Remark on the Dunne-Ünsal relation in exact semiclassics

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Recently, it was realized that nonperturbative instanton effects can be generated to all orders by perturbation theory around a degenerate minima via the Dunne-Ünsal relation in several quantum mechanical systems. In this work we verify the Dunne-Ünsal relation for resonance energy levels of one-dimensional polynomial anharmonic oscillators. We show that the relation is applicable to cubic and quartic anharmonic oscillators which are genus-one potentials. However for higher order (higher genus) anharmonic potentials the relation is not satisfied and is subject to a certain extension.

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I. INTRODUCTION

It is well known that in quantum mechanical systems perturbation theory around a degenerate minima has energy expansion of the form

$$E(g) = \sum_{n=0}^{\infty} E_n g^n, \tag{1}$$

which often diverges asymptotically [1–3]. One way to resolve this issue is to apply Borel summation and give a physical meaning to these divergent series. However, in the process of analytic continuation of the Borel transform, singularities arise on the integration contour, and hence the Borel sum includes imaginary terms due to the deformation of the contour (see, e.g., Refs. [4–10]). Moreover, the choice of the contour also affects the imaginary contribution and hence gives rise to ambiguities on the energy eigenvalue [10,11]. Another ambiguity arises from the fluctuations around the n-instanton sector which again asymptotically diverges and leads to ambiguous imaginary terms. So by the inclusion of these nonperturbative effects, we confront ambiguous imaginary terms coming from both perturbative and nonperturbative sectors which, at first, make the problem even more subtle. Since any physical observable must be real and ambiguity free, a further analysis is needed to resolve these issues.

Recently, important progress has been made in studying the question of the relation between the pertubative and nonperturbative contributions in quantum theories [12–17] (see also earlier works [3,18–21]). Much of this progress is due to resurgence theory, developed by Ecalle in the early 1980s [22] (see also Refs. [23,24]). Rather than the usual perturbative expansion (1) for the energy eigenvalue, resurgence analysis connects perturbative contributions with nonperturbative effects through "resurgent trans-series," where the imaginary terms with ambiguities coming from the Borel summation systematically cancel each other to all orders[1] [28,29]. For instance, imaginary terms arising from the perturbative vacuum cancels the imaginary term arising from the 2-instanton sector, the imaginary term from the 1-instanton sector is canceled by an imaginary term in the 3-instanton sector, and so on. Hence, this leaves us with a real and unambiguous result for our observable (in this paper, energy). This cancellation has been carried out to all orders by using the following resurgent expansion form of the Nth energy level [12,13,28,29]:

$$E^{(N)}(g) = \sum_{\pm} \sum_{n=0}^{\infty} \sum_{l=1}^{n-1} \sum_{m=0}^{\infty} c^{\pm}_{n,l,m} \frac{e^{-n\frac{S}{g}}}{g^{n(N+\frac{1}{2})}} \left(\ln\left[\mp\frac{2}{g}\right]\right)^l g^m, \tag{2}$$

which takes into account the n-instanton contributions, generated by $e^{-\frac{S}{g}}$, where S is the coefficient of the instanton action, with the fluctuations around them as well as the quasi-zero modes generated by the logarithmic term. One should note that for $n = 0$ the term in the sum is the usual perturbative expansion of the form (1), and the logarithmic terms start to appear at the 2-instanton sector. So the expansion (1) is actually not the complete expansion of a real unambiguous eigenvalue; one needs to extend it by adding nonperturbative effects. On the other hand, the expansion (2) handles these nonperturbative effects with the right coefficients $c^{\pm}_{n,l,m}$ such that the total sum is real and

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[1]For an introduction to resurgence in physics, see recent reviews on the topic [25–27].

ambiguity free. These cancellations imply a deep relationship between perturbative and nonperturbative sectors, and this relationship lies behind the resurgence analysis [12,13,29].

In recent years resurgence techniques have also been applied to different branches of physics and mathematics, including quantum mechanics [12,13,15,28–32], quantum field theory [11,33–37], string theory [38–45], hydrodynamic gradient expansion [46–48], and supersymmetric theories [49–52].

The organization of the paper is as follows. In order to put the results into context, in Sec. II we briefly discuss some aspects of [28] and [15], focusing, in particular, on the Dunne-Ünsal relation. In Secs. III and IV we verify the relation for cubic and quartic anharmonic oscillators, respectively. While the present work is mainly devoted to the verification of the Dunne-Ünsal relation, we also briefly recall some properties of the cubic and quartic potentials. In Sec. V we discuss the fact that the formula is not satisfied in its current form for higher degree polynomial oscillators; in particular, we show the calculations for the quintic potential. For the sake of completeness, in Appendixes A, B and C, we write down the related data for sextic, septic and octic potentials, although they will not be discussed here. In this paper the generalized quantization conditions and expressions for the functions B and A in terms of E and g are collected from Ref. [15].

