Multitrace approach to noncommutative Φ_2^4 theory

Badis Ydri*

Institute of Physics, BM Annaba University, BP 12, 23000 Annaba, Algeria

(Received 16 September 2015; published 21 March 2016)

In this article we provide a multitrace analysis of the theory of noncommutative Φ^4 in two dimensions on the fuzzy sphere $\mathbf{S}^2_{N,\Omega}$, and on the Moyal-Weyl plane $\mathbf{R}^2_{\theta,\Omega}$, with a nonzero harmonic oscillator term added. The double-trace matrix model symmetric under $M \to -M$ is solved in closed form. An analytical prediction for the disordered-to-nonuniform-ordered phase transition and an estimation of the triple point, from the termination point of the critical boundary, are derived and compared with previous Monte Carlo measurement.

DOI: 10.1103/PhysRevD.93.065041

I. INTRODUCTION

A scalar phi-four theory on a nondegenerate noncommutative Euclidean spacetime is a matrix model of the form

$$S = \mathrm{Tr}_H(a\Phi\Delta\Phi + b\Phi^2 + c\Phi^4). \tag{1.1}$$

The Laplacian Δ defines the underlying geometry, i.e. the metric, of the noncommutative Euclidean spacetime in the sense of [1,2]. This is a three-parameter model with the following three known phases:

(i) The usual second order Ising phase transition between disordered $\langle\Phi\rangle = 0$ and uniform ordered $\langle\Phi\rangle \sim \mathbf{1}$ phases. This appears for small values of c. This is the only transition observed in commutative phi-four, and thus it can be accessed in a small noncommutativity parameter expansion, using the conventional Wilson renormalization group equation [3]. See [4] for an analysis along this line applied to the $O(N)$ version of the noncommutative phi-four theory.

(ii) A matrix transition between disordered $\langle\Phi\rangle = 0$ and nonuniform ordered $\langle\Phi\rangle \sim \Gamma$ phases with $\Gamma^2 = \mathbf{1}_H$. For a finite dimensional Hilbert space H, this transition coincides, for very large values of c, with the third order transition of the real quartic matrix model, i.e. the model with $a = 0$, which occurs at $b = -2\sqrt{Nc}$. In terms of $\tilde{b} = bN^{-3/2}$ and $\tilde{c} = cN^{-2}$ this reads

$$\tilde{b} = -2\sqrt{\tilde{c}}. \tag{1.2}$$

This is therefore a transition from a one-cut (disk) phase to a two-cut (annulus) phase [5,6].

(iii) A transition between uniform ordered $\langle\Phi\rangle \sim \mathbf{1}_H$ and nonuniform ordered $\langle\Phi\rangle \sim \Gamma$ phases. The nonuniform phase, in which translational/rotational invariance is spontaneously broken, is absent in the commutative theory. The nonuniform phase is essentially the stripe phase observed originally on Moyal-Weyl spaces in [7,8].

Let us discuss a little further the phase structure of the pure potential model $V = \mathrm{Tr}_H(b\Phi^2 + c\Phi^4)$, in the case when the Hilbert space H is N-dimensional, in some more detail. The ground state configurations are given by the matrices

$$\Phi_0 = 0, \tag{1.3}$$

$$\Phi_\gamma = \sqrt{-\frac{b}{2c}}U\gamma U^+, \qquad \gamma^2 = \mathbf{1}_N,$$
$$UU^+ = U^+U = \mathbf{1}_N. \tag{1.4}$$

We compute $V[\Phi_0] = 0$ and $V[\Phi_\gamma] = -b^2/4c$. The first configuration corresponds to the disordered phase characterized by $\langle\Phi\rangle = 0$. The second solution makes sense only for $b < 0$, and it corresponds to the ordered phase characterized by $\langle\Phi\rangle \neq 0$. As mentioned above, there is a nonperturbative transition between the two phases which occurs quantum mechanically, not at $b = 0$, but at $b = b_* = -2\sqrt{Nc}$, which is known as the one-cut to two-cut transition. The idempotent γ can always be chosen such that $\gamma = \gamma_k = \mathrm{diag}(\mathbf{1}_k, -\mathbf{1}_{N-k})$. The orbit of γ_k is the Grassmannian manifold $U(N)/(U(k) \times U(N-k))$ which is d_k-dimensional where $d_k = 2kN - 2k^2$. It is not difficult to show that this dimension is maximum at $k = N/2$, assuming that N is even, and hence from entropy argument, the most important two-cut solution is the so-called stripe configuration given by $\gamma = \mathrm{diag}(\mathbf{1}_{N/2}, -\mathbf{1}_{N/2})$. In this real quartic matrix model, we have therefore three possible phases characterized by the following order parameters:

$$\langle\Phi\rangle = 0 \quad \text{disordered phase.} \tag{1.5}$$

$$\langle\Phi\rangle = \pm\sqrt{-\frac{b}{2c}}\mathbf{1}_N \qquad \text{Ising (uniform) phase.} \tag{1.6}$$

$$\langle\Phi\rangle = \pm\sqrt{-\frac{b}{2c}}\gamma \qquad \text{matrix (nonuniform or stripe) phase.} \tag{1.7}$$

*ydri@stp.dias.ie

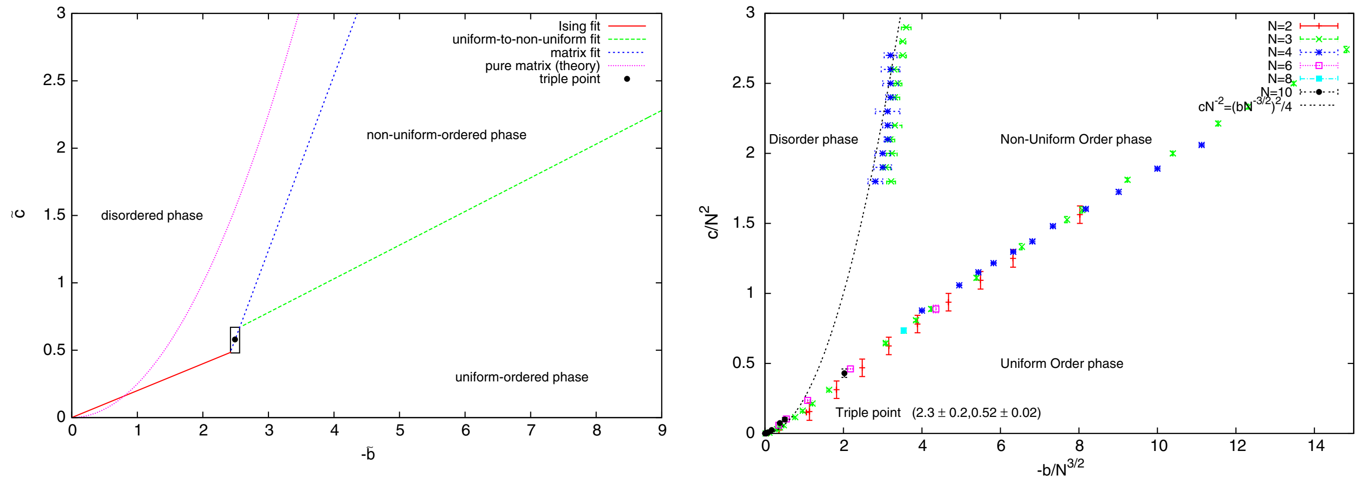

FIG. 1. The phase diagram of phi-four theory on the fuzzy sphere. In the first part the fits are reproduced from actual Monte Carlo data [11]. The second part is reproduced from [9] with the gracious permission of D. O'Connor.

However, as one can explicitly check by calculating the free energies of the respective phases, the uniform ordered phase is not stable in the real quartic matrix model $V = \mathrm{Tr}_H(b\Phi^2 + c\Phi^4)$.

The above picture is expected to hold for noncommutative/fuzzy phi-four theory in any dimension, and the three phases are all stable and are expected to meet at a triple point. This structure was confirmed in two dimensions by means of Monte Carlo simulations on the fuzzy sphere in [9,10]. The phase diagram is shown in Fig. 1. Both parts of Fig. 1 were generated using the Metropolis algorithm on the fuzzy sphere. In the first part coupling of the scalar field Φ to a U(1) gauge field on the fuzzy sphere is included, and as a consequence, we can employ the U(N) gauge symmetry to reduce the scalar sector to only its eigenvalues.

The problem of the phase structure of fuzzy scalar phi-four was also studied in [12–15]. The analytic derivation of the phase diagram of noncommutative phi-four on the fuzzy sphere was attempted in [16–19]. The related problem of Monte Carlo simulation of noncommutative phi-four on the fuzzy torus and the fuzzy disk was considered in [8,20], and [21], respectively. For a recent study see [22].

In this paper, we are interested in studying, by means of the multitrace approach initiated in [16], the theory of noncommutative Φ^4 in two dimensions on the fuzzy sphere $\mathbf{S}^2_{N,\Omega}$ and the Moyal-Weyl plane $\mathbf{R}^2_{\theta,\Omega}$, with a nonzero harmonic oscillator term. The construction of the harmonic oscillator term on the Moyal-Weyl plane can be found in [23], whereas the analogue construction on the fuzzy sphere is done in [11]. The multitrace expansion is the analogue of the Hopping parameter expansion on the lattice in the sense that we perform a small kinetic term expansion, i.e. expanding in the parameter a of (1.1), as opposed to the small potential expansion of the usual perturbation theory [24,25]. This technique is expected to capture the matrix transition between disordered $\langle\Phi\rangle = 0$ and nonuniform ordered $\langle\Phi\rangle \sim \gamma$ phases with arbitrarily increasing accuracy by including more and more terms in the expansion in a. From this we can then infer and/or estimate the position of the triple point. Capturing the Ising transition requires, in our opinion, the whole expansion in a, or at least a very large number of terms in the expansion. This is because it is not obvious how a small number of terms in the expansion in a approximates the geometry encoded in the kinetic term, and as a consequence, the Ising phase $\langle\Phi\rangle \sim \mathbf{1}$ will more likely be seen as metastable within this scheme. There is, of course, the expectation that the uniform ordered phase will become stable at some order of this approximation.

This article is organized as follows:

(1) Section 2: The Model and The Method.
(2) Section 3: The Real Multitrace Quartic Matrix Model.
(3) Section 4: Matrix Model Solutions.
(4) Section 5: Monte Carlo Results
    (a) Summary of models and algorithm.
    (b) Monte Carlo tests of multitrace approximations.
    (c) Phase diagrams and other physics.
(5) Section 6: The Nonperturbative Effective Potential Approach.
(6) Section 7: Conclusion.

We also include three appendixes for the benefit of interested readers.

