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Mandelstam-Leibbrandt prescription

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The light-cone gauge is used frequently in string theory as well as gauge theories and gravitation. Loop integrals, however, have to be infrared regulated to remove spurious poles. The most popular and consistent of these infrared regulators is the Mandelstam-Leibbrandt prescription (ML). The calculations with the ML are rather cumbersome, though. In this work, we show that the ML can be replaced by a symmetry of the regulator. This symmetry simplifies the calculations, reducing them to conventional dimensional regularization integrals.

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I. INTRODUCTION

Computations in superstring theory as well as gauge theories, supersymmetry, gravitation, and Chern-Simons theories are often simplified by recurring to the light-cone gauge. The light-cone gauge is termed one of the physical gauges because ghosts decouple in these gauges.[1] Computing loop corrections in the light-cone gauge has some peculiarities, though: spurious infrared poles appear, nonlocal terms are present, and Lorentz invariance is explicitly broken. To deal with these problems, an infrared regulator is needed. The most popular and internally consistent regulator used at present is the Mandelstam-Leibbrandt regulator (ML) [2,3]. The ML has very nice properties: the poles in the k_0 complex plane are situated such that the Wick rotation from Euclidean to Minkowsky space is justified; it preserves naive power counting of loop integrals; and in gauge theories, it maintains the Ward identities of the gauge symmetry [1,4]. Explicit computations with the ML are long and cumbersome, though.

Here, we present a method to evaluate the loop integrals that appear in the light-cone gauge based on a scale symmetry of the regulator. No new integrals are required, aside from the standard dimensionally regularized integrals. In fact, the ML prescription can be safely replaced by the scale symmetry and a regularity condition. We do not have to specify the exact value of the two null vectors of the ML, but merely its mutual relations. The results coincide with the answer given by the ML for the same integral.

II. NEW PRESCRIPTION

Let us compute the simple integral

$$A_\mu = \int dp \frac{f(p^2)p_\mu}{(n\cdot p)},$$

where f is an arbitrary function, dp is the integration measure in d dimensional space, and n_μ is a fixed null vector $[(n\cdot n)=0]$. This integral is infrared divergent when $(n\cdot p)=0$.

The ML is

$$\frac{1}{(n\cdot p)} = \lim_{\varepsilon\to 0}\frac{(p\cdot\bar{n})}{(n\cdot p)(p\cdot\bar{n})+i\varepsilon}, \tag{1}$$

where $\bar{n}_\mu$ is a new null vector with the property $(n\cdot\bar{n})=1$.

To compute A_μ, we have to know the specific form of f, provide a specific form of n_μ and $\bar{n}_\mu$, and evaluate the residues of all poles of $\frac{f(p^2)}{(n\cdot p)}$ in the p_0 complex plane, a rather formidable task for an arbitrary f.

Instead, we want to point out the following symmetry:

$$n_\mu \to \lambda n_\mu, \qquad \bar{n}_\mu \to \lambda^{-1}\bar{n}_\mu, \qquad \lambda\neq 0, \qquad \lambda\varepsilon R. \tag{2}$$

It preserves the definitions of n_μ and $\bar{n}_\mu$:

$$0=(n\cdot n)\to\lambda^2(n\cdot n)=0$$
$$0=(\bar{n}\cdot\bar{n})\to\lambda^{-2}(\bar{n}\cdot\bar{n})=0$$
$$1=(n\cdot\bar{n})\to(n\cdot\bar{n})=1.$$

We see from Eq. (1) that

$$\frac{1}{(n\cdot p)}\to\frac{1}{(n\cdot p)}\lambda^{-1}.$$

Now, we compute A_μ, based on its symmetries. It is a Lorentz vector which scales under Eq. (2) as λ^{-1}. The only Lorentz vectors we have available in this case are n_μ and $\bar{n}_\mu$. But Eq. (2) forbids n_μ. That is,

$$A_\mu = a\bar{n}_\mu.$$

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[1] There are some subtleties related to this point. Please see Ref. [1], chapter 4.4.