II. CONNECTING PERTURBATIVE AND NONPERTURBATIVE SECTORS

Zinn-Justin *et al.* obtained the resurgent expansion (2) by a small systematic g expansion of an exact quantization condition [3,12,13,15] in a given system. In their approach, this quantization condition includes two functions, $B(E,g)$ and $A(E,g)$, which are related to the perturbative expansion of the energy eigenvalues and instanton contributions to the system, respectively [5,6]. Schematically, this generalized quantization condition has the following form:

$$\frac{1}{\Gamma(1-B(E,g))}\sim\left(\frac{2}{g}\right)^{B(E,g)}e^{-A(E,g)}. \tag{3}$$

One can compute the perturbative expansions of $A(E,g)$ and $B(E,g)$ functions by using the WKB approximation [6,12,13]. Alternatively, if one knows the energy in the one-instanton approximation to all orders, then from the one-instanton approximation of the quantization condition[2] (3), it is easy to determine the function $A(E,g)$ [5].

In order to calculate exact energy eigenvalues of a quantum-mechanical system as in the form of (2), one has to calculate the $B(E,g)$ and $A(E,g)$ functions separately and then expand this quantization condition for a small coupling parameter g.

Recently, Dunne and Ünsal have revealed a remarkable simple relation between these two functions. For given $B(E,g)$ and $A(E,g)$ functions of several physical systems [like double-well, sine-Gordon, Fokker-Planck, $O(d)$ symmetric potential], it was shown that by converting the functions $B(E,g)$ and $A(E,g)$ into $E(B,g)$ and $A(B,g)$, they satisfy the following relation [28]:

$$\frac{\partial E(B,g)}{\partial B}=-\frac{g}{2S}\left(2B+g\frac{\partial A(B,g)}{\partial g}\right), \tag{4}$$

where S is the instanton action coefficient.

The relation (4) provides a powerful computational tool: By knowing the perturbative expansion about a degenerate minima with a global boundary condition, one can derive the function A rather than calculating them separately. In other words, resurgent trans-series for the energy can actually be generated only from the perturbative expansion of E with a global boundary condition. The Dunne-Ünsal relation (4) shows a close connection between perturbative and nonperturbative sectors which is not very obvious in the Zinn-Justin *et al.* approach [3,12,13,15].

Although the Dunne-Ünsal relation (4) is a powerful equation, under which conditions this equation holds is still unclear [53,54].

In this paper, we consider a set of one-dimensional anharmonic oscillators with polynomial potentials [14,18,19,55,56] and verify the Dunne-Ünsal relation for cubic and quartic anharmonic potentials which correspond to genus-one potentials. However, as we go into higher order (higher genus) anharmonic potentials we observe that the relation is not satisfied.

We use the notations of [15] and denote the Hamiltonian of an even oscillator by

$$H_N(g)=-\frac{1}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}q^2+gq^N, \tag{5}$$

and we use the convention H_M for the Hamiltonian of an odd oscillator[3]

$$H_M(g)=-\frac{1}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}q^2+\sqrt{g}q^M. \tag{6}$$

For N even, instanton configurations exist for $g<0$, and the generalized Bohr-Sommerfeld quantization condition for each potential has the following form:

[2]Note that in the case of locally harmonic oscillators, taking $B(E,g)=\frac{1}{2}+N$ gives the usual perturbative expansion of the energy.

[3]There are several reasons for choosing the coupling constant as $\sqrt{g}$ for odd oscillators. We will not stress these aspects in this paper, referring the reader to [15] for complete details.

$$\frac{1}{\Gamma(\frac{1}{2}-B_N(E,g))}=\frac{1}{\sqrt{2\pi}}\left(-\frac{2^{\frac{N}{N-2}}}{(-g)^{2/(N-2)}}\right)^{B_N(E,g)}e^{-A_N(E,g)}. \tag{7}$$

For M odd, instantons exist for $g>0$, and the quantization condition reads

$$\frac{1}{\Gamma(\frac{1}{2}-B_M(E,g))}=\frac{1}{\sqrt{8\pi}}\left(-\frac{2^{\frac{M}{M-2}}}{g^{1/(M-2)}}\right)^{B_M(E,g)}e^{-A_M(E,g)}. \tag{8}$$

Here $\Gamma(z)$ is the Euler gamma function.

In [15–17], Zinn-Justin *et al.* discuss contributions of instanton related effects in one-dimensional anharmonic oscillators of arbitrary even and odd degrees; in particular, they present expressions for $B(E,g)$ and $A(E,g)$ for the anharmonic oscillators with polynomial potentials of degrees $M=3$, 5, 7 and $N=4$, 6, 8. In this work we check the Dunne-Ünsal relation for those potentials by using their generalized quantization conditions.

III. HARMONIC OSCILLATOR WITH CUBIC POTENTIAL

In this section we consider an anharmonic oscillator with cubic potential. The Hamiltonian of the cubic anharmonic oscillator has the following form:

$$H_3(g)=-\frac{1}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}q^2+\sqrt{g}q^3. \tag{9}$$

Note that for a positive and real coupling parameter g the one-dimensional cubic oscillator possesses resonances [15,57].