II. THE MODEL AND THE METHOD

The model studied in this paper, on the fuzzy sphere $\mathbf{S}^2_{N,\Omega}$ and on the regularized Moyal-Weyl plane $\mathbf{R}^2_{\theta,\Omega}$, can be rewritten coherently as the following matrix model:

$$S[M] = r^2 K[M] + \mathrm{Tr}[bM^2 + cM^4], \qquad (2.1)$$

$$K[M] = \mathrm{Tr}\left[\sqrt{\omega}\Gamma^{+}M\Gamma M - \frac{\epsilon}{N+1}\Gamma_3 M\Gamma_3 M + EM^2\right]. \tag{2.2}$$

The first term is precisely the kinetic term. The parameter ϵ takes one of two possible values corresponding to the topology/metric of the underlying geometry, viz.

$$\epsilon = 1, \quad \text{sphere}, \qquad \epsilon = 0, \quad \text{plane}. \tag{2.3}$$

The parameters b, c, r^2, and $\sqrt{\omega}$ are related to the mass parameter m^2, the quartic coupling constant λ, the noncommutativity parameter θ, and the harmonic oscillator parameter Ω, of the original model, by the equations

$$b = \frac{1}{2}m^2, \qquad c = \frac{\lambda}{4!}\frac{1}{2\pi\theta},$$
$$r^2 = \frac{2(\Omega^2+1)}{\theta}, \qquad \sqrt{\omega} = \frac{\Omega^2-1}{\Omega^2+1}. \tag{2.4}$$

The matrices Γ, Γ_3, and E are given by

$$(\Gamma_3)_{lm} = l\delta_{lm}, \quad (\Gamma)_{lm} = \sqrt{(m-1)\left(1-\epsilon\frac{m}{N+1}\right)}\delta_{lm-1},$$
$$(E)_{lm} = \left(l-\frac{1}{2}\right)\delta_{lm}. \tag{2.5}$$

Let us discuss the connection between the actions (1.1) and (2.1). We note first that the original action (1.1) on the fuzzy sphere, with a nonzero harmonic oscillator term, is defined by the Laplacian [11]

$$\Delta = [L_a,[L_a,\ldots]] + \Omega^2[L_3,[L_3,\ldots]] + \Omega^2\{L_i,\{L_i,\ldots\}\}. \tag{2.6}$$

Explicitly we have

$$S = \frac{4\pi R^2}{N+1}\mathrm{Tr}\left(\frac{1}{2R^2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4\right). \tag{2.7}$$

Equivalently this action with the substitution $\Phi = \mathcal{M}/\sqrt{2\pi\theta}$, where $\mathcal{M} = \sum_{i,j=1}^{N} M_{ij}|i\rangle\langle j|$, reads

$$S = \mathrm{Tr}(a\mathcal{M}\Delta_{N,\Omega}\mathcal{M} + b\mathcal{M}^2 + c\mathcal{M}^4). \tag{2.8}$$

This is identical to (2.1). The relationship between the parameters[1] $a = 1/(2R^2)$ and r^2 is given by $r^2 = 2a(\Omega^2+1)N$.

[1]The noncommutativity parameter on the fuzzy sphere is related to the radius of the sphere by $\theta = 2R^2/\sqrt{N^2-1}$.

We start from the path integral

$$Z = \int dM \exp(-S[M]). \tag{2.9}$$

First, we will diagonalize the scalar matrix as $M = U\Lambda U^{-1}$. The measure becomes $dM = d\Lambda dU \Delta^2(\Lambda)$, where dU is the usual Haar measure over the group SU(N) which is normalized such that $\int dU = 1$, whereas the Jacobian $\Delta^2(\Lambda)$ is precisely the so-called Vandermonde determinant defined by $\Delta^2(\Lambda) = \prod_{i>j}(\lambda_i - \lambda_j)^2$. The path integral becomes the eigenvalues problem

$$Z = \int d\Lambda \Delta^2(\Lambda)\exp(-\mathrm{Tr}(b\Lambda^2 + c\Lambda^4))$$
$$\times \int dU \exp(-aK[U\Lambda U^{-1}]). \tag{2.10}$$

The fundamental question we want to answer is, can we integrate the unitary group completely?

The answer, which is the straightforward and obvious one, is to expand the kinetic term in powers of $a = r^2$, perform the integral over U, and then resume the sum back into an exponential to obtain an effective potential. This is very reminiscent of the hopping parameter expansion on the lattice. This approximation will clearly work if, for whatever reason, the kinetic term is indeed small compared to the potential term which, as it turns out, is true in the matrix phase of noncommutative phi-four theory.

Toward this end, we will take the following steps:

(i) We expand the scalar field M in the basis formed by the Gell-Mann matrices t_a and the identity matrix $t_0 = \mathbf{1}_N/\sqrt{2N}$ as $M = \sum_A M^A t_A$. The path integral becomes

$$Z = \int d\Lambda \Delta^2(\Lambda)\exp(-\mathrm{Tr}(b\Lambda^2 + c\Lambda^4))$$
$$\times \int dU \exp(-aK_{AB}(\mathrm{Tr}U\Lambda U^{-1}t_A)$$
$$\times (\mathrm{Tr}U\Lambda U^{-1}t_B)). \tag{2.11}$$

The kinetic matrix K is given explicitly by

$$K_{AB} = 2\sqrt{\omega}\mathrm{Tr}\Gamma^{+}t_A\Gamma t_B + 2\sqrt{\omega}\mathrm{Tr}\Gamma^{+}t_B\Gamma t_A$$
$$-\frac{4\epsilon}{N+1}\mathrm{Tr}\Gamma_3 t_A\Gamma_3 t_B + 2\mathrm{Tr}E\{t_A,t_B\}. \tag{2.12}$$

(ii) We expand in powers of a. In this paper we only go up to the second order in a. This can be extended to any order in an obvious way as we will see.

(iii) We use $(\mathrm{Tr}A)(\mathrm{Tr}B) = \mathrm{Tr}_{N^2}(A \otimes B)$ and $(A \otimes C) \times (B \otimes D) = AB \otimes CD$.

(iv) We decompose the N^2-dimensional and the N^4-dimensional Hilbert spaces, under the SU(N) action, into the direct sums of subspaces corresponding to the irreducible representations ρ contained in $N \otimes N$ and $N \otimes N \otimes N \otimes N$, respectively. The tensor products of interest are

$$\boxed{A} \otimes \boxed{B} = \begin{array}{|c|c|}\hline A & B \\ \hline\end{array} \oplus \begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}. \tag{2.13}$$

$$\begin{aligned}\boxed{A} \otimes \boxed{B} \otimes \boxed{C} \otimes \boxed{D} = {} & \begin{array}{|c|c|c|c|}\hline A & B & C & D \\ \hline\end{array} \oplus \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline D \\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline A & B & C \\ \hline D \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline A & B & D \\ \hline C \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline A & C & D \\ \hline B \\ \cline{1-1}\end{array} \\ & \oplus \begin{array}{|c|c|}\hline A & D \\ \hline B \\ \cline{1-1} C \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline A & C \\ \hline B \\ \cline{1-1} D \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline A & B \\ \hline C \\ \cline{1-1} D \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline A & C \\ \hline B & D \\ \hline\end{array}.\end{aligned} \tag{2.14}$$

(v) We use the orthogonality relation

$$\int dU \rho(U)_{ij} \rho(U^{-1})_{kl} = \frac{1}{\dim(\rho)} \delta_{il} \delta_{jk}. \tag{2.15}$$

We obtain (see [16,17] and [26] for more detail)

$$\begin{aligned}&\int dU \exp(-aK[U\Lambda U^{-1}]) \\ &\quad = 1 - aK_{AB} \sum_\rho \frac{1}{\dim(\rho)} \mathrm{Tr}_\rho \Lambda \otimes \Lambda . \mathrm{Tr}_\rho t_A \otimes t_B \\ &\qquad + \frac{1}{2!} a^2 K_{AB} K_{CD} \sum_\rho \frac{1}{\dim(\rho)} \mathrm{Tr}_\rho \Lambda \otimes \Lambda . \mathrm{Tr}_\rho t_A \\ &\qquad \otimes \cdots \otimes t_D + \cdots .\end{aligned} \tag{2.16}$$

Thus, the calculation of the first and second order corrections reduces to the calculation of the traces $\mathrm{Tr}_\rho t_A \otimes t_B$ and $\mathrm{Tr}_\rho t_A \otimes t_B \otimes t_C \otimes t_D$, respectively. It is then obvious that generalization to higher order corrections will involve the traces $\mathrm{Tr}_\rho t_{A_1} \otimes \cdots \otimes t_{A_n}$ and $\mathrm{Tr}_\rho \Lambda \otimes \cdots \otimes \Lambda$. Explicitly the nth order correction should read

$$\begin{aligned}n\text{th order} = {} & \frac{1}{n!} K_{A_1 A_2} \cdots K_{A_{2n-1} A_{2n}} \\ & \times \sum_\rho \frac{1}{\dim(\rho)} \mathrm{Tr}_\rho \Lambda \otimes \cdots \otimes \Lambda . \mathrm{Tr}_\rho t_{A_1} \\ & \otimes t_{A_2} \otimes \cdots \otimes t_{A_{2n-1}} \otimes t_{A_{2n}}.\end{aligned} \tag{2.17}$$

(vi) By substituting the dimensions of the various irreducible representations and the relevant SU(N) characters we arrive at the formula (with $t_i = \mathrm{Tr}\Lambda^i$)

$$\begin{aligned}\int dU \exp(-aK[U\Lambda U^{-1}]) = {} & 1 - \frac{a}{2}[(s_{1,2} + s_{2,1})t_1^2 + (s_{1,2} - s_{2,1})t_2] + \frac{a^2}{2}\left[\frac{1}{4}(s_{1,4} - s_{4,1} - s_{2,3} + s_{3,2})t_4 \right. \\ & + \frac{1}{3}(s_{1,4} + s_{4,1} - s_{2,2})t_1 t_3 + \frac{1}{8}(s_{1,4} + s_{4,1} - s_{2,3} - s_{3,2} + 2s_{2,2})t_2^2 \\ & \left. + \frac{1}{4}(s_{1,4} - s_{4,1} + s_{2,3} - s_{3,2})t_2 t_1^2 + \frac{1}{24}(s_{1,4} + s_{4,1} + 3s_{2,3} + 3s_{3,2} + 2s_{2,2})t_1^4\right] + \cdots .\end{aligned} \tag{2.18}$$

(vii) There remains the explicit calculation of the coefficients s, taking the large N limit, and finally reexponentiating the series back to obtain the effective potential. This is a considerably long calculation that is done originally on the fuzzy sphere in [16] and extended to the current case, which includes a harmonic oscillator term, in [26]. We will skip here the lengthy detail.

The definition of the coefficients s in terms of the kinetic matrix K and characters and dimensions of various SU(N)/U(N) representations, the explicit calculation of these coefficients, as well as extraction of the large N behavior are sketched in Appendix A.

