PHYSICAL REVIEW D **93,** 065010 (2016)

Nonlocal dynamics and infinite nonrelativistic conformal symmetries

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(Received 23 December 2015; published 4 March 2016)

We study the symmetry of the class of nonlocal models which includes the nonlocal extension of the Pais-Uhlenbeck oscillator. As a consequence, we obtain an infinite-dimensional symmetry algebra, containing the Virasoro algebra, which can be considered as a generalization of the nonrelativistic conformal symmetries to the infinite order. Moreover, this nonlocal extension resembles to some extent the string model, and on the quantum level, it leads to the centrally extended Virasoro algebra.

DOI: 10.1103/PhysRevD.93.065010

I. INTRODUCTION

Because of the importance of the noncommutative field theories and the string field theory, the nonlocal models have been extensively investigated in last two decades (see, e.g., Refs. [1–7] as a very brief list of references). However, the nonlocality in physics has a long history (see Ref. [8] and the references in Ref. [9]). The nonlocal systems provide an extension of the higher-derivatives ones to the case of the infinite-order derivatives. This fact is reflected in the form of the Hamiltonian formalism for the nonlocal theories, which is a generalization of the Ostrogradski approach to the higher-derivatives theory [9–13].

The basic example of the theory with higher derivatives is the Pais-Uhlenbeck (PU) oscillator of order N (which generalizes the standard Lagrangian of the harmonic oscillator to the case of the Nth-order time derivative) proposed in the classical paper [8]. This model has attracted interest through the years (for the last few years, see, e.g., Refs. [14–30]). In particular, it has been shown (see Ref. [30]) that when the relevant eigenfrequencies are proportional to consecutive odd integers then the maximal symmetry group of the PU oscillator is the l-conformal Newton-Hooke group with half-integer $l = N - \frac{1}{2}$, which is isomorphic to the l-conformal Galilei group (for this reason, we will simply refer to it as the l-conformal nonrelativistic group in what follows). These groups are natural conformal extensions of the Newton-Hooke (Galilei) group (in particular, for $N = 1$, we obtain the Schrödinger one) and originally have been introduced in the context of the nonrelativistic gravity [31,32]; however, there are many dynamical models, both classical (including also the second-order dynamical equations) and quantum, which exhibit such symmetries [33–41]. Furthermore, in a number of papers [42–49], infinite-dimensional extensions of those symmetries, for fixed order l, have also been considered in the context of some more sophisticated models. Let us note that neither the l-conformal nonrelativistic algebra nor its infinite extension are well defined as N (or l) tends to infinity.

In the present paper, we show that a very simple nonlocal theory (proposed in Ref. [8]) enjoys symmetry which can be considered as an infinite-dimensional generalization of the nonrelativistic conformal symmetries to the case of N going to infinity. The starting point is the mentioned observation that the $(N - \frac{1}{2})$-conformal nonrelativistic group appears as a symmetry of the PU oscillator of order N. Next, we note that the $N = \infty$ extension of the PU oscillator introduced in Ref. [8] (slightly formally) in the case of odd frequencies leads, after some manipulations, to the quadratic nonlocal Lagrangian (see also Ref. [13]). We find the symmetry algebra of this nonlocal Lagrangian and its central extension on the Hamiltonian level. The symmetry algebra obtained in this way contains two representations of the Virasoro algebra. In the case of many spatial dimensions, we obtain the Virasoro-Kac-Mody–type algebra. Finally, we show that on the quantum level the Virasoro generators do form the centrally extended algebra.

The paper is organized as follows. In Sec. II, we describe in some detail a class of nonlocal Lagrangians and derive the explicit relation, on the Lagrangian level, between the symmetry and the integral of motion. In Sec. III, we briefly recall the Hamiltonian formalism for the nonlocal PU model; in the next section, we find the symmetries and corresponding integrals of motion for this model in the Lagrangian framework. Section V is devoted to the symmetry realizations on the Hamiltonian and quantum levels with special emphasis put on the Virasoro algebra. Finally, Sec. VI contains the final discussion and sketches the possible ways of further development.

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II. NONLOCAL SYSTEMS

Let us consider the following class of nonlocal Lagrangians:

$$L = L(q(t), q(t+\alpha)), \qquad \alpha > 0. \tag{2.1}$$

The action principle

$$\delta S \equiv \int_{-\infty}^{\infty} \frac{\delta L(s)}{\delta q(t)} ds = 0 \tag{2.2}$$

applied to the Lagrangian (2.1) implies the following equation of motion:

$$\frac{\partial L(q(t), q(t+\alpha))}{\partial q(t)} + \frac{\partial L(q(t-\alpha), q(t))}{\partial q(t)} = 0. \tag{2.3}$$

To better understand the structure of the nonlocal theory and to derive the Hamiltonian formalism (and, eventually, the quantum theory), let us expand the Lagrangian (2.1) in powers of the parameter α; then, L becomes a function of all time derivatives of $q(t)$. Thus, it requires an infinite number of initial conditions or the knowledge of a finite piece of the trajectory. Consequently, we need a Hamiltonian formalism which would be a generalization of the Ostrogradski one, which is valid only for a finite-order derivatives. Such a formalism has been proposed and discussed in Refs. [9–13]. In this approach, we consider, instead of the trajectory $q(t)$, the $1+1$-dimensional field $Q(t,\lambda)$ such that

$$Q(t,\lambda) = q(t+\lambda). \tag{2.4}$$

Our Lagrangian L is rewritten as

$$\tilde{L}(Q(t,0), Q(t,\alpha)) = \int_{-\infty}^{\infty} d\lambda \delta(\lambda) \tilde{L}(Q(t,\lambda), Q(t,\lambda+\alpha)), \tag{2.5}$$

where $\tilde{L}(Q(t,\lambda), Q(t,\lambda+\alpha))$ is obtained from the original Lagrangian (2.1) by making the following replacements:

$$q(t) \to Q(t,\lambda), \qquad q(t+\alpha) \to Q(t,\lambda+\alpha). \tag{2.6}$$

Now, the Hamiltonian for the field Q is provided by

$$H = \int_{-\infty}^{\infty} d\lambda P(t,\lambda) Q'(t,\lambda) - \tilde{L}, \tag{2.7}$$

where P is the canonical momentum of Q and the prime stands for ∂_λ. The Poisson bracket is of the form

$$\{Q(t,\lambda), P(t,\lambda')\} = \delta(\lambda - \lambda'). \tag{2.8}$$

To recover the original dynamics, one has to impose a new constraint,

$$\phi \equiv P(t,\lambda) - \frac{1}{2}\int_{-\infty}^{\infty} d\sigma (\mathrm{sgn}(\lambda) - \mathrm{sign}(\sigma)) E(t,\sigma,\lambda) \approx 0, \tag{2.9}$$

where

$$E(t,\sigma,\lambda) = \frac{\delta L(Q(t,\sigma), Q(t,\sigma+\alpha))}{\delta Q(t,\lambda)}. \tag{2.10}$$

The constraint ϕ implies the secondary constrains which can be written collectively in the following form:

$$\psi \equiv \int_{-\infty}^{\infty} d\sigma E(t,\sigma,\lambda) \approx 0. \tag{2.11}$$

The constraint (2.11) is equivalent to the Euler-Lagrange equation for the nonlocal Lagrangian (2.1), and together with the constraint (2.9), they form the second-class constraints. So, we can use the Dirac method for constrained systems and solve ϕ for P. Then, the Hamiltonian

$$\begin{aligned} H = &-\frac{1}{2}\int_{-\infty}^{\infty} d\lambda (\mathrm{sgn}(\lambda - \alpha) \\ &- \mathrm{sign}(\lambda)) \frac{\partial L(Q(t,\lambda-\alpha), Q(t,\lambda))}{\partial Q(t,\lambda)} Q'(t,\lambda) \\ &- L(Q(t,0), Q(t,\alpha)) \end{aligned} \tag{2.12}$$

is expressed only in terms of Q, which is subjected to the single condition (2.11) and satisfies commutation relations following from the relevant Dirac bracket.

Apart from the Hamiltonian, we can look for other integrals of motion (and the corresponding symmetries) related to the Lagrangian (2.1). To do this, let us consider the infinitesimal transformation of the time t and the coordinate q,

$$t' = t + \delta t(t), \qquad q'(t') = q(t) + \delta q(t), \tag{2.13}$$

satisfying the following condition:

$$\delta t(t+\alpha) = \delta t(t). \tag{2.14}$$

Then,

$$\begin{aligned} t' + \alpha &= t + \alpha + \delta t(t+\alpha), \\ q'(t'+\alpha) &= q(t+\alpha) + \delta q(t+\alpha). \end{aligned} \tag{2.15}$$

Next, let us define

$$\begin{aligned} \delta_0 q(t) &= \delta q(t) - \dot{q}(t)\delta(t), \\ \delta_0 q(t+\alpha) &= \delta q(t+\alpha) - \dot{q}(t+\alpha)\delta t(t+\alpha). \end{aligned} \tag{2.16}$$

The action principle is invariant under the transformations (2.13), provided

$$L(q'(t'),q'(t'+\alpha))\frac{dt'}{dt}=L(q(t),q(t+\alpha))+\frac{dF}{dt}, \tag{2.17}$$

which infinitesimally takes the form

$$\frac{d}{dt}(L\delta t-\delta F)+\frac{\partial L(q(t),q(t+\alpha))}{\partial q(t)}\delta_0 q(t) + \frac{\partial L(q(t),q(t+\alpha))}{\partial q(t+\alpha)}\delta_0 q(t+\alpha)=0. \tag{2.18}$$

Using Eq. (2.18), one can check, by direct calculations, that the function C, defined as

$$C=L\delta t+\int_t^{t+\alpha}d\lambda\frac{\partial L(q(\lambda-\alpha),q(\lambda))}{\partial q(\lambda)}\delta_0 q(\lambda)-\delta F, \tag{2.19}$$

is an integral of motion associated with the symmetry (2.13). This result can be also obtained following Ref. [11] directly within the Hamiltonian framework, keeping in mind that the constraint (2.9), being of the second kind, can be used to eliminate the momentum variable $P(t,\lambda)$. Let us note that in the case of the time translation, i.e.,

$$\delta t=-\epsilon,\qquad \delta q=0, \tag{2.20}$$

C coincides with H after making the replacement (2.6)

III. NONLOCAL PAIS-UHLENBECK MODEL

The PU oscillator of order N,

$$L_{\rm PU}=-\frac{1}{2}q\prod_{k=1}^{N}\left(\frac{d^2}{dt^2}+\omega_k^2\right)q, \tag{3.1}$$

studied in the original paper [8] is considered there as a starting point toward the nonlocal theory. Namely, it has been shown (rather formally) that taking the limit $N\to\infty$ in (3.1) we obtain, in the case of the odd frequencies $\omega_k=(2k-1)\frac{\pi}{2\alpha}$, $k=1,2,\ldots$, a quadratic nonlocal Lagrangian which (after some manipulations) can be written in the form

$$L=-\frac{m}{\alpha^2}q(t)q(t+\alpha), \tag{3.2}$$

where m and α are some constants of the dimension of mass and time, respectively. The equation of motion (2.3) in this case yields

$$q(t-\alpha)+q(t+\alpha)=0. \tag{3.3}$$

The slightly surprising interpretation of the Lagrangian (3.2) as a generalization of the PU oscillator with the odd frequencies to the case of the infinite time derivatives (or, equivalently, with an infinite number of oscillators) has been recently confirmed, in two ways, in Ref. [13]. First, it was shown that, expanding the right-hand side of Eq. (3.2) in a Taylor series in α and then restricting ourselves to the first k terms, we obtain a Lagrangian which is proportional to the PU oscillator with some frequencies, which tends to the odd ones in the limit $k\to\infty$. The authors of Ref. [13] applied also the Hamiltonian formalism discussed in the previous section and arrived at the Hamiltonian dynamics