First we calculate the instanton action of the system (9). The instanton action for odd anharmonic oscillators can be computed using the following general formula [15,56]:

$$s_M[q]=\left(\frac{1}{g}\right)^{1/(M-2)}\int_0^{2^{\frac{1}{2-M}}}2\sqrt{q^2-2q^M}dq, \tag{10}$$

where M stands for the degree of the polynomial. In the case of the cubic potential, i.e., for $M=3$, we have

$$s_3=\frac{1}{g}\int_0^{\frac{1}{2}}2q\sqrt{1-2q}dq. \tag{11}$$

By defining $u=1-2q$, we obtain

$$s_3=\frac{1}{g}\int_0^1\frac{(1-u)}{2}\sqrt{u}du. \tag{12}$$

Now one can easily compute the integral and get the final result

$$s_3=\frac{2}{15g}, \tag{13}$$

which is positive for $g>0$. This quantity determines the leading contribution to the ground-state energy of order $\exp(-\frac{2}{15g})$. The generalized Bohr-Sommerfeld quantization condition for the cubic potential reads

$$\frac{1}{\Gamma(\frac{1}{2}-B_3(E,g))}=\frac{1}{\sqrt{8\pi}}\left(-\frac{8}{g}\right)^{B_3(E,g)}e^{-A_3(E,g)}, \tag{14}$$

with the following characteristic functions:

$$B_3(E,g)=E+\sum_{i=1}^{\infty}g^ib_{i+1}(E), \tag{15}$$

$$A_3(E,g)=\frac{2}{15g}+\sum_{i=1}^{\infty}g^ia_{i+1}(E), \tag{16}$$

where b_i and a_i are polynomials of degree i in E and the term $\frac{2}{15g}$ is the instanton action (13). The expression (14) is a relation for the resonance energies of the anharmonic oscillator with the cubic potential for $g>0$. It is worth mentioning here that for $g<0$ the right-hand side of the expression (14) is equal to zero.

The expressions for $B(E,g)$ and $A(E,g)$ for the cubic anharmonic oscillator were calculated in [15,56,57]. The expansion of the perturbative function $B_3(E,g)$ in g is given by

$$\begin{aligned}B_3(E,g)=&E+g\left(\frac{7}{16}+\frac{15}{4}E^2\right)+g^2\left(\frac{1365}{64}E+\frac{1155}{16}E^3\right)+g^3\left(\frac{119119}{2048}+\frac{285285}{256}E^2+\frac{255255}{128}E^4\right)\\&+g^4\left(\frac{156165009}{16384}E+\frac{121246125}{2048}E^3+\frac{66927861}{1024}E^5\right)\\&+g^5\left(\frac{10775385621}{262144}+\frac{67931778915}{65536}E^2\right.\\&\left.+\frac{51869092275}{16384}E^4+\frac{9704539845}{4096}E^6\right)+\cdots,\end{aligned} \tag{17}$$

and the nonperturbative function $A_3(E,g)$ has the following expansion:

$$A_3(E,g) = \frac{2}{15g} + g\left(\frac{77}{32} + \frac{141}{8}E^2\right) + g^2\left(\frac{15911}{128}E + \frac{11947}{32}E^3\right) + g^3\left(\frac{49415863}{122880} + \frac{6724683}{1024}E^2 + \frac{5481929}{512}E^4\right) + g^4\left(\frac{2072342055}{32768}E + \frac{44826677}{128}E^3 + \frac{733569789}{2048}E^5\right) + g^5\left(\frac{404096853629}{1310720} + \frac{1100811938289}{163840}E^2 + \frac{307346388279}{16384}E^4 + \frac{134713909947}{10240}E^6\right) + \cdots. \tag{18}$$

In order to verify the Dunne-Ünsal relation, we need to rewrite expansions of the function A and the energy E in terms of B and g. We use an ansatz $E(B,g) = B - \sum_{j=1}^{\infty} p_{j+1}(B)g^j$, where $p_{j+1}(B)$ are polynomials of degree $(j+1)$ in B. By inserting the ansatz into (17) and by comparison of coefficients in each order of g, we get

$$E_3(B,g) = B - g\left(\frac{7}{16} + \frac{15}{4}B^2\right) - g^2\left(\frac{1155}{64}B + \frac{705}{16}B^3\right) - g^3\left(\frac{101479}{2048} + \frac{209055}{256}B^2 + \frac{115755}{128}B^4\right) - g^4\left(\frac{129443349}{16384}B + \frac{77300685}{2048}B^3 + \frac{23968161}{1024}B^5\right) - g^5\left(\frac{2375536317}{65536} + \frac{26541790065}{32768}B^2 + \frac{3601649205}{2048}B^4 + \frac{1412410545}{2048}B^6\right) + \cdots. \tag{19}$$

