-charged Skyrmion solutions take the shape of  $B$  spheres situated at the vertices of a face-centered cubic (FCC) lattice. This is quite in contrast to the Skyrmions of the normal Skyrme model that prefer to sit in a lattice of alpha particles. The latter is quite a welcomed feature from the point of view of nuclear clusters [16] (see e.g. Fig. 6 in Ref. [16]), which indeed hint at the importance of the alpha particles or the  $B = 4$  solutions in baryons with higher baryon numbers.

A remarkable achievement in the Skyrme model is the description of the Hoyle state in  $^{12}\text{C}$  (carbon-12) and its corresponding band of rotationally excited states [17]. In this normal formulation of the Skyrme model, two classical solutions with baryon number 12 are found to have almost the same classical energy, but very different shapes, resulting in moments of inertia whose ratio is about 2.5—in perfect agreement with experimental data [17]. The

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ratio is indeed observable from the slopes of the rotational bands coming from the ground state and the Hoyle states, respectively.

The reconciliation of the two above-mentioned results is however hard to meet. The lightly bound Skyrme model, in contrast to the normal Skyrme model, predicts 12 spheres situated at the vertices of the FCC lattice with nearly the same energies of all its different configurations (this is of course just a simple argument from the fact that the overlap of the spheres is marginal and thus the energy is roughly independent of where the spheres are placed on the nearby vertices). It is easy to convince oneself that there are many different configurations with almost the same energy, but different moments of inertia. This degeneracy is observed already at the classical level for  $B = 6, 7, 8$  in Ref. [15] (for instance, five different configurations with  $B = 8$  and nearly the same energy were found) and so it is expected to be even higher for  $B = 12$ . Although there might exist one classical Skyrmion configuration with approximately 7 MeV higher energy than the global minimizer—the ground state—and possibly giving rise to a slope that is 2.5 times lower than that of the ground state, there will still be too many other states with different slopes. Whether quantization or some other mechanism can solve this puzzle is beyond the scope of the present paper.

The mechanism at work in the lightly bound Skyrme model [15] is a repulsive force due to the nonlinear potential of the form  $(1 - \text{Tr}[U]/2)^4$  that acts at short distances and is strong enough to separate the  $B$ -Skyrmion into  $B$  identifiable spheres that are still bound together. Notice that due to the nonlinearity of the potential, it does not alter the linear force present in the Skyrme model without the addition of this potential. The long-range attractive forces present in the normal Skyrme model thus remain. Exactly this type of potential was studied long ago in the baby Skyrme model [18], see also Refs. [19–22].

In this paper the scope is to study (a part of) the parameter space of a class of potentials

$$V_n \propto \frac{1}{n} \left( 1 - \frac{1}{2} \text{Tr}[U] \right)^n, \quad n > 2 \quad (1)$$

exhibiting repulsive forces and determine how low binding energies can be attained without losing the  $B = 4$  cube that is a welcomed feature of the Skyrme model in light of clustering into alpha particles. As the parameter space of the linear superposition of several potentials is obviously huge, we limit ourselves to a slice in the parameter space spanned by  $V_2$  and  $V_4$ .  $V_4$  is exactly the holomorphic type of potential of the lightly bound Skyrme model [15], whereas  $V_2$  is a similar potential with a smaller repulsive force.

We find that both  $V_2$  and  $V_4$  decrease the classical binding energies, but  $V_2$  is able to lower the classical binding energies further without breaking the platonic

symmetries of the Skyrmions, however, not quite enough to reach the experimentally observed values of nuclei. The inclusion of the pion mass was originally thought to be a minor effect but its effect is studied over the entire selected region of parameter space. It turns out that although it lowers the classical binding energies when the potentials  $V_2$  and  $V_4$  are turned off, it actually increases the classical binding energies when a sizable value of the coefficient of either one of the potentials is turned on. Although this effect is less welcome, it also has the effect of maintaining the platonic symmetries to larger values of the coefficients of said potentials. After finding the optimal point in the parameter space—which turns out to be at  $(m_2, m_4) \sim (0.7, 0)$ —a calibration to physical units is done and an estimate of the contributions due to spin and isospin quantization is taken into account. The result is that the  $V_2$  model can retain platonic symmetries and have total binding energies at the 6% level (whereas the classical contribution is near the 3% level).

The paper is organized as follows. We introduce the Skyrme model with the additional potentials in Sec. II and present numerical results in Sec. III. Finally, we conclude with a discussion in Sec. IV.

## II. THE MODEL

The Lagrangian density of the model under study is given by

$$\mathcal{L} = \frac{c_2}{4} \text{Tr}[L_\mu L^\mu] + \frac{c_4}{32} \text{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]) - V(U), \quad (2)$$

where  $L_\mu \equiv U^\dagger \partial_\mu U$  is the left-invariant  $\mathfrak{su}(2)$ -valued current,  $c_2 > 0$  and  $c_4 > 0$  are positive-definite real constants,  $\mu, \nu = 0, 1, 2, 3$  are spacetime indices,  $U$  is the Skyrme field related to the pions as

$$U = \mathbf{1}_2 \sigma + i\tau^a \pi^a \quad (3)$$

obeying  $U^\dagger U = \mathbf{1}_2$  which translates into  $\sigma^2 + \pi^a \pi^a = 1$ ,  $\tau^a$  are the Pauli matrices and finally, the potential is taken to be a function of  $\text{Tr}U$  with the vacuum expectation value of  $U$  being at  $U = \mathbf{1}_2$ . This vacuum breaks  $\text{SU}(2) \times \text{SU}(2)$  spontaneously down to a diagonal  $\text{SU}(2)$ , but it keeps the latter  $\text{SU}(2)$ —corresponding to isospin—unbroken.

The target space of the Skyrme model  $\mathcal{M} \simeq \text{SU}(2) \simeq \mathcal{S}^3$  has a nontrivial homotopy group

$$\pi_3(\mathcal{M}) = \mathbb{Z}, \quad (4)$$

which admits solitons called Skyrmions. The topological degree  $B \in \pi_3(\mathcal{S}^3)$  is defined as

$$B = \frac{1}{2\pi^2} \int d^3x \mathcal{B}^0, \quad (5)$$

where the baryon charge density is given by

$$\mathcal{B}^0 = -\frac{1}{12} \epsilon^{ijk} \text{Tr}[L_i L_j L_k]. \quad (6)$$

$B$  is often called the baryon number.

The model is a nonlinear sigma model, which means that a lot of ambiguity is left in the potential. The vacuum is at  $U = \mathbf{1}_2$  around which small excitations of the field correspond to physical pions. Therefore one physical parameter that is known in the pion vacuum is the pion mass, which is given by

$$m_\pi^2 = -2 \left. \frac{\partial V}{\partial \text{Tr}[U]} \right|_{U=\mathbf{1}_2}. \quad (7)$$

Hence the traditional pion mass term is written as

$$V_1 = m_1^2 \left( 1 - \frac{1}{2} \text{Tr}[U] \right), \quad (8)$$

giving rise to a pion mass

$$m_\pi^2 = m_1^2. \quad (9)$$

However, another potential called the modified pion mass term is given by [23–26]

$$V_{02} = \frac{1}{2} m_{02}^2 \left( 1 - \frac{1}{4} \text{Tr}[U]^2 \right), \quad (10)$$

which also yields Eq. (9), see also Refs. [27–31].<sup>1</sup> By just knowing the pion mass, we cannot distinguish between the potentials  $V_1$  and  $V_{02}$  given in Eq. (8) and (10), respectively. The difference is that  $V_{02}$  gives *exactly* the pion mass term, whereas  $V_1$  gives the pion mass term as well as higher-order pion interactions, such as  $(\pi^a \pi^a)^2$  and higher powers.

In fact, from just the pion mass term, *any* normalized linear combination of the terms<sup>2</sup>

$$V_{0n} = \frac{1}{n} m_{0n}^2 \left( 1 - \frac{1}{2^n} \text{Tr}[U]^n \right) \quad (11)$$

gives rise to the physical pion mass around the vacuum  $U = \mathbf{1}_2$ .

One aspect of this argument is that the pion mass is only the sum of any of the terms  $V_{0n}$  in Eq. (11); the other side of the same coin is that there is an enormous ambiguity in the nonlinearity of the potential.

<sup>1</sup>This potential has two degenerate vacua allowing for a domain wall interpolating between them.

<sup>2</sup>See also Refs. [23,27].

In particular, we can write a class of potentials

$$V_n = \frac{1}{n} m_n^2 \left( 1 - \frac{1}{2} \text{Tr}[U] \right)^n, \quad (12)$$

which for  $n \geq 2$  gives no contribution to the pion mass in the vacuum  $U = \mathbf{1}_2$ .<sup>3</sup>

As we mentioned in the Introduction, one of these potentials, namely  $V_4$  has received some attention recently, due to the fact that it saturates a lower bound on the energy, giving a Skyrmion mass proportional to the baryon number [14]. Unfortunately, only the solution for  $B = 1$  (a single baryon) saturates the energy bound [14]. However, solutions with baryon numbers larger than 1 have masses quite close to the bound, yielding the possibility for relatively small classical binding energies. The model is therefore dubbed the lightly bound Skyrme model [15].

Let us contemplate for a moment what happens when adding a potential  $V_n$  of Eq. (12) to the Skyrme Lagrangian density. Since the Skyrmion is a map from the target space  $S^3$  to space  $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ , of positive degree, then at least  $B > 0$  points in configuration space ( $\mathbb{R}^3$ ) will attain the value  $U = -\mathbf{1}_2$ , i.e. the antipodal point to the vacuum on the target space. At these points, all the potentials  $V_n$  (for any  $n > 0$ ) have their maximum value. Since the map is topological, the Skyrmion cannot avoid going over the points, but the effect is clear. The Skyrmion field wants to get away from the antipodal points as quickly as possible, but due to the presence of the kinetic term, this induces an effective repulsion between the antipodal points of the Skyrmion. The implication is a reduction of the binding energy. A similar effect was observed for the same potential in the baby Skyrme model, where the authors called the baby Skyrmions aloof due to the latter effect [20].