$$H=\frac{m}{2\alpha^2}\int_{-\infty}^{\infty}d\lambda(\mathrm{sgn}(\lambda-\alpha)-\mathrm{sign}(\lambda))Q(t,\lambda-\alpha)Q'(t,\lambda) + \frac{m}{\alpha^2}Q(t,0)Q(t,\alpha), \tag{3.4}$$

where $Q(t,\lambda)$ is subjected to the condition

$$\psi\equiv Q(t,\lambda-\alpha)+Q(t,\lambda+\alpha)\approx 0, \tag{3.5}$$

and the Dirac bracket reads

$$\{Q(t,\lambda),Q(t,\lambda')\}=\frac{\alpha^2}{m}\sum_{k=-\infty}^{\infty}(-1)^k\delta(\lambda-\lambda'+(2k+1)\alpha). \tag{3.6}$$

The constraint ψ can be explicitly solved,

$$Q(t,\lambda)=\sum_{k=-\infty}^{\infty}a_k(t)\psi_k(\lambda), \tag{3.7}$$

where

$$\psi_k=\frac{1}{\sqrt{2\alpha}}e^{\frac{i\pi}{2\alpha}(2k+1)\lambda}, \tag{3.8}$$

forms an orthonormal and complete set in $L^2[-\alpha,\alpha]$. Then, the Hamiltonian (3.4), expressed in terms of the new variables, takes the form

$$H=\frac{\pi m}{4\alpha^3}\sum_{k=-\infty}^{\infty}(-1)^k(2k+1)a_k a_{-(k+1)}, \tag{3.9}$$

while the Dirac bracket reads

$$\{a_k,a_n\}=\frac{i\alpha^2}{m}(-1)^k\delta_{k+n+1,0}. \tag{3.10}$$

After a simple redefinition of a's,

$$a_k = \frac{\alpha}{\sqrt{m}} \begin{cases} c_k & \text{k-odd positive,} \\ c_{-(k+1)} & \text{k-odd negative,} \\ \bar{c}_k & \text{k-even non-negative,} \\ \bar{c}_{-(k+1)} & \text{k-even negative,} \end{cases} \tag{3.11}$$

we arrive at the infinite alternating sum of oscillators with the odd frequencies

$$H = \sum_{k=0}^{\infty} (-1)^k (2k+1) \frac{\pi}{2\alpha} \bar{c}_k c_k, \tag{3.12}$$

with the standard brackets

$$\{c_k, \bar{c}_n\} = -i\delta_{kn}. \tag{3.13}$$

Thus, the model (3.2) describes the PU oscillator of infinite order with the odd frequencies. This fact is interesting from the symmetry point of view because, as it was shown in Ref. [30], for the odd frequencies, the symmetry group of the Nth-order PU oscillator is larger than for the case of generic frequencies; it contains the conformal and dilatation generators which, together with the remaining symmetry generators (related to the time translation and the change of initial conditions), form the $(N - \frac{1}{2})$-conformal nonrelativistic algebra. Consequently, the question arises about the symmetry of the nonlocal model (3.2).

IV. SYMMETRIES OF THE NONLOCAL PU MODEL ON THE LAGRANGIAN LEVEL

We start with the observation that the change of the initial conditions produces new solutions. Because of the fact that any solution of the equation of motion (3.3) is an antiperiodic function on the interval $[-\alpha, \alpha]$, we postulate the infinitesimal symmetry

$$t' = t, \qquad q' = q + \delta q, \\ \text{where } \delta q(t-\alpha) + \delta q(t+\alpha) = 0. \tag{4.1}$$

Indeed, taking

$$\delta F = -\frac{m}{\alpha^2} \int_t^{t+\alpha} d\lambda \delta q(\lambda - \alpha) q(\lambda), \tag{4.2}$$

one can check that the symmetry condition (2.18) holds. The corresponding integral of motion reads

$$C = -\frac{m}{\alpha^2} \int_{-\alpha}^{\alpha} q(\lambda + t - \alpha) \delta q(\lambda + t). \tag{4.3}$$

Expanding δq in the Fourier series [after substitution $q(t) \to q(t) e^{-\frac{i\pi t}{2\alpha}}$], we can identify the symmetry generators corresponding to the coefficients in the Fourier expansion

$$\delta q_k^c = \epsilon \cos\left(\frac{2k+1}{2}\omega t\right), \qquad \delta q_k^s = \epsilon \sin\left(\frac{2k+1}{2}\omega t\right), \tag{4.4}$$

where $\omega = \frac{\pi}{\alpha}$ and $k = 0, 1, 2\ldots$ (for simplicity of the further notation, we can extend k to the negative integers). Consequently, the symmetry generators take the form

$$C_k^c = -\frac{m}{\alpha^2} \int_{-\alpha}^{\alpha} q(\lambda + t - \alpha) \cos\left(\frac{2k+1}{2}\omega(\lambda + t)\right), \\ C_k^s = -\frac{m}{\alpha^2} \int_{-\alpha}^{\alpha} q(\lambda + t - \alpha) \sin\left(\frac{2k+1}{2}\omega(\lambda + t)\right), \tag{4.5}$$

or, in the differential realization,

$$C_k^c = -\cos\left(\frac{2k+1}{2}\omega t\right) \frac{d}{dq}, \\ C_k^s = -\sin\left(\frac{2k+1}{2}\omega t\right) \frac{d}{dq}. \tag{4.6}$$

Note that with our convention concerning k (positive and negative) some generators are linearly dependent.