One can verify this result for the nonalternating perturbation series for the ground-state energy without instanton effects, i.e., by inserting $B_3 = \frac{1}{2}$ in (19),

$$E^{(3)}_{\text{ground}}(g) = \frac{1}{2} - \frac{11}{8}g - \frac{465}{32}g^2 - \frac{39709}{128}g^3 - \frac{19250805}{2048}g^4 + \cdots. \tag{20}$$

This is exactly the result obtained by the Rayleigh-Schrödinger perturbation theory. Similarly writing A_3 in terms of B and g will yield

$$A_3(B,g) = \frac{2}{15g} + g\left(\frac{77}{32} + \frac{141}{8}B^2\right) + g^2\left(\frac{13937}{128}B + \frac{7717}{32}B^3\right) + g^3\left(\frac{43147783}{122880} + \frac{5153379}{1024}B^2 + \frac{2663129}{512}B^4\right) + g^4\left(\frac{1769452671}{32768}B + \frac{240109947}{1024}B^3 + \frac{282482109}{2048}B^5\right) + g^5\left(\frac{724731745353}{2621440} + \frac{3555387349941}{655360}B^2 + \frac{359377601583}{32768}B^4 + \frac{168844301703}{40960}B^6\right) + \cdots. \tag{21}$$

After these conversions, the new series obey the following relation:

$$\frac{\partial E_3(B,g)}{\partial B} = -\frac{15}{2}Bg - \frac{15}{2}g^2\frac{\partial A_3(B,g)}{\partial g}. \tag{22}$$

This means that the Dunne-Ünsal relation [28,29] is satisfied for the cubic potential in the following form[4]:

$$\frac{\partial E(B,g)}{\partial B} = -\frac{g}{S}\left(B + g\frac{\partial A}{\partial g}\right). \tag{23}$$

From the relation (23) it is obvious that the nonperturbative function $A_3(E,g)$ could actually be determined by the perturbative function $B_3(E,g)$.

Note that the relationship between the A and B functions in the quantization conditions provided by [12,13] may differ from each other. In particular, the function A in quantization conditions (7) and (8) appears as $\exp(-A)$ rather than $\exp(-A/2)$, which is the case for double-well and sine-Gordon potentials. This is the reason why we have the factor $-g^2/S$ instead of $-g^2/2S$ in front of $\frac{\partial A}{\partial g}$.

[4]The S stands for the coefficient of (13).

IV. QUARTIC POTENTIAL CASE

In this section we consider the anharmonic oscillator with a quartic potential. The Hamiltonian for the quartic potential is

$$H_4(g) = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 + gq^4. \tag{24}$$

Note that for $g < 0$ the system has resonances.

The instanton action for even anharmonic oscillators has a slightly different formula than the expression (10) for odd potentials [15,56]:

$$S_N[q] = \left(-\frac{1}{g}\right)^{2/(N-2)} \int_0^{2^{\frac{1}{2-N}}} 2\sqrt{q^2 - 2q^N}\,dq. \tag{25}$$

Here the label N stands for the degree of the polynomial. In the case of a quartic potential, i.e., for $N = 4$, we have

$$S_4 = -\frac{1}{g}\int_0^{\frac{1}{\sqrt{2}}} 2q\sqrt{1-2q^2}\,dq. \tag{26}$$

By defining $u = 1 - 2q^2$, we get

$$S_4 = -\frac{1}{g}\int_0^1 \frac{\sqrt{u}}{2}\,du. \tag{27}$$

Then the instanton action is

$$S_4 = -\frac{1}{3g}. \tag{28}$$

The generalized Bohr-Sommerfeld quantization condition in the case of the quartic potential reads [15]

$$\frac{1}{\Gamma(\frac{1}{2} - B_4(E,g))} = \frac{1}{\sqrt{2\pi}}\left(\frac{4}{g}\right)^{B_4(E,g)} e^{-A_4(E,g)}, \tag{29}$$

with the following perturbative B and nonperturbative A functions:

$$B_4(E,g) = E + \sum_{j=1}^{\infty} g^j b_{j+1}(E), \tag{30}$$

$$A_4(E,g) = -\frac{1}{3g} + \sum_{j=1}^{\infty} g^j a_{j+1}(E). \tag{31}$$

The coefficients b_j and a_j are odd or even polynomials in E of degree j. The evaluation of B_4 and A_4 in terms of a series in variables E and g has been described in [15]. The first five orders of the B and A functions are given by

$$\begin{aligned} B_4(E,g) = {} & E - g\left(\frac{3}{8} + \frac{3}{2}E^2\right) + g^2\left(\frac{85}{16}E + \frac{35}{4}E^3\right) \\ & - g^3\left(\frac{1995}{256} + \frac{2625}{32}E^2 + \frac{1155}{16}E^4\right) \\ & + g^4\left(\frac{400785}{1024}E + \frac{165165}{128}E^3 + \frac{45045}{64}E^5\right) \\ & + \cdots, \end{aligned} \tag{32}$$

$$\begin{aligned} A_4(E,g) = {} & -\frac{1}{3g} - g\left(\frac{67}{48} + \frac{17}{4}E^2\right) + g^2\left(\frac{671}{32}E + \frac{227}{8}E^3\right) \\ & - g^3\left(\frac{372101}{9216} + \frac{125333}{384}E^2 + \frac{47431}{192}E^4\right) \\ & + g^4\left(\frac{3839943}{2048}E + \frac{82315}{16}E^3 + \frac{317629}{128}E^5\right) \\ & + \cdots. \end{aligned} \tag{33}$$