III. THE REAL MULTITRACE QUARTIC MATRIX MODEL

The end result of the above steps is the effective potential [26]

$$\Delta V = \frac{r^2}{4}\left(v_{2,1}T_2 + \frac{2N}{3}w_1 t_2\right) + \frac{r^4}{24}\left(v_{4,1}T_4 - \frac{4}{N^2}v_{2,2}T_2^2 + 4w_2(t_1t_3 - t_2^2) + \frac{4}{N}w_3 t_2 T_2\right) + O(r^6). \tag{3.1}$$

The complete effective action in terms of the eigenvalues is the sum of the classical potential, the Vadermonde determinant, and the above effective potential. The operators T_2 and T_4 are defined below. The coefficients v and w are given by [26]

$$v_{2,1} = 2 - \epsilon - \frac{2}{3}(\sqrt{\omega}+1)(3-2\epsilon), \tag{3.2}$$

$$v_{4,1} = -(1-\epsilon), \tag{3.3}$$

$$v_{2,2} = w_3 + (\sqrt{\omega}+1)(1-\epsilon) + \frac{1}{12}(\omega-1)(9-8\epsilon) - \frac{1}{8}(2-3\epsilon), \tag{3.4}$$

$$w_1 = (\sqrt{\omega}+1)(3-2\epsilon), \tag{3.5}$$

$$w_2 = -(\sqrt{\omega}+1)(1-\epsilon), \tag{3.6}$$

$$w_3 = (\sqrt{\omega}+1)(1-\epsilon) - \frac{1}{15}(\sqrt{\omega}+1)^2(15-14\epsilon). \tag{3.7}$$

Three important remarks are now in order:

(i) *Zero mode*: We know that, in the limit $\Omega^2 \to 0$ ($\sqrt{\omega} \to -1$), the trace part of the scalar field drops from the kinetic action, and as a consequence, the effective potential can be rewritten solely in terms of the differences $\lambda_i - \lambda_j$ of the eigenvalues. Furthermore, in this limit, the effective potential must also be invariant under any permutation of the eigenvalues, as well as under the parity $\lambda_i \to -\lambda_i$, and hence it can only depend on the following functions [16]:

$$T_4 = Nt_4 - 4t_1t_3 + 3t_2^2 = \frac{1}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^4, \tag{3.8}$$

$$T_2^2 = \left[\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\right]^2 = t_1^4 - 2Nt_1^2t_2 + N^2t_2^2, \tag{3.9}$$

$$T_n^m = \left[\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^n\right]^m. \tag{3.10}$$

It is clear that only the functions T_2 and T_4 can appear at the second order in $a = r^2$. We also observe that the quadratic part of the resulting effective potential can be expressed, modulo a term that vanishes as $\sqrt{\omega}+1$ in the limit $\sqrt{\omega} \to -1$, in terms of the function

$$T_2 = Nt_2 - t_1^2 = \frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2. \tag{3.11}$$

In general, it is expected that for generic values of $\sqrt{\omega}$, away from the zero harmonic oscillator case $\sqrt{\omega} = -1$, the effective potential will contain terms proportional to $\sqrt{\omega}+1$ that cannot be expressed solely in terms of the functions T_2, T_4, etc. This is obvious from the result (3.1).

(ii) *The case* $\epsilon = 1$, $\Omega^2 = 0$: In this case $v_{2,1} = 1$, $v_{4,1} = 0$, $v_{2,2} = 1/8$ while all the w coefficients vanish. We get then the effective potential

$$\Delta V = \frac{r^2}{4}T_2 - \frac{r^4}{48N^2}T_2^2 + O(r^6). \tag{3.12}$$

This result is very different from the one obtained in [16][2] in which the coefficient $v_{4,1}$ was found to be nonzero, more precisely $v_{4,1} = 3/2$, while the correction associated with the coefficient $v_{2,2}$ was suppressed, i.e. they set $v_{2,2} = 0$, and hence the effective potential, in their case, is given by

$$\Delta V = \frac{r^2}{4}T_2 + \frac{r^4}{16}T_4 + O(r^6). \tag{3.13}$$

A detailed discussion of this point can be found in [26], while a concise description of the discrepancy is included in Appendix B. However, we should note that although these two results are quantitatively different, the resulting physics is qualitatively the same.

(iii) *Scaling*: From the Monte Carlo results of [9,10] on the fuzzy sphere, we know that the scaling behavior of the parameters a, b, and c appearing in the action (2.8) is given by

$$\bar{a} = \frac{a}{N^{\delta_a}}, \qquad \bar{b} = \frac{bN^{2\delta_\lambda}}{aN^{3/2}}, \qquad \bar{c} = \frac{cN^{4\delta_\lambda}}{a^2N^2}. \tag{3.14}$$

In the above equation we have also included a possible scaling of the field/matrix M, which is

[2] Compare with Eq. (4.4) of [16].

not included in [9,10], given by δ_λ. The scaling of the parameter a encodes the scaling of the radius R^2 or equivalently the noncommutativity parameter θ. There is, of course, an extra parameter in the current case given by $d = a\Omega^2$, or equivalently $\sqrt{\omega} = (\Omega^2 - 1)/(\Omega^2 + 1)$, which comes with another scaling δ_d not discussed altogether in Monte Carlo simulations.

We will assume, in most of this paper, that the four parameters b, c, r^2, and $\sqrt{\omega}$ of the matrix model (2.1) scale as

$$\tilde{b} = \frac{b}{N^{\delta_b}}, \qquad \tilde{c} = \frac{c}{N^{\delta_c}},$$
$$\tilde{r}^2 = \frac{r^2}{N^{\delta_r}}, \qquad \sqrt{\tilde{\omega}} = \frac{\sqrt{\omega}}{N^{\delta_\omega}}. \tag{3.15}$$

Obviously $\delta_r = \delta_a + 1$. Further, we will assume a scaling δ_λ of the eigenvalues λ, viz.

$$\tilde{\lambda} = \frac{\lambda}{N^{\delta_\lambda}}. \tag{3.16}$$

Hence, in order for the effective action to come out of order N^2, we must have the following values:

$$\delta_b = 1 - 2\delta_\lambda, \qquad \delta_c = 1 - 4\delta_\lambda,$$
$$\delta_r = -2\delta_\lambda, \qquad \delta_\omega = 0. \tag{3.17}$$

By substituting in (3.14) we obtain the collapsed exponents

$$\delta_\lambda = -\frac{1}{4}, \qquad \delta_a = -\frac{1}{2},$$
$$\delta_b = \frac{3}{2}, \qquad \delta_c = 2,$$
$$\delta_d = -\frac{1}{2}, \qquad \delta_r = \frac{1}{2}. \tag{3.18}$$

In simulations, it is found that the scaling behavior of the mass parameter b and the quartic coupling c is precisely given by $3/2$ and 2, respectively. We will assume, for simplicity, the same scaling on the Moyal-Weyl plane.

IV. MATRIX MODEL SOLUTIONS

The saddle point equation corresponding to the sum $V_{r^2,\Omega}$ of the classical potential and the effective potential (3.1), which also includes the appropriate scaling,[3] takes the form

$$\frac{1}{N} S'_{\rm eff} = V'_{r^2,\Omega} - \frac{2}{N}\sum_i \frac{1}{\lambda - \lambda_i} = 0. \tag{4.1}$$

Next, we will assume a symmetric support of the eigenvalues distributions, and as a consequence, all odd moments vanish identically [16]. This is motivated by the fact that the expansion of the effective action employed in the current paper, i.e. the multitrace technique, is expected to probe, very well, the transition between the disordered phase and the nonuniform ordered phase.

We will, therefore, assume here that across the transition line between disordered phase and nonuniform ordered phase, the matrix M remains massless, the eigenvalues distribution $\rho(\lambda)$ is always symmetric, and hence all odd moments m_q vanish identically, viz.

$$m_q = \int_a^b d\lambda \rho(\lambda)\lambda^q = 0, \qquad q = \text{odd}. \tag{4.2}$$

The derivative of the generalized potential $V'_{r^2,\Omega}$ is therefore given by

$$V'_{r^2,\Omega}(\lambda) = 2\tilde{b}\lambda + 4\tilde{c}\lambda^3 + \tilde{r}^2\left(\frac{v_{2,1}}{2} + \frac{w_1}{3}\right)\lambda$$
$$+ \tilde{r}^4\left[\frac{1}{6}v_{4,1}(\lambda^3 + 3m_2\lambda) - \frac{2}{3}w_2 m_2\lambda - \frac{2}{3}v_{2,2}m_2\lambda + \frac{1}{3}w_3 m_2\lambda\right] + \cdots. \tag{4.3}$$

The corresponding matrix model potential and effective action are given, respectively, by the following

$$V_{r^2,\Omega} = N\int d\lambda\rho(\lambda)\left[\left(\tilde{b} + \frac{\tilde{r}^2}{2}\left(\frac{v_{2,1}}{2} + \frac{w_1}{3}\right)\right)\lambda^2 + \left(\tilde{c} + \frac{\tilde{r}^4}{24}v_{4,1}\right)\lambda^4\right] - \frac{\tilde{r}^4 N}{6}\eta\left[\int d\lambda\rho(\lambda)\lambda^2\right]^2, \tag{4.4}$$

$$S_{\rm eff} = NV_{r^2,\Omega} - \frac{N^2}{2}\int d\lambda d\lambda'\rho(\lambda)\rho(\lambda')\ln(\lambda - \lambda')^2. \tag{4.5}$$

The coefficient η is defined by

$$\eta = v_{2,2} - \frac{3}{4}v_{4,1} + w_2 - \frac{1}{2}w_3$$
$$= \frac{1}{8}(4 - 3\epsilon) - \frac{1}{6}(\sqrt{\omega} + 1)(6 - 5\epsilon) + \frac{1}{20}(\sqrt{\omega} + 1)^2(5 - 4\epsilon). \tag{4.6}$$

These can be derived from the matrix model given by

[3]The eigenvalues here are also scaled only we suppress the tilde for ease of notation.

$$V_{r^2,\Omega} = \mu_0 \mathrm{Tr} M^2 + g_0 \mathrm{Tr} M^4 - \frac{\tilde{r}^4}{6N}\eta[\mathrm{Tr} M^2]^2. \tag{4.7}$$

The parameters μ_0 and g_0 are defined by

$$\mu_0 = \tilde{b} + \frac{\tilde{r}^2}{2}\left(\frac{v_{2,1}}{2} + \frac{w_1}{3}\right) = \tilde{b} + \frac{\tilde{r}^2}{4}(2-\epsilon),$$
$$g_0 = \tilde{c} + \frac{\tilde{r}^4}{24}v_{4,1} = \tilde{c} - \frac{\tilde{r}^4}{24}(1-\epsilon). \tag{4.8}$$

This matrix model was studied originally in [27] within the context of $c > 1$ string theories. The dependence of this result on the harmonic oscillator potential is fully encoded in the parameter η, which is the strength of the double-trace term since μ_0 and g_0 are independent of $\sqrt{\omega}$. For later purposes we rewrite the derivative of the generalized potential $V'_{r^2,\Omega}$ in the suggestive form

$$V'_{r^2,\Omega}(\lambda) = 2\mu\lambda + 4g\lambda^3, \tag{4.9}$$

$$\mu = \mu_0 - \frac{\tilde{r}^4}{3}\eta m_2, \qquad g = g_0. \tag{4.10}$$

The above saddle point equation (4.1) can be solved using the approach outlined in [28] for real single trace quartic matrix models. We only need to account here for the fact that the mass parameter μ depends on the eigenvalues through the second moment m_2. In other words, besides the normalization condition that the eigenvalues distribution must satisfy, we must also satisfy the requirement that the computed second moment m_2, using this eigenvalues density, will depend on the mass parameter μ that itself is a function of the second moment m_2.

The phase structure of the real quartic matrix model is described concisely in [6]. The two stable phases of the theory are the one-cut (disk) and the two-cut (annulus) phases that are separated by the critical line (1.2).[4] There exists also an asymmetric (uniform) one-cut solution that corresponds to a metastable phase. Here, for our real multitrace quartic matrix model (4.7), the phase diagram will consist of the same stable phases, separated by a deformation of the critical line (1.2), as well as an analogous metastable asymmetric one-cut phase.

In the remainder, we discuss further the two stable phases of the real multitrace quartic matrix model (4.7), the critical boundary between them, as well as a lower estimation of the triple point. More detail can be found in [26].