Now, the main observation is that, except the quite natural symmetry (4.1), the Lagrangian (3.2) possesses a more interesting set of symmetries related to the time transformation. Namely, let us consider an arbitrary time transformation $t' = t'(t)$ satisfying the condition

$$t'(t + \alpha) = t'(t) + \alpha \tag{4.7}$$

and define the following transformation of the coordinate q:

$$q'(t') = q(t) \left(\frac{dt'}{dt}\right)^{-\frac{1}{2}}. \tag{4.8}$$

Then, one can check that the invariance condition (2.17) for the nonlocal Lagrangian (3.2) is satisfied (with $F = 0$). To identify this symmetry, let us take its infinitesimal form

$$t' = t + \epsilon f(t), \qquad f(t + \alpha) = f(t). \tag{4.9}$$

Then,

$$\delta t = \epsilon f(t), \qquad \delta_0 q = -\frac{\epsilon}{2} \dot{f} q; \tag{4.10}$$

because of Eq. (2.19), the corresponding integral of motion reads

$$C_f = -\frac{m}{\alpha^2}q(t)q(t+\alpha)f(t) + \frac{m}{\alpha^2}\int_0^\alpha d\lambda q(\lambda+t-\alpha) \times \left(\frac{1}{2}f'(\lambda+t)q(\lambda+t) + q'(\lambda+t)f(\lambda+t)\right) = \frac{m}{2\alpha^2}\int_{-\alpha}^{\alpha} q(\lambda+t-\alpha)q'(\lambda+t)f(\lambda+t), \tag{4.11}$$

where in the second equality we have used Eq. (3.3). Since the function f is periodic with the period α, it can be expanded in terms of the functions

$$f_k^c(t) = \frac{1}{2\omega}\cos(2k\omega t), \quad f_k^s(t) = \frac{1}{2\omega}\sin(2k\omega t), \tag{4.12}$$

for $k = 0, 1, 2\ldots$ Substituting Eq. (4.12) into Eq. (4.11), we obtain a family of generators L_k^c, L_k^s (as before, we may assume k can be extended to all integers (positive and negative); then, $L_{-k}^s = -L_k^s$ and $L_{-k}^c = L_k^c$). One can easily find the differential realizations of those generators,

$$L_k^c = \frac{-1}{2\omega}\cos(2k\omega t)\frac{d}{dt} - \frac{k}{2}\sin(2k\omega t)q\frac{d}{dq}, \quad L_k^s = \frac{-1}{2\omega}\sin(2k\omega t)\frac{d}{dt} + \frac{k}{2}\cos(2k\omega t)q\frac{d}{dq}, \tag{4.13}$$

as well as the commutation rules

$$[L_k^s, L_n^s] = \frac{k-n}{2}L_{n+k}^s + \frac{n+k}{2}L_{n-k}^s, \quad [L_k^c, L_n^c] = \frac{n-k}{2}L_{n+k}^s + \frac{n+k}{2}L_{n-k}^s, \quad [L_k^s, L_n^c] = \frac{k-n}{2}L_{n+k}^c + \frac{n+k}{2}L_{n-k}^c. \tag{4.14}$$

Moreover, the commutators of L's with C's read

$$[C_k^s, L_n^s] = \frac{1}{8}((2n-2k-1)C_{k-2n}^s + (2n+2k+1)C_{k+2n}^s),$$
$$[C_k^s, L_n^c] = \frac{1}{8}((-2n+2k+1)C_{k-2n}^c + (2n+2k+1)C_{k+2n}^c),$$
$$[C_k^c, L_n^s] = \frac{1}{8}((2n-2k-1)C_{k-2n}^c + (2n+2k+1)C_{k+2n}^c),$$
$$[C_k^c, L_n^c] = \frac{1}{8}((2n-2k-1)C_{k-2n}^s - (2n+2k+1)C_{k+2n}^s). \tag{4.15}$$

Summarizing, the symmetry algebra is defined by Eqs. (4.14) and (4.15); as usual, on the Lagrangian level, there is no central change.

To have a better insight into this algebra, let us note that the generators L_0^c, L_1^c, L_1^s form a three-dimensional subalgebra,

$$[L_0^c, L_1^c] = L_1^s, \qquad [L_1^s, L_0^c] = L_1^c, \qquad [L_1^s, L_1^c] = L_0^c, \tag{4.16}$$

in which, after the substitution,

$$\tilde{H} = -(L_1^c + L_0^c), \qquad K = L_1^c - L_0^c, \qquad D = L_1^s, \tag{4.17}$$

we recognize the $sl(2,R)$ algebra

$$[D, \tilde{H}] = \tilde{H}, \quad [K, D] = K, \quad [\tilde{H}, K] = -2D. \tag{4.18}$$

Now, let us remind the reader that the free classical motion (in general, in the sense of higher-derivatives theory) possesses the conformal symmetry (i.e., $SL(2,\mathbb{R})$ symmetry; see Refs. [36] and [37]) which acts on time variable $\tilde{t}$ according to the formula

$$\tilde{t}' = \frac{a\tilde{t}+b}{c\tilde{t}+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{4.19}$$

This symmetry can be transformed to the harmonic oscillator case (the PU oscillator with the odd frequencies in the case of higher derivatives) by means of the Niederer transformation (or its generalization to the higher derivatives; see Refs. [23,30] for further references), in which the time of the free motion $\tilde{t}$ is related to the "oscillator" time t by the formula

$$\tilde{t} = \tan(\omega t). \tag{4.20}$$

Since our model is a natural extension of the harmonic oscillator, we can try to induce directly the symmetry from the action of $SL(2,\mathbb{R})$ on $\tilde{t}$. Indeed, taking the standard realization of the conformal generators

$$\tilde{H} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad D = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.21}$$

we obtain, by virtue of Eq. (4.19), the infinitesimal action on t,

$$\tilde{H}\colon\ t' = t - \frac{\epsilon}{2\omega}(\cos(2\omega t) + 1), \quad K\colon\ t' = t + \frac{\epsilon}{2\omega}(\cos(2\omega t) - 1), \quad D\colon\ t' = t + \frac{\epsilon}{2\omega}\sin(2\omega t), \tag{4.22}$$

which perfectly agrees with the functions f_0^c, f_1^c, f_1^s and the relation (4.17).