Note that the leading term of the nonperturbative function $A(E,g)$ contains the instanton action as given in (28). In order to check the Dunne-Ünsal relation, we convert the series $B(E,g)$ into $E(B,g)$ as

$$\begin{aligned} E_4(B,g) = {} & B + g\left(\frac{3}{8} + \frac{3}{2}B^2\right) - g^2\left(\frac{67}{16}B + \frac{17}{4}B^3\right) \\ & + g^3\left(\frac{1539}{256} + \frac{1707}{32}B^2 + \frac{375}{16}B^4\right) \\ & - g^4\left(\frac{305141}{1024}B + \frac{89165}{128}B^3 + \frac{10689}{64}B^5\right) \\ & + \cdots. \end{aligned} \tag{34}$$

The usual alternating perturbation series for the ground state can be derived by taking $B_4 = \frac{1}{2}$ in (34),

$$E^{(4)}_{\text{ground}} = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \cdots, \tag{35}$$

which agrees with results in [3,18].

By converting $A(E,g)$ into $A(B,g)$ we get

$$\begin{aligned} A_4(B,g) = {} & -\frac{1}{3g} - g\left(\frac{67}{48} + \frac{17}{4}B^2\right) + g^2\left(\frac{569}{32}B + \frac{125}{8}B^3\right) \\ & - g^3\left(\frac{305141}{9216} + \frac{89165}{384}B^2 + \frac{17815}{192}B^4\right) \\ & - g^4\left(\frac{91745}{256}B + \frac{133505}{64}B^3 + \frac{3595}{2}B^5\right) \\ & + \cdots. \end{aligned} \tag{36}$$

One can then see that these series satisfy the following equation:

$$\frac{\partial E_4(B,g)}{\partial B} = 3Bg + 3g^2\frac{\partial A_4(B,g)}{\partial g}, \tag{37}$$

which is in the form of

$$\frac{\partial E(B,g)}{\partial B} = -\frac{g}{S}\left(B + g\frac{\partial A}{\partial g}\right). \tag{38}$$

Thus, we verify the Dunne-Ünsal relation for the quartic potential.

V. COMMENT ON HIGHER DEGREE POTENTIALS

The problem arises when we consider higher power polynomial potentials, which correspond to genus > 1. So far, only genus-one potentials have been approved to satisfy the Dunne-Ünsal relation.[5] However, in this section we observe that for a higher genus case, the Dunne-Ünsal relation is not satisfied in its current form.

As an example, let us consider the anharmonic oscillator with the polynomial potential of degree five (quintic). The Hamiltonian in this case is

$$H_5(g) = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 + \sqrt{g}q^5, \tag{39}$$

and the generalized quantization condition reads

$$\frac{1}{\Gamma(\frac{1}{2} - B_5(E,g))} = \frac{1}{\sqrt{8\pi}}\left(-\frac{2^{5/3}}{g^{1/3}}\right)^{B_5(E,g)} e^{-A_5(E,g)}. \tag{40}$$

The instanton action for the ground state of the quintic potential can be calculated from the expression (10), and one gets the following result:

$$s_5[q] = \left(\frac{1}{g}\right)^{\frac{1}{3}}\int_0^{2^{-1/3}} 2q\sqrt{1-2q^3}\,dq = \frac{3\sqrt{3}\Gamma^3(\frac{2}{3})}{7\pi(2g)^{1/3}}. \tag{41}$$

The perturbative function $B(E,g)$ has the following expansion:

$$B_5(E,g) = E + g\left(\frac{1107}{256} + \frac{1085}{32}E^2 + \frac{315}{16}E^4\right) + g^2\left(\frac{118165905}{8192}E + \frac{96201105}{2048}E^3 + \frac{15570555}{512}E^5 + \frac{692835}{128}E^7\right) + g^3\left(\frac{36358712597025}{4194304} + \frac{142306775756145}{1048576}E^2 + \frac{30926063193025}{131072}E^4 + \frac{4140194663605}{32768}E^6 + \frac{456782651325}{16384}E^8 + \frac{9704539845}{4096}E^{10}\right) + \cdots. \tag{42}$$

By converting E as a function of B and g for the first three terms, we find that

$$E(B,g) = B - g\left(\frac{1107}{256} + \frac{1085}{32}B^2 + \frac{315}{16}B^4\right) - g^2\left(\frac{115763715}{8192}B + \frac{90794795}{2048}B^3 + \frac{13519905}{512}B^5 + \frac{494385}{128}B^7\right) - g^3\left(\frac{36099752507685}{4194304} + \frac{140162880546045}{1048576}B^2 + \frac{29646883011725}{131072}B^4 + \frac{3708489756265}{32768}B^6 + \frac{351124790625}{16384}B^8 + \frac{5590822545}{4096}B^{10}\right) + \cdots. \tag{43}$$

