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Topological solitons in a gauged $CP(2)$ model

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$(2+1)$-dimensional Abelian gauged $CP(2)$ model with a self-interaction potential is considered. It is shown that there are topological solitons in this model. The magnetic flux of these solitons can be either quantized or nonquantized. Properties of the topological soliton with quantized magnetic flux are investigated as well as properties of the topological soliton with nonquantized magnetic flux. A comparative analysis of the properties is performed for the topological solitons of both types. Solutions of the model field equations are obtained numerically for the topological solitons of both types. The dependencies on the model parameters are presented for the energy and magnetic flux of the solitons. The stability of the topological solitons of both types to the decay into solitons with smaller topological charges is studied numerically. Possible generalizations of the investigated topological solitons are discussed.

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I. INTRODUCTION

There are many $(2+1)$-dimensional field models that have in their spectra static soliton solutions [1]. The existence and stability of these soliton solutions are based on the nontrivial topology of the corresponding field models. Two-dimensional topological solitons play an important role in various areas of physics: field theory, condensed matter physics, astrophysics, and hydrodynamics. The Derrick theorem [2], however, imposes severe restrictions on the existence of static two-dimensional solitons.

Let us first consider as an example $(2+1)$-dimensional pure scalar field models. If the target manifold of such a model is topologically trivial, then static two-dimensional solitons do not exist in this model. If the target manifold of such a model is topologically nontrivial and there is no potential term in the model Lagrangian, then the existence of static two-dimensional solitons is possible only if the model Lagrangian is quadratic in the derivatives of scalar fields. The classic examples of such solitons are the static solitons of the nonlinear O(3) σ-model [3] and the $CP(N-1)$ model [4,5]. If a pure scalar model with a topologically nontrivial target manifold has a potential term, then the existence of static two-dimensional solitons is possible only if the model Lagrangian contains, along with the quadratic terms, higher-order terms in the derivatives of scalar fields. An example of such a soliton is the static soliton of the baby Skyrme model [6,7].

Now let us consider $(2+1)$-dimensional field models in which an Abelian gauge field interacts minimally with a scalar field. Note that by a static field configuration in a Maxwell gauge theory, we mean one that does not depend on time in the temporal gauge. By a static field configuration in a Chern-Simons gauge theory, we mean one that does not depend on time up to a gauge transformation. Then the existence of static two-dimensional solitons is possible only if the model Lagrangian has a potential term, while the model target manifold can be either trivial or nontrivial. Examples of solitons of $(2+1)$-dimensional gauged models with trivial target manifolds are the vortices of the effective theory of superconductivity [8] and of the Abelian Higgs model [9]. Among solitons of $(2+1)$-dimensional gauged models with nontrivial target manifolds, we should mention the soliton of the gauged O(3) σ-model [10] and the soliton of the gauged baby Skyrme model [11].

Two-dimensional topological solitons of Abelian gauged models can be divided into two types: solitons with quantized magnetic flux and solitons with nonquantized magnetic flux. This is because the requirement of the finiteness of the soliton energy leads to different boundary conditions for the gauge fields of solitons of both types. The gauge field of a soliton with quantized magnetic flux satisfies the Dirichlet boundary condition at spatial infinity, while that of a soliton with nonquantized magnetic flux satisfies the Neumann boundary condition. Examples of topological solitons with quantized magnetic flux are the Abrikosov-Nielsen-Olesen vortex [8,9] and its non-Abelian generalizations [12–14], the electrically charged vortex of the Abelian Higgs model with the Chern-Simons term [15], the semilocal string [16], and vortices of many other models. Examples of topological solitons with nonquantized magnetic flux are the soliton of the gauged O(3) σ-model [10], the solitons of the gauged O(3) σ-model with the Chern-Simons term [17,18], the soliton of the gauged baby Skyrme model [11], and the soliton of the gauged baby Skyrme model with the Chern-Simons term [19].

Static topological solitons of $(2+1)$-dimensional field models can be considered as instantons of the corresponding two-dimensional Euclidean field models. One such model is the $CP(N-1)$ model. Since its appearance in the

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late 1970s [20–23], the $CP(N-1)$ model has attracted undiminishing interest. This interest is primarily based on the fact that the $CP(N-1)$ model is an indispensable instrument for the study of the nonperturbative effects in four-dimensional Yang-Mills models. The two-dimensional $CP(N-1)$ models share many common properties with four-dimensional Yang-Mills models, including asymptotic freedom in the ultraviolet regime [24], strong coupling in the infrared regime, and the existence of instantons [4,5]. The lower dimensionality of the $CP(N-1)$ models facilitates the analysis of nonperturbative effects in the strong coupling regime, compared to four-dimensional Yang-Mills models. Note also that $CP(N-1)$ models are the effective field models, which describes the low-energy dynamics on the world sheet of vortex solutions of four-dimensional Yang-Mills models [25]. It should also be noted that for $N=2$ the $CP(N-1)$ model is equivalent to the O(3) σ-model; in particular, both models have instanton solutions. This equivalence, however, is absent for larger N. The $CP(N-1)$ model continues to have the nontrivial vacuum structure and instanton solutions for $N>2$, while the O(N) σ-model has the trivial vacuum structure and no instanton solutions for $N>3$.

Unlike many other field models the $CP(N-1)$ model possesses a local Abelian invariance that is not connected to any physical gauge field. The $CP(N-1)$ model can be gauged with the help of a physical gauge field interacting minimally with the scalar $CP(N-1)$ field. It was shown in [26,27] that the topological soliton exists in the gauged $CP(1)$ model with the Chern-Simons term and properties of this soliton were studied. Properties of the topological soliton of the gauged $CP(1)$ model with the Chern-Simons term were also studied in [28–31]. The topological soliton of the gauged $CP(1)$ model with the Maxwell and Skyrme terms was considered in [32]. The solitons [26–32] possess the quantized magnetic flux taking on discrete values related to the vorticity of the soliton.

In the present paper, we research the topological solitons of the $(2+1)$-dimensional Abelian gauged $CP(2)$ model with the Maxwell term. In particular, it was found that the topological soliton with quantized magnetic flux and the topological soliton with nonquantized magnetic flux simultaneously exist in this model. The paper is structured as follows. In Sec. II we describe briefly the Lagrangian, the symmetry groups, and the field equations of the gauged $CP(2)$ model. Section III is divided into three subsections. In Secs. III A and III B, the soliton solutions corresponding to different charge matrices are considered. We give the ansatz used for solving the model field equations and obtain the boundary conditions for the topological soliton solutions. It is shown that in the gauged $CP(2)$ model, the topological soliton with quantized magnetic flux and the topological soliton with nonquantized magnetic flux simultaneously exist. The systems of nonlinear differential equations for the ansatz functions are derived. The expressions for the energy and Noether current densities are obtained in terms of the ansatz functions. The asymptotic properties of the soliton solutions of both types are investigated as $r\to 0$ and $r\to\infty$. In Sec. III C, research of soliton properties is continued. First, we consider the properties that are common for the topological solitons of both types, then the properties that are not common for them. In Sec. IV we describe the procedure for numerically solving the system of nonlinear differential equations for the ansatz functions. We present numerical results for the ansatz functions, the energy density, and the magnetic field strength for both types of the topological solitons. The dependence of the soliton energy on the model parameters is also presented. Then we numerically study the possibility of the decay for the soliton with a given topological charge into solitons with smaller topological charges. Finally, in Sec. V, we discuss possible generalizations of the considered topological solitons.

Throughout the paper the natural units $c=1$, $\hbar=1$ are used.