Moreover, the Hamiltonian (3.4) coincides with the generator of the time translation, i.e., $L_0^c = -\frac{1}{2\omega}H$, which again agrees with the fact that the dynamics of the considered model is related to the different choice of the basis of the conformal algebra $H = \omega(\tilde{H} + K)$ (as in the case of the harmonic and PU oscillators; see Refs. [23,48] and the references therein).

We see that L's define an infinite-dimensional extension of the $sl(2,\mathbb{R})$ algebra. Things become more transparent if we use the functions $f_k(t) = \frac{1}{2\omega}e^{2ik\omega t}$ with a k integer, instead of the functions (4.12), and ψ_k [Eq. (3.8)] with a k integer, instead of the functions appearing in Eq. (4.4). Then, the corresponding generators L_k and C_k (in the complexified algebra) can be expressed in the form

$$L_k = L_k^c + iL_k^s, \qquad C_k = \frac{1}{\sqrt{2\alpha}}(C_k^c + iC_k^s), \quad k - \text{integer} \tag{4.23}$$

and satisfy the following commutation relations:

$$[L_k, L_n] = i(k-n)L_{k+n}, \tag{4.24}$$

$$[C_k, L_n] = \frac{i}{4}(2k+1+2n)C_{k+2n}. \tag{4.25}$$

From Eq. (4.27), we infer that the symmetry algebra can be split in two parts spanned by C_k's with k even or odd; thus, we define

$$C_k^+ = C_{2k}, \qquad C_k^- = C_{2k-1}, \qquad k - \text{integer}. \tag{4.26}$$

As a result, the commutation rules (4.27) are converted into

$$[C_k^\pm, L_n] = i\left(k + \frac{n}{2} \pm \frac{1}{4}\right)C_{k+n}^\pm, \tag{4.27}$$

in which we recognize two representations of the Virasoro algebra acting on $C^\pm$'s (with the nonhomogeneous part $\pm\frac{1}{4}$; see, e.g., Ref. [50]). At this point, let us recall that the PU oscillator with the odd frequencies enjoys the $(N-\frac{1}{2})$-conformal nonrelativistic symmetry, which, for fixed N, possesses infinite-dimensional extensions (see Refs. [42–49]). However, there is no direct limit in these algebras as N tends to infinity (in contrast to the Lagrangian or the Hamiltonian of the PU oscillator).

Moreover, similar algebras (i.e., with the similar semidirect form) appear in the context of asymptotically flat space-time at null infinity (called the Bondi-Metzner-Sachs algebras; see Refs. [51–53]). To better understand this situation, let us recall that BMS_3 is actually isomorphic (see Ref. [54]) to the infinite-dimensional extension of Galilean conformal algebra (GCA) in two space-time dimensions (sometimes, this extension is also referred to as the GCA algebra since it arises precisely from a nonrelativistic contraction of the relativistic conformal algebra). In dimension 4, one can check that a subalgebra of BMS_4 (its one component) is isomorphic to the infinite extension of the Schrödinger algebra [55]; see also Ref. [54] for further relations of BMS_4 with nonrelativistic symmetries. Let us note that for both mentioned infinite-dimensional extensions of the nonrelativistic algebras the action of the generators L_0, $L_{\pm 1}$ [corresponding to the $sl(2,\mathbb{R})$] defines a reducible (but indecomposable) representation of the $SL(2,\mathbb{R})$ group (it follows directly from Theorem 1.2.7, point 2a in Ref. [56]). Let us now go back to the semidirect product obtained in our paper. In this case, the action (4.27) of the Virasoro generators corresponding to $sl(2,\mathbb{R})$ algebra on C^+'s and C^-'s is much more subtle. One can check (using, e.g., the results of Ref. [57]) that this action defines two irreducible representations of the $sl(2,\mathbb{R})$ algebra (characterized by $(q,\tau) = (\frac{1}{4},\frac{1}{4})$ and $(q,\tau) = (\frac{1}{4},\frac{3}{4})$ in the notation of Ref. [57]) belonging to the continuous series of representations of the universal covering of the $SL(2,\mathbb{R})$ group. So, the difference is not only related to the fact that in our case the action of L_0, $L_{\pm 1}$ is irreducible but also corresponds to the universal covering of the $SL(2,\mathbb{R})$ group. Moreover, both sets of $sl(2,\mathbb{R})$ generators (in Ref. [55] and ours) are related by the composition of the (local) change of variables (a kind of Niederer's transformation for the oscillator)

$$\tau = \tan(\omega t), \qquad x = q\cos(\omega t) \tag{4.28}$$

with the transformation $y = 1/x$. These two observations strongly suggest that the action of the Virasoro algebra [Eq. (4.27)] is not equivalent to the one appearing in the infinite extensions of the nonrelativistic algebras mentioned above.

Finally, let us note that the model (3.2) as well as our former considerations can be extended to the case of coordinates taking their values in three-dimensional space $\vec{q} = (q^\alpha)$ (or, in generally, in d-dimensional space). Then, the symmetry generators (4.6) and (4.13) extend naturally to the vector cases, C^α's and L's, respectively. Moreover, we have an additional family of generators,

$$J_k^{\alpha c} = f_k^c J^\alpha, \qquad J_k^{\alpha s} = f_k^s J^\alpha, \tag{4.29}$$

related to the symmetry

$$t' = t, \qquad \vec{q}' = R(t)\vec{q}, \tag{4.30}$$

where R is a time-dependent rotation satisfying $R(t+\alpha) = R(t)$. As earlier [see Eq. (4.23)], it is instructive to pass to the complexification of the algebra and define the new generators $J_k^\alpha = J_k^{\alpha c} + iJ_k^{\alpha s}$. Then, the commutation rules containing J's take the Kac-Moody form

$$[J_k^\alpha, J_n^\beta] = \frac{1}{2\omega}\epsilon^{\alpha\beta\gamma}J^\gamma_{k+n},$$
$$[J_k^\alpha, C_n^{\beta\pm}] = \frac{1}{2\omega}\epsilon^{\alpha\beta\gamma}C^{\gamma\pm}_{n+k},$$
$$[J_k^\alpha, L_n] = ikJ^\alpha_{k+n} \tag{4.31}$$

and coincide with the ones described in Refs. [42–48] (the difference is in the action of the Virasoro subalgebra).