The first few terms of the perturbative expansion of the function $A(E,g)$ for the quintic potential are given by [15]

$$A_5(E,g) = \frac{3\sqrt{3}\Gamma^3(\frac{2}{3})}{7\pi(2g)^{1/3}} - g^{1/3}\frac{3\sqrt{3}\Gamma^3(\frac{1}{3})}{2^{2/3}8\pi}\left(\frac{11}{54} + \frac{14}{27}E^2\right) + g^{2/3}\frac{\Gamma^3(\frac{2}{3})}{2^{1/3}\sqrt{3}\pi}\left(\frac{385}{32}E + \frac{935}{72}E^3\right) + g\left(\frac{21171}{1024} + \frac{132245}{1152}E^2 + \frac{10865}{192}E^4\right) + \cdots. \tag{44}$$

The function A can be written in terms of variables B and g as follows:

[5]This issue has been discussed by several authors [53,54]. For instance, in [54] it was claimed that for genus = 1 potentials, the Dunne-Ünsal relation coincides with the equation of motion in the Whitham dynamics.

$$A_5(B,g) = \frac{3\sqrt{3}\Gamma^3(\frac{2}{3})}{7\pi(2g)^{1/3}} - g^{1/3}\frac{3\sqrt{3}\Gamma^3(\frac{1}{3})}{2^{2/3}8\pi}\left(\frac{11}{54}+\frac{14}{27}B^2\right) + g^{2/3}\frac{\Gamma^3(\frac{2}{3})}{2^{1/3}\sqrt{3}\pi}\left(\frac{385}{32}B+\frac{935}{72}B^3\right) + g\left(\frac{21171}{1024}+\frac{132245}{1152}B^2+\frac{10865}{192}B^4\right)+\cdots. \tag{45}$$

One can easily see that this example differs from the preceding ones since the expansion includes Euler gamma functions and fractional orders of the coupling parameter g.

By using these data, the left-hand side of the Dunne-Ünsal relation yields

$$\frac{\partial E}{\partial B} = 1 - g\left(\frac{1085}{16}B+\frac{315}{4}B^3\right) - g^2\left(\frac{115763715}{8192}+\frac{272384385}{2048}B^2+\frac{67599525}{512}B^4+\frac{3460695}{128}B^6\right) - g^3\left(\frac{140162880546045}{524288}B+\frac{29646883011725}{32768}B^3 + \frac{11125469268795}{16384}B^5+\frac{351124790625}{2048}B^7+\frac{27954112725}{2048}B^9\right)+\cdots \tag{46}$$

and from the right-hand side, we again get fractional powers of g as well as gamma functions which clearly do not match the left-hand side of the Dunne-Ünsal relation (46).

Other higher order (higher genus) potentials have similar fractional terms in the expansion of $A(E,g)$ and they do not match the left-hand side of the Dunne-Ünsal relation (4). We provide A and B functions of these potentials for $N=6$, 8 and $M=7$ in the Appendices A, B and C for convenience.

VI. CONCLUSIONS

To conclude, trans-series expansion of an energy eigenvalue gives us a real and unambiguous result due to the cancellation of ambiguous imaginary terms arising from the perturbative expansion around perturbative vacuum and nonperturbative saddles. This cancellation mechanism implies a close relationship between perturbative and nonperturbative sectors. This cannot be easily seen in the Zinn-Justin *et al.* approach where one needs to separately calculate the functions $A(E,g)$ and $B(E,g)$. However, this relationship can be seen by the Dunne-Ünsal relation [28,29], and the nonperturbative sector can be generated purely from the perturbative sector. Rather than calculating the functions $A(E,g)$ and $B(E,g)$ separately, it is actually enough to generate the trans-series expansion of energy using the knowledge of the perturbative function $B(E,g)$ with a global boundary condition. The Dunne-Ünsal relation has been shown to apply for several genus-one potentials including double-well, periodic sine-Gordon (periodic cosine), $O(d)$ symmetric and Fokker-Planck potentials.

In our current study we confirmed that the relation also holds for resonance energy levels of unified even- and odd-degree anharmonic complexified potentials. We verified the relation for cubic and quartic anharmonic potentials, which are genus-one potentials. However, for higher order (higher genus) potentials we observed that the formula is not satisfied and needs to be generalized.

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APPENDIX A: SEXTIC POTENTIALS

Here we list expansions of the A and B functions in terms of a series in variables E and g, as well as generalized quantization conditions for sextic, septic and octic potentials. All the expressions for $B(E,g)$ and $A(E,g)$ listed in the appendixes were taken from [15].