II. THE LAGRANGIAN AND THE FIELD EQUATIONS

The Lagrangian density for the pure $CP(N-1)$ model in $2+1$ dimensions can be written in the form

$$\mathcal{L}=(\partial_\mu\phi_a)^*\partial^\mu\phi_a-h^{-1}\phi_a(\partial_\mu\phi_a)^*\phi_b^*\partial^\mu\phi_b, \quad (1)$$

where the $CP(N-1)$ field ϕ is the set of N complex scalar fields satisfying the condition

$$\phi_a^*\phi_a=h. \quad (2)$$

Let us define the Hermitian projection operator

$$\mathcal{P}_{ab}=\delta_{ab}-h^{-1}\phi_a\phi_b^*. \quad (3)$$

Then the Lagrangian density (1) can be written in the condensed form

$$\mathcal{L}=(\mathcal{P}_{ab}\partial_\mu\phi_b)^*\mathcal{P}_{ac}\partial^\mu\phi_c. \quad (4)$$

The Lagrangian density of the pure $CP(N-1)$ model is invariant under the global SU(N) transformations

$$\phi_a(x)\to U_{ab}\phi_b(x), \quad (5)$$

and under the local U(1) transformations

$$\phi_a(x)\to\exp(i\Lambda(x))\phi_a(x). \quad (6)$$

Let us consider the $(2+1)$-dimensional Abelian gauged $CP(N-1)$ model with the Maxwell term and a self-interacting potential. The Lagrangian density for this model is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathcal{P}_{ab}D_{\mu}\phi_b)^{*}\mathcal{P}_{ac}D^{\mu}\phi_c - V(|\phi|), \quad (7)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor of the Abelian gauge field A_μ,

$$D_\mu\phi_a = \partial_\mu\phi_a - igA_\mu Q_{ab}\phi_b \quad (8)$$

is the covariant derivative of the $CP(N-1)$ field ϕ, and $V(|\phi|)$ is a self-interaction potential. The charge matrix Q in Eq. (8) is some real diagonal matrix: $Q_{ab} = \mathrm{diag}(q_1, \ldots, q_N)$. Note that the gauging of the $CP(N-1)$ model used in the present paper is different from the gauging used in [26–32]. The feature of gauging (7)–(8) is that the Lagrangian (7) of the gauged $CP(N-1)$ model becomes the Lagrangian (4) of the pure $CP(N-1)$ model as the gauge coupling constant g tends to zero.

The Lagrangian (7) is invariant under the local U(1) gauge transformations,

$$A_\mu(x) \to A_\mu(x) + g^{-1}\partial_\mu\tilde{\Lambda}(x),$$
$$\phi_a(x) \to \exp(iq_a\tilde{\Lambda}(x))\phi_a(x), \quad (9)$$

where q_a is the ath element of the diagonal charge matrix Q. Note that the Lagrangian (7) continues to be invariant under local U(1) transformations (6). It can be shown that, if the charge matrix Q is a multiple of the unit matrix, then the Abelian gauge field A_μ is decoupled from the $CP(N-1)$ field ϕ. This fact is the consequence of the invariance of the Lagrangian (7) under local U(1) transformations (6). Therefore, in what follows, we assume that the charge matrix Q is traceless:

$$\mathrm{Tr}Q = 0. \quad (10)$$

By varying the action $S = \int \mathcal{L}d^3x$ in A_μ and ϕ_a^* subject to constraint (2), we obtain the field equations of the model,

$$\partial_\nu F^{\nu\mu} = -ig\phi_a^* Q_{ab}\overset{\leftrightarrow}{D}{}^{\mu}\phi_b + ih^{-1}g(\phi_a^*\overset{\leftrightarrow}{D}{}^{\mu}\phi_a) \times (\phi_b Q_{bc}\phi_c^*), \quad (11)$$

$$\mathcal{P}_{ab}\left[2\mathcal{P}_{bc}D_\mu(\mathcal{P}_{cd}D^\mu\phi_d) - D_\mu D^\mu\phi_b + \frac{\partial V}{\partial\phi_b^*}\right] = 0, \quad (12)$$

where

$$u^*\overset{\leftrightarrow}{D}{}^{\mu}v \equiv u^*(D^\mu v) - (D^\mu u)^* v.$$

Note that the right-hand side of Eq. (11) is the Noether (electromagnetic) current corresponding to gauge transformations (9). Note also that the Noether current corresponding to transformations (6) vanishes identically. Using the well-known formula $T_{\mu\nu} = 2\partial\mathcal{L}/\partial g^{\mu\nu} - g_{\mu\nu}\mathcal{L}$, we obtain the symmetric energy-momentum tensor of the model:

$$T_{\mu\nu} = -F_{\mu\lambda}F^{\lambda}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} + (D_\mu\phi_a)^*\mathcal{P}_{ab}(D_\nu\phi_b) + (D_\nu\phi_a)^*\mathcal{P}_{ab}(D_\mu\phi_b) - g_{\mu\nu}((D_\lambda\phi_a)^*\mathcal{P}_{ab}(D^\lambda\phi_b) - V(|\phi|)). \quad (13)$$

From Eq. (13) the expression for the energy density of the model can be obtained,

$$\mathcal{E} = T_{00} = \frac{1}{2}E_iE_i + \frac{1}{2}B^2 + (D_0\phi_a)^*\mathcal{P}_{ab}(D_0\phi_b) + (D_i\phi_a)^*\mathcal{P}_{ab}(D_i\phi_b) + V(|\phi|), \quad (14)$$

where

$$E^i = -E_i = F_{0i}, \qquad B = F_{12}$$

are the electric and magnetic field strengths in $2+1$ dimensions, respectively.

III. THE ANSATZ AND SOME PROPERTIES OF THE $CP(2)$ SOLITONS

In this paper we investigate properties of the topological solitons of Abelian gauged $CP(N-1)$ model (7) for the case $N = 3$. We use the following self-interaction potential in Eq. (7):

$$V(|\phi|) = \lambda|\phi_3|^2. \quad (15)$$

To find the solutions of field equations (11)–(12), we use the following ansatz for the normalized $CP(2)$ field ϕ,

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = h^{\frac{1}{2}}\begin{pmatrix}\exp(im_1\theta)\sin(\alpha(r))\cos(\beta(r))\\ \exp(im_2\theta)\sin(\alpha(r))\sin(\beta(r))\\ \exp(im_3\theta)\cos(\alpha(r))\end{pmatrix}, \quad (16)$$

where $m_i \in \mathbb{Z}$. Ansatz (16) is axially symmetric under the combined action of spatial SO(2) rotations, diagonal SU(3) transformations (5), and U(1) transformations (6). For the $(2+1)$-dimensional Abelian gauge field A^μ the well-known vortex ansatz is used,

$$A_0 = 0, \qquad A_i = -A^i = -\frac{1}{gr}\epsilon^{ij}n^j A(r), \quad (17)$$

where ϵ^{ij} and n^j are the components of the two-dimensional antisymmetric tensor and unit vector, respectively. It can be shown that Eqs. (16)–(17) are compatible with the Lagrangian (7) and field equations (11)–(12).

From the regularity condition at $r = 0$ it follows that $m_3 = 0$ in Eq. (16) and the functions $\alpha(r)$ and $A(r)$ in Eqs. (16)–(17) satisfy the boundary conditions:

$$\alpha(0) = 0, \qquad A(0) = 0. \quad (18)$$

The finiteness of the soliton energy leads us to the boundary conditions as $r \to \infty$:

$$\alpha(r) \underset{r\to\infty}{\longrightarrow} \frac{\pi}{2}, \tag{19}$$

$$\mathcal{P}_{ab}(D_r\phi_b) = h^{\frac{1}{2}} \begin{pmatrix} -\exp(im_1\theta)\sin(\beta)\beta' \\ \exp(im_2\theta)\cos(\beta)\beta' \\ -\alpha' \end{pmatrix} \underset{r\to\infty}{\longrightarrow} 0, \tag{20}$$

$$\mathcal{P}_{ab}(D_\theta\phi_b) = \frac{i}{2}h^{\frac{1}{2}} \begin{pmatrix} -\exp(im_1\theta)(\Delta m - \Delta q A)\sin(2\beta)\sin(\beta) \\ \exp(-im_2\theta)(\Delta m - \Delta q A)\sin(2\beta)\cos(\beta) \\ 0 \end{pmatrix} \underset{r\to\infty}{\longrightarrow} 0, \tag{21}$$

where $\Delta m = m_2 - m_1$, $\Delta q = q_2 - q_1$. Boundary conditions (20)–(21) require the vanishing of only the transverse part (with respect to ϕ) of the covariant derivatives. At the same time, the covariant derivatives themselves can be different from zero as $r \to \infty$. The reason for this is the invariance of the Lagrangian (7) under local U(1) transformations (6). Such a situation differs considerably from the general case in which covariant derivatives always tend to zero as $r \to \infty$.