The disordered phase: The one-cut (disk) solution is given by the equation

$$\rho(\lambda) = \frac{1}{\pi}\left(2g_0\lambda^2 + \frac{2}{\delta^2} - \frac{g_0\delta^2}{2}\right)\sqrt{\delta^2 - \lambda^2}. \tag{4.11}$$

The radius $\delta^2 = x$ is the solution of a depressed quartic equation given by

$$\tilde{r}^4\eta g_0 x^4 - 72\left(g_0 - \frac{\tilde{r}^4}{18}\eta\right)x^2 - 48\mu_0 x + 96 = 0. \tag{4.12}$$

This eigenvalues distribution is always positive definite for

$$x^2 \le x_*^2 = \frac{4}{g_0}. \tag{4.13}$$

Obviously, x_* must also be a solution of the quartic equation (4.12). By substitution, we get the solution

$$\mu_{0*} = -2\sqrt{g_0} + \frac{\eta\tilde{r}^4}{3\sqrt{g_0}}. \tag{4.14}$$

This critical value μ_{0*} is negative for $g_0 \ge \eta\tilde{r}^4/6$. As expected this line is a deformation of the real quartic matrix model critical line $\mu_{0*} = -2\sqrt{g_0}$. In terms of the original parameters, we have

$$\tilde{b}_* = -\frac{\tilde{r}^2}{4}(2-\epsilon) - 2\sqrt{\tilde{c} - \frac{\tilde{r}^4}{24}(1-\epsilon)} + \frac{\eta\tilde{r}^4}{3\sqrt{\tilde{c} - \frac{\tilde{r}^4}{24}(1-\epsilon)}}. \tag{4.15}$$

This result, to our knowledge, is completely new. By assuming that the parameter η is positive, the range of this solution is found to be $\mu_0 \ge \mu_{0*}$. The second moment m_2 corresponding to this solution is given by the equation

$$m_2 = \frac{9g_0}{2\tilde{r}^4\eta x}(x - x_+)(x - x_-), \tag{4.16}$$

$$x_\pm = \frac{1}{3g_0}\left(-\mu_0 \pm \sqrt{\mu_0^2 + 12g_0}\right). \tag{4.17}$$

This is always positive since $x > x_+ > 0 > x_-$.

The nonuniform ordered phase: The two-cut (annulus) solution is given by

$$\rho(\lambda) = \frac{2g_0}{\pi}|\lambda|\sqrt{(\lambda^2 - \delta_1^2)(\delta_2^2 - \lambda^2)}. \tag{4.18}$$

The radii δ_1 and δ_2 are given by

$$\delta_1^2 = \frac{3}{6g_0 - \tilde{r}^4\eta}\left(-\mu_0 - 2\sqrt{g_0} + \frac{\tilde{r}^4\eta}{3\sqrt{g_0}}\right),$$
$$\delta_2^2 = \frac{3}{6g_0 - \tilde{r}^4\eta}\left(-\mu_0 + 2\sqrt{g_0} - \frac{\tilde{r}^4\eta}{3\sqrt{g_0}}\right). \tag{4.19}$$

[4] At $\tilde{b} = -2\sqrt{\tilde{c}}$, the eigenvalues density approaches the same behavior from both sides of the transition. This is the sense in which this phase transition is termed critical although it is actually third order as seen from the behavior of the specific heat.

We have $\delta_1^2 \geq 0$, and by construction then $\delta_2^2 \geq \delta_1^2$, iff

$$g_0 \geq \frac{\tilde{r}^4 \eta}{6}, \qquad \mu_0 \leq \mu_{0*}. \tag{4.20}$$

The critical value μ_{0*} is still given by (4.14), i.e. the range of μ of this phase meshes exactly with the range of μ of the previous phase.

The triple point: In the rest of this paper, we will concentrate only on the case of the fuzzy sphere, while we will leave the case of the Moyal-Weyl plane as an exercise.

In the case of the fuzzy sphere, i.e. $\epsilon = 1$, we have the following critical line:

$$\tilde{b}_* = -\frac{\tilde{r}^2}{4} - 2\sqrt{\tilde{c}} + \frac{\eta \tilde{r}^4}{3\sqrt{\tilde{c}}}. \tag{4.21}$$

We recall that $r^2 = 2a(\Omega^2 + 1)N$ or equivalently $\tilde{r}^2 = 2\tilde{a}(\Omega^2 + 1)$. The above critical line in terms of the scaled parameters (3.14) reads then

$$\bar{b}_* = -\frac{\Omega^2 + 1}{2} - 2\sqrt{\bar{c}} + \frac{4\eta(\Omega^2 + 1)^2}{3\sqrt{\bar{c}}}. \tag{4.22}$$

This should be compared with (1.2). The range $g_0 \geq \tilde{r}^4\eta/6$ of this critical line reads now

$$\bar{c} \geq \frac{2\eta(\Omega^2 + 1)^2}{3}. \tag{4.23}$$

The termination point of this line provides a lower estimate of the triple point, and it is located at

$$(\bar{b}, \bar{c})_T = \left(-\frac{\Omega^2 + 1}{2}, \frac{2\eta(\Omega^2 + 1)^2}{3}\right). \tag{4.24}$$

We have verified numerically the consistency of the above analytic solution extensively. The starting point is the quartic equation (4.12). We have checked, among other things, that for all $\bar{b} \geq \bar{b}_*$ there exists a positive solution x of (4.12) that satisfies $x \leq x_*$ and $x > x_+$, i.e. with positive second moment m_2. From the other side, i.e. for $\bar{b} < \bar{b}_*$, there ceases to exist any solution of (4.12) with these properties. This behavior extends down until around $\bar{c}_T$. The basics of the algorithm used are explained in Appendix C.

Recall that in the case of the fuzzy sphere with a harmonic oscillator term the coefficient η is given by

$$\eta = \frac{1}{8} - \frac{1}{6}(\sqrt{\omega} + 1) + \frac{1}{20}(\sqrt{\omega} + 1)^2. \tag{4.25}$$

For the zero harmonic oscillator, i.e. for the ordinary noncommutative phi-four theory on the fuzzy sphere with $\Omega^2 = 0$ and $\sqrt{\omega} = -1$, we have then the results

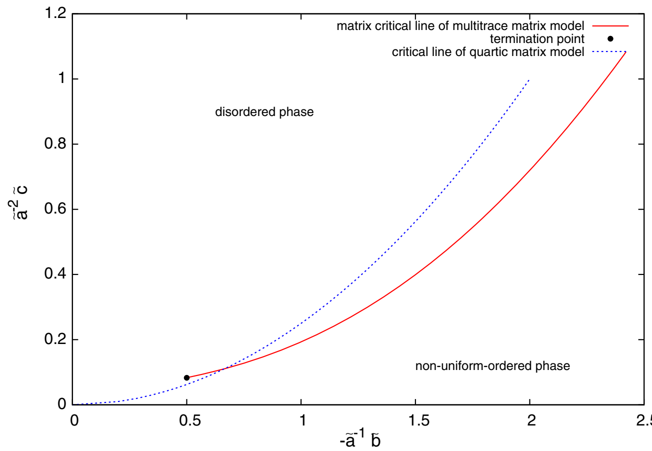

FIG. 2. The disordered-to-nonuniform-ordered (matrix) transition of phi-four theory on the fuzzy sphere.

$$\bar{b}_* = -\frac{1}{2} - 2\sqrt{\bar{c}} + \frac{1}{6\sqrt{\bar{c}}}, \tag{4.26}$$

$$\bar{c} \geq \frac{1}{12}. \tag{4.27}$$

This line is shown in Fig. 2. The limit for large $\bar{c}$ is essentially given by (1.2). As discussed above, the termination point of this line, which is located at

$$(\bar{b}, \bar{c})_T = (-1/2, 1/12), \tag{4.28}$$

yields a lower estimation of the triple point. This is quite far from the actual value of the triple point found in [16] to lie at $\sim(-2.3, 0.5)$, but it provides an explicit and robust indication that the disordered to nonuniform-ordered transition line does not extend to zero as in the case of the real quartic matrix model.

In any event, the above prediction hinges on the calculated value of the parameter η, which is expected to increase in value if we include higher order corrections. Furthermore, the inclusion of other multitrace terms, which will arise in higher order calculations, will also affect this result.

It is obvious that the above behavior should hold, essentially unchanged, on the regularized Moyal-Weyl plane.

V. MONTE CARLO RESULTS

A. Summary of models and algorithm

We start by rewriting the effective action on the fuzzy sphere without a harmonic oscillator term in a form suited for Monte Carlo results. The multitrace matrix models of interest are of the form

$$V = V_0 + \Delta V = V_0 + V_2 + V_4. \tag{5.1}$$

The quartic matrix model V_0 and the quadratic and quartic corrections V_2 and V_4 are given explicitly by

$$V_0 = b\mathrm{Tr}M^2 + c\mathrm{Tr}M^4, \tag{5.2}$$

$$V_2 = F'\mathrm{Tr}M^2 + B'(\mathrm{Tr}M)^2, \tag{5.3}$$

$$\begin{aligned} V_4 &= E'\mathrm{Tr}M^4 + C'\mathrm{Tr}M\mathrm{Tr}M^3 + D'(\mathrm{Tr}M)^4 \\ &\quad + A'\mathrm{Tr}M^2(\mathrm{Tr}M)^2 + D(\mathrm{Tr}M^2)^2. \end{aligned} \tag{5.4}$$

The primed parameters of the model are

$$\begin{aligned} F' &= \frac{aN^2 v_{2,1}}{2}, \qquad B' = -\frac{aN}{2} v_{2,1}, \\ E' &= \frac{a^2N^3 v_{4,1}}{6}, \qquad C' = -\frac{2a^2N^2}{3} v_{4,1}, \\ D' &= -\frac{2a^2}{3} v_{2,2}, \qquad A' = \frac{4a^2N}{3} v_{2,2}. \end{aligned} \tag{5.5}$$

The remaining parameter D is given by

$$D = -\frac{2a^2N^2}{3}\eta, \qquad \eta = v_{2,2} - \frac{3}{4} v_{4,1}. \tag{5.6}$$

The parameters F' and E' can be reabsorbed into b and c as

$$B = b + \frac{aN^2 v_{2,1}}{2}, \qquad C = c + \frac{a^2N^3 v_{4,1}}{6}. \tag{5.7}$$

The effective action we want to study becomes

$$\begin{aligned} V &= \mathrm{Tr}(BM^2 + CM^4) + D(\mathrm{Tr}M^2)^2 \\ &\quad + B'(\mathrm{Tr}M)^2 + C'\mathrm{Tr}M\mathrm{Tr}M^3 + D'(\mathrm{Tr}M)^4 \\ &\quad + A'\mathrm{Tr}M^2(\mathrm{Tr}M)^2. \end{aligned} \tag{5.8}$$

As we have shown in this article, the coefficients $v_{2,1}$, $v_{4,1}$, and $v_{2,2}$ are given by the following two competing calculations found in Eqs. (3.13) (Model I) and (3.12) (Model II):

$$\begin{aligned} &v_{2,1} = -1, \qquad v_{4,1} = \frac{3}{2}, \qquad v_{2,2} = 0, \quad \text{Model I} \\ &v_{2,1} = +1, \qquad v_{4,1} = 0, \qquad v_{2,2} = \frac{1}{8}, \quad \text{Model II}. \end{aligned} \tag{5.9}$$

Explicitly we have

Model I:

$$\begin{aligned} V &= \mathrm{Tr}(BM^2 + CM^4) + D(\mathrm{Tr}M^2)^2 \\ &\quad + B'(\mathrm{Tr}M)^2 + C'\mathrm{Tr}M\mathrm{Tr}M^3, \\ B &= b - \frac{aN^2}{2}, \quad C = c + \frac{a^2N^3}{4}, \\ B' &= \frac{aN}{2}, \quad C' = -a^2N^2, \quad D = \frac{3a^2N^2}{4}. \end{aligned} \tag{5.10}$$

Model II:

$$\begin{aligned} V &= \mathrm{Tr}(BM^2 + CM^4) + D(\mathrm{Tr}M^2)^2 + B'(\mathrm{Tr}M)^2 \\ &\quad + D'(\mathrm{Tr}M)^4 + A'\mathrm{Tr}M^2(\mathrm{Tr}M)^2, \\ B &= b + \frac{aN^2}{2}, \qquad C = c, \qquad B' = -\frac{aN}{2}, \\ D' &= -\frac{a^2}{12}, \qquad A' = \frac{a^2N}{6}, \qquad D = -\frac{a^2N^2}{12}. \end{aligned} \tag{5.11}$$

There are two independent parameters in these models that we take to be the usual ones B and C. It is found that the scaling of the parameters in Monte Carlo simulation is given approximately by

$$\tilde{B} = BN^{-3/2}, \quad \tilde{C} = CN^{-2}, \quad \tilde{D} = DN^{-1}, \text{ etc.} \tag{5.12}$$

Since only two of these parameters are independent, we must choose $\tilde{a}$, for the consistency of the large N limit, to be any fixed number. We then choose for simplicity $\tilde{a} = 1$ or equivalently $D = -2\eta N/3$.