The sextic anharmonic oscillator is described by the following Hamiltonian:

$$H_6(g) = -\frac{1}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}q^2+gq^6. \tag{A1}$$

The generalized Bohr-Sommerfeld quantization condition in this case reads

$$\frac{1}{\Gamma(\frac{1}{2}-B_6(E,g))} = \frac{1}{\sqrt{2\pi}}\left(\frac{2^{3/2}}{g^{1/2}}\right)^{B_6(E,g)} e^{-A_6(E,g)}. \tag{A2}$$

The first few terms of the perturbative expansions of the functions A and B for the sextic potential are given by

$$B_6(E,g) = E - g\left(\frac{25}{8}E + \frac{5}{2}E^3\right) + g^2\left(\frac{21777}{256}E + \frac{5145}{32}E^3 + \frac{693}{16}E^5\right) - g^3\left(\frac{12746305}{2048}E + \frac{8703695}{512}E^3 + \frac{1096095}{128}E^5 + \frac{36465}{32}E^7\right) + \cdots, \tag{A3}$$

$$A_6(E,g) = \frac{\pi}{2^{5/2}(-g)^{1/2}} - g\left(\frac{221}{24}E + \frac{17}{3}E^3\right) + g^2\left(\frac{2504899}{7680}E + \frac{45769}{96}E^3 + \frac{17527}{160}E^5\right) + \cdots, \tag{A4}$$

where again the first term of the function A has fractional order of the coupling parameter g.

Here, we also present expansions of the energy E and the nonperturbative function A in terms of B and g for the sextic anharmonic oscillator:

$$E_6(B,g) = B + g\left(\frac{25}{2}B + \frac{5}{2}B^3\right) - g^2\left(-\frac{18223}{256}B + \frac{1145}{32}B^3 + \frac{93}{32}B^5\right) + g^3\left(\frac{12390905}{2048}B + \frac{6152155}{512}B^3 + \frac{789495}{128}B^5 + \frac{28605}{32}B^7\right) + \cdots, \tag{A5}$$

$$A_6(B,g) = \frac{\pi}{2^{5/2}(-g)^{1/2}} - g\left(\frac{221}{24}B + \frac{17}{3}B^3\right) + g^2\left(\frac{1620899}{7680}B + \frac{23159}{96}B^3 + \frac{10727}{160}B^5\right) + \cdots. \tag{A6}$$

From these expressions one can easily see that the Dunne-Ünsal formula is not satisfied. By taking $B = \frac{1}{2}$ in (A5), one obtains the following perturbative series for the ground-state energy:

$$E^{(6)}_{\text{ground}}(g) = \frac{1}{2} + \frac{15}{8}g - \frac{3495}{64}g^2 + \frac{1239675}{256}g^3 + \cdots. \tag{A7}$$

APPENDIX B: SEPTIC POTENTIAL

The Hamiltonian of the septic anharmonic oscillator is

$$H_7(g) = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 + \sqrt{g}q^7. \tag{B1}$$

The generalized quantization condition has the following form:

$$\frac{1}{\Gamma(\frac{1}{2} - B_7(E,g))} = \frac{1}{\sqrt{8\pi}}\left(-\frac{2^{7/5}}{g^{1/5}}\right)^{B_7(E,g)} e^{-A_7(E,g)}. \tag{B2}$$

The first terms of the expansions for $B_7(E,g)$ and $A_7(E,g)$ read as follows:

$$B_7(E,g) = E + g\left(\frac{180675}{2048} + \frac{444381}{512}E^2 + \frac{82005}{128}E^4 + \frac{3003}{32}E^6\right) + g^2\left(\frac{182627818702875}{2097152}E + \frac{156916927352185}{524288}E^3 + \frac{13513312267455}{65536}E^5 + \frac{824707412529}{16384}E^7 + \frac{43689020375}{8192}E^9 + \frac{456326325}{2048}E^{11}\right) + \cdots, \tag{B3}$$

$$A_7(E,g) = \frac{5^{1/4}\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{2^{1/10}(\sqrt{5}+1)^{1/2}9\pi g^{1/5}} + g^{1/5}\frac{5^{1/4}\Gamma^2(\frac{3}{5})\Gamma(\frac{4}{5})}{2^{9/10}(\sqrt{5}+1)^{1/2}\pi}\left(\frac{5}{8}+\frac{9}{10}E^2\right) - g^{2/5}\frac{5^{1/4}(\sqrt{5}+1)^{1/2}\Gamma^2(\frac{1}{5})\Gamma(\frac{3}{5})}{2^{3/10}\pi}\left(\frac{377}{1600}E+\frac{299}{2000}E^3\right) + g^{3/5}\frac{5^{1/4}\Gamma(\frac{2}{5})\Gamma^2(\frac{4}{5})}{2^{7/10}(\sqrt{5}-1)^{1/2}\pi}\left(\frac{59143}{9600}+\frac{15351}{400}E^2+\frac{13209}{1000}E^4\right) + \cdots. \tag{B4}$$