Multiply by n_μ to find $(A\cdot n) = a$. Thus, $a = \int dp f(p^2)$. Finally,

$$\int dp \frac{f(p^2)p_\mu}{(n\cdot p)} = \bar{n}_\mu \int dp f(p^2).$$

By the same token, we find

$$A_{\mu\nu\lambda} = \int dp \frac{f(p^2)p_\mu p_\nu p_\lambda}{(n\cdot p)} = a(\bar{n}_\mu g_{\nu\lambda})_S + b(\bar{n}_\mu \bar{n}_\nu n_\lambda)_S,$$

where $()_S$ means symmetric in all Lorentz indices.

We get:

$$A_{\mu\nu\lambda} n^\lambda = \frac{1}{d} g_{\mu\nu} \int dp f(p^2) p^2.$$

Therefore

$$a = \frac{1}{d} \int dp f(p^2) p^2 = -b.$$

The integrals on p_μ are dimensionally regularized.

That is,

$$\int dp \frac{f(p^2)p_\mu p_\nu p_\lambda}{(n\cdot p)} = \frac{1}{d} \int dp f(p^2) p^2 \{(\bar{n}_\mu g_{\nu\lambda})_S - (\bar{n}_\mu \bar{n}_\nu n_\lambda)_S\}.$$

A. Generic integrals

We consider now a more general integral. We will see here that regularity of the answer will determine it uniquely.

Consider

$$A = \int dp \frac{F(p^2, p\cdot q)}{(n\cdot p)} = (\bar{n}\cdot q) f(q^2, (n\cdot q)(\bar{n}\cdot q)). \quad (3)$$

q_μ is an external momentum, a Lorentz vector. F is an arbitrary function. The last relation follows from Eq. (2), for a certain f we will find in the following.

We get

$$\frac{\partial A}{\partial q^\mu} = \int dp \frac{F_{,u} p_\mu}{(n\cdot p)} = \bar{n}_\mu f(x,y) + 2(\bar{n}\cdot q) q_\mu \frac{\partial}{\partial x} f(x,y) + [(\bar{n}\cdot q)^2 n_\mu + (n\cdot q)(\bar{n}\cdot q)\bar{n}_\mu] \frac{\partial}{\partial y} f(x,y).$$

We defined $u = p\cdot q$, $x = q^2$, $y = (n\cdot q)(\bar{n}\cdot q)$. $()_{,u}$ means a derivative with respect to u,

$$\frac{\partial A}{\partial q^\mu} n_\mu = \int dp F_{,u} = g(x) = f(x,y) + 2y \frac{\partial}{\partial x} f(x,y) + y \frac{\partial}{\partial y} f(x,y). \quad (4)$$

Assuming that the solution and its partial derivatives are finite in the neighborhood of $y = 0$, it follows from the equation that $f(x,0) = g(x)$. That is the partial differential equation has a unique regular solution.

We will find the solution of Eq. (4) using the method of characteristics [5],

$$\begin{aligned} &\dot{x} = 2y && \dot{y} = y \\ &\dot{f} + f = g(x(t)) && \\ &y = Ce^t && \dot{x} = 2Ce^t,\ x = 2Ce^t + D \\ &x - 2y = D. && \end{aligned}$$

The most general solution of the system is

$$f = be^{-t} + e^{-t} \int_{-\infty}^{t} dt' e^{t'} g(x(t'))$$

for b arbitrary corresponding to the solution of Eq. (4) with $g = 0$ (homogeneous solution). The regular solution of Eq. (4), f_0, is obtained imposing that $b = 0$, the reason being that the homogeneous solution is $f = \Pi(x - 2y) y^{-1}$, with Π an arbitrary function. We readily see that f will diverge at $y = 0$, unless $\Pi(x) = 0$, for all x.

Moreover,

$$\lim_{t\to-\infty} f_0 = \lim_{t\to-\infty} e^{-t} \int_{-\infty}^{t} dt' e^{t'} g(x(t')) = g(D).$$

That is $f_0(x,0) = g(x)$. f_0 is the unique regular solution of Eq. (4).

What we have developed up to here shows that the scale transformation (2) plus the regularity condition determines uniquely the value of the integral (3).