As the charge matrices Q we use the generators of the Cartan subgroup of SU(3) group

$$Q = \frac{1}{2}\lambda_3 = \frac{1}{2}\mathrm{diag}(1, -1, 0), \tag{22}$$

and

$$Q = \frac{1}{2}\lambda_8 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1, 1, -2), \tag{23}$$

where λ_3, λ_8 are the diagonal Gell-Mann matrices. In this paper we consider the two most interesting combinations of the charge matrices and winding numbers: $Q = \lambda_3/2$, $m_1 = -m_2 = m$, and $Q = \lambda_8/2$, $m_1 = m_2 = m$.

A. The case $Q = \frac{1}{2}\lambda_3$, $m_1 = -m_2 = m$, $m_3 = 0$

In this case, the differential equation for the ansatz function $\beta(r)$ is current corresponding to gauge transformations

$$\beta''(r) + \left(\frac{1}{r} + 2\cot(\alpha(r))\alpha'(r)\right)\beta'(r) - \frac{\sin^2(\alpha(r))\sin(4\beta(r))}{r^2}\left(m - \frac{A(r)}{2}\right)^2 = 0. \tag{24}$$

It is readily seen that Eq. (24) has the two types of solutions:

$$\beta(r) = \frac{\pi}{4} + \frac{\pi}{2}k, \qquad k \in \mathbb{Z}, \tag{25}$$

and

$$\beta(r) = \frac{\pi}{2}k, \qquad k \in \mathbb{Z}. \tag{26}$$

It is clear that constant solutions (25)–(26) satisfy condition (20). Now let us consider cases (25)–(26) separately.

1. The case $\beta(r) = \frac{\pi}{4} + \frac{\pi}{2}k$, $k \in \mathbb{Z}$

In this case, condition (21) can be satisfied only if $A(r)$ satisfies the boundary condition of the vortex type

$$A(r) \underset{r\to\infty}{\to} 2m, \qquad m \in \mathbb{Z}. \tag{27}$$

The system of differential equations for the ansatz functions $\alpha(r)$ and $A(r)$ can be written as

$$\alpha''(r) + \frac{\alpha'(r)}{r} - \frac{\sin(2\alpha(r))}{2r^2}\left(m - \frac{A(r)}{2}\right)^2 + \frac{\lambda}{2}\sin(2\alpha(r)) = 0, \tag{28}$$

$$A''(r) - \frac{A'(r)}{r} + g^2 h \sin^2(\alpha(r))\left(m - \frac{A(r)}{2}\right) = 0. \tag{29}$$

The functions $\alpha(r)$ and $A(r)$ satisfy boundary conditions (18)–(19) and (27). Note that boundary condition (27) leads to the magnetic flux quantization

$$\Phi = 2\pi\int B(r) r dr = \frac{2\pi}{g}A_\infty = \frac{4\pi}{g}m, \tag{30}$$

where

$$B(r) = \frac{A'(r)}{gr} \tag{31}$$

is the magnetic field strength of the soliton.

Substituting Eqs. (15)–(17), (22), and (25) into Eqs. (11) and (14), we obtain the expressions for the electromagnetic current density and the energy density in terms of the ansatz functions $\alpha(r)$, $A(r)$:

$$j_0(r) = 0,$$
$$j_r(r) = 0,$$
$$j_\theta(r) = gh\sin^2(\alpha(r))\left(m - \frac{A(r)}{2}\right), \tag{32}$$

$$\mathcal{E}(r) = \frac{A'(r)^2}{2g^2r^2} + h\left(\alpha'(r)^2 + \frac{\sin^2(\alpha(r))}{r^2} \times \left(m - \frac{A(r)}{2}\right)^2 + \lambda\cos^2(\alpha(r))\right). \tag{33}$$

Substituting the power expansions for $\alpha(r)$, $A(r)$ into Eqs. (28)–(29), we obtain the behavior of the solution as $r \to 0$,

$$\alpha(r) = \alpha_m r^m + \alpha_{m+2} r^{m+2} + O(r^{m+4}), \qquad (34)$$

$$A(r) = A_2 r^2 + A_{2m+2} r^{2m+2} + O(r^{2m+4}), \qquad (35)$$

where

$$\alpha_{m+2} = -\frac{\alpha_m (m A_2 + \lambda) + 2\alpha_1^3 \delta_m^1/3}{4(m+1)},$$
$$A_{2m+2} = -\frac{g^2 h}{4} \frac{\alpha_m^2}{m+1}. \qquad (36)$$

It follows from Eqs. (34)–(35) that $A(r)$ is an even function of r, while $\alpha(r)$ is an even function of r for an even m and an odd function of r for an odd m. From Eqs. (32)–(36) we obtain the power expansions for the electromagnetic current and energy densities as $r \to 0$:

$$j_\theta(r) = c_{2m} r^{2m} + c_{2m+2} r^{2m+2} + O(r^{2m+4}), \qquad (37)$$

$$\mathcal{E}(r) = d_0 + d_{2m-2} r^{2m-2} + d_{2m} r^{2m} + O(r^{2m+2}), \qquad (38)$$

where

$$\begin{aligned} c_{2m} &= ghm\alpha_m^2, \\ c_{2m+2} &= gh\alpha_m (2m\alpha_{m+2} - \alpha_m A_2/2), \\ d_0 &= 2A_2^2 g^{-2} + h\lambda, \\ d_{2m-2} &= 2hm^2 \alpha_m^2, \\ d_{2m} &= 4hm(m+1)\alpha_m \alpha_{m+2} - h\alpha_m^2 (mA_2 + \lambda) \\ &\quad + 4(m+1) g^{-2} A_2 A_{2m+2} - \delta_m^1 h \alpha_1^4/3. \end{aligned} \qquad (39)$$

Equations (34)–(39) are valid only for $m > 0$. The corresponding expressions for a negative m can be obtained from the relations

$$\begin{aligned} \alpha(r, -m) &= \alpha(r, m), \\ A(r, -m) &= -A(r, m), \\ j_\theta(r, -m) &= -j_\theta(r, m), \\ \mathcal{E}(r, -m) &= \mathcal{E}(r, m). \end{aligned} \qquad (40)$$

Note that Eqs. (40) follow from the invariance of the Lagrangian (7) under the charge conjugation.