Let us define a rescaled mass

$$\tilde{m}_n \equiv \frac{2^{\frac{n}{2}} m_n}{\sqrt{n}}. \quad (13)$$

At the antipodal point on the target space,  $V_n/\tilde{m}_n^2$  tends to unity. Therefore, if we now hold  $\tilde{m}_n$  fixed and increase  $n$ , nothing changes at the antipodal point, but the function goes to zero faster the larger  $n$  is. It is now clear that the potential  $V_n$  with larger  $n$  induces stronger repulsion than  $V_n$  with a smaller  $n$ . In particular, the repulsion is a monotonically increasing function of  $n$ . Figure 1 shows the potentials  $V_n/\tilde{m}_n^2$  for various values of  $n$ .

The potentials  $V_n$  for  $n > 1$  are basically free parameters of the theory as they are not directly measured (and are not related to the pion mass). This is not the case for the

<sup>3</sup>A recent paper considers this class of potentials in the BPS Skyrme model [32].

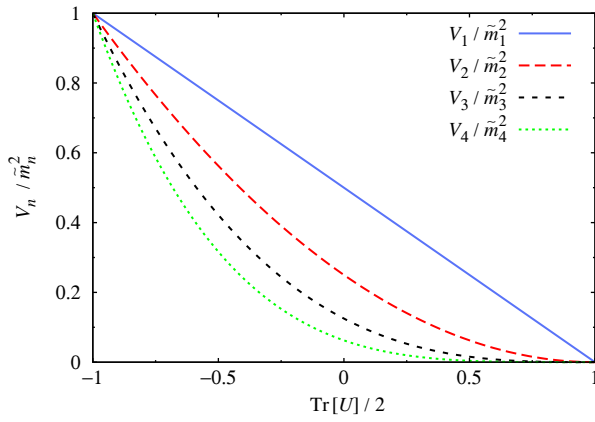


FIG. 1. Potentials  $V_n/\tilde{m}_n^2$  normalized by the rescaled masses as functions of  $\text{Tr}[U]/2$  for  $n = 1, 2, 3, 4$ .

potentials  $V_{0n}$  whose sum is constrained to be within reasonable range of the measured pion mass.<sup>4</sup>

As we mentioned in the Introduction, the reduction of the binding energy is of course more than welcome. However, the repulsion—if too excessive—also leads to Skyrmions with different symmetries than the platonic symmetries and in particular not preferring crystals of alpha particles. Reference [15] found that the Skyrmion in the limit of large  $m_4$  consists of  $B$  spheres located at the vertices of an FCC lattice.

In this paper, we will consider a more complicated potential

$$V = V_1 + V_2 + V_4, \quad (14)$$

which depends on the parameters  $m_1, m_2$  and  $m_4$ . In light of the above discussion, it is clear that  $V_4$  induces more repulsion than  $V_2$  which in turn induces more repulsion than  $V_1$ . The value of  $m_1$  is, however, not quite a free parameter, but  $m_2$  and  $m_4$  are.

Now let us consider the coefficients  $c_2$  and  $c_4$ . The Skyrme units correspond to  $c_2 = c_4 = 2$  where energies and lengths are given in units of  $f_\pi/(4e)$  and  $2/(ef_\pi)$ , respectively, see Ref. [33]. As the region where the repulsion is large, corresponding to smaller binding energies, is where the parameters  $m_2$  and  $m_4$  are large, we choose to use different values for the coefficients  $c_2$  and  $c_4$ , namely

$$c_2 = \frac{1}{4}, \quad c_4 = 1. \quad (15)$$

Now the energies and lengths are given in units of  $f_\pi/e$  and  $1/(ef_\pi)$ , respectively. When the normal Skyrme model units

<sup>4</sup>The reason for not fixing the pion mass to the exact value measured in experiment is that the latter value is the pion mass in the pion vacuum, appropriate for describing pion physics. The pion mass relevant for the Skyrmion is the renormalized effective pion mass inside the baryon. This value is not necessarily the same, but is expected to be within a factor of a few within the measured value.

are used, a common choice of the pion mass is  $m_\pi = 1$ , which in our rescaled units corresponds to  $m_\pi = 1/4$ .

A mathematical problem is to find an energy bound for the Skyrme model with the potential (14) and the closer the energies for various  $B$ -Skyrmions are to the bound, the smaller the classical binding energy must be.

Here, we are instead interested in a more difficult problem. We want to get as close to the (best possible) energy bound as we can *and* at the same time keep the symmetries of the strongly bound Skyrmions. In particular, we want the binding energy per nucleon of  $B = 4$  to be larger than that of  $B = 5$  (and also that of  $B = 8$ ). This latter condition implies that higher- $B$  Skyrmions are composed by crystals of alpha particles.

This problem is of course somewhat difficult to address from a purely mathematical angle. We therefore turn to numerical methods and calculate numerical solutions in the next section.

### III. NUMERICAL SOLUTIONS

In this section we embark on a large-scale numerical calculation of many series of Skyrmion solutions in the parameter space spanned by  $\{m_1, m_2, m_4\}$  for  $B = 1, 2, 3, 4, 5$ . We do not consider  $B > 5$  in this paper due to the amount of computing resources needed for this investigation. However, our analysis should be sufficient for having only  $B$  equal 1 through 5.

Let us first mention the numerical method we will use to calculate the numerical Skyrmion solutions. We will discretize space with the finite-difference method using a fourth-order stencil and then cool the partial differential equations with the relaxation method until a static solution has been found to the accuracy that we require. The relaxation method of course requires an initial condition (configuration), for which we will use an appropriate rational map *Ansatz* with the given baryon number  $B$ . We will use the rational maps given in Ref. [6].

Let us define the observables that we calculate for each solution. Of course the classical mass of the Skyrmion is an important value. However, it will be convenient to evaluate the classical binding energy

$$\Delta_B = BE_1 - E_B, \quad (16)$$

and in particular the *relative* (classical) binding energy, which we define as

$$\delta_B \equiv \frac{\Delta_B}{BE_1} = 1 - \frac{E_B}{BE_1}. \quad (17)$$

This observable is very easy to compare to experimental data as the units drop out. Comparing all solutions for all values of  $B$ , we define

$$\varepsilon_\delta(a) \equiv \sum_B (\delta_B^{\text{exp}} - \delta_B - a)^2, \quad (18)$$

which measures the overall discrepancy between the solutions and the experimental data for a given parameter-space point. The parameter  $a$  is introduced as an overall bias, reflecting the fact that the ground state energy of the quantized 1-Skyrmion is the classical mass plus spin- $\frac{1}{2}$  and isospin- $\frac{1}{2}$  contributions, whereas e.g. the ground state energy of the 4-Skyrmion is simply the classical energy.<sup>5</sup> The ground states of the 2- and 3-Skyrmions are the spin-1, isospin-0 and spin- $\frac{1}{2}$ , isospin- $\frac{1}{2}$  states, respectively. Nevertheless, the additional contribution to the ground state energy for the 1-Skyrmion typically turns out to be larger than both that of the 2- and 3-Skyrmions. This can be understood from the fact that the 2- and 3-Skyrmions are larger resulting in larger moments of inertia and in turn smaller quantum contribution to their energies.

A more rigorous method would be to identify the symmetries of the  $B$ -Skyrmions for each point in the parameter space and then quantize their zero modes, incorporating the Finkelstein-Rubinstein constraints for each of them, evaluating the moments of inertia tensors and calculating their ground state energies. For now, we will stick to just evaluating the classical binding energies, knowing that they should be somewhat smaller than the experimental values, but still in the ballpark.

Another observable is the size of the Skyrmion, which we define in terms of the baryon charge density (6) as

$$r_B^2 \equiv \frac{1}{2\pi^2 B} \int d^3x r^2 \mathcal{B}^0, \quad (20)$$

where  $r^2 = x^2 + y^2 + z^2$  is a radial coordinate measured from the center of the charge distribution.<sup>6</sup> The length unit is just fitted to experimental data; therefore it will prove convenient to use a *relative* size

$$\rho_B \equiv \frac{r_B}{r_1}, \quad (21)$$

where  $r_B = \sqrt{r_B^2}$  and  $\rho_B$  is given in units of the size of the  $B = 1$  solution. Comparing again all solutions for all values of  $B$ , we define

<sup>5</sup>The contribution from the spin and isospin quantization of the 1-Skyrmion to the energy modifies  $\delta_B$  as

$$\delta_B \rightarrow 1 - \frac{E_B}{B(E_1 + \varepsilon_1)} = 1 - \frac{E_B}{BE_1} + \frac{\varepsilon_1 E_B}{BE_1^2} + \mathcal{O}(\varepsilon_1^2), \quad (19)$$

where we for simplicity use the parameter  $a$  instead of the physical parameter  $\varepsilon_1$ . There is also a contribution  $\varepsilon_B$ , but it is typically a smaller effect.

<sup>6</sup>A recent paper argues that using the baryon charge density for the volume/size is in some sense the natural way in Skyrme-like models [34] (as opposed to using e.g. the energy density).

$$\varepsilon_\rho \equiv \sum_B \varepsilon_{\rho_B}, \quad \varepsilon_{\rho_B} \equiv \rho_B^{\text{exp}} - \rho_B. \quad (22)$$

Notice that we do not square the summands so that the sign will be evident (negative if the solutions are too large and positive if not).<sup>7</sup>

Finally, an observable which gives a good handle on the accuracy, is the numerically integrated baryon number (5). Our solutions will be equal to the integer  $B$  with an accuracy in the range of [0.16%, 0.019%] (with an overall average around 0.052%).