These models can be simulated using the ordinary Metropolis algorithm applied to the eigenvalues of the matrix M; i.e. we diagonalize the matrix M, add to the above action the contribution coming from the Vandermonde determinant, and then simulate the resulting effective action. This method is free from ergodic problems.

Our first test for the validity of this algorithm, or any other algorithm for that matter, is to look at the Schwinger-Dyson identity given for the above multitrace matrix models by

$$\langle (2b\mathrm{Tr}M^2 + 4c\mathrm{Tr}M^4 + 2V_2 + 4V_4) \rangle = N^2. \tag{5.13}$$

The second powerful test is to look at the conventional quartic matrix model with $a = 0$, viz. $V = V_0$. The eigenvalues distributions in the two stable phases [disorder (one-cut) and nonuniform order (two-cut)] as well as the demarcation of their boundary and the behavior of the specific heat across the transition are all well known analytically given, respectively, by the formulas (5.14), (5.15), (5.16), and (5.17) and (5.18) below.

$$\rho(\lambda) = \frac{1}{N\pi}(2C\lambda^2 + B + Cr^2)\sqrt{r^2 - \lambda^2},$$
$$r^2 = \frac{1}{3C}(-B + \sqrt{B^2 + 12NC}), \quad (5.14)$$

$$\rho(\lambda) = \frac{2C|\lambda|}{N\pi}\sqrt{(\lambda^2 - r_-^2)(r_+^2 - \lambda^2)},$$
$$r_\mp^2 = \frac{1}{2C}(-B \mp 2\sqrt{NC}), \quad (5.15)$$

$$B_c^2 = 4NC \leftrightarrow B_c = -2\sqrt{NC}, \quad (5.16)$$

$$\frac{C_v}{N^2} = \frac{1}{4}, \qquad \bar{B} = \frac{B}{B_c} < -1, \quad (5.17)$$

$$\frac{C_v}{N^2} = \frac{1}{4} + \frac{2\bar{B}^4}{27} - \frac{\bar{B}}{27}(2\bar{B}^2 - 3)\sqrt{\bar{B}^2 + 3},$$
$$\bar{B} > -1. \quad (5.18)$$

B. Monte Carlo tests of multitrace approximations

The quartic multitrace approximations can be tested and verified directly in Monte Carlo simulations in order to resolve the ambiguity in the coefficients v given in Eq. (5.9). We must have as the identity the two equations

$$\langle a\int dU\mathrm{Tr}[L_a, U\Lambda U^{-1}]^2\rangle_{V_0} = \langle -V_2(\Lambda)\rangle_{V_0}, \quad (5.19)$$

$$\left\langle\frac{1}{2}(a\int dU\mathrm{Tr}[L_a, U\Lambda U^{-1}]^2)^2\right\rangle_{V_0}$$
$$= \left\langle -V_4(\Lambda) + \frac{1}{2}V_2^2(\Lambda)\right\rangle_{V_0}. \quad (5.20)$$

The coefficients v appear in the potentials V_2 and V_4. The expectation values are computed with respect to the conventional quartic matrix model $V_0 = V_0(\Lambda)$.

This test clearly requires the computation of the kinetic term and its square, which means in particular that we need to numerically perform the integral over U in the term $\int dU\mathrm{Tr}[L_a, U\Lambda U^{-1}]^2$, which is not obvious how to do in any direct way. Equivalently, we can undo the diagonalization in the terms involving the kinetic term to obtain instead the equations

$$\langle a\mathrm{Tr}[L_a, M]^2\rangle_{V_0} = \langle -V_2\rangle_{V_0}, \quad (5.21)$$

$$\left\langle\frac{1}{2}(a\mathrm{Tr}[L_a, M]^2)^2\right\rangle_{V_0} = \left\langle -V_4 + \frac{1}{2}V_2^2\right\rangle. \quad (5.22)$$

Now the expectation values in the left hand side must be computed with respect to the conventional quartic matrix model $V_0 = V_0(M)$ with the full matrix $M = U\Lambda U^{-1}$ instead of the eigenvalues matrix Λ, i.e. the eigenvalues + angles. The expectation values in the right hand side can be computed either way.

In other words, the eigenvalues Metropolis algorithm discussed above, which can compute terms such as $\langle -V_2\rangle_{V_0}$ and $\langle -V_4 + V_2^2/2\rangle_{V_0}$, cannot be used to compute the terms $\langle a\mathrm{Tr}[L_a, M]^2\rangle_{V_0}$ and $\langle a\mathrm{Tr}[L_a, M]^2)^2/2\rangle_{V_0}$. We use instead the hybrid Monte Carlo algorithm to compute these terms as well as the terms $\langle -V_2\rangle_{V_0}$ and $\langle -V_4 + V_2^2/2\rangle_{V_0}$ in order to verify the above equations. This also should be viewed as a countercheck for the hybrid Monte Carlo algorithm since we can compare the values of $\langle -V_2\rangle_{V_0}$ and $\langle -V_4 + V_2^2/2\rangle_{V_0}$ obtained using the hybrid Monte Carlo algorithm with those obtained using our eigenvalues Metropolis algorithm. We note, in passing, that the Metropolis algorithm employed for the eigenvalues problem here is far more efficient than the hybrid Monte Carlo algorithm applied to the same problem without diagonalization.

In summary, we need to show that Eqs. (5.21) and (5.22) hold as identities in the correct calculation. To solve this problem we need to Monte Carlo sample both the eigenvalues and the angles of the matrix M, using the quartic matrix model.

$$V_0 = b\mathrm{Tr}M^2 + c\mathrm{Tr}M^4. \quad (5.23)$$

Clearly, we can choose without any loss of generality c such that $\tilde{c} = 1$. Monte Carlo simulations of this model can also be compared to the exact solution outlined in the previous subsection so calibration in this case is easy. A sample of this calculation including the eigenvalues distributions and the specific heat across the transition point are shown in Fig. 3. We can be satisfied from these results that the algorithm and simulations are working properly.

The two identities (5.21) and (5.22) are shown in Fig. 4 for $N = 10$ and $N = 17$ with $\tilde{c} = 1$. It is decisively shown that the calculation of the coefficients v reported in this article (Model II) gives the correct approximation of noncommutative scalar Φ_2^4 on the fuzzy sphere. Indeed, the data points for the expectation values $\langle -V_2\rangle_{V_0}$ and $\langle -V_4 + V_2^2/2\rangle_{V_0}$ in Model II coincide, within the best statistical errors, with the data points of the kinetic terms $\langle a\mathrm{Tr}[L_a, M]^2\rangle_{V_0}$ and $\langle a\mathrm{Tr}[L_a, M]^2)^2/2\rangle_{V_0}$, respectively. The discrepancy with Model II is obvious in Fig. 4.

C. Phase diagrams and other physics

We can now turn to the more serious study of the phase diagrams, critical boundaries, triple point and critical exponents of the multitrace matrix Models I and II using Monte Carlo simulation. This is a long calculation that can only be reported elsewhere [29,30]. Here we summarize some of our results, which include:

(i) The phase diagram of Model I contains three stable phases: (i) disordered (symmetric, one-cut, disk)

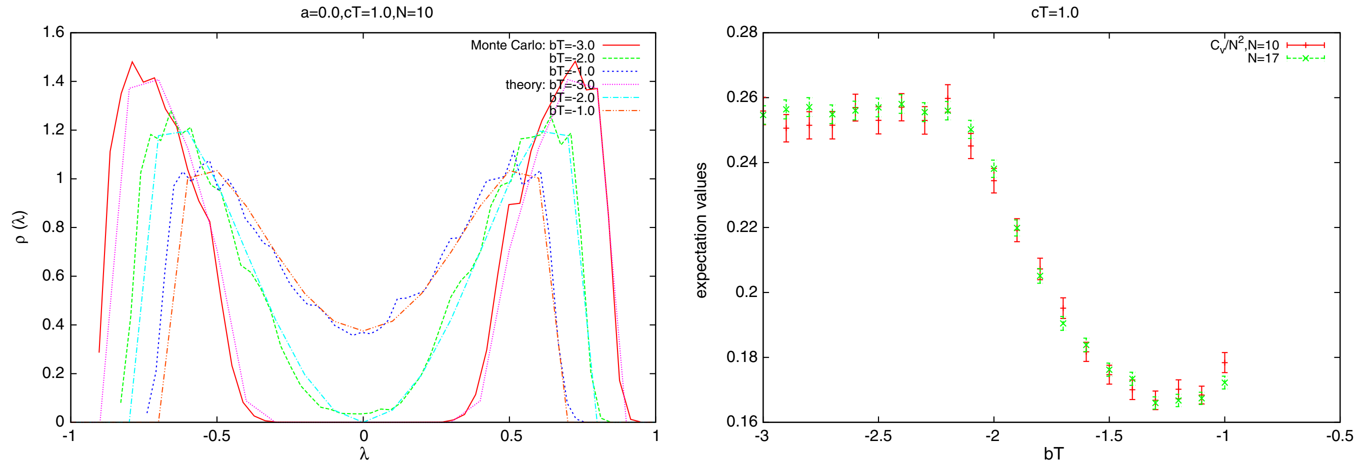

FIG. 3. The eigenvalues distributions and the specific heat of the quartic matrix model. These are used to calibrate the algorithm against known exact solutions.

phase; (ii) uniform ordered (Ising, broken, asymmetric one-cut) phase; and (iii) nonuniform ordered (matrix, stripe, two-cut, annulus) phase that meet at a triple point. The nonuniform ordered phase is a full blown nonperturbative manifestation of the perturbative UV-IR mixing effect, which is due to the underlying highly nonlocal matrix degrees of freedom of the noncommutative scalar field. The