Boundary conditions (19) and (27) lead us to the following asymptotics of $\alpha(r)$ and $A(r)$ as $r \to \infty$:

$$\alpha(r) \sim \frac{\pi}{2} + a_\infty \frac{\exp(-\sqrt{\lambda} r)}{\sqrt{\sqrt{\lambda} r}} \left(1 + \frac{1}{16\sqrt{\lambda} r}\right), \qquad (41)$$

$$A(r) \sim 2m + b_\infty \sqrt{g\sqrt{h} r} \exp\left(-\frac{g\sqrt{h}}{\sqrt{2}} r\right) \times \left(1 - \frac{3}{8\sqrt{2}} \frac{1}{g\sqrt{h} r}\right). \qquad (42)$$

From Eqs. (32)–(33) and (41)–(42), we obtain the asymptotic expressions for the electromagnetic current and energy densities of the soliton as $r \to \infty$:

$$j_\theta(r) \sim -\frac{b_\infty}{2} gh \sqrt{g\sqrt{h} r} \exp\left(-\frac{g\sqrt{h}}{\sqrt{2}} r\right), \qquad (43)$$

$$\mathcal{E}(r) \sim \frac{2a_\infty^2 h \sqrt{\lambda}}{r} \exp(-2\sqrt{\lambda} r) + \frac{b_\infty^2 g h^{\frac{3}{2}}}{2r} \exp(-\sqrt{2h} g r). \qquad (44)$$

It is understood that the asymptotics of the energy density $\mathcal{E}(r)$ is determined by the smallest of the two exponents $2\sqrt{\lambda}$ and $\sqrt{2h}g$ in Eq. (44).

2. *The case* $\beta(r) = \frac{\pi}{2} k$, $k \in \mathbb{Z}$

In this case boundary condition (21) is satisfied identically. Therefore, this condition imposes no restrictions on the function $A(r)$ as $r \to \infty$. However, the finiteness of the soliton energy leads to the Neumann boundary condition for $A(r)$,

$$A'(r) \underset{r \to \infty}{\longrightarrow} 0. \qquad (45)$$

The system of differential equations for the functions $\alpha(r)$ and $A(r)$ can be written as

$$\alpha''(r) + \frac{\alpha'(r)}{r} - \frac{\sin(4\alpha(r))}{4r^2} \left(m - \frac{A(r)}{2}\right)^2 + \frac{\lambda}{2} \sin(2\alpha(r)) = 0, \qquad (46)$$

$$A''(r) - \frac{A'(r)}{r} + \frac{g^2 h}{4} \sin^2(2\alpha(r)) \left(m - \frac{A(r)}{2}\right) = 0. \qquad (47)$$

The functions $\alpha(r)$ and $A(r)$ satisfy boundary conditions (18)–(19) and (45). Now the limiting value of $A(r)$ is nonquantized as $r \to \infty$, as is the magnetic flux of the soliton.

By analogy with Eqs. (32)–(33), we obtain the expressions for the electromagnetic current density and the energy density:

$$\begin{aligned} j_0(r) &= 0, \\ j_r(r) &= 0, \\ j_\theta(r) &= \frac{gh}{4} \sin^2(2\alpha(r)) \left(m - \frac{A(r)}{2}\right), \end{aligned} \qquad (48)$$

$$\mathcal{E}(r) = \frac{A'(r)^2}{2g^2r^2} + h\left(\alpha'(r)^2 + \frac{\sin^2(2\alpha(r))}{4r^2} \times \left(m - \frac{A(r)}{2}\right)^2 + \lambda\cos^2(\alpha(r))\right). \quad (49)$$

As $r \to 0$ the behavior of $\alpha(r)$, $A(r)$, $j_\theta(r)$, and $\mathcal{E}(r)$ is also determined by Eqs. (34)–(40). The only difference is that the formulas for the coefficients α_{m+2} and d_{2m} in Eqs. (36) and (39) must be slightly changed:

$$\alpha_{m+2} = -\frac{\alpha_m(mA_2 + \lambda) + 8\alpha_1^3\delta_m^1/3}{4(m+1)}, \quad (50)$$

$$d_{2m} = 4hm(m+1)\alpha_m\alpha_{m+2} - h\alpha_m^2(mA_2 + \lambda) + 4(m+1)g^{-2}A_2A_{2m+2} - \delta_m^1 4h\alpha_1^4/3. \quad (51)$$

Boundary conditions (19) and (45) lead us now to the following asymptotics of $\alpha(r)$ and $A(r)$ as $r \to \infty$:

$$\alpha(r) \sim \frac{\pi}{2} + a_\infty \frac{\exp(-\sqrt{\lambda}r)}{\sqrt{\sqrt{\lambda}r}} \times \left(1 + \frac{(A_\infty - 2m)^2 - 1}{8}\frac{1}{\sqrt{\lambda}r}\right), \quad (52)$$

$$A(r) \sim A_\infty - \frac{g^2h}{8\lambda}(2m - A_\infty)a_\infty^2 \frac{\exp(-2\sqrt{\lambda}r)}{\sqrt{\lambda}r}. \quad (53)$$

The asymptotic expressions for the electromagnetic current and energy densities follow from Eqs. (32)–(33) and (52)–(53):

$$j_\theta(r) \sim -a_\infty^2 gh(A_\infty - 2m)\frac{\exp(-2\sqrt{\lambda}r)}{2\sqrt{\lambda}r}, \quad (54)$$

$$\mathcal{E}(r) \sim 4a_\infty^2\lambda h\frac{\exp(-2\sqrt{\lambda}r)}{2\sqrt{\lambda}r}. \quad (55)$$

Note that unlike Eqs. (43)–(44), the ratio of energy density (55) to electromagnetic current density (54) tends to some constant as $r \to \infty$:

$$\frac{\mathcal{E}(r)}{j_\theta(r)} \underset{r\to\infty}{\to} \frac{4\lambda}{g(2m - A_\infty)}. \quad (56)$$

B. The case $Q = \frac{1}{2}\lambda_8$, $m_1 = m_2 = m$, $m_3 = 0$

Now the boundary condition (21) vanishes identically, so it imposes no restrictions on the functions $A(r)$ and $\beta(r)$ as $r \to \infty$. The differential equation for $\beta(r)$ is simplified for this case,

$$\beta''(r) + \left(\frac{1}{r} + 2\cot(\alpha(r))\alpha'(r)\right)\beta'(r) = 0, \quad (57)$$

and has the general solution

$$\beta(r) = \beta_0 + \beta_1\int_1^r \frac{d\varrho}{\varrho\sin^2(\alpha(\varrho))}. \quad (58)$$

The finiteness of the soliton energy leads us to boundary condition (45) for $A(r)$ and to the trivial solution $\beta(r) = \beta_0$, where β_0 is an arbitrary constant. The system of differential equations for $\alpha(r)$ and $A(r)$ now is

$$\alpha''(r) + \frac{\alpha'(r)}{r} - \frac{\sin(4\alpha(r))}{4r^2}\left(m - \frac{\sqrt{3}}{2}A(r)\right)^2 + \frac{\lambda}{2}\sin(2\alpha(r)) = 0, \quad (59)$$

$$A''(r) - \frac{A'(r)}{r} + \frac{\sqrt{3}g^2h}{4}\sin^2(2\alpha(r))\left(m - \frac{\sqrt{3}}{2}A(r)\right) = 0, \quad (60)$$

and the expressions for the current and energy densities are

$$j_0(r) = 0,$$
$$j_r(r) = 0,$$
$$j_\theta(r) = \frac{\sqrt{3}gh}{4}\sin^2(2\alpha(r))\left(m - \frac{\sqrt{3}}{2}A(r)\right), \quad (61)$$

$$\mathcal{E}(r) = \frac{A'(r)^2}{2g^2r^2} + h\left(\alpha'(r)^2 + \frac{\sin^2(2\alpha(r))}{4r^2} \times \left(m - \frac{\sqrt{3}}{2}A(r)\right)^2 + \lambda\cos^2(\alpha(r))\right). \quad (62)$$

It is readily seen that the solution of Eqs. (59)–(60) and that of Eqs. (46)–(47) are related to each other. Let us denote the solution of Eqs. (59)–(60) by subscript B and that of Eqs. (46)–(47) by subscript $A2$. Then we can write the relations:

$$A_B(r, g) = \frac{A_{A2}(r, \sqrt{3}g)}{\sqrt{3}},$$
$$\alpha_B(r, g) = \alpha_{A2}(r, \sqrt{3}g). \quad (63)$$

In Eqs. (63) the dependence on the gauge coupling is explicitly shown, and the values of the parameters h, λ, and m are assumed to be equal for both sides of the relations. From Eqs. (48)–(49) and (61)–(63), we obtain similar relations for the electromagnetic current and energy densities:

$$j_{\theta B}(r,g) = j_{\theta A2}(r,\sqrt{3}g),$$
$$\mathcal{E}_B(r,g) = \mathcal{E}_{A2}(r,\sqrt{3}g). \tag{64}$$

The asymptotics of the ansatz functions, the current density, and the energy density follow directly from Eqs. (50)–(55) and (63)–(64). We see that the cases $Q = \lambda_8/2$, $m_1 = m_2 = m, m_3 = 0$, $\beta(r) = \beta_0$ and $Q = \lambda_3/2$, $m_1 = -m_2 = m, m_3 = 0$, $\beta(r) = \pi k/2$ are essentially equivalent to each other.