For the  $B = 1$  sector, we calculate all the solutions with very high accuracy using the ordinary differential equation (ODE) derived from the Lagrangian density (2) with the hedgehog Ansatz:  $U = \mathbf{1}_2 \cos f(r) + i\tau \cdot \hat{x} \sin f(r)$ . The ODE reads

$$\begin{aligned} & c_2 \left( f_{rr} + \frac{2}{r} f_r - \frac{\sin 2f}{r^2} \right) + c_4 \left( \frac{2\sin^2(f) f_{rr}}{r^2} \right. \\ & \quad \left. + \frac{\sin(2f) f_r^2}{r^2} - \frac{\sin(2f) \sin^2 f}{r^4} \right) \\ & = m_1^2 \sin f + m_2^2 (1 - \cos f) \sin f \\ & \quad + m_4^2 (1 - \cos f)^3 \sin f, \end{aligned} \quad (23)$$

where  $f_r \equiv \partial_r f$ , etc. The solution of the above equation yields  $E_1(m_1, m_2, m_4)$  with very high accuracy (better than the  $10^{-6}$  level). Let us now comment on how we calculate the energy for the  $B > 1$  solutions. As the  $B = 1$  sector is very accurate, we need a precise estimate of the energy for the higher- $B$  solutions in order to calculate the classical binding energy (16) and in turn the relative classical binding energy (17) (otherwise we will underestimate them). First, we find our solution relaxed down to the accuracy level such that all equations of motion are satisfied better than the  $10^{-3}$  level locally. From this point on, the energy as a function of relaxation time  $\tau$  (steps) is then fitted to an exponential curve and this process is continued until the accuracy of the exponential fit has converged to a given accuracy. Then we take the  $\tau \rightarrow \infty$  limit of the exponential as an estimate of the asymptotic energy value. This trick is very precise and saves some computation time. Now, since our finite-difference lattice is also just an approximation to the continuous field and the fact that the Skyrmion charge is a convex function (resulting in  $B_{\text{numerical}} < B$ ), we compensate the final result by  $B/B_{\text{numerical}}$ . The final result has the form

<sup>7</sup>Although this definition allows for the caveat that some cancellation between different  $B$ -Skyrmion sizes takes place, this will not be an issue as all the  $B$ -Skyrmions are generally too small compared to nuclei.

$$E_B \approx \frac{B}{B_{\text{numerical}}} \times \frac{E_{B,\text{numerical}}(\tau_0)E_{B,\text{numerical}}(\tau_2) - E_{B,\text{numerical}}^2(\tau_1)}{E_{B,\text{numerical}}(\tau_0) - 2E_{B,\text{numerical}}(\tau_1) + E_{B,\text{numerical}}(\tau_2)}, \quad (24)$$

where  $\tau_0$  is the relaxation time where the solution is good enough for the initial accuracy level (equations of motion at the  $10^{-3}$  level, locally),  $\tau_2$  is the final relaxation time where the exponential fit is precise enough and  $\tau_1 = (\tau_0 + \tau_2)/2$ . After this complicated process of estimating the energy of the Skyrmion solution, we check for the  $B = 1$  sector that we obtain the energies within an accuracy of about  $2.7 \times 10^{-4}$  or better.

We are now ready to present the results in the next subsections, for vanishing and nonvanishing pion mass, respectively, and finally the effect of semiclassical zero-modes quantization.

### A. Zero pion mass

We will begin with taking a vanishing pion mass  $m_1 = 0$  and scan (a part of) the  $(m_2, m_4)$  parameter space. In the next subsection we will consider the inclusion of the pion mass.

We start by calculating the Skyrmion energies in the  $B = 1$  sector, for which as we mentioned above use simply the ODE. This is very precise and we will use these energies as the basis to calculate the binding energies for the higher- $B$  Skyrmion solutions. Figure 2 shows the energies in our units (which are normalized differently than the normal Skyrme units) for solutions in the  $(m_2, m_4)$ -parameter space. For comparison Fig. 2 has the normal Skyrme units on the right-hand scale. Throughout this section the ranges of the masses in the parameter space

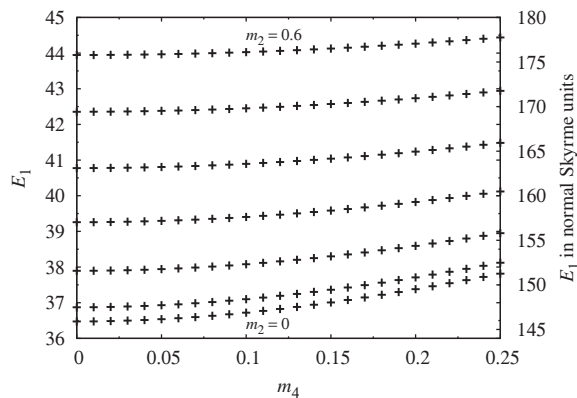


FIG. 2. Energy of the  $B = 1$  Skyrmion with various values in the  $(m_2, m_4)$ -parameter space. The series of points is for  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  with  $m_2$  increasing from bottom to top. The left-hand scale shows the units we are using in this paper while the right-hand scale shows the normal Skyrme units.

will be chosen as  $m_4$  from 0 to 0.25 with steps of 0.01 and for  $m_2$  from 0 to 0.6 with steps of 0.1.

Now we are ready to calculate the higher- $B$  Skyrmions. We use very small increasing/decreasing steps for  $m_4$  and use the latest data point as an initial condition for the next one. We tried going both from the  $(m_2, m_4) = (0, 0)$  point and upwards in masses and the reverse in order to check that the solutions found are really the minimizers of the energy for the given value of  $(m_2, m_4)$ . As we mentioned already, the  $(m_2, m_4) = (0, 0)$  point is calculated with the initial conditions constructed from the rational map *Ansätze* of Ref. [6]. If the steps in, for instance  $m_4$ , are too large then the direction (increasing or decreasing of the mass) may give different solutions to the approximated accuracy levels chosen for numerical calculations. Therefore we use quite small steps and check that the results do not change much by reversing the direction (we found some critical points in parameter space where the solutions did shift a bit, but it will not have essential consequences for our purpose here). Figures 13–16 show isosurfaces of the baryon charge density at half-maximum values for the chosen part of parameter space in the  $(m_2, m_4)$  plane (only every second solution in the  $m_4$  direction is shown in these figures due to space limitations). The coloring adapted here is chosen such that the pions are normalized to a unit vector ( $\hat{\pi}^2 = 1$ ) and  $\hat{\pi}_1$  determines the lightness whereas  $\hat{\pi}_3 + i\hat{\pi}_2$  is mapped to the hue of the color circle (the coloring scheme is similar to that adapted in Refs. [35,36], see also Ref. [37]).

Now that we have the data for a bunch of Skyrmion solutions, we begin by calculating the classical binding energies for the different points. Figure 3 shows the relative classical binding energies for all the solutions and the blue crosses represent connected Skyrmions (for the baryon charge density at half-maximum values), whereas the red  $x$ 's are disconnected. Of course it is a bit arbitrary to choose connectedness at half the maximum value of the baryon charge density; any other reasonable value may be just as good and shift the connected/disconnected lines of the figures. Nevertheless, it is clear that in the far blue area the platonic symmetries are still unbroken, whereas in the far red area the Skyrmions are spheres at the vertices of an FCC lattice.

What we seek is to find a region in parameter space where the binding energy is decreased with respect to that of the normal Skyrme model and where the platonic symmetries are more or less still present. At least the symmetries of the  $B = 4$  cubic Skyrmion would be preferable to maintain, as it provides a number of phenomenologically appealing properties as we mentioned in the Introduction.

The lesson we learn from all these data points is that increasing  $m_4$  (from zero) does indeed lower the binding energy as expected. However, long before the binding energies of experimental data are reached, the symmetries of the Skyrmions change from platonic symmetries to the