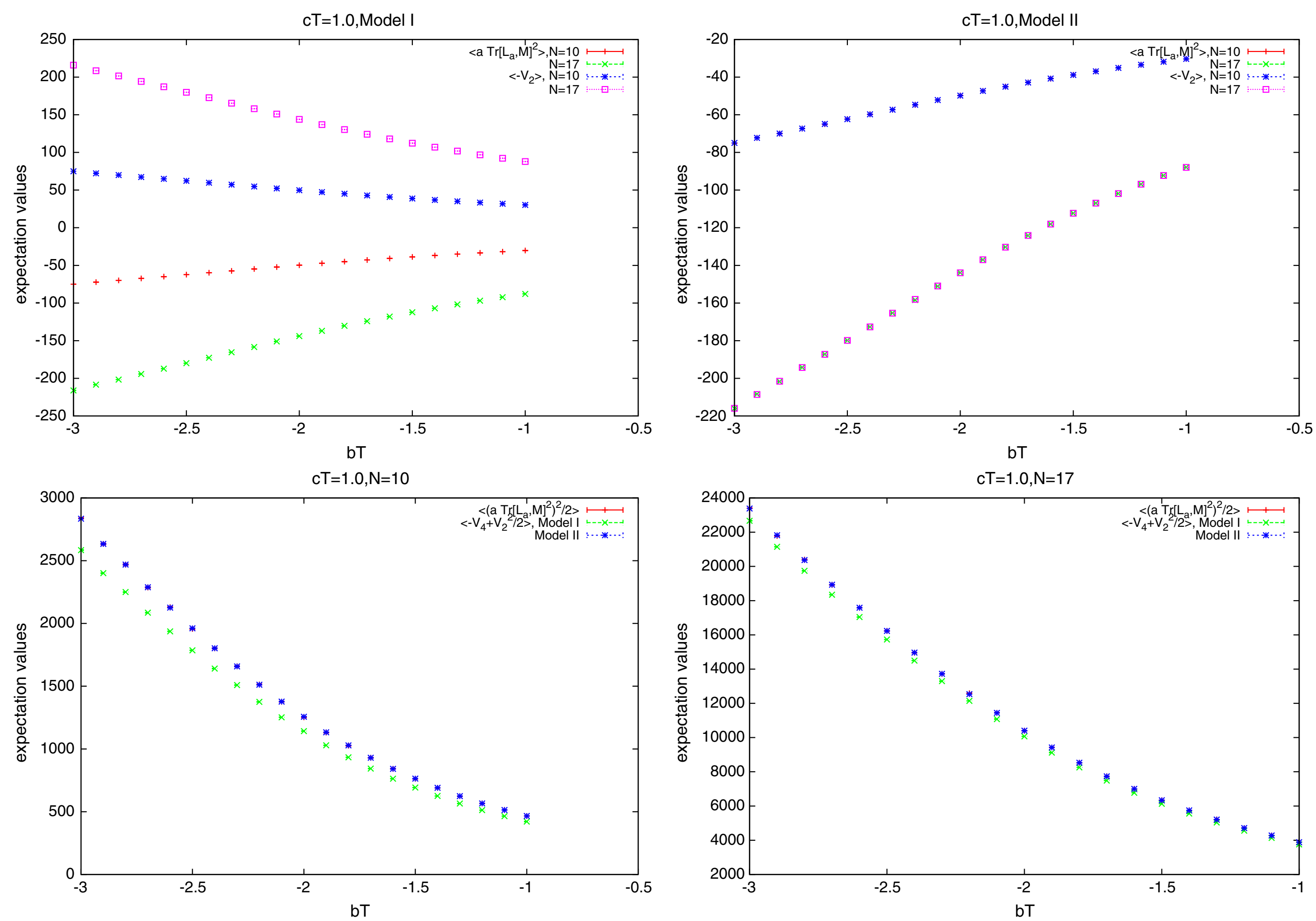

FIG. 4. The behaviors of multitrace matrix Models I and II against the behavior of noncommutative scalar phi-four on the fuzzy sphere.

critical boundaries are determined, and the triple point is located.

(ii) The uniform ordered phase exists in the Model I only with the odd terms included. If we assume the symmetry $M \to -M$, then all odd terms can be set to zero and the uniform ordered phase disappears. This is at least true in the domain studied in this article, which includes the triple point of fuzzy Φ^4 on the fuzzy sphere and extends to all its phase diagram probed in [9,10].

(iii) The delicate computation of the critical exponents of the Ising transition is discussed, and our estimate of the critical exponents ν, α, γ, β agrees very well with the Onsager values.

(iv) The phase diagram of Model II, with or without odd terms, does not contain the uniform ordered phase.

(v) The one-cut-to-two-cut transition line does not extend to the origin in Model II, which gives us an estimation of the triple point in this case.

(vi) As we have shown in this article, in Model II without odd terms, the termination point can be computed from the requirement that the critical point $\tilde{B}_*$ remains always negative. The result is given in Eq. (4.28), which agrees with what is obtained in Monte Carlo simulation.

(vii) In Model II with odd terms the termination point is found numerically to be located at $(\tilde{B},\tilde{C}) = (-1.05, 0.4)$. This is our measurement of the triple point.

(viii) In all cases the one-cut-to-two-cut matrix transition line agrees better with the double-trace matrix theory, studied in this article, than with the quartic matrix model. We recall that the double-trace matrix theory is given by $D \neq 0$ while all primed parameters are zero.

(ix) The model of Grosse-Wulkenhaar can also be discussed along the same lines using a combination of the multitrace approach and the Monte Carlo approach.

We note in passing that other far more important physics can also be obtained from these multitrace matrix models [30]. More precisely, a novel scenario for the emergence of geometry in generic random multitrace matrix models of a single Hermitian matrix M with unitary $U(N)$ invariance, i.e. without a kinetic term, can be formulated as follows. If the multitrace matrix model under consideration does not sustain the uniform ordered phase, then there is no emergent geometry. On the other hand, if the uniform ordered phase is sustained, then there is an underlying or emergent geometry with dimension determined from the critical exponents of the uniform-to-disordered (Ising) phase transition and a metric (Laplacian, propagator) determined from the Wigner semicircle law behavior of the eigenvalues distribution of the matrix M.

VI. THE NONPERTURBATIVE EFFECTIVE POTENTIAL APPROACH

The formalism due to Nair, Polychronakos, and Tekel [18,19,31,32] will allow us to compute the even part of the nonperturbative effective potential, i.e. the part of the potential symmetric under $M \to -M$, as a multitrace matrix model. This will also allow us to compare our multitrace matrix models obtained here, at least in this special case, to an independent exact result. We slightly change notation and start with the action

$$S = \mathrm{Tr}\left(\frac{1}{2} r M^2 + g M^4\right). \tag{6.1}$$

We define the moments m_n as usual by $m_n = \mathrm{Tr} M^n = \sum_i x_i^n$. By assuming that the kinetic operator $\mathcal{K}$ satisfies $\mathcal{K}(\mathbf{1}) = 0$ and that odd moments are zero we get immediately

$$\int dU \exp\left(-\frac{1}{2}\mathrm{Tr} M\mathcal{K}M\right) = \exp(-S_{\mathrm{eff}}(t_{2n})),$$
$$t_{2n} = \mathrm{Tr}\left(M - \frac{1}{N}\mathrm{Tr}M\right)^{2n}. \tag{6.2}$$

Let us first consider the free theory $g = 0$. In the limit $N \to \infty$ we know that planar diagrams dominates, and thus the eigenvalues distribution of M, obtained via the calculation of $\mathrm{Tr}M^n$, is a Wigner semicircle law [31,33]

$$\rho(x) = \frac{2N}{\pi R_W^2}\sqrt{R_W^2 - x^2}, \qquad R_W^2 = \frac{4f(r)}{N},$$
$$f(r) = \sum_{l=0}^{N-1} \frac{2l+1}{\mathcal{K}(l) + r}. \tag{6.3}$$

We consider now $g \neq 0$. The equation of motion of the eigenvalue x_i arising from the effective action S_{eff} contains a linear term in x_i+ the Vandermonde contribution + higher order terms. Explicitly, we have

$$\sum_n \frac{\partial S_{\mathrm{eff}}}{\partial t_{2n}} 2n x_i^{2n-1} = 2\sum_{i\neq j} \frac{1}{x_i - x_j}. \tag{6.4}$$

The semicircle distribution is a solution for $g \neq 0$ since it is a solution for $g = 0$ [18]. The term $n = 1$ alone will give the semicircle law. Thus the terms $n > 1$ are cubic and higher order terms that cause the deformation of the semicircle law. These terms must vanish when evaluated on the semicircle distribution in order to guarantee that the semicircle distribution remains a solution. We rewrite the action S_{eff} as the following power series in the eigenvalues:

$$S_{\mathrm{eff}} = a_2 t_2 + (a_4 t_4 + a_{22} t_2^2) + (a_6 t_6 + a_{42} t_4 t_2 + a_{222} t_2^3)$$
$$+ (a_8 + a_{62} t_6 t_2 + a_{422} a_4 t_2^2 + a_{2222} t_2^4) + \cdots. \tag{6.5}$$

We impose then the condition

$$\frac{\partial S_{\rm eff}}{\partial t_{2n}}\bigg|_{\rm Wigner} = 0, \quad n > 1, \tag{6.6}$$

and use the fact that the moments in the Wigner distribution satisfy

$$t_{2n} = C_n t^n, \qquad C_n = \frac{(2n)!}{n!(n+1)!}, \tag{6.7}$$

to get immediately the conditions

$$\begin{aligned} &a_4 = 0, \qquad a_6 = a_{42} = 0, \\ &a_8 = a_{62} = 0, \qquad 4a_{44} + a_{422} = 0, \ldots. \end{aligned} \tag{6.8}$$

By plugging these values back into the effective action we obtain the form

$$\begin{aligned} S_{\rm eff} = &\frac{1}{2}F(t_2) + (b_1 + b_2 t_2)(t_4 - 2t_2^2)^2 \\ &+ c(t_6 - 5t_2^3)(t_4 - 3t_2^2) + \cdots. \end{aligned} \tag{6.9}$$

Thus the effective action is still an arbitrary function $F(t_2)$ of t_2, but it is fully fixed in the higher moments $t_4, t_6, \ldots$. We note that the extra terms vanish for the Wigner semicircle law. The action up to sixth order in the eigenvalues depends therefore only on t_2, viz.

$$S_{\rm eff} = \frac{1}{2}F(t_2) + \cdots. \tag{6.10}$$

The equations of motion of the eigenvalues for $g = 0$ read now explicitly

$$(F'(t_2) + r)x_i = 2\sum_{i\neq j}\frac{1}{x_i - x_j}. \tag{6.11}$$

The radius of the semicircle distribution is immediately obtained by

$$R_W^2 = \frac{4N}{F'(t_2) + r}. \tag{6.12}$$

By comparing (6.3) and (6.12) we obtain the self-consistency equation

$$\frac{4f(r)}{N} = \frac{4N}{F'(t_2) + r}. \tag{6.13}$$

Another self-consistency condition is the fact that t_2 computed using the effective action $S_{\rm eff}$ for $g = 0$, i.e. using the Wigner distribution, should give the same value, viz.

$$t_2 = {\rm Tr}M^2 = \int_{-R_W}^{R_W} dx x^2 \rho(x) = \frac{N}{4}R_W^2 = \frac{N^2}{F'(t_2) + r}. \tag{6.14}$$

We have then the two conditions

$$F'(t_2) + r = \frac{N^2}{t_2}, \qquad t_2 = f(r). \tag{6.15}$$

The solution is given by

$$F(t_2) = N^2 \int dt_2 \left(\frac{1}{t_2} - \frac{1}{N^2} g(t_2)\right). \tag{6.16}$$

$g(t_2)$ is the inverse function of $f(r)$, viz. $f(g(t_2)) = t_2$.