We conclude this section with the following remark. The considered Lagrangian (3.2) provides the simplest generalization of the PU oscillator of the finite order. However, one could consider adding some additional terms which do not spoil at least the part of the symmetry discussed above. For example, one can add the local quadratic term to the Lagrangian (3.2) yielding

$$L = -\frac{m}{\alpha^2}(\vec{q}(t)\vec{q}(t+\alpha) + \sigma\vec{q}^2(t)). \tag{4.32}$$

Obviously, it continues to exhibit the symmetry described by Eqs. (4.7) and (4.8) as well as Eq. (4.30). Equation (2.3) gives the following equation of motion:

$$\vec{q}(t+\alpha) + \vec{q}(t-\alpha) + 2\sigma\vec{q}(t) = 0. \tag{4.33}$$

To solve it, we define

$$\lambda_\pm = -\sigma \pm \sqrt{\sigma^2 - 1}, \tag{4.34}$$

and, for $\sigma^2 \neq 1$,

$$\vec{r}_\pm(t) = \frac{\vec{q}(t) - \lambda_\mp\vec{q}(t+\alpha)}{\lambda_\pm - \lambda_\mp} e^{-\frac{t}{\alpha}\ln(-2\sigma-\lambda_\mp)}. \tag{4.35}$$

Then, by Eq. (4.33),

$$\vec{r}_\pm(t+\alpha) = \vec{r}_\pm(t). \tag{4.36}$$

Note that Eq. (4.35) becomes singular for $\sigma^2 = 1$. The case $\sigma = -1$ corresponds to the α expansion of the Lagrangian (4.32) starting with the derivative term. Then, the solution contains the pieces linear in time instead of the exponential ones entering Eq. (4.35). However, such terms do appear also for $\sigma = 1$, while the counterpart of $r_\pm$ is then antiperiodic instead of being periodic [cf. Eq. (4.36)].

We see that the process of solving the generalized equations of motion (4.33) is basically reduced to the Fourier analysis so the corresponding symmetries could be found by proceeding along similar lines as previously.

V. SYMMETRY REALIZATIONS ON THE HAMILTONIAN AND QUANTUM LEVELS

In this section, we discuss the symmetry algebra in Hamiltonian approaches and show that it forms a natural central extension of the algebra appearing on the Lagrangian level; on the quantum level, further central extension for the Virasoro algebra is obtained.

A. Constrained approach

First, let us note that the Hamiltonian (3.4) as well as the symmetry generators C's and L's can be rewritten, due to the correspondence (2.6), as follows:

$$H = -\frac{m}{2\alpha^2}\int_{-\alpha}^{\alpha} d\lambda Q(t,\lambda-\alpha)Q'(t,\lambda),$$
$$L_k^c = \frac{m\omega}{4\pi^2}\int_{-\alpha}^{\alpha} d\lambda Q(t,\lambda-\alpha)Q'(t,\lambda)\cos(2\omega k(\lambda+t)),$$
$$L_k^s = \frac{m\omega}{4\pi^2}\int_{-\alpha}^{\alpha} d\lambda Q(t,\lambda-\alpha)Q'(t,\lambda)\sin(2\omega k(\lambda+t)),$$
$$C_k^c = -\frac{m}{\alpha^2}\int_{-\alpha}^{\alpha} d\lambda Q(t,\lambda-\alpha)\cos\left(\frac{2k+1}{2}\omega(\lambda+t)\right),$$
$$C_k^s = -\frac{m}{\alpha^2}\int_{-\alpha}^{\alpha} d\lambda Q(t,\lambda-\alpha)\sin\left(\frac{2k+1}{2}\omega(\lambda+t)\right). \tag{5.1}$$

Now, using the functional Dirac bracket (3.6), we can find all commutation rules and consequently the symmetry algebra on the Hamiltonian level. The computations are straightforward but rather tedious, due to the fact that we have to carefully take into account the support of each delta function appearing in the Dirac bracket (3.6). As a consequence, we obtain almost the same relations on the Hamiltonian level as on the Lagrangian one—the only difference appears in commutators between C's; in this case, we obtain a central extension, namely,

$$[C_k^c, C_n^s] = \frac{m}{\alpha}(-1)^{n-1}\delta_{nk}, \qquad k,n = 1,2,3\ldots \tag{5.2}$$

B. Unconstrained approach

As we saw in Sec. III, solving the constraint (3.5) on the Hamiltonian level allows us to identify our model with the PU oscillator of the infinite order [see Eq. (3.12)]. Thus we will find the form of the remaining symmetry generators in terms of classical counterparts the creation and annihilation operators c's (equivalently a's). We will see that the symmetry generators (except the Hamiltonian) are not direct extensions of the ones for the classical PU oscillator [22]. To do this, we can express, by virtue of Eq. (3.7), the symmetry generators in terms of a's. After some troublesome calculations, we arrive at the form of the generators

$$C_k^c = \sqrt{\frac{\alpha}{2}}(C_k + \bar{C}_k), \qquad C_k^s = i\sqrt{\frac{\alpha}{2}}(\bar{C}_k - C_k),$$
$$L_k^c = \frac{1}{2}(L_k + \bar{L}_k), \qquad L_k^s = \frac{i}{2}(\bar{L}_k - L_k), \tag{5.3}$$

where

$$C_k = \frac{m}{\alpha^2} e^{\frac{i\pi}{2\alpha}(2k+1)(t-\alpha)} \bar{a}_k, \tag{5.4}$$

and

$$\begin{aligned} L_n &= -\frac{m}{8\alpha^2} e^{2in\omega t} \sum_{k=-\infty}^{\infty} (-1)^k (2k+1) a_{-k-2n-1} a_k \\ &= \frac{m}{4\alpha^2} (-1)^{n+1} e^{2in\omega t} \sum_{k=0}^{\infty} (-1)^k (2k+1) \bar{a}_{k+n} a_{k-n}. \end{aligned} \tag{5.5}$$