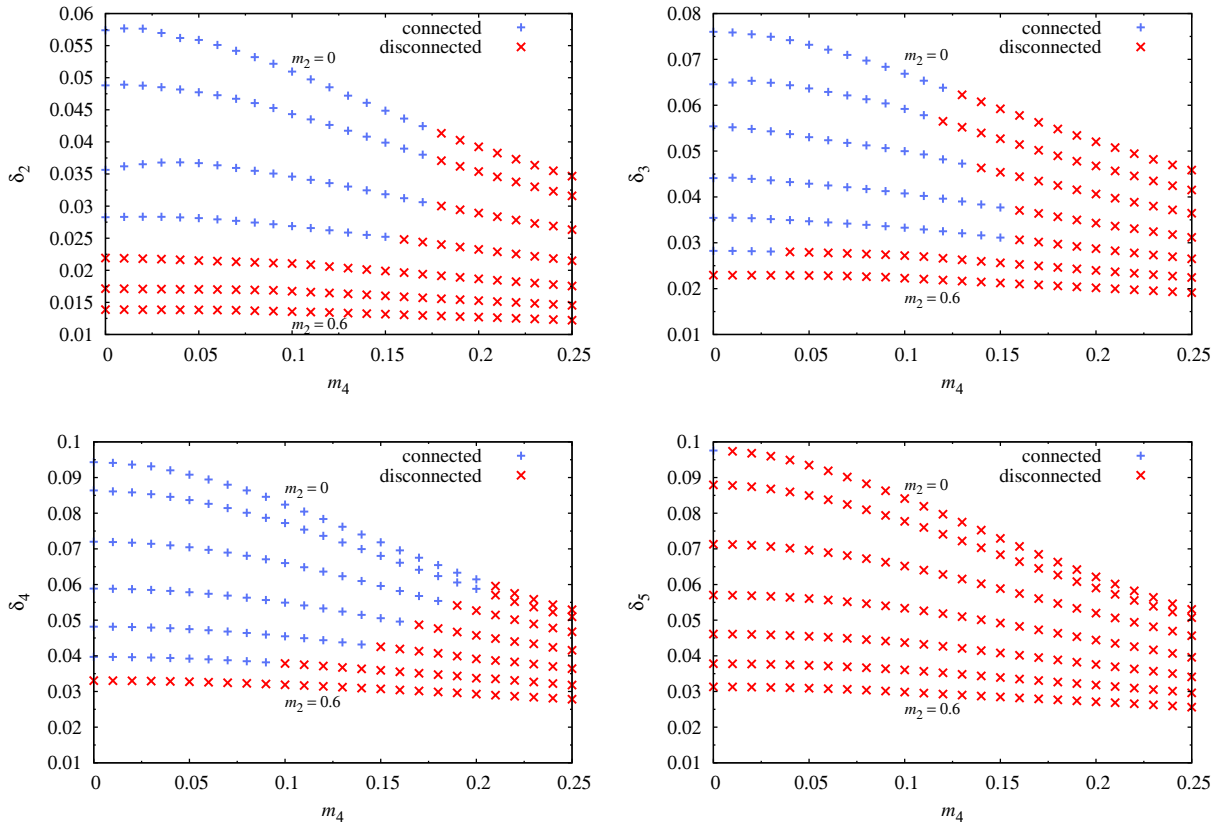


FIG. 3. Relative classical binding energies  $\delta_B$  for  $B = 2, 3, 4, 5$ . The series of points is for  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  with  $m_2$  increasing from top to bottom. The blue crosses (+) are connected isosurfaces at half-maximum baryon charge density while the red  $\times$ 's are disconnected.

FCC lattice. On the other hand, increasing  $m_2$  (again from zero) has the same qualitative effect; namely it decreases the binding energy and eventually breaks the platonic symmetries to the same FCC lattice structure of aloof Skyrmions. The difference, however, is that the binding energies obtained before the symmetries change are far lower when using  $V_2$  than when using  $V_4$ . Consider the  $B = 4$  sector in Fig. 3. If we regard the boundary between the blue and red dots as some sort of measure of change of symmetry, then the  $m_2 = 0$  branch reaches classical binding energies of about 6%, whereas the  $m_4 = 0$  branch goes down below 4%.

Moreover, it is observed from Fig. 3 that when  $m_2 = 0$ , the binding energy does go down when increasing  $m_4$ . However, when  $m_2$  is large, increasing  $m_4$  does not lower the binding energy substantially before it breaks the platonic symmetries down to the FCC lattice symmetries. Therefore, if we insist on keeping the old symmetries of the normal Skyrme model, then we can basically turn off the potential  $V_4$  and work with just  $V_2$ . If however we prefer the FCC lattice symmetries, then  $V_4$  is a suitable potential that lowers the classical binding energies, but so is  $V_2$ .

Considering now the function (18). This function is a least-squares-fit function of the parameter space to the experimental data for the nuclear binding energies. We use the experimental values shown in Table I. Figure 4 shows

the fit in the calculated part of parameter space. The black line shows where the  $B = 4$  Skyrmion splits up into disconnected pieces at the level of the isosurfaces at the half-maximum value of the baryon charge density. The left panel of the figure is the real fit of the classical binding energies to the experimental data, whereas the right panel shows a fit where the value  $a$  has been optimized to improve the fit (shape fit only). The physical meaning is that if the energy of the  $B = 1$  Skyrmion is reduced about 4%, then the preferred region of the fit is within the boundary of the black line and thus the platonic symmetries remain while the classical binding energies of the higher- $B$  Skyrmions match reasonably well the experimental values. Had the best value for  $a$  been a positive value, then semiclassical quantization could be a fix to this problem, but since it is negative then quantization will only exacerbate the problem.

TABLE I. Experimental values for nuclear masses.

$^1\text{H}$	1.007825
$^2\text{H}$	2.014101
$^3\text{He}$	3.016029
$^4\text{He}$	4.002603
$^5\text{He}$	5.012057

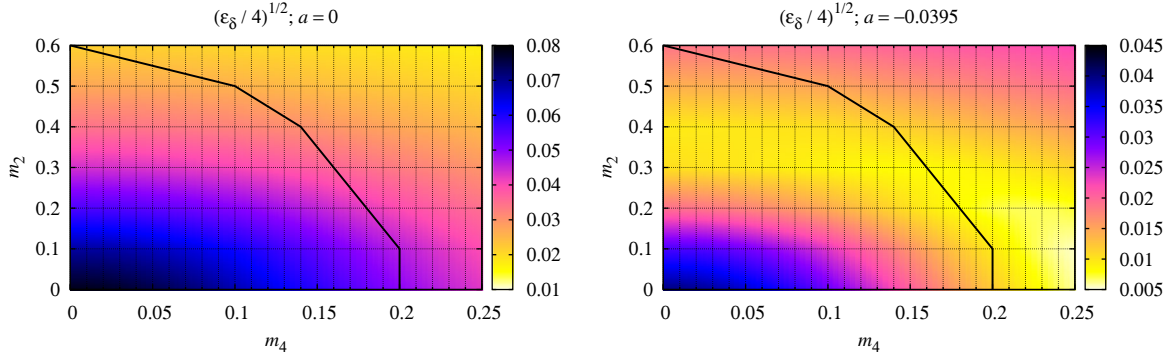


FIG. 4. Fits of the relative binding energies  $\delta_B$  summed up in the function  $\varepsilon_\delta$ .  $\sqrt{\varepsilon_\delta/4}$  corresponds to the average discrepancy of the classical binding energy, which ranges from about 8% to 1%. The black line shows where the  $B = 4$  Skyrmion splits up into disconnected pieces at the level of the isosurfaces at the half-maximum value of the baryon charge density. The zero point  $a$  is fitted in the right panel of the figure, which corresponds to ignoring the  $B = 1$  Skyrmion's energy (that is, fitting just the shape of the remaining binding energies).

Next we will consider a rough fit of the sizes of the Skyrmion solutions to the experimental values of charge radii of nuclei. The experimental values used here are shown in Table II. Of course the charge radius is not quite the size of the nucleus, but we take that as a good approximation to the latter. Figure 5 shows the fits of the Skyrmion sizes to the experimental data for the  $B = 2, 3, 4$  sectors as well as the average fit of all three sectors.

The qualitative information that can be read off of Fig. 5 is that the 2-Skyrmion and the 3-Skyrmion are generally too small. The 4-Skyrmion has about the right size when the potentials are turned off, but then the binding energies are too large. It is interesting to note that the Skyrmion size is increased by the addition of the sixth-order potential, which is the backbone of the BPS Skyrme model [12,13], see also Ref. [37].

### B. Nonzero pion mass

Now we consider a physical value of the pion mass, which corresponds to  $m_1 = 1/4$  (this is equal to  $m_\pi = 1$  in the normal Skyrme units). This value is commonly used in Skyrmion calculations, but other values could also be considered. Here we are mostly interested in the qualitative effect on our results with the addition of the pion mass.

As the common lore is that for  $B \leq 7$  the qualitative effect of the addition of the pion mass is rather small, we would *a priori* not expect big changes with respect to the last subsection. However, as we will see shortly, some changes do occur.

TABLE II. Experimental values for charge radii [38].

$^1\text{H}$	0.8783 fm
$^2\text{H}$	2.1421 fm
$^3\text{He}$	1.9661 fm
$^4\text{He}$	1.6755 fm
$^5\text{He}$	

We will start by computing the relative classical binding energies on the same parameter space as used in Fig. 3. The color code is used in the same way such that blue indicates a connected Skyrmion at the level of half-maximum baryon charge isosurfaces and red indicates a disconnected Skyrmion. The plots in the figure are arrows from the dots (without pion mass) to the heads of the arrows (with pion mass). It is interesting to note that the change due to the inclusion of the pion mass is not monotonic over the parameter space; for small  $m_2 \lesssim 0.1 - 0.2$  the binding energies decrease (more drastically for smaller values of  $m_4$  than larger values), while for  $m_2 \gtrsim 0.1 - 0.2$  the binding energies increase. The same effect occurs for  $m_4 \sim 0.1$  and larger. Another feature that we can read off the figure is that the  $B = 3$  and  $B = 4$  Skyrmons become more persistent not to deform as functions of increasing  $m_2$ . One may naively think that it may imply that smaller binding energies may be reached before the Skyrmons split up and change their symmetries, but the pion mass also increases the binding energies in that region of parameter space. Therefore there are two competing forces at play here.