C. More on properties of the $CP(2)$ solitons

Let us continue the consideration of properties of the topological solitons of the gauged $CP(2)$ model. We begin with the properties that are shared by the soliton with quantized magnetic flux and the soliton with nonquantized magnetic flux. First of all note that Eqs. (33), (49), and (62) for the energy densities can be represented as the sum of the three parts:

$$\mathcal{E}(r) = \mathcal{E}_{\mathrm{em}}(r) + \mathcal{E}_{\mathrm{p}}(r) + \mathcal{E}_{\mathrm{g}}(r), \tag{65}$$

where

$$\mathcal{E}_{\mathrm{em}}(r) = \frac{A'(r)^2}{2g^2r^2} \tag{66}$$

is the electromagnetic part,

$$\mathcal{E}_{\mathrm{p}}(r) = h\lambda\cos^2(\alpha(r)) \tag{67}$$

is the potential part, and the remainder $\mathcal{E}_{\mathrm{g}}(r)$ is the gradient part. Then it follows from Derrick's theorem [2] that on a two-dimensional static soliton solution, the following relation holds between the electromagnetic and potential parts of energy:

$$E_{\mathrm{em}} = E_{\mathrm{p}}, \tag{68}$$

where $E_{\mathrm{em}} = \int \mathcal{E}_{\mathrm{em}} d^2x$ and $E_{\mathrm{p}} = \int \mathcal{E}_{\mathrm{p}} d^2x$.

It follows from Eqs. (29) and (47) that the function $a(r) = m - A(r)/2$ has no positive maxima or negative minima for finite values of r. From this fact it follows easily that $A(r)$ is a monotonically increasing function for $m > 0$ and a monotonically decreasing function for $m < 0$. In the topologically trivial sector with $m = 0$ the function $A(r)$ must monotonically increase if $A(r) > 0$, and monotonically decrease if $A(r) < 0$. It can be shown from Eqs. (29) and (47) that the increase or decrease of $A(r)$ is unbounded for $m = 0$. Hence, boundary conditions (18), (27), and (45) lead us to the conclusion that $A(r)$ must vanish for the topologically trivial solutions. In this case, the only solution of Eqs. (24), (28), (46), (57), and (59) with a finite energy is the vacuum one: $\alpha(r) = \pi/2$, $\beta(r) = \beta_0$, where β_0 is an arbitrary constant.

Systems of differential equations (28)–(29) and (46)–(47) depend on the four parameters: g, h, λ, and m. From the three dimensional parameters we can form the dimensionless combination $\kappa = g^2h/\lambda$. Then it can be shown that the general dependence of the ansatz functions on the parameters g, h, λ, and m can be written as $\alpha(\sqrt{\lambda}r,\kappa,m)$ and $A(\sqrt{\lambda}r,\kappa,m)$. From this general dependence and Eqs. (30)–(31), (33), and (49), we can conclude that the energy density $\mathcal{E}$, the energy E, the magnetic flux Φ, and the magnetic field strength B must have the following general dependence on g, h, λ, and m:

$$\mathcal{E} = h\lambda\tilde{\mathcal{E}}(\sqrt{\lambda}r,\kappa,m), \tag{69}$$

$$E = h\tilde{E}(\kappa,m), \tag{70}$$

$$\Phi = \frac{2\pi}{g}A_\infty(\kappa,m), \tag{71}$$

$$B = \frac{\lambda}{g}\tilde{B}(\sqrt{\lambda}r,\kappa,m), \tag{72}$$

where $\tilde{\mathcal{E}}(\sqrt{\lambda}r,\kappa,m)$, $\tilde{E}(\kappa,m)$, $A_\infty(\kappa,m)$, and $\tilde{B}(\sqrt{\lambda}r,\kappa,m)$ are dimensionless quantities. Note once again that $A_\infty(\kappa,m) = 2m$ for the solitons with quantized magnetic flux. Later we need the explicit expression for dimensionless energy density (69)

$$\tilde{\mathcal{E}}(\rho,\kappa,m) = \frac{A'(\rho)^2}{2\kappa\rho^2} + \alpha'(\rho)^2 + \frac{\sin^2(n\alpha(\rho))}{n^2\rho^2} \times \left(m - \frac{A(\rho)}{2}\right)^2 + \cos^2(\alpha(\rho)), \tag{73}$$

where $\rho = \sqrt{\lambda}r$, $n = 1$ for Eq. (33), $n = 2$ for Eq. (49), and the dependence of the ansatz functions on κ and m is not shown.

Let the ansatz functions $\alpha(\rho)$ and $A(\rho)$ satisfy field equations (28)–(29) [or field equations (46)–(47)], and variations $\delta\alpha(\rho)$ and $\delta A(\rho)$ of these functions vanish as $\rho \to 0$ and as $\rho \to \infty$. The first variation of the energy functional $\tilde{E} = 2\pi\int\tilde{\mathcal{E}}(\rho,\kappa,m)\rho d\rho$ vanishes on these variations. Now we consider variations of $\alpha(\rho)$ and $A(\rho)$ that correspond to a variation of the dimensionless parameter κ:

$$\delta_\kappa\alpha(\rho,\kappa) = \frac{\partial\alpha(\rho,\kappa)}{\partial\kappa}\delta\kappa,$$
$$\delta_\kappa A(\rho,\kappa) = \frac{\partial A(\rho,\kappa)}{\partial\kappa}\delta\kappa. \tag{74}$$

For the topological soliton with quantized magnetic flux the variations $\delta_\kappa\alpha(\rho,\kappa)$ and $\delta_\kappa A(\rho,\kappa)$ vanish as $\rho \to 0$ and $\rho \to \infty$ because of boundary conditions (18)–(19) and (27). Thus the corresponding first variation of the energy functional $\tilde{E}$ vanishes in this case. However for the topological soliton with nonquantized magnetic flux the variation $\delta_\kappa A(\rho,\kappa)$ does not vanish as $\rho \to \infty$, because boundary condition (45) does not fix the value of $A(\rho,\kappa)$ at spatial infinity. In this case, the corresponding first variation of the energy functional can be written as

$$\delta_\kappa \tilde{E} = \lim_{\rho\to\infty} \frac{\partial A(\rho,\kappa)}{\rho\partial\rho}\frac{\partial A(\rho,\kappa)}{\kappa\partial\kappa}\delta\kappa. \tag{75}$$

It follows from Eq. (53) that the first factor in Eq. (75) vanishes exponentially as $\rho \to \infty$. Thus we conclude that the first variation $\delta_\kappa \tilde{E}$ vanishes on field configurations of the topological soliton with nonquantized magnetic flux. Note that the factor κ in the denominator of the first term in Eq. (73) is kept fixed when calculating the first variation $\delta_\kappa \tilde{E}$. Hence the first derivative of $\tilde{E}$ with respect to κ can be written as

$$\begin{aligned}\frac{d\tilde{E}}{d\kappa} &= -2\pi\kappa^{-2}\int \frac{A'(\rho)^2}{2\rho}d\rho + \frac{\delta_\kappa \tilde{E}}{\delta\kappa} \\ &= -\kappa^{-1}\tilde{E}_{\text{em}}(\kappa),\end{aligned} \tag{76}$$

where

$$\tilde{E}_{\text{em}}(\kappa) = 2\pi \int \frac{A'(\rho)^2}{2\kappa\rho}d\rho \tag{77}$$

is the electromagnetic part of the dimensionless soliton energy $\tilde{E}$. It follows from Eqs. (76)–(77) that the energy of the topological solitons of both types decreases monotonically with the increase of κ.