The following identity appears to be useful:

$$\bar{a}_k = a_{-(k+1)}. \tag{5.6}$$

Now, using the bracket (3.10), we can confirm the commutation relations (4.14), (4.15), and (5.2) which were previously obtained within the Hamiltonian framework with constraints. The calculations are straightforward but rather long, so we only mention that, using Eqs. (3.10) and (5.6), one can check the relation (4.24), which, in turn, easily implies Eq. (4.14).

By virtue of Eq. (3.11), one can express the generators (5.4) and (5.5) in terms of $c's$ [for the Hamiltonian, see Eq. (3.12)]. Consequently, we obtain a nonstandard realization of the Virasoro algebra (4.24) which is reflected in the form of the central extension on the quantum level [see Eq. (5.11) or (5.14) below]. Moreover, let us note that

$$\begin{aligned} &\bar{C}_n = C_{-(n+1)}, \qquad \bar{L}_n = L_{-n} \\ &L_0 = \bar{L}_0 = L_0^c = -\frac{1}{2\omega} H. \end{aligned} \tag{5.7}$$

Finally, the generalization of our considerations on the Hamiltonian level to the three-dimensional (in general d-dimensional) case is straightforward; the main difference is that we have additional generators corresponding to the rotational symmetry [cf. Eqs. (4.29) and (4.30)],

$$\begin{aligned} J_n^\alpha &= \frac{m\omega}{4\pi^2} \epsilon^{\alpha\beta\gamma} \int_{-\alpha}^{\alpha} d\lambda Q^\beta(t, \lambda - \alpha) Q^\gamma(t, \lambda) e^{2i\omega n(t+\lambda)} \\ &= \frac{-im\omega}{4\pi^2} \epsilon^{\alpha\beta\gamma} e^{2i\omega tn} \sum_{k=-\infty}^{\infty} (-1)^k a^\gamma_{-k-2n-1} a^\beta_k, \end{aligned} \tag{5.8}$$

which satisfy the commutation rules (4.31).

C. Quantum level

In this section, we analyze the symmetries of the nonlocal PU model on the quantum level with special emphasis put on the Virasoro algebra. Following the standard reasoning, we replace the classical variables c_k, $\bar{c}_k$, satisfying Eq. (3.13), by the creation and annihilation operators $\hat{c}_k$, $\hat{c}_k^+$,

$$[\hat{c}_k, \hat{c}_n^+] = \delta_{kn}, \tag{5.9}$$

for $k, n = 0, 1, 2\ldots$ Next, using the quantum counterpart of the relation (3.11), we define the operators $\hat{a}_k, \hat{a}_{-(k+1)} = \hat{a}_k^+$. Then, we have the following commutator rules:

$$[\hat{a}_k, \hat{a}_n] = \frac{-\alpha^2}{m} (-1)^k \delta_{k+n+1,0}. \tag{5.10}$$

Now, we can translate the symmetry generators to the quantum level. In the case of C's, J's, and L_n with $n \neq 0$, it can be done directly [there is no problem with ordering of the operators; see Eqs. (5.4), (5.5), and (5.8)]; however, the generator L_0 requires more careful analysis. To this end, let us compute the following commutator $[\hat{L}_n, \hat{L}_{-n}]$ for $n \neq 0$. Using the second form in Eqs. (5.5) and (5.10), we arrive, after some computations, at the result

$$[\hat{L}_n, \hat{L}_{-n}] = -2n\hat{L}_0 + \frac{1}{12}\left(n^3 - \frac{1}{4}n\right), \tag{5.11}$$

where

$$\hat{L}_0 = \frac{-m}{4\alpha^2} \sum_{k=0}^{\infty} (-1)^k (2k+1) \hat{a}_k^+ \hat{a}_k. \tag{5.12}$$

Now, let us express $\hat{L}_0$ in terms of the creation and annihilation operators:

$$\hat{L}_0 = \frac{1}{4} \sum_{\substack{k=0 \\ k\text{-odd}}}^{\infty} (2k+1) \hat{c}_k^+ \hat{c}_k - \frac{1}{4} \sum_{\substack{k=0 \\ k\text{-even}}}^{\infty} (2k+1) \hat{c}_k \hat{c}_k^+. \tag{5.13}$$

Performing the regularization procedure by means of the Riemann zeta function, $\hat{L}_0$ becomes a normally ordered operator (proportional to the quantum Hamiltonian) and the commutation rules

$$[\hat{L}_n, \hat{L}_k] = (k-n)\hat{L}_{n+k} + \frac{1}{12}\left(n^3 + \frac{3}{4}n\right)\delta_{k+n,0} \tag{5.14}$$

are satisfied; in the case of d dimensions, there is the additional factor d in front of the central charge in Eq. (5.14). Finally, by adding a constant to the normally ordered $\hat{L}_0$ and rescaling $\hat{L}_n \to -\hat{L}_n$, one obtains the standard form of the centrally extended Virasoro algebra.