For the case of the fuzzy sphere with a kinetic term given by the canonical formula $\mathcal{K}(l) = l(l+1)$ we have the result

$$f(r) = \ln\left(1 + \frac{N^2}{r}\right). \tag{6.17}$$

Thus the corresponding solution is explicitly given by

$$F(t_2) = N^2 \ln\frac{t_2}{1 - \exp(-t_2)}. \tag{6.18}$$

The full effective action on the sphere is then

$$\begin{aligned} S_{\rm eff} &= \frac{N^2}{2}\ln\frac{t_2}{1 - \exp(-t_2)} + {\rm Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \cdots \\ &= \frac{N^2}{2}\left(\frac{t_2}{2} - \ln\frac{\exp(t/2) - \exp(-t/2)}{t}\right) \\ &\quad + {\rm Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \cdots \\ &= \frac{N^2}{2}\left(\frac{t_2}{2} - \frac{1}{24}t_2^2 + \frac{1}{2880}t_2^4 + \cdots\right) \\ &\quad + {\rm Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \cdots. \end{aligned} \tag{6.19}$$

This should be compared with our result in this article with action given by $a{\rm Tr}M\mathcal{K}M + b{\rm Tr}M^2 + c{\rm Tr}M^4$ and effective action given by our Eq. (3.12) or equivalently

$$\begin{aligned} V_0 + \Delta V_0 = &\left(\frac{aN^2}{2}{\rm Tr}M^2 - \frac{a^2N^2}{12}({\rm Tr}M^2)^2 + \cdots\right) \\ &+ {\rm Tr}(bM^2 + cM^4) + \cdots. \end{aligned} \tag{6.20}$$

It is very strange that the author of [18] notes that their result (6.19) is in agreement with the result of [16], given by Eq. (4.5), which involves the term $T_4 = \sum_{i\neq j}(x_i - x_j)^4/2$. It is very clear that T_4 is not present in the above Eq. (6.19), which depends instead on the term T_2^2 where $T_2 = \sum_{i\neq j}(x_i - x_j)^2/2$. The work [17] contains the correct calculation, which agrees with both the results of [18] and our result here.

The one-cut-to-two-cut phase transition derived from the effective action $S_{\rm eff}$ will be appropriately shifted. The equation determining the critical point is still given, as before, by the condition that the eigenvalues distribution becomes negative. We get [18]

$$r = -5\sqrt{g} - \frac{1}{1-\exp(1/\sqrt{g})}. \tag{6.21}$$

For large g we obtain

$$r = -\frac{1}{2} - 4\sqrt{g} + \frac{1}{12\sqrt{g}} + \cdots. \tag{6.22}$$

This is precisely the result obtained in this article given by Eq. (4.26) with the identification $a = 1$, $b = r$, and $c = 4g$.

VII. CONCLUSION

In this article we have extended the multitrace approach of [16] to two-dimensional noncommutative phi-four theory with a nonzero harmonic oscillator term on the fuzzy sphere and on the Moyal-Weyl plane. We computed the corresponding real multitrace quartic matrix model up to the second order in the kinetic term parameter and then derived explicitly, in the case of the even double-trace matrix models, the critical transition line between the one-cut (disordered,disk) phase with $\langle\Phi\rangle = 0$ and the two-cut (nonuniform ordered, annulus) phase with $\langle\Phi\rangle = \gamma$. A robust prediction of the triple point, identified as a termination point of the matrix transition line, is derived and compared with the previous Monte Carlo result of [9,10]. Our estimation is improved considerably by including odd moments in the effective multitrace action as evidenced by Monte Carlo simulations of this multitrace matrix model.

The multitrace matrix model of [16] as well as the one obtained in this article are tested for their correctness using Monte Carlo simulation where it is decisively shown that our calculation here gives the correct approximation of noncommutative scalar Φ_2^4 on the fuzzy sphere to the second order. This was also confirmed using the nonperturbative multitrace approach of [18,19,31,32]. The rich phase diagrams of both of these models, obtained in Monte Carlo simulations, are also described in some detail together with some unexpected physics, such as emergent geometry, of generic multitrace matrix models.

ACKNOWLEDGMENTS

This research was supported by CNEPRU: "The National (Algerian) Commission for the Evaluation of University Research Projects" under Contract No. DO1120130009.

APPENDIX A: LARGE N BEHAVIOR

The coefficients s are defined, in terms of the kinetic matrix K and characters and dimensions of various SU(N)/U(N) representations, by the following equations:

$$s_{1,2} = \frac{1}{\dim(1,2)} K_{AB} {\rm Tr}_{(1,2)} t_A \otimes t_B, \qquad s_{2,1} = \frac{1}{\dim(2,1)} K_{AB} {\rm Tr}_{(2,1)} t_A \otimes t_B. \tag{A1}$$

$$s_{1,4} = \frac{1}{\dim(1,4)} K_{AB} K_{CD} {\rm Tr}_{(1,4)} t_A \otimes t_B \otimes t_C \otimes t_D, \tag{A2}$$

$$s_{4,1} = \frac{1}{\dim(4,1)} K_{AB} K_{CD} {\rm Tr}_{(4,1)} t_A \otimes t_B \otimes t_C \otimes t_D, \tag{A3}$$

$$s_{2,3} = \frac{1}{\dim(2,3)} (2K_{AB} K_{CD} + K_{AD} K_{BC}) {\rm Tr}_{(2,3)} t_A \otimes t_B \otimes t_C \otimes t_D, \tag{A4}$$

$$s_{3,2} = \frac{1}{\dim(3,2)} (K_{AB} K_{CD} + 2K_{AC} K_{BD}) {\rm Tr}_{(3,2)} t_A \otimes t_B \otimes t_C \otimes t_D, \tag{A5}$$

$$s_{2,2} = \frac{1}{\dim(2,2)} (K_{AB} K_{CD} + K_{AC} K_{BD}) {\rm Tr}_{(2,2)} t_A \otimes t_B \otimes t_C \otimes t_D. \tag{A6}$$

On the other hand, the combinations that appear in the quartic part of the effective potential are given by the following expressions:

$$\frac{1}{8N}(s_{1,4}-s_{4,1}-s_{2,3}+s_{3,2})=\frac{1}{16N^6}\left(-2-\frac{25}{N^2}+O_4\right)K_{ii,jj}^2+\frac{1}{8N^5}\left(1+\frac{15}{N^2}+O_4\right)K_{ii,kl}K_{jj,lk}, \tag{A7}$$

$$\frac{1}{6}(s_{1,4}+s_{4,1}-s_{2,2})=\frac{1}{4N^6}\left(2+\frac{25}{N^2}+O_4\right)K_{ii,jj}^2-\frac{1}{2N^5}\left(1+\frac{15}{N^2}+O_4\right)K_{ii,kl}K_{jj,lk} -\frac{1}{4N^5}\left(1+\frac{15}{N^2}+O_4\right)K_{ii,jj}K_{kl,lk}+\frac{1}{4N^4}\left(1+\frac{17}{N^2}+O_4\right)K_{ij,jl}K_{kk,li}, \tag{A8}$$

$$\frac{1}{8N}(s_{1,4}-s_{4,1}+s_{2,3}-s_{3,2}-2(s_{1,2}^2-s_{2,1}^2)) =\frac{1}{8N^6}(-1+O_2)K_{ij,kl}K_{ji,lk}+\frac{1}{16N^6}\left(-\frac{18}{N^2}+O_4\right)K_{ii,jj}^2+\frac{5}{4N^7}(1+O_2)K_{ii,kl}K_{jj,lk}+\frac{1}{16N^5}\left(\frac{12}{N^2}+O_4\right)K_{ii,jj}K_{kl,lk} +\frac{1}{8N^5}(1+O_2)K_{ij,ki}K_{lk,jl}-\frac{3}{4N^6}(1+O_2)K_{ij,jl}K_{kk,li}+\frac{1}{8N^6}(-1+O_2)K_{ij,ji}^2, \tag{A9}$$

$$\frac{1}{16N^2}(s_{1,4}+s_{4,1}-s_{2,3}-s_{3,2}+2s_{2,2}-2(s_{1,2}-s_{2,1})^2) =\frac{1}{16N^6}(1+O_2)K_{ij,kl}K_{ji,lk}+\frac{1}{32N^6}\left(\frac{6}{N^2}+O_4\right)K_{ii,jj}^2-\frac{1}{8N^7}(2+O_2)K_{ii,kl}K_{jj,lk}, \tag{A10}$$

$$\frac{1}{48}(s_{1,4}+s_{4,1}+3s_{2,3}+3s_{3,2}+2s_{2,2}-6(s_{1,2}+s_{2,1})^2) =\frac{1}{16N^6}(1+O_2)K_{ij,kl}K_{ji,lk}+\frac{1}{32N^6}\left(\frac{18}{N^2}+O_4\right)K_{ii,jj}^2-\frac{5}{8N^7}(1+O_2)K_{ii,kl}K_{jj,lk}+\frac{1}{16N^5}\left(-\frac{8}{N^2}+O_4\right)K_{ii,jj}K_{kl,lk} -\frac{1}{8N^5}(1+O_2)K_{ij,ki}K_{lk,jl}+\frac{1}{4N^6}(2+O_2)K_{ij,jl}K_{kk,li}+\frac{1}{32N^4}\left(\frac{4}{N^2}+O_4\right)K_{ij,ji}^2. \tag{A11}$$

The kinetic matrix $K_{ij,kl}$ is defined in terms of the kinetic matrix K_{AB}, which is defined by Eq. (2.12), by

$$K_{AB}=(t_A)_{jk}(t_B)_{li}K_{ij,kl}. \tag{A12}$$

The large N behavior of the different operators is given by

$$\frac{1}{N^5}K_{ii,kl}K_{jj,lk}=\frac{r^4}{3}(13-10\epsilon)+\cdots, \tag{A13}$$

$$K_{ij,kl}K_{li,jk}=16r^4(\sqrt{\omega}+1)^2\left(\frac{1}{3}N^3-\epsilon\frac{3}{10}N^3\right)+\cdots, \tag{A14}$$

$$K_{ij,jl}K_{kk,li}=8r^4(\sqrt{\omega}+1)\left(\frac{7}{12}N^4-\epsilon\frac{1}{2}N^4\right)+\cdots, \tag{A15}$$

$$K_{ij,kl}K_{ji,lk}=\frac{2r^4N^4}{9}(21-16\epsilon)+\frac{2r^4\omega N^4}{9}(9-8\epsilon)+\cdots, \tag{A16}$$

$$\frac{1}{4N^6}K_{ii,jj}^2=r^4\left(1-3\frac{\epsilon}{4}\right)+\cdots, \tag{A17}$$

$$K_{ii,jj}K_{kl,lk}=4r^4(\sqrt{\omega}+1)\left(N^5-\epsilon\frac{5}{6}N^5\right)+\cdots, \tag{A18}$$

$$K_{mj,km}K_{nk,jn}=16r^4(\sqrt{\omega}+1)^2\left(\frac{1}{3}N^3-\epsilon\frac{3}{10}N^3\right)+\cdots, \tag{A19}$$

$$K_{ij,ji}^2=16r^4(\sqrt{\omega}+1)^2\left(\frac{1}{4}N^4-\epsilon\frac{2}{9}N^4\right)+\cdots. \tag{A20}$$

APPENDIX B: RESULT OF [16] REVISITED

The starting point is the result given by Eq. (3.25) of [16], which in our notation reads [with $a=2\pi/(N+1)$]

$$\int dU \exp(a\mathrm{Tr}[U^{-1}L_aU,\Lambda]^2)$$
$$= 1 + 2a\cdot\frac{T_2}{N(N^2-1)}K_{aa} + 8a^2\cdot\frac{T_2^2-2T_4}{4N^2(N^2-1)(N^2-9)}X_1 + 8a^2\cdot\frac{-5T_2^2+(N^2+1)T_4}{2N(N^2-1)(N^2-4)(N^2-9)}X_2+\cdots. \quad \text{(B1)}$$