By making use of Eq. (29) or (47) we can obtain two integral relations for the soliton solutions. The first integral relation is

$$A_\infty = \frac{g}{2}\int_0^\infty j_\theta(r) r dr. \tag{78}$$

Equation (78) relates the limiting value of $A(r)$ at spatial infinity to the first moment of the covariant θ-component of the current density. The second integral relation is

$$A''|_{r=0} = g\int_0^\infty \frac{j_\theta}{r}dr = -gI. \tag{79}$$

Equation (79) establishes the relationship between the second derivative of $A(r)$ at $r=0$ and the first inverse moment of the covariant θ-component of the current density. Note that the integral $-\int_0^\infty j_\theta/r dr$ is equal to the electromagnetic current I flowing perpendicular to the half-line $\theta = \text{const}$, $r \in [0,\infty)$. From Eqs. (28) and (46) we can obtain the third integral relation

$$\int_0^\infty \left(X(r) + \frac{\lambda}{2}\sin(2\alpha(r))\right) r dr = 0, \tag{80}$$

where

$$X(r) = -\frac{\sin(k\alpha(r))}{kr^2}\left(m - \frac{A(r)}{2}\right)^2. \tag{81}$$

The integer number k is equal to 2 for Eq. (28) and 4 for Eq. (46).

If the values of the parameters g, h, λ, and m are fixed, then the behavior of the solution $\alpha(r)$, $A(r)$ as $r \to 0$ is determined by the two parameters α_m and A_2 in Eqs. (34)–(35). The behavior of the solution $\alpha(r)$, $A(r)$ as $r \to \infty$ is also determined by the two parameters a_∞ and b_∞ in Eqs. (41)–(42), or a_∞ and A_∞ in Eqs. (52)–(53). Thus we have the four free parameters in all. The continuity condition for $\alpha(r)$ and $A(r)$ and their derivatives $\alpha'(r)$ and $A'(r)$ at arbitrary r give us four equations. Therefore, we have the four equations for determining the four parameters: α_m, A_2, a_∞, and b_∞ (or α_m, A_2, a_∞, and A_∞). According to [33], this fact is an argument in favor of the existence of the solution for the boundary value problem in some range of the parameters g, h, λ, and m.

It can be shown that the topological $CP(2)$ solitons of two types are not separated by a topological barrier of infinite energy. Indeed, a continuous transition from the soliton with quantized magnetic flux to the soliton with nonquantized magnetic flux can be realized in two steps. At the first step the limiting value A_∞ of the ansatz function $A(r)$ is set equal to its quantized value $2m$, while the constant ansatz function $\beta(r)$ varies continuously from $\pi/4 + \pi k/2$ to $\pi l/2$, where k and l are integers. At the second step the ansatz function $\beta(r)$ is set equal to $\pi l/2$, while the limiting value A_∞ varies continuously from $2m$ to some nonquantized value corresponding to the topological soliton with nonquantized magnetic flux. At any point of this continuous transition the transverse part of the covariant derivative $\mathcal{P}_{ab}(D_i\phi_b)$ vanishes at spatial infinity. Consequently, the static energy of a field configuration is finite at any point of the transition, so there is no topological barrier between the $CP(2)$ solitons of two types. However, the topological $CP(2)$ solitons of two types are separated by a kinetic barrier of infinite energy. Indeed, at the second step of transition the gauge field at large r can be written as

$$A_0 = 0, \qquad A_i = -A^i \underset{r\to\infty}{\longrightarrow} -\frac{2m}{gr}\epsilon^{ij}n^j\xi(t), \tag{82}$$

where $\xi(t)$ is a function of time varying from 1 to some fixed value. From Eq. (82) we obtain the electric field strength $E^i = -\partial_t A^i$ and the energy density of the electric field

$$\mathcal{E}_E(r)\underset{r\to\infty}{\longrightarrow}\frac{2m^2}{g^2r^2}(\partial_t\xi)^2. \qquad (83)$$

If the transition occurs within a finite period of time, then the time derivative $\partial_t\xi$ is different from zero. Then the integral $2\pi\int\mathcal{E}_E(r)rdr$ diverges logarithmically at the upper limit. Hence there is a kinetic barrier of infinite energy, so the topological soliton of one type cannot decay into the topological soliton of the other type.

Now we consider the properties that are not common for the topological solitons of two types. From Eqs. (28)–(29) it follows that the regions, where the functions $\alpha(r)$ and $A(r)$ are appreciably different from their limiting values $\pi/2$ and $2m$, are of the order $1/\sqrt{\lambda}$ and $1/(g\sqrt{h})$, respectively. Let $\sqrt{\lambda}$ be much greater than $g\sqrt{h}$. In this case the soliton with quantized magnetic flux has a small scalar core and a large vector core. Then for $r\gtrsim 1/\sqrt{\lambda}$ we can replace $\alpha(r)$ in Eq. (29) by its limiting value $\pi/2$:

$$A''(r)-\frac{A'(r)}{r}+g^2h\left(m-\frac{A(r)}{2}\right)=0. \qquad (84)$$

Equation (84) is the equation for the free massive vector field with the mass $m_A=g\sqrt{h/2}$, which has the solution

$$A(r)=2m(1-m_A r\mathrm{K}_1(m_A r)), \qquad (85)$$

where K_1 is the modified Bessel function of the second kind of order 1. Note that this solution satisfies boundary conditions (18) and (27). Substituting Eqs. (85) and $\alpha(r)=\pi/2$ into Eq. (33) for $\mathcal{E}(r)$ and integrating the expression $2\pi r\mathcal{E}(r)$ from $\sim 1/\sqrt{\lambda}$ to ∞, we obtain the expression for the soliton energy with logarithmic accuracy

$$E\approx-\pi hm^2\ln(\kappa)+O(1). \qquad (86)$$

We see that the energy of the soliton with quantized magnetic flux diverges logarithmically as $\kappa=g^2h/\lambda\to 0$. It follows from Eqs. (68), (70), (76), and (86) that the electromagnetic and potential parts E_{em} and E_{p} of the soliton energy E tend to the constant πhm^2 as $\kappa\to 0$. Note that as κ tends to zero, the energy of the soliton with winding number m is proportional to m^2, while the energy of the m widely separated solitons with winding numbers 1 is proportional to m. Therefore the soliton with quantized magnetic flux is unstable to the decay $m\to m_1+m_2+\ldots,\sum_i m_i=m$ as $\kappa\to 0$.