In Fig. 7 we display the least-squares-fit function  $\varepsilon_\delta$  which is the average mismatch of the classical binding energies of all the Skyrmion sectors ( $B = 2, 3, 4, 5$ ) compared with the experimental data. It is seen from the figure that in this part of parameter space, the dependence on  $m_4$  is rather weak, whereas the increase of  $m_2$  decreases the average classical binding energies to about 3%. The right-hand-side panel of Fig. 7 shows a fit to the shape of the binding energies ignoring the  $B = 1$  Skyrmion's energy. This fit prefers points in the parameter space around  $(m_2, m_4) = (0.4, 0)$  (and along a line extending in the  $m_4$  direction). This shape fit corresponds to the situation where the  $B = 2, 3, 4, 5$  Skyrmons do not receive extra contribution upon semiclassical quantization and the  $B = 1$  Skyrmion has about 4.6% lower energy. Since its ground state is a spin- $\frac{1}{2}$



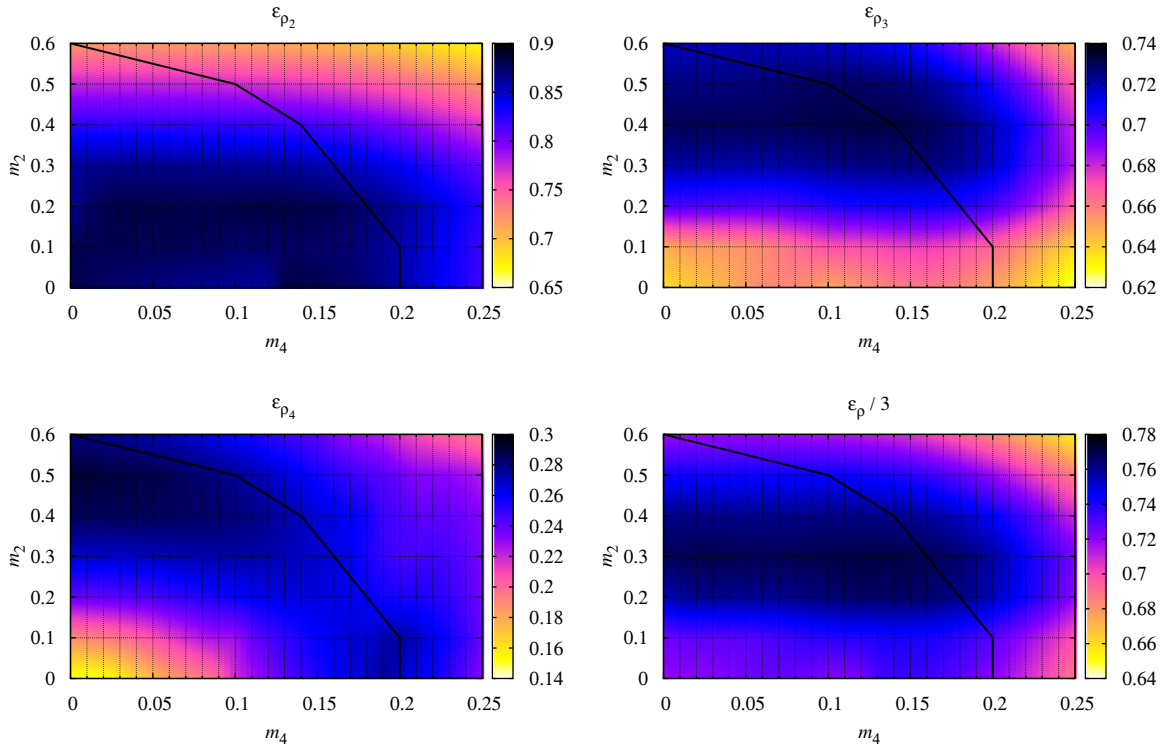


FIG. 5. Fits of relative sizes, separately for  $B = 2, 3, 4$  and at last the mean fit of the same three Skymion sectors. Positive values indicate that the Skymion size is too small compared with the experimental value for the nucleus [see Eq. (22)]. The value of  $\varepsilon_p$  corresponds roughly to the relative mismatch with data, which is in the range of 14% to 90%. The black line shows again where the  $B = 4$  Skymion splits up into disconnected pieces at the level of the isosurfaces at the half-maximum value of the baryon charge density.

state, the quantum contribution will only worsen the problem. The quantum contribution for the  $\text{spin-}\frac{1}{2}$ ,  $\text{isospin-}\frac{1}{2}$  state found in Ref. [39] is about 2.2% for the  $B = 1$  Skymion.

The effect of turning on the pion mass is evident in Fig. 6; however, the parameter space is unfortunately too small in order to see the effect of the Skymions with the pion mass turned on, breaking up into disconnected pieces and eventually situating themselves in an FCC lattice. Therefore we show a larger part of the parameter space for the  $B = 4$  sector in Fig. 8. The situation is now quite clear. The effect of increasing  $m_4$  (from zero) is a decrease in binding energy, but long before the binding energies of realistic nuclei are reached, the Skymion breaks up into disconnected pieces and soon prefers the FCC lattice structure. The effect of  $m_2$ , on the other hand, is also a decrease in binding energy, but much lower binding energies can be reached before the symmetries of the Skymion (in the  $B = 4$  sector) change. Another lesson that can be drawn from Fig. 8 is that once  $m_2$  takes on a sizable nonzero value, then the effect of  $m_4$  is rather weak (other than breaking up the Skymion), i.e. meaning that the binding energies do not drop quickly with the increase of  $m_4$ . Due to this latter fact, we will consider only  $m_4 = 0$  in the remainder of the paper.

In Fig. 9 we consider all  $B = 2, 3, 4, 5$  sectors and display the relative classical binding energies for various values of  $m_2$  ranging from zero to 1 in steps of 0.1. The isosurfaces of their baryon charge densities at half-maximum values are displayed in Fig. 10. It is seen from Fig. 9 that the larger the values of  $m_2$  are, the closer the classical binding energies come to those experimentally observed. However, for  $m_2 \sim 0.7 - 0.9$  the Skymions start to split up into disconnected pieces and soon begin the transformation from platonic symmetries to FCC lattice symmetries. Note that since these binding energies are purely classical binding energies, we are not seeking an exact match between the lines of the model calculation and the experimental data. We are merely seeking the right ballpark value and acceptable shapes of the curves. The experimental data for the nuclear binding energies should instead be compared to those of the semi-classically quantized Skymions. We will consider this in the next subsection.

Figure 11 shows the least-squares-fit function  $\varepsilon_\delta$  as a function of  $m_2$  for Skymions with  $B = 2, 3, 4, 5$  and the pion mass  $m_1 = 1/4$  turned on. The value of  $a$  that would make the model fit the experimental data is about 2%–3%, whereas the fit prefers negative values for  $a$ . This means that even the classical value of the 1-Skymion energy is too large by 0.5%–6.5%.

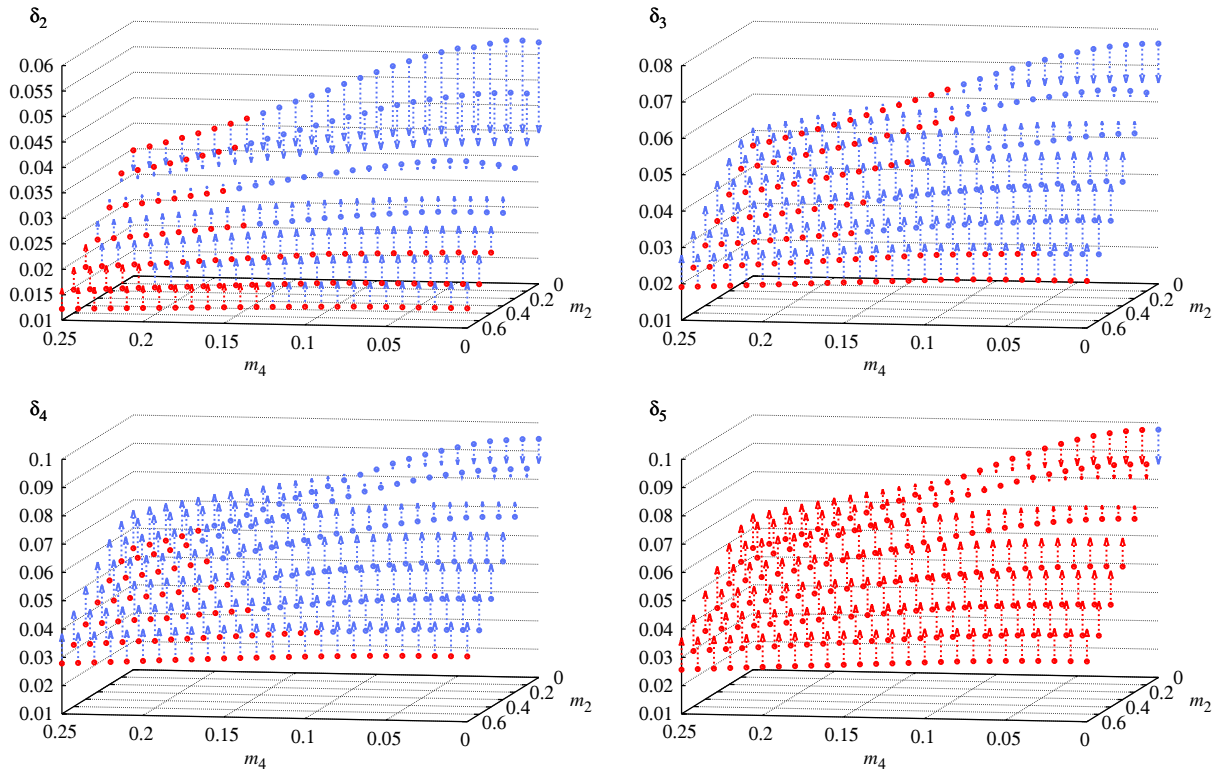


FIG. 6. Relative classical binding energies  $\delta_B$  for  $B = 2, 3, 4, 5$  with pion masses turned on  $m_1 = 1/4$ . The circles are the relative classical binding energies  $\delta_B$  without the inclusion of pion masses, i.e. the data from Fig. 3, whereas the heads of the arrows denote the new classical binding energies after inclusion of the pion mass. Blue dots and blue arrows denote connected isosurfaces at the half-maximum baryon charge level, while red dots and red arrows are disconnected.