The expressions of E and A in terms of B and g are then

$$E_7(B,g) = B - g\left(\frac{180675}{2048}+\frac{444381}{512}B^2+\frac{82005}{128}B^4+\frac{3003}{32}B^6\right) - g^2\left(\frac{182306664554175}{2097152}B+\frac{156008499432541}{524288}B^3+\frac{13291408081875}{65536}B^5 + \frac{787132323285}{16384}B^7+\frac{38763800075}{8192}B^9+\frac{348110217}{2048}B^{11}\right)+\ldots, \tag{B5}$$

$$A_7(B,g) = \frac{5^{1/4}\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{2^{1/10}(\sqrt{5}+1)^{1/2}9\pi g^{1/5}} + g^{1/5}\frac{5^{1/4}\Gamma^2(\frac{3}{5})\Gamma(\frac{4}{5})}{2^{9/10}(\sqrt{5}+1)^{1/2}\pi}\left(\frac{5}{8}+\frac{9}{10}B^2\right) - g^{2/5}\frac{5^{1/4}(\sqrt{5}+1)^{1/2}\Gamma^2(\frac{1}{5})\Gamma(\frac{3}{5})}{2^{3/10}\pi}\left(\frac{377}{1600}B+\frac{299}{2000}B^3\right) + g^{3/5}\frac{5^{1/4}\Gamma(\frac{2}{5})\Gamma^2(\frac{4}{5})}{2^{7/10}(\sqrt{5}-1)^{1/2}\pi}\left(\frac{59143}{9600}+\frac{15351}{400}B^2+\frac{13209}{1000}B^4\right) + \cdots. \tag{B6}$$

Taking $B = \frac{1}{2}$ in (B5) gives rise to the following perturbative series for the ground-state energy:

$$E^{(7)}_{\text{ground}}(g) = \frac{1}{2} - \frac{44379}{128}g - \frac{715842493569}{8192}g^2 + \cdots. \tag{B7}$$

APPENDIX C: OCTIC POTENTIAL

The Hamiltonian of the octic anharmonic oscillator is

$$H_8(g) = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 + gq^8. \tag{C1}$$

The generalized quantization condition is given by

$$\frac{1}{\Gamma(\frac{1}{2}-B_8(E,g))} = \frac{1}{\sqrt{2\pi}}\left(\frac{2^{4/3}}{g^{1/3}}\right)^{B_8(E,g)} e^{-A_8(E,g)}. \tag{C2}$$

The first few terms of the functions $B_8(E,g)$ and $A_8(E,g)$ are

$$B_8(E,g) = E - g\left(\frac{315}{128}+\frac{245}{16}E^2+\frac{35}{8}E^4\right) + g^2\left(\frac{5604849}{2048}E+\frac{3209745}{512}E^3+\frac{291291}{128}E^5+\frac{6435}{32}E^7\right)+\cdots, \tag{C3}$$

$$A_8(E,g) = \frac{\sqrt{3}\Gamma^3(\frac{1}{3})}{2^{2/3}10\pi(-g)^{1/3}} + (-g)^{1/3}\frac{\Gamma^3(\frac{2}{3})}{2^{1/3}\sqrt{3}\pi}\left(\frac{17}{16}+\frac{5}{4}E^2\right) - (-g)^{\frac{2}{3}}\frac{\Gamma^3(\frac{1}{3})}{2^{\frac{2}{3}}\sqrt{3}\pi}\left(\frac{77}{96}+\frac{91}{216}E^2\right) - g\left(\frac{28007}{2560}+\frac{22669}{576}E^2+\frac{2587}{288}E^4\right)+\cdots. \tag{C4}$$

The expressions of E and A in terms of B and g then become

$$E(B,g) = B + g\left(\frac{315}{128}+\frac{245}{16}B^2-\frac{35}{8}B^4\right) - g^2\left(\frac{5450499}{2048}B+\frac{2947595}{512}B^3+\frac{239841}{128}B^5+\frac{3985}{32}B^7\right)+\cdots, \tag{C5}$$

$$A_8(B,g) = \frac{\sqrt{3}\Gamma^3(\frac{1}{3})}{2^{2/3}10\pi(-g)^{1/3}} + (-g)^{1/3}\frac{\Gamma^3(\frac{2}{3})}{2^{1/3}\sqrt{3}\pi}\left(\frac{17}{16}+\frac{5}{4}B^2\right) - (-g)^{\frac{2}{3}}\frac{\Gamma^3(\frac{1}{3})}{2^{\frac{2}{3}}\sqrt{3}\pi}\left(\frac{77}{96}+\frac{91}{216}B^2\right) - g\left(\frac{28007}{2560}+\frac{22669}{576}B^2+\frac{2587}{288}B^4\right)+\cdots. \tag{C6}$$

By taking $B = \frac{1}{2}$ in (C5) we find the following perturbative series for the ground-state energy for the octic anharmonic oscillator,

$$E^{(8)}_{\mathrm{ground}}(g) = \frac{1}{2} + \frac{105}{16}g - \frac{67515}{32}g^2 + \cdots. \tag{C7}$$

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