VI. CONCLUSIONS AND OUTLOOK

We have discussed the symmetry properties of a simple nonlocal theory which provides an extension, to the case of an infinite number of degrees of freedom, of the PU oscillator with frequencies proportional to consecutive odd integers. Since the PU oscillator of order N with such frequencies enjoys $(N - \frac{1}{2})$-conformal nonrelativistic symmetry, the resulting symmetry algebra can be viewed as an extension of the latter to infinite order, $N = \infty$. It can be described within the framework of the Noether theorem both in its Lagrangian and Hamiltonian forms: the original $sl(2,\mathbb{R})$ subalgebra of conformal nonrelativistic algebra is extended to the Virasoro algebra; the additional charges (outside the Virasoro subalgebra) form two representations of the latter under its adjoint action. They commute with each other on the Lagrangian level, while on the Hamiltonian level, we are dealing with the central extension defined by Eq. (5.2). Moreover, on the quantum level, the Virasoro algebra itself is also centrally extended [Eq. (5.11) or (5.14)].

Let us note that the model described here resembles to some extent the string model. In the Hamiltonian framework, we are dealing with a string obeying antiperiodic boundary conditions [cf. Eq. (3.5)] with the dynamics restricted to left movers only [Eq. (2.4)]. There are, however, some important differences. First, Q's (or q's) take their values in Euclidean space, and there are no additional constraints on them. Moreover, the energy is the alternating sum of independent oscillators energies [Eq. (3.12)], which makes the system unstable.

The model, although basically noninteracting, exhibits an interesting structure. It is worthy further study in several directions. First, one can consider the supersymmetric extensions following the construction proposed for finite order in Refs. [24,29].

Next, the above-mentioned problem of (in)stability of the system might be studied. There is a number of options concerning this problem. First, one should keep in mind that higher-derivative theories are, in general, the effective ones obtained by integrating out some degrees of freedom which are irrelevant ("frozen") in the energy regime under consideration. Then, one can argue that switching them on will stabilize the dynamics, making the full Hamiltonian bounded from below. Let us note that, from this point of view, our model seems to provide a consistent truncation of full dynamics because the motion is bounded (in spite of the unboundeness of the Hamiltonian). It can become unstable when the interaction is switched on. However, it comes from other degrees of freedom which were frozen in the low-energy domain; thus, it should rather preserve the stability of motion (as suggested above). Another possibility is to use an alternative Hamiltonian formalism with a positive definite Hamiltonian. However, this is only possible if the system admits additional integral(s) of motion, which can serve as alternative Hamiltonians; for PU-like models, this is the case because they are integrable. More serious is the problem of symmetries. In general, not all symmetries of the equations of motion can be lifted on the Lagrangian/Hamiltonian level; this can depend on the choice of Hamiltonian formalism. Finally, another way out is to use the indefinite metric in the space of states; the price to be paid is that we have to exclude some states from the space of states as unphysical (ghosts). This can be done by generalizing the approach proposed in Ref. [14] and suggested by the Euclidean path integral approach [58]. The latter is based on the observation that in the Euclidean space-time (imaginary time) the Lagrangian for the PU model becomes positively defined for the range of its parameters corresponding to oscillatory motion (actually, in Ref. [58], it is demonstrated for the fourth-order oscillator). Therefore, the Euclidean path integral for the propagator is well defined. One can read off from it the underlying algebraic structure [14]. It appears that the resulting Hamiltonian is complex (non-Hermitian). However, it still possesses a complete set of discrete eigenvalues and eigenvectors which are nonorthogonal. One can make them orthogonal by introducing a new scalar product. It appears then that some states (described by some eigenvectors of the Hamiltonian) have negative norm and should be viewed as ghosts. On the other hand, the subspace of physical states is stable under the dynamical evolution, and, moreover, the correspondence principle is obeyed. This idea can be extended to the PU oscillator of an arbitrary finite order and next to the nonlocal case; however, even in the fourth-order case, this is technically quite involved; in consequence, an explicit generalization of this method is a little bit complicated.

The problem of extending the model to the interacting case is also worthy of attention and seems to be rather nontrivial. Let us note that a large part of symmetry generators (those related to C's generators) is the ones which leave the Lagrangian invariant up to a total time derivative [cf. Eqs. (2.17) and (4.2)] only. Therefore, the most straightforward idea of taking some polynomials in the initial free Lagrangian does not work. This situation resembles that one encountered in supersymmetric theories where the supersymmetry invariance also holds up to a total time derivative; in latter case, it is, however, an advantage as it makes, for example, the ultraviolet divergences milder. On the other hand, one can follow the way the space-time symmetries emerge. For instance, the Galilean symmetry acts only on center-of-mass variables and leave large freedom for the form of interparticle interaction because the (squares of) relative coordinates are Galilean invariant. Analogously, here, one can start with (say) two generalized coordinates and define the action of our symmetry group as follows: one defines two independent linear combinations as the initial variables; one of them, x_1 (the "center of mass variable"), transforms as the q variable we are dealing with in this paper, while the second x_2 transforms according to the homomorphic representation corresponding to all C's vanishing. It would be then much easier to construct the invariant F which depends on x_2 only. The Lagrangian

$$L = \frac{-m}{\alpha^2} x_1(t)x_1(t+\alpha) + F(x_2) \tag{6.1}$$

is then invariant (up to a total derivative) and describes, when expressed in terms of initial coordinates, their interaction. However, such a model is, in a sense, trivial unless we find good physical reasons for a particular choice of x_1 and x_2 (as in the case of space-time symmetries).

Finally, an interpretation of the obtained symmetry algebra seems to be (especially in the context of the asymptotic symmetry) a tempting task; perhaps, it would be instructive to consider the nonrelativistic space-time (such as Newton-Cartan gravity). To this end, some methods developed in the relativistic case can be very helpful.

ACKNOWLEDGMENTS

Special thanks are due to Piotr Kosiński, who suggested the problem studied here. The discussions with Joanna and Cezary Gonera as well as Paweł Maślanka are gratefully acknowledged. The anonymous referee's comments and suggestions, which allowed us to improve the paper, are also gratefully acknowledged. The research was supported by the National Science Center Grant No. DEC-2013/09/B/ST2/02205.

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