### C. Quantization

We will now attempt to make a crude estimate of the semiclassically quantized energy contributions to the Skyrmions for  $m_2 = 0.7$ ,  $m_4 = 0$  and the pion mass  $m_1 = 1/4$  turned on. In order to carry out a rigorous job, one should establish their symmetries and probably not rely on the rigid body quantization because we are working on the borderline where the Skyrmions are trying to split up and change their symmetries. Instead of the rigid body quantization, one should consider the procedure carried out in

Ref. [40], where the isospinning of the Skyrmion is taken into account dynamically. This may reveal the symmetry to be used for the quantization. The first row of Fig. 10 shows the Skyrmions for  $m_2 = 0.7$  and  $m_4 = 0$ . For the  $B = 2$  Skyrmion, there are two options; it may break up into two localized (possibly deformed) spheres or it may restore axial symmetry upon taking isospinning into account dynamically. The  $B = 3$  and  $B = 4$  Skyrmions retain their platonic symmetries, namely tetrahedral and cubic symmetry, respectively. The symmetry of the  $B = 5$  Skyrmion

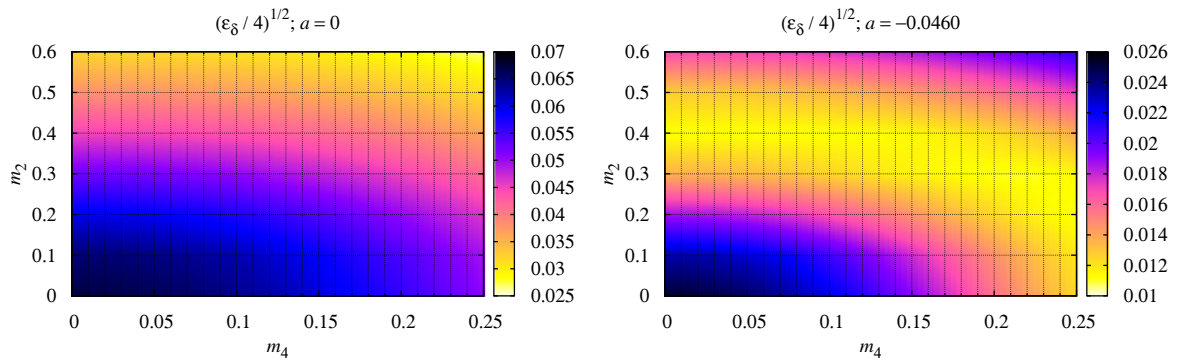


FIG. 7. Fits of the relative classical binding energies  $\delta_B$  for Skyrmions with pion masses  $m_1 = 1/4$ , summed up in the function  $\varepsilon_\delta$ .  $\sqrt{\varepsilon_\delta/4}$  corresponds to the average discrepancy of binding energy, which ranges from about 7% to 2.5%. The zero point  $a$  is fitted in the right panel of the figure, which corresponds to ignoring the  $B = 1$  Skyrmion's energy (shape fit).

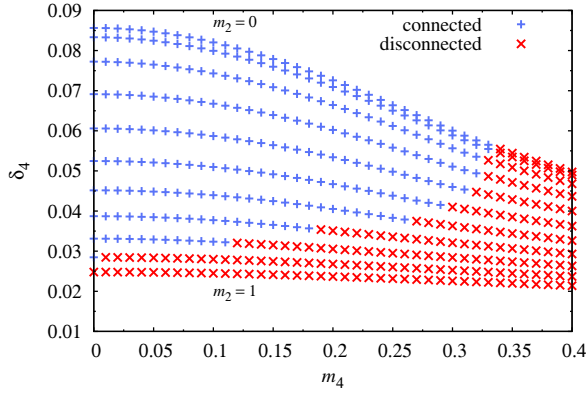


FIG. 8. Relative classical binding energies  $\delta_B$  for  $B = 4$  with the pion mass turned on  $m_1 = 1/4$ . The series of points is for  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  with  $m_2$  increasing from top to bottom. The blue crosses (+) are connected isosurfaces at half-maximum baryon charge densities while the red  $x$ 's ( $\times$ ) are disconnected.

is somewhat harder to determine at this stage. Since we are only interested in a ballpark estimate of the contribution from semiclassical quantization to their ground state energies, we will (possibly unjustified) assume that they can be quantized with the platonic symmetries used for the quantization in Ref. [39]. As we will see shortly, the mistake of this assumption (if wrong) will be negligible.

In order to add the classical Skyrmion mass and the semiclassically quantized energy contribution, we can no longer ignore the calibration of the model and have to make a choice. Fitting the  $B = 4$  sector gives rise to

$$m_2 = 0.7: e = 3.45, \quad f_\pi = 69.80 \text{ MeV}, \\ \Rightarrow m_\pi = 120.25 \text{ MeV}, \quad (25)$$

$$m_2 = 0.5: e = 3.49, \quad f_\pi = 75.65 \text{ MeV}, \\ \Rightarrow m_\pi = 132.14 \text{ MeV}, \quad (26)$$

$$m_2 = 0: e = 3.62, \quad f_\pi = 88.00 \text{ MeV}, \\ \Rightarrow m_\pi = 159.34 \text{ MeV}, \quad (27)$$

where we have used the nuclear mass of  ${}^4\text{He}$ : 3727 MeV and the charge radius of  ${}^4\text{He}$ : 1.6755 fm. As per usual in the Skyrme model, the physical values used in the  $B = 0$  sector, i.e. pion physics are not quite captured by the fits to experimental nuclear data.

As can readily be seen from the above calibrations, the choice of  $m_1 = 1/4$  is not an accurate choice and in order to match the physical pion mass, one should recalibrate the system for each  $(m_2, m_4)$  point in the parameter space and adjust  $m_1$  accordingly. In this paper, we have merely chosen an average value that fits in the ballpark of the physical value.

Using the results of Ref. [39], the semiclassical quantum contributions to the ground state energies are given by

$$E_1^{J=\frac{1}{2}, I=\frac{1}{2}} = \frac{f_\pi}{e} E_1 + \frac{3e^3 f_\pi}{8V_{11}}, \quad (28)$$

$$E_2^{J=1, I=0} = \frac{f_\pi}{e} E_2 + \frac{e^3 f_\pi}{V_{11}}, \quad (29)$$

$$E_3^{J=\frac{1}{2}, I=\frac{1}{2}} = \frac{f_\pi}{e} E_3 + \frac{3e^3 f_\pi}{8} \frac{U_{11} + V_{11} - 2W_{11}}{U_{11}V_{11} - W_{11}^2}, \quad (30)$$

$$E_4^{I=0, J=0} = \frac{f_\pi}{e} E_4, \quad (31)$$

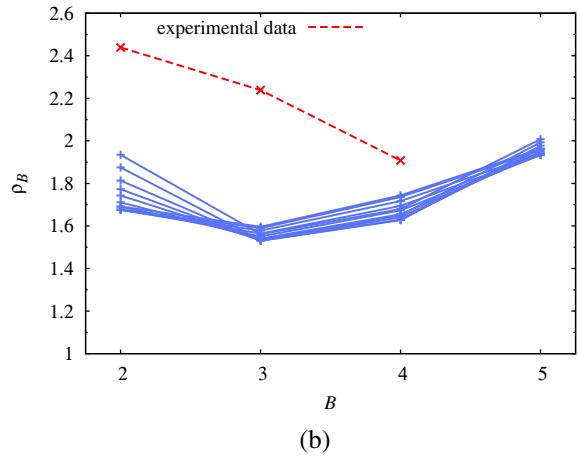
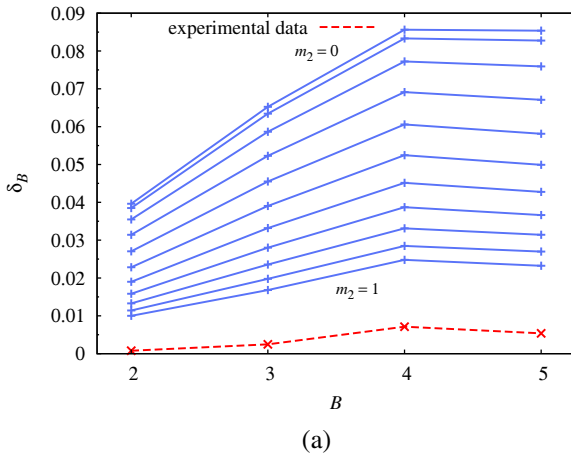


FIG. 9. (a) Relative classical binding energies  $\delta_B$  and (b) relative charge radii for  $B$ -Skyrmions with the pion mass  $m_1 = 1/4$  turned on. The series of points is for  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  with  $m_2$  increasing from top to bottom. The reddashed line is connecting the experimental data from (a) Table I and (b) Table II.