B. Application to loop integrals

We consider now integrals that appear in gauge theory loops:

$$\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{1}{(n\cdot p)} = (\bar{n}\cdot q) f(x,y)$$

In this case

$$g(x) = -2a \int dp \frac{1}{[p^2 - x - m^2]^{a+1}}.$$

Therefore

$$\begin{aligned} f &= e^{-t}\int_{-\infty}^{t} dt' e^{t'} g(x(t')) \\ &= \int dp e^{-t}[p^2 - 2Ce^{t'} - D - m^2]^{-a} \frac{1}{-C}\Big|_{-\infty}^{t} \\ &= -\frac{1}{y}\left\{\int dp[p^2 - x - m^2]^{-a} - \int dp[p^2 - x + 2y - m^2]^{-a}\right\}. \end{aligned}$$

We readily verify that

$$f(x,0) = -2a\int dp[p^2 - x - m^2]^{-a-1} = g(x).$$

In the same way, we get

$$\int dp \tfrac{1}{[p^2+2p\cdot q-m^2]^a}\tfrac{1}{((n\cdot p))^2} = (\bar{n}\cdot q)^2 f(x,y)$$

with

$$f(x,y) = \frac{1}{y^2}\int dp\{[p^2 - x - m^2]^{-a} - [p^2 - x + 2y - m^2]^{-a} - 2ay[p^2 - x + 2y - m^2]^{-a-1}\}.$$

Following the same procedure, we can get an answer for the whole family of loop integrals:

$$\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a}\frac{1}{((n\cdot p))^b} = (\bar{n}\cdot q)^b(-2)^b\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dt t^{b-1}\int dp[p^2 - q^2 + 2(n.q)(\bar{n}.q)t - m^2]^{-a-b}.$$

Using dimensional regularization, we obtain

$$\begin{aligned} \int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a}\frac{1}{((n\cdot p))^b} &= (-1)^{a+b} i(\pi)^{\omega}(-2)^b\frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)}(\bar{n}\cdot q)^b \\ &\quad\times\int_0^1 dt t^{b-1}\frac{1}{(m^2 + q^2 - 2(n\cdot q)(\bar{n}\cdot q)t)^{a+b-\omega}}, \\ \omega &= d/2. \end{aligned} \tag{5}$$

We sketch the proof of Eq. (5):

$$\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a}\frac{1}{((n\cdot p))^b} = (\bar{n}\cdot q)^b f(b,a,x,y)$$

with

$$-2af(b-1,a+1,x,y) = bf(b,a,x,y) + 2y\frac{\partial}{\partial x}f(b,a,x,y) + y\frac{\partial}{\partial y}f(b,a,x,y), \tag{6}$$

$$f(b,a,x,0) = -\frac{2a}{b}f(b-1,a+1,x,0). \tag{7}$$

It is easy to check that Eq. (5) satisfies the partial differential equation (6) and the boundary condition (7), so it is the unique regular solution and thus determines the value of the integral.

Other integrals can be obtained computing partial derivatives with respect to q_μ:

$$\int dp \frac{p_\mu}{[p^2 + 2p\cdot q - m^2]^{a+1}}\frac{1}{((n\cdot p))^b} = \begin{array}{l} (-1)^{a+b} i(\pi)^{\omega}(-2)^{b-1}\frac{\Gamma(a+b-\omega)}{\Gamma(a+1)\Gamma(b)}(\bar{n}\cdot q)^{b-1} b\bar{n}_\mu\int_0^1 dt t^{b-1}\frac{1}{(m^2+x-2yt)^{a+b-\omega}} + \\ (-1)^{a+b} i(\pi)^{\omega}(-2)^{b}\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+1)\Gamma(b)}(\bar{n}\cdot q)^{b}\int_0^1 dt t^{b-1}\frac{q_\mu - t(n\cdot q\bar{n}_\mu + \bar{n}\cdot q n_\mu)}{(m^2+x-2yt)^{a+b+1-\omega}} \end{array} \tag{8}$$