In the above equation we have multiplied by the appropriate factor and also included, for completeness, the first and second order correction terms. We should make the identification $\Xi = T$ between our notation and the notation of [16]. The operators X_1 and X_2 are defined by

$$X_1 = 2K_{ab}^2 + K_{aa}^2, \quad \text{(B2)}$$

$$X_2 = K_{ab}K_{cd}\left(\frac{1}{2}d_{abk}d_{cdk} + d_{adk}d_{bck}\right). \quad \text{(B3)}$$

In other words, X_1 and X_2 are essentially the operators $2trK^2 + (trK)^2$ and $K^{\perp}K$ of [16]. We must furthermore take into account the different normalizations for the Gell-Mann matrices employed in the two cases. The kinetic matrix in this case is defined by

$$K_{ab} = \mathrm{Tr}[L_i,t_a][L_i,t_b]. \quad \text{(B4)}$$

The source of the discrepancy between our result (3.12) and the result obtained in [16] was traced to the operator $K^{\perp}K$, i.e. X_2, defined in Eq. (3.18) of [16], which was neglected in the large N limit in their analysis. The operators X_1 and X_2 can be computed in closed form. We find

$$X_1 = \frac{N^4(N^2-1)^2}{16} + \frac{N^2(N^2-1)^2}{6}, \quad \text{(B5)}$$

$$X_2 = \frac{N^3(N^2-1)^2}{16} - \frac{N(N^2-1)^2}{6} - \frac{N(N^2-1)}{4}. \quad \text{(B6)}$$

Clearly X_1 is of order N^8 while X_2 is of order N^7. As a consequence the coefficient of T_4 in (B1) comes out to be subleading. This can be inferred quite easily from the exact result[5]

[5]This can be derived directly for $N = 2$.

$$\int dU \exp(a\mathrm{Tr}[U^{-1}L_aU,\Lambda]^2)$$
$$= 1 - \frac{aN}{2}T_2 + \frac{a^2}{24}(T_2^2-2T_4)\frac{N^2-1}{N^2-9}(3N^2+8) + \frac{a^2}{12}(-5T_2^2+(N^2+1)T_4)\frac{3N^2+1}{N^2-9}+\cdots. \quad \text{(B7)}$$

This leads to (3.12). Furthermore, it is obvious from this result that although the subleading coefficient of the operator T_2^2 is of order N^0, the contribution associated with this term cannot be suppressed, since this operator is actually of order N^4.

APPENDIX C: QUARTIC EQUATION

The quartic equation (4.12) reads in terms of the scaled parameters $\bar{b}$, $\bar{c}$, and $\bar{x} = \tilde{a}\gamma^{-1/4}x$ as follows:

$$\bar{x}^4 + \alpha\bar{x}^2 + \beta\bar{x} + 1 = 0, \quad \text{(C1)}$$

$$\alpha = \gamma^{\frac{1}{2}}\bar{\alpha}, \qquad \beta = \gamma^{\frac{1}{4}}\bar{\beta}, \quad \text{(C2)}$$

$$\gamma = \frac{24}{(\Omega^2+1)^2\eta\bar{c}}, \quad \text{(C3)}$$

$$\bar{\alpha} = -\frac{1}{12}(9\bar{c} - 2\eta(\Omega^2+1)^2), \quad \bar{\beta} = -\frac{1}{4}(2\bar{b} + \Omega^2 + 1). \quad \text{(C4)}$$

The range (4.23) of $\bar{c}$ translates to a range of $\bar{\alpha}$ given by

$$\bar{\alpha} \le -\frac{\eta(\Omega^2+1)^2}{3}. \quad \text{(C5)}$$

The four solutions of our depressed quartic equation can be rewritten as

$$\bar{x} = \frac{1}{2}\left[\pm_1\sqrt{z} \pm_2 \sqrt{-\left(z + 2\alpha \pm_1 \frac{2\beta}{\sqrt{z}}\right)}\right], \quad \text{(C6)}$$

where z is a solution of the cubic equation

$$z^3 + 2\alpha z^2 + (\alpha^2 - 4)z - \beta^2 = 0. \quad \text{(C7)}$$

Define

$$z = t - \frac{2\alpha}{3}. \quad \text{(C8)}$$

The corresponding depressed cubic equation is

$$t^3 + 3Qt - 2R = 0, \quad \text{(C9)}$$

$$Q = -\frac{\alpha^2}{9} - \frac{4}{3}, \qquad R = -\frac{4\alpha}{3} + \frac{\alpha^3}{27} + \frac{\beta^2}{2}. \quad \text{(C10)}$$

Next we reduce to a quadratic equation. We start from the identity

$$(t^3 - B^3) + C(t - B) = (t - B)(t^2 + Bt + B^2 + C). \tag{C11}$$

By comparison we get

$$C = 3Q, \qquad B^3 + CB = 2R. \tag{C12}$$

In other words, B solves the cubic equation

$$B^3 + 3QB - 2R = 0. \tag{C13}$$

The solution is immediately given by

$$B = [R + \sqrt{Q^3 + R^2}]^{1/3} + [R - \sqrt{Q^3 + R^2}]^{1/3}. \tag{C14}$$

We have then

$$t^3 + 3Qt - 2R = (t - B)(t^2 + Bt + B^2 + 3Q). \tag{C15}$$

In other words, $t = B$ is a solution of the cubic equation (C9). The other two solutions solve the quadratic equation $t^2 + Bt + B^2 + 3Q = 0$, viz.

$$t = \frac{-B \pm \sqrt{-3B^2 - 12Q}}{2}. \tag{C16}$$

This can be rewritten also as

$$t = \frac{-B \pm i\sqrt{3}A}{2}, \tag{C17}$$

$$A = [R + \sqrt{Q^3 + R^2}]^{1/3} - [R - \sqrt{Q^3 + R^2}]^{1/3}. \tag{C18}$$

Define

$$D = Q^3 + R^2, \quad S = [R + \sqrt{D}]^{1/3}, \quad T = [R - \sqrt{D}]^{1/3}. \tag{C19}$$

In summary the real solutions of interest of our cubic equation are given by

$$D > 0 \Rightarrow t = \{S + T\}, \tag{C20}$$

$$D \le 0 \Rightarrow t = \Big\{S + T = \mathrm{real}(2S), \frac{1}{2}(-1 \pm i\sqrt{3})S + \frac{1}{2}(-1 \mp i\sqrt{3})T = \mathrm{real}((-1 \pm i\sqrt{3})S)\Big\}. \tag{C21}$$

Our numerical approach is based on the solutions (C6)–(C8)–(C10) and (C19)–(C20)–(C21).

[1] A. Connes, *Noncommutative Geometry* (Academic Press, London, 1994).
[2] J. Frohlich and K. Gawedzki, Conformal field theory and geometry of strings, arXiv:hep-th/9310187.
[3] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).
[4] B. Ydri and A. Bouchareb, The fate of the Wilson-Fisher fixed point in non-commutative phi-four, J. Math. Phys. (N.Y.) **53**, 102301 (2012).
[5] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, Planar diagrams, Commun. Math. Phys. **59**, 35 (1978).
[6] Y. Shimamune, On the phase structure of large N matrix models and gauge models, Phys. Lett. **108B**, 407 (1982).
[7] S. S. Gubser and S. L. Sondhi, Phase structure of non-commutative scalar field theories, Nucl. Phys. **B605**, 395 (2001).
[8] J. Ambjorn and S. Catterall, Stripes from (noncommutative) stars, Phys. Lett. B **549**, 253 (2002).
[9] F. Garcia Flores, X. Martin, and D. O'Connor, Simulation of a scalar field on a fuzzy sphere, Int. J. Mod. Phys. A **24**, 3917 (2009).
[10] F. Garcia Flores, D. O'Connor, and X. Martin, Simulating the scalar field on the fuzzy sphere, *Proc. Sci.*, LAT2005 (2006) 262 [arXiv:hep-lat/0601012].
[11] B. Ydri, New algorithm and phase diagram of noncommutative ϕ^4 on the fuzzy sphere, J. High Energy Phys. 03 (2014) 065.
[12] X. Martin, A matrix phase for the phi**4 scalar field on the fuzzy sphere, J. High Energy Phys. 04 (2004) 077.
[13] M. Panero, Numerical simulations of a non-commutative theory: The scalar model on the fuzzy sphere, J. High Energy Phys. 05 (2007) 082.
[14] J. Medina, W. Bietenholz, and D. O'Connor, Probing the fuzzy sphere regularisation in simulations of the 3d lambda phi**4 model, J. High Energy Phys. 04 (2008) 041.
[15] C. R. Das, S. Digal, and T. R. Govindarajan, Finite temperature phase transition of a single scalar field on a fuzzy sphere, Mod. Phys. Lett. A **23**, 1781 (2008).
[16] D. O'Connor and C. Saemann, Fuzzy scalar field theory as a multitrace matrix model, J. High Energy Phys. 08 (2007) 066.
[17] C. Saemann, The multitrace matrix model of scalar field theory on fuzzy *CP***n, SIGMA **6**, 050 (2010).

[18] A. P. Polychronakos, Effective action and phase transitions of scalar field on the fuzzy sphere, Phys. Rev. D **88**, 065010 (2013).
[19] J. Tekel, Uniform order phase and phase diagram of scalar field theory on fuzzy CP^n, J. High Energy Phys. 10 (2014) 144.
[20] W. Bietenholz, F. Hofheinz, and J. Nishimura, Phase diagram and dispersion relation of the noncommutative lambda phi**4 model in $d = 3$, J. High Energy Phys. 06 (2004) 042.
[21] F. Lizzi and B. Spisso, Noncommutative field theory: Numerical analysis with the fuzzy disc, Int. J. Mod. Phys. A **27**, 1250137 (2012).
[22] H. Meja-Daz, W. Bietenholz, and M. Panero, The continuum phase diagram of the 2d non-commutative lambda phi**4 model, J. High Energy Phys. 10 (2014) 56.
[23] E. Langmann and R. J. Szabo, Duality in scalar field theory on noncommutative phase spaces, Phys. Lett. B **533**, 168 (2002).
[24] I. Montvay and G. Munster, *Quantum Fields on a Lattice, Cambridge Monographs on Mathematical Physics* (Cambridge University Press, Cambridge, UK, 1994).
[25] J. Smit, Introduction to quantum fields on a lattice: A robust mate, Cambridge Lect. Notes Phys. **15**, 1 (2002).
[26] B. Ydri, Matrix field theory (unpublished).
[27] S. R. Das, A. Dhar, A. M. Sengupta, and S. R. Wadia, New critical behavior in $d = 0$ large N matrix models, Mod. Phys. Lett. A **05**, 1041 (1990).
[28] B. Eynard, Random matrices, Cours de Physique Theorique de Saclay.
[29] B. Ydri, K. Ramda, and A. Rouag, Phase diagrams of the multitrace quartic matrix models of noncommutative Φ^4 (work in progress).
[30] B. Ydri, A. Rouag, and K. Ramda, Emergent geometry from random multitrace matrix models (work in progress).
[31] V. P. Nair, A. P. Polychronakos, and J. Tekel, Fuzzy spaces and new random matrix ensembles, Phys. Rev. D **85**, 045021 (2012).
[32] J. Tekel, Random matrix approach to scalar fields on fuzzy spaces, Phys. Rev. D **87**, 085015 (2013).
[33] H. Steinacker, A non-perturbative approach to noncommutative scalar field theory, J. High Energy Phys. 03 (2005) 075.