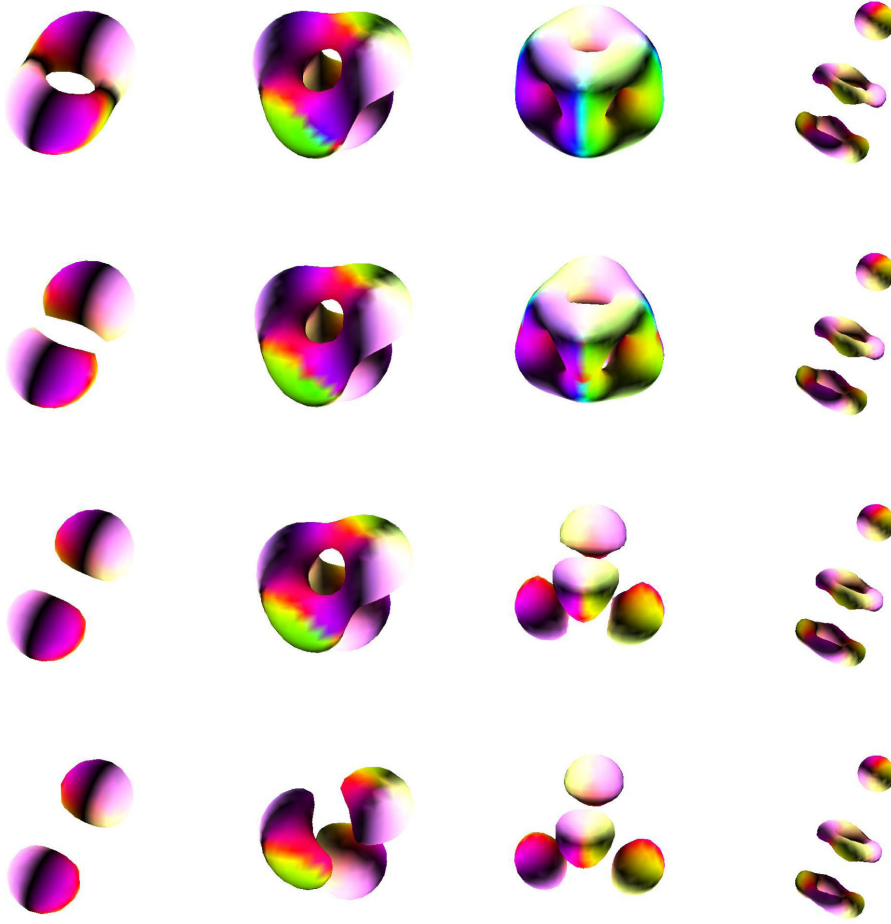


FIG. 10. Isosurfaces of baryon charge density for Skyrmion solutions with baryon number  $B = 2$  through  $B = 5$  for  $m_4 = 0$  as functions of  $m_2 = 0.7, 0.8, 0.9, 1$  (from top to bottom). The coloring is described in the text.

$$E_5^{J=\frac{3}{2}, I=\frac{1}{2}} = \frac{f_\pi}{e} E_5 + \frac{e^3 f_\pi}{4} \frac{3U_{11} + V_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{e^3 f_\pi}{8} \frac{9U_{33} + V_{33} + 6W_{33}}{U_{33}V_{33} - W_{33}^2}, \quad (32)$$

where we have restored the physical units and the tensors in our notation are given by [39]

$$U_{ij} = -\frac{1}{2} \int d^3x \text{Tr} \left( c_2 T_i T_j + \frac{c_4}{4} [L_k, T_i][L_k, T_j] \right), \quad (33)$$

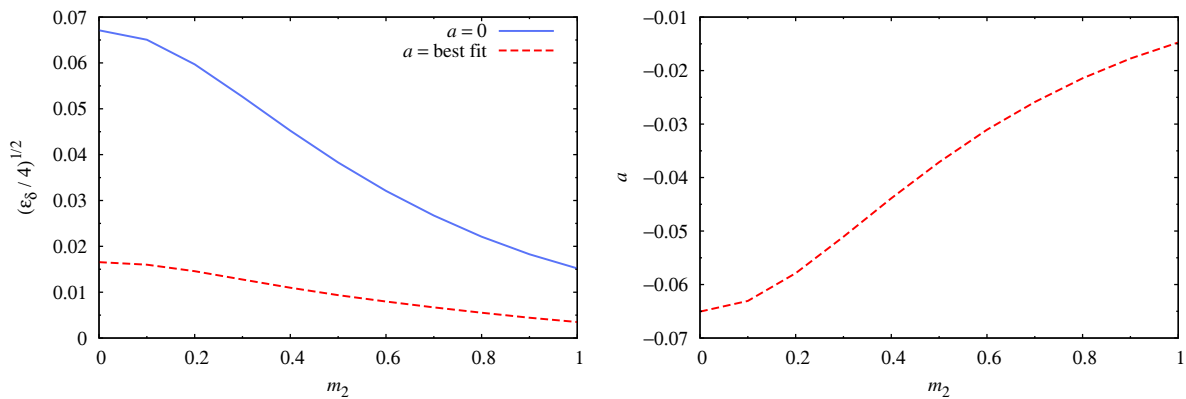


FIG. 11. Least-squares fit of the relative classical binding energy to experimental data  $\sqrt{\epsilon_8/4}$  as a function of  $m_2$  (left panel) for  $B = 2, 3, 4, 5$  Skyrmions with the pion mass  $m_1 = 1/4$  turned on.  $a$  is the offset constant corresponding to an extra contribution to the energy in the  $B = 1$  sector.  $a$  is fitted for each value of  $m_2$  and is shown in the right panel.

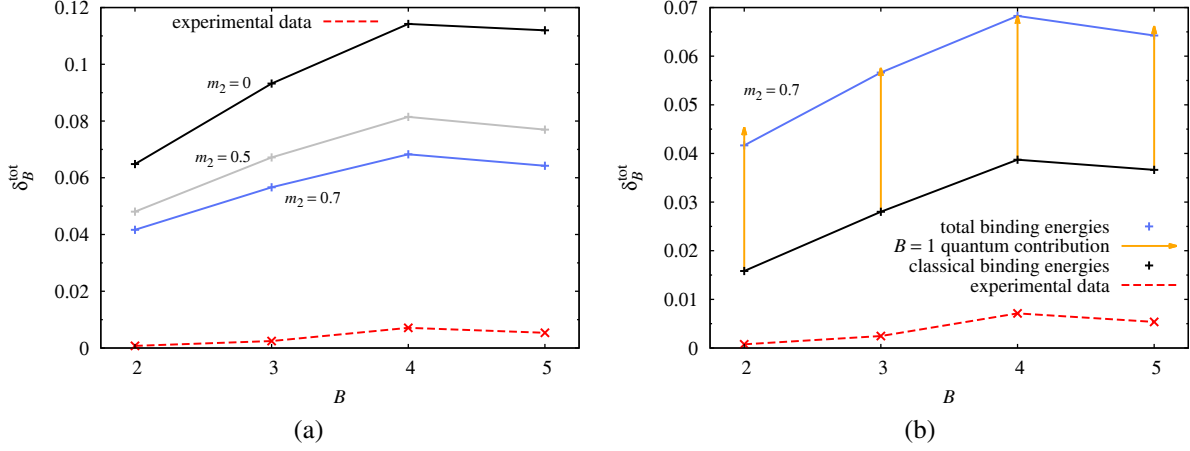


FIG. 12. (a) Relative total binding energies  $\delta_B^{\text{tot}}$  with semiclassical quantum contributions from spin and isospin included for  $B$ -Skyrmions with the pion mass  $m_1 = 1/4$  turned on. The series of points shown is for  $m_2 = 0, 0.5, 0.7$  with  $m_2$  increasing from top to bottom. The red-dashed line is again the experimental data from Table I. (b) Breakdown of the semiclassical quantum contribution to the  $m_2 = 0.7$  series from spin and isospin quantization. The black line shows the classical binding energies whereas the blue line is the total binding energies. The orange arrows represent the  $B = 1$  quantum contribution and the difference between the arrow heads and the blue line is the  $B$  quantum contribution (which vanishes for  $B = 4$  as it should).

$$V_{ij} = -\frac{1}{2} \int d^3x \epsilon_{ilm} \epsilon_{jnp} x_l x_n \text{Tr} \times \left( c_2 L_m L_p + \frac{c_4}{4} [L_k, L_m][L_k, L_p] \right), \quad (34)$$

$$W_{ij} = \frac{1}{2} \int d^3x \epsilon_{jlm} x_l \text{Tr} \left( c_2 T_i L_m + \frac{c_4}{4} [L_k, T_i][L_k, L_m] \right), \quad (35)$$

and  $T_i \equiv \frac{i}{2} U^\dagger [\tau_i, U]$ .

The binding energies for the quantum states—that is, the classical Skyrmion masses with the addition of the spin and isospin contribution—are shown in Fig. 12(a) for  $m_2 = 0, 0.5, 0.7$ ,  $m_4 = 0$  and the pion mass turned on  $m_1 = 1/4$ . Although the  $m_2 = 0.7$  series (roughly) retains the platonic symmetries of the Skyrmions and lowers the binding energies till about the 6% level, there is still some way to go in order for the model to reproduce the experimentally measured binding energies of nuclei. In Fig. 12(b) is shown the breakdown of the binding energies of the  $m_2 = 0.7$  series. As can be seen from the figure, now the problem of the classical binding energies is at the same level as the quantum contributions to the masses. Since the ground state of the  ${}^4\text{He}$  nucleus is a spin-0, isospin-0 state and the experimentally measured binding energy is  $\lesssim 1\%$ , the spin contribution of almost 3% to the  $B = 1$  Skyrmion energy presents an equally big problem as the classical counterparts in the quest for low binding energies in the Skyrme model. We also note that the contributions from the semiclassical quantization to the higher-charged Skyrmions  $B = 2, 3, 4, 5$  is so low that although they lower the binding energies, their importance is somewhat academic at this stage. We should

remind the reader of the possibly unjustified calculation for the spin and isospin contribution to the  $B = 2$  and  $B = 5$  Skyrmions. The proper identification of the relevant symmetries and rigorous quantization is an interesting problem which however is beyond the scope of this paper.

Let us sum up what we learned so far. The classical binding energies for the Skyrmion are generically too large (as very well known) and the 1-Skyrmion is too small giving rise to a large spin contribution upon semiclassical quantization (large means 2%–3%). The lightly bound model which uses  $V_4$  as well as our new potential  $V_2$  can both decrease the classical binding energies to the level of the experimentally observed values, but at the same time the quantum contribution to the spin- $\frac{1}{2}$  state—identified as the ground state—of the 1-Skyrmion increases and is by no means negligible. The lightly bound model (which uses  $V_4$ ) cannot retain the platonic symmetries and lower the classical binding energies below about 5.5%, whereas the  $V_2$  model can obtain classical binding energies below 3% (approximately) maintaining the platonic symmetries of the Skyrmions. In the  $V_2$  model, the classical binding energies and the quantum contributions are thus of the same order of magnitude, yielding total binding energies near the 6% level.