It can be shown that for $\lambda=g^2h/2$ the energy of the soliton with quantized magnetic flux can be written in Bogomolny-Prasad-Sommerfield (BPS) form

$$\begin{aligned}E&=2\pi\int\frac{1}{2}\left(\frac{A'(r)}{gr}\pm gh\cos(\alpha(r))\right)^2rdr\\&\quad+2\pi h\int\left(\alpha'(r)\pm\frac{\sin(\alpha(r))}{r}\left(m-\frac{A(r)}{2}\right)\right)^2rdr\\&\quad+4\pi|m|h.\end{aligned} \qquad (87)$$

From Eq. (87), we read off Bogomolny equations for the ansatz functions of the soliton:

$$\alpha'(r)\mp\frac{\sin(\alpha(r))}{r}\left(m-\frac{A(r)}{2}\right)=0, \qquad (88)$$

$$A'(r)\mp g^2hr\cos(\alpha(r))=0. \qquad (89)$$

The upper and lower signs in Eqs. (88)–(89) correspond to the cases $m>0$ and $m<0$. It can be shown that if the ansatz functions $\alpha(r)$ and $A(r)$ satisfy Eqs. (88)–(89), then these functions also satisfy field equations (28)–(29). Consequently, for $\lambda=\lambda_{\mathrm{BPS}}=g^2h/2$, the energy of the soliton with quantized magnetic flux is equal to

$$E_{\mathrm{BPS}}=4\pi|m|h \qquad (90)$$

in the topological sector with a given m. Expressions (87)–(90) are valid only for $\lambda=\lambda_{\mathrm{BPS}}=g^2h/2$ (i.e., for $\kappa=\kappa_{\mathrm{BPS}}=2$). Now let us consider the case $\lambda\neq\lambda_{\mathrm{BPS}}$. For the potential part of the energy density $\mathcal{E}_{\mathrm{p}}=h\lambda\cos^2(\alpha(r))$ we use the representation $\mathcal{E}_{\mathrm{p}}=h\lambda_{\mathrm{BPS}}\cos^2(\alpha(r))+h(\lambda-\lambda_{\mathrm{BPS}})\cos^2(\alpha(r))$ for $\lambda>\lambda_{\mathrm{BPS}}$, and $\mathcal{E}_{\mathrm{p}}=h\lambda_{\mathrm{BPS}}(\lambda/\lambda_{\mathrm{BPS}})\cos^2(\alpha(r))$ for $\lambda<\lambda_{\mathrm{BPS}}$. By completing the squares in the expression $E=2\pi\int\mathcal{E}(r)rdr$ similarly to Eq. (87) and discarding some positive terms, we obtain the inequality for the energy of the soliton with quantized magnetic flux in the topological sector with a given m:

$$E\geq\begin{cases}4\pi h|m| & \text{if } \kappa\leq 2\\ \frac{8\pi h}{\kappa}|m| & \text{if } \kappa>2.\end{cases} \qquad (91)$$

Inequality (91) is saturated at $\kappa=\kappa_{\mathrm{BPS}}=2$. Note also that there is no BPS representation similar to Eq. (87) for the energy of the soliton with nonquantized magnetic flux.

IV. NUMERICAL RESULTS

The system of differential equations (28)–(29) with boundary conditions (18)–(19) and (27) is the boundary value problem with Dirichlet boundary conditions, while the system of differential equations (46)–(47) with boundary conditions (18)–(19) and (45) is the mixed boundary value problem. Both problems are defined on the semi-infinite interval $r\in[0,\infty)$ and can be solved only by numerical methods. In this paper, these boundary value problems were solved using the MAPLE package [34] by the method of finite differences and subsequent Newtonian iterations. The point $r = 0$ is the regular singular point for both problems, so we applied difference schemes that do not use the boundary values of the functions. Richardson extrapolation was used to accelerate the convergence of the numerical procedure to the exact solution. Equations (68), (78), and (80) were used to check the correctness of the numerical solutions.

The boundary value problems depend on the parameters g, h, λ, and m. The general dependence of the ansatz functions and basic physical quantities on these parameters is given by Eqs. (69)–(72). In the present paper, numerical results are presented for the soliton with quantized magnetic flux and the soliton with nonquantized magnetic flux in the topological sector with $m = 1$. We use the dimensionless ansatz functions α and A, the dimensionless quantities $\tilde{\mathcal{E}}$, $\tilde{E}$, A_∞, and $\tilde{B}$, and the dimensionless radial variable $\rho = \sqrt{\lambda} r$ for presenting numerical results.

At first, let us present the numerical results for the soliton with quantized magnetic flux. Figure 1 shows the numerical solution for the ansatz functions $\alpha(\rho)$ and $A(\rho)$, corresponding to the dimensionless parameter $\kappa = 1$. The functions $\alpha(\rho)$ and $A(\rho)$ have the typical vortex form. Their behavior at small and large distances corresponds to asymptotic expressions (34)–(35) and (41)–(42). Note, in particular, that $\alpha(\rho) \propto \rho$ and $A(\rho) \propto \rho^2$ as $\rho \to 0$. Note also that the functions $\alpha(\rho)$ and $A(\rho)$ tend exponentially to their limiting values $\pi/2$ and 2 as $\rho \to \infty$. Figure 2 shows the energy density $\tilde{\mathcal{E}}(\rho)$ and the magnetic field strength $\tilde{B}(\rho)$, corresponding to the solution in Fig. 1. Let us consider the behavior of $\tilde{\mathcal{E}}(\rho)$ and $\tilde{B}(\rho)$ at small ρ. From Eqs. (31), (35), and (72) we obtain the power expansion for $\tilde{B}(\rho)$ as $\rho \to 0$,

$$\tilde{B}(\rho) = \frac{2A_2}{\lambda} - \frac{\alpha_m^2 \kappa}{2\lambda^m} \rho^{2m} + O(\rho^{2m+2}). \tag{92}$$

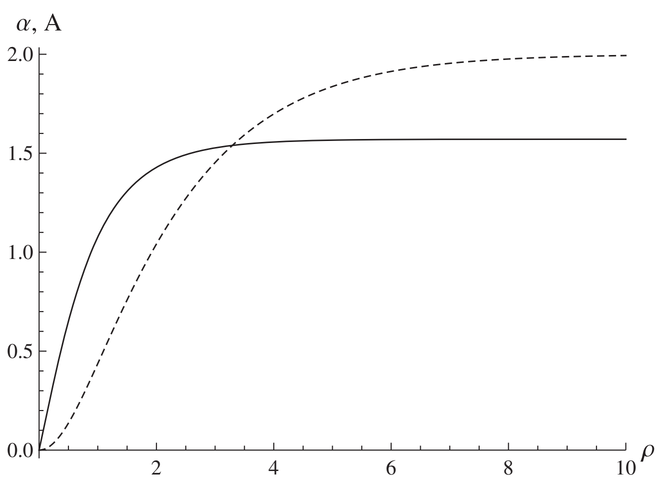

FIG. 1. The numerical solution $\alpha(\rho)$, $A(\rho)$ for the soliton with quantized magnetic flux, corresponding to $\kappa = 1$. The solid curve is for $\alpha(\rho)$; the dashed curve is for $A(\rho)$.

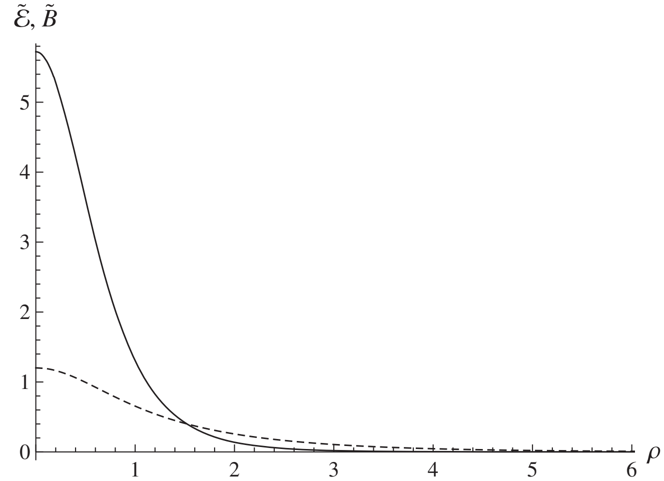

FIG. 2. The dependence of the energy density $\tilde{\mathcal{E}}$ and the magnetic field strength $\tilde{B}$ of the soliton with quantized magnetic flux on ρ. The solid curve is for $\tilde{\mathcal{E}}$; the dashed curve is for $\tilde{B}$. The parameter κ is the same as in Fig. 1.