and

$$\int dp \frac{p_\mu p_\nu}{[p^2+2p\cdot q-m^2]^{a+2}} \frac{1}{((n\cdot p))^b} = (-1)^{a+b} i(\pi)^\omega (-2)^{b-2} \Bigg\{ \frac{\Gamma(a+b-\omega)}{\Gamma(a+2)\Gamma(b-1)} (\bar{n}\cdot q)^{b-2} b \bar{n}_\mu \bar{n}_\nu \\ \times \int_0^1 dt t^{b-1} \frac{1}{(m^2+x-2yt)^{a+b-\omega}} - 2\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b-1} b \bar{n}_\mu \\ \times \int_0^1 dt t^{b-1} \frac{(q_\nu - t(n\cdot q \bar{n}_\nu + \bar{n}\cdot q n_\nu))}{(m^2+x-2yt)^{a+b+1-\omega}} - 2\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b-1} b \bar{n}_\nu \\ \times \int_0^1 dt t^{b-1} \frac{q_\mu - t(n\cdot q \bar{n}_\mu + \bar{n}\cdot q n_\mu)}{(m^2+x-2yt)^{a+b+1-\omega}} + 4\frac{\Gamma(a+b+2-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b} \\ \times \int_0^1 dt t^{b-1} \frac{[q_\mu - t(n\cdot q \bar{n}_\mu + \bar{n}\cdot q n_\mu)][q_\nu - t(n\cdot q \bar{n}_\nu + \bar{n}\cdot q n_\nu)]}{(m^2+x-2yt)^{a+b+2-\omega}} \Bigg\}. \tag{9}$$

The right-hand sides of Eqs. (5), (8), and (9) are analytic in the parameters a, b, and ω almost everywhere in their respective complex planes, so they provide the analytic extension of the integral to these wider domains.

III. COMPARISON WITH THE ML

The simpler integral A_μ, $A_{\mu\nu\lambda}$ of Sec. II, agrees with the ML prescriptions [1]. But in this section, we want to compute a more involved integral, in order to compare both finite and divergent results with the MLs.

We want to compute

$$A(\sigma,q) = \int dp \frac{(p^2)^{\sigma-1}}{(p-q)^2((n\cdot p))^2}.$$

We introduce Feynman parameters [6]

$$\frac{1}{A_1^{m_1} A_2^{m_2}} = \int_0^1 dx \frac{x^{m_1-1}(1-x)^{m_2-1}}{[xA_1+(1-x)A_2]^{m_1+m_2}} \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)}$$

to get

$$A(\sigma,q) = (1-\sigma)\int_0^1 dx \\ \times \int dp \frac{x^{-\sigma}}{[p^2+(1-x)(-2p\cdot q+\bar{q}^2)]^{2-\sigma}((n\cdot p))^2}.$$

Using Eq. (5), we finally get

$$A(\sigma,q) = 4(\bar{n}\cdot q)^2(-1)^\sigma i(\pi)^\omega \frac{\Gamma(4-\sigma-\omega)}{\Gamma(1-\sigma)} \int_0^1 dx x^{-\sigma}(1-x)^{\sigma+\omega-2} \int_0^1 dt t [q^2 x + 2t(n\cdot q)(\bar{n}\cdot q)(1-x)]^{\sigma+\omega-4}.$$

This coincides with Ref. [7], Eq. (C4), by the change of variable $t=1-y$ and $\sigma=\omega$.

Notice that we have considered $\bar{q}^2 \neq q^2$ and taken the limit $\bar{q}^2 = q^2$ after evaluating the integral. This is justified because the integral is a regular function of q^2. If we expand $A(\sigma,q)$ in powers of q^2, each term of the series can be evaluated using Eq. (5). The summation of the series is equivalent to the procedure we followed above.

IV. CONCLUSIONS

We have developed a way of evaluating the light-cone loop integrals based on the scale symmetry (2) and the condition of regularity of the solution. We do not have to specify the exact value of the two null vectors of the ML but merely its mutual relations. The answer is the same as in the ML prescription, but a significant simplification of the calculation is available now.

For future work, we want to mention that the scale transformation (2) is also a symmetry of the uniform prescription introduced by Leibbrandt [1] to treat the spurious infrared poles in light-cone, axial, planar, and temporal gauges. The application of the method presented here to these more general gauges will be done elsewhere.

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