#### IV. DISCUSSION

In this paper we have studied the Skyrme model with the addition of two scalar potentials that do not contribute to the pion mass, but yield repulsive forces at short range—thus reducing the classical binding energies. The two potentials we have considered are  $V_2 \propto (1 - \text{Tr}U/2)^2$  and  $V_4 \propto (1 - \text{Tr}U/2)^4$ , where the latter was considered in Refs. [14,15]. Both potentials are able to lower the classical binding energies, but  $V_2$  can lower them further

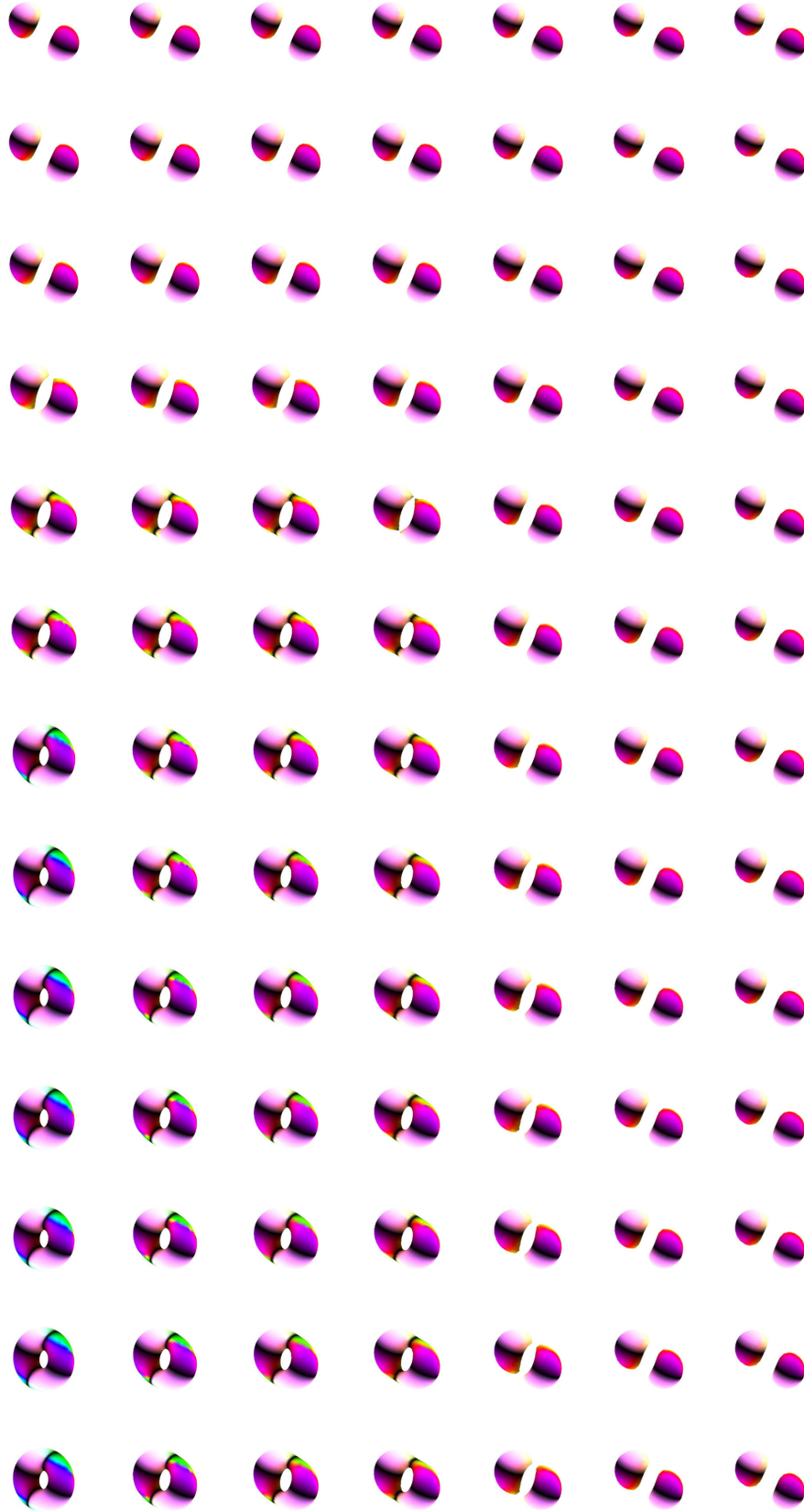


FIG. 13. Isosurfaces of baryon charge densities for Skymion solutions with baryon number  $B = 2$  in the  $(m_2, m_4)$ -parameter space. The values of  $m_4 = 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2, 0.22, 0.24$  (increasing from bottom to top) while  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  (increasing from left to right). The coloring is described in the text.



FIG. 14. Isosurfaces of baryon charge densities for Skyrmion solutions with baryon number  $B = 3$  in the  $(m_2, m_4)$ -parameter space. The values of  $m_4 = 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2, 0.22, 0.24$  (increasing from bottom to top) while  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  (increasing from left to right). The coloring is described in the text.



FIG. 15. Isosurfaces of baryon charge densities for Skymion solutions with baryon number  $B = 4$  in the  $(m_2, m_4)$ -parameter space. The values of  $m_4 = 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2, 0.22, 0.24$  (increasing from bottom to top) while  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  (increasing from left to right). The coloring is described in the text.





FIG. 16. Isosurfaces of baryon charge densities for Skymion solutions with baryon number  $B = 5$  in the  $(m_2, m_4)$ -parameter space. The values of  $m_4 = 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2, 0.22, 0.24$  (increasing from bottom to top) while  $m_2 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$  (increasing from left to right). The coloring is described in the text.

without breaking the platonic symmetries well known to describe the lowest-energy configurations for  $V_2 = V_4 = 0$ , i.e. the normal Skyrme model. Although the potential  $V_2$  is able to lower the classical binding energies to about the 3% level, semiclassical quantization of the 1-Skyrmion—corresponding to taking the proton or neutron spin into account—yields another 3% contribution such that the total binding energies of the model is about 6%—if platonic symmetries are wished intact. If we give up on the platonic symmetries, both  $V_2$  and  $V_4$  can lower the classical binding energies further, but since both potentials have the effect of shrinking the 1-Skyrmion, the  $V_{11}$  inertia tensor decreases, yielding an increasing spin contribution to the energy. This thus increases the total binding energies of all the  $B$ -Skyrmions (since the higher- $B$  Skyrmions do not have sizable contributions from quantization). The question of whether the experimental values of the binding energies can be reached with either one of the two potentials is beyond the scope of this paper—but an interesting future problem.

One of the aims of this paper has been to retain the platonic symmetries of the Skyrme model, which may or may not be necessary. The simple argument in favor of keeping the symmetries is to keep the successes of the Skyrme model, including the description of the Hoyle state in  $^{12}\text{C}$  [17]. Further studies on this problem are however required.

It was argued in Ref. [20] that the aloof property that comes hand in hand with the lightly bound Skyrme model is welcome for two reasons. The first is obviously the reduction of the classical binding energies and the second is that the normal Skyrmions are claimed to be too symmetric. The argument of Ref. [20] is based on the fact that the  $B = 7$  Skyrmion fits poorly the experimentally observed data because the Skyrmion has a very large symmetry that eliminates the states with spin- $\frac{1}{2}$ , spin- $\frac{3}{2}$ , and spin- $\frac{7}{2}$  which is in conflict with the experimental observation that the ground state of  $^7\text{Li}$  is a spin- $\frac{3}{2}$  state. The recent paper [41], however, remedies the failure of the Skyrme model to include the spin- $\frac{3}{2}$  state by considering quantization of the vibrational modes of the 7-Skyrmion. The result is that a spin- $\frac{3}{2}$  state is present in the normal Skyrme model enjoying the platonic symmetries.

Since our model does not quite achieve the requirement of very low binding energies observed experimentally in nuclei, further improvements are needed. It has been

observed in this paper that the pion mass term actually increases the classical binding energies and thus exacerbates the problem at hand. One possibility is to switch the traditional pion mass term for another potential also yielding the pion mass, but with different nonlinear realization. One candidate here is the modified pion mass term ( $V_{02}$ ), which was studied in Refs. [23–31]. As discussed in Sec. II, a large class of potentials gives rise to the pion mass, but may have different effects on the Skyrmions—including their classical binding energies.

Another direction that may be considered in the search for improvement of the model is to include the sixth-order derivative term of the BPS Skyrme model [12,13]. This obviously introduces another parameter in the model, but may yield properties that are more than welcome, for instance its near-perfect fluid properties [42–45]. It has been observed in several contexts that the BPS Skyrme term increases the size of the Skyrmion [15,37], which is very welcome in light of the fact that the Skyrmions are too small and that the moment of inertia of the 1-Skyrmion is too small.

One approximation that when relaxed may ameliorate the problem of the total binding energies is the unbroken isospin symmetry. In the setting we are working in now, the proton and the neutron are the same object and so the 1-Skyrmion ground state should be considered as an average of the two. Taking the splitting in energy into account due to the isospin breaking may improve the model.

Finally, we have not exhausted the possibilities for potential terms. Other powers and also noninteger powers may be considered. Other potentials than those we have considered may have interesting and important effects that have not yet been explored.

The quest for finding a high-precision Skyrme-like model that can capture important features for many (all?) nuclei is certainly interesting and important. We will end on this remark: whether the symmetries of the Skyrmions should be platonic or FCC. The jury is still out.

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