Note that the coefficient A_2 in Eqs. (35) and (92) is always positive, since $A(r)$ is a monotonically increasing function for $m > 0$. Thus for $m > 0$ the magnetic field strength $\tilde{B}(\rho)$ has a positive maximum at $\rho = 0$, since the parameters g, h, λ, and the coefficient A_2 are all positive. Then it follows from Eqs. (40) and (92) that for $m < 0$ the magnetic field strength $\tilde{B}(\rho)$ has a negative minimum at $\rho = 0$. From Eqs. (38)–(40) and (69) it follows that for $|m| \geq 2$ the energy density $\tilde{\mathcal{E}}(\rho)$ has a positive minimum at $\rho = 0$, since the coefficient d_{2m-2} in Eq. (38) is positive in this case. However, the situation is quite different for $|m| = 1$. Indeed, from Eqs. (38)–(39) and (69) we can obtain the power expansion of $\tilde{\mathcal{E}}(\rho)$ for $m = 1$,

$$\tilde{\mathcal{E}}(\rho) = 1 + 2\frac{\alpha_1^2}{\lambda} + \frac{2}{\kappa}\frac{A_2^2}{\lambda^2} - \frac{\alpha_1^2}{\lambda}\left(2 + \frac{\alpha_1^2}{\lambda} + 3\frac{A_2}{\lambda}\right)\rho^2 + O(\rho^4). \tag{93}$$

From Eq. (40) it follows that Eq. (93) is valid also for $m = -1$. Thus we see from Eq. (93) that the energy density $\tilde{\mathcal{E}}(\rho)$ has a positive maximum at $\rho = 0$ for $|m| = 1$. We see also from Fig. 2 that in the neighborhood of $\rho = 0$ the behavior of $\tilde{B}(\rho)$ and $\tilde{\mathcal{E}}(\rho)$ corresponds to Eqs. (92)–(93). Note that this behavior of $\tilde{B}(\rho)$ and $\tilde{\mathcal{E}}(\rho)$ is similar to that of the classical vortex solution [8,9].

Figure 3 presents the dependence of the dimensionless energy $\tilde{E}$ of the soliton with quantized magnetic flux on the logarithm of the dimensionless combination $\kappa = g^2 h/\lambda$. The soliton energy decreases monotonically with the increase of κ. It was checked numerically that the dependence $\tilde{E}(\kappa)$ satisfies Eq. (76). Also it was found that the electromagnetic and potential parts of the soliton energy $\tilde{E}$

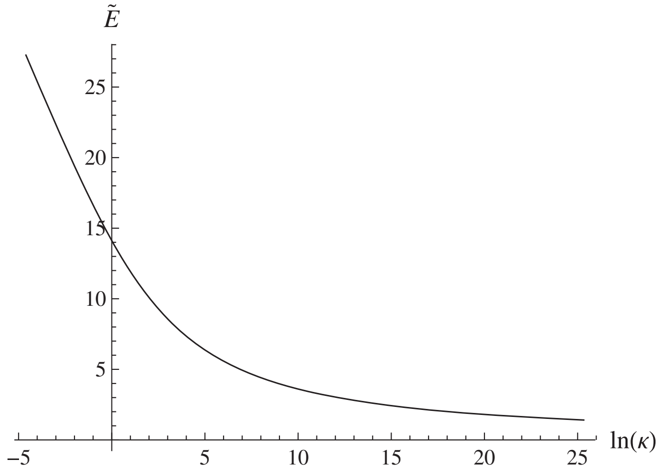

FIG. 3. The energy $\tilde{E}$ of the soliton with quantized magnetic flux as a function of the logarithm of the dimensionless combination κ.

tend to the constant π as $\kappa \to 0$ in accordance with Eqs. (76) and (86). It follows from Fig. 3 that $d\tilde{E}/d\ln(\kappa) \approx -\pi$ for small values of κ. This fact is also in accordance with Eq. (86). It follows from Eq. (76) that the electromagnetic part $\tilde{E}_{\rm em}$ of the soliton energy $\tilde{E}$ must tend to zero as $\kappa \to \infty$. Moreover, $\tilde{E}_{\rm em}$ must tend to zero faster than $\ln^{-1}(\kappa)$. Indeed, if $\tilde{E}_{\rm em}$ tends to some nonzero positive value or $\tilde{E}_{\rm em}$ tends to zero like $\ln^{-1}(\kappa)$ or more slowly, then according to Eq. (76) the soliton energy $\tilde{E}$ becomes negative and diverges logarithmically as $\kappa \to \infty$. If $\tilde{E}_{\rm em}$ tends to zero faster than $\ln^{-1}(\kappa)$, then the soliton energy $\tilde{E}$ can be written as

$$\tilde{E}(\kappa) \underset{\kappa\to\infty}{\longrightarrow} \tilde{E}_\infty + \Delta\tilde{E}(\kappa), \qquad (94)$$

where $\tilde{E}_\infty$ is some non-negative constant and $\Delta\tilde{E}(\kappa)$ is some positive function of κ that tends to zero as $\kappa \to \infty$. The exact determination of $\tilde{E}_\infty$ becomes problematic because the function $\Delta\tilde{E}(\kappa)$ decreases slowly and there are difficulties in the numerical solution of the boundary value problem for values of $\kappa \gtrsim 10^{12}$. The existence of the nonzero limiting value $\tilde{E}_\infty$ can be shown as follows. Since $\tilde{E}_{\rm p} = \tilde{E}_{\rm em}$, the potential part $\tilde{E}_{\rm p}$ of the soliton energy $\tilde{E}$ tends to zero as $\kappa \to \infty$. It follows from this that the variation of the ansatz function $\alpha(\rho)$ from 0 to $\pi/2$ is concentrated in a small interval $[0, \bar{\rho})$, where $\bar{\rho}$ tends to zero as $\kappa \to \infty$. Then we can approximate $\alpha(\rho)$ by the first term in its power expansion: $\alpha(\rho) \approx \alpha_m \rho^m$, where $\alpha_m \sim \bar{\rho}^{-m}$. Substituting this approximation into Eq. (73) and integrating the term $\alpha'(\rho)^2$ over a disk of radius $\bar{\rho}$, we obtain an approximate estimate for the gradient part of the soliton energy: $\tilde{E}_{\rm g} \sim m$. We see that $\tilde{E}_{\rm g}$ does not vanish as $\kappa \to \infty$. Thus we can conclude that the soliton energy $\tilde{E}$ does not vanish as $\kappa \to \infty$. From Eq. (90) it follows that the energy $\tilde{E}$ of the soliton with a given m is equal to the BPS value $4\pi|m|$ for $\kappa = \kappa_{\rm BPS} = 2$. Indeed, from Fig. 3 we see that $\tilde{E} = 4\pi \approx 12.57$ for $\ln(\kappa) = \ln(2) \approx 0.69$.

Now we present the numerical results for the soliton with nonquantized magnetic flux in the topological sector with $m = 1$. Figure 4 shows the numerical solution for the ansatz functions $\alpha(\rho)$ and $A(\rho)$, corresponding to the dimensionless parameter $\kappa = 60$. Note that the limiting value A_∞ in Fig. 4 is not equal to the quantized value 2 as it is in Fig. 1. Consequently the magnetic flux of the soliton is nonquantized. The behavior of $\alpha(\rho)$ and $A(\rho)$ at small and large distances corresponds to asymptotic expressions (34)–(35) and (52)–(53). Figure 5 presents the energy density $\tilde{\mathcal{E}}(\rho)$ and the magnetic field strength $\tilde{B}(\rho)$, corresponding to the solution in Fig. 4. As well as for the soliton with quantized magnetic flux, the power expansion of $\tilde{B}(\rho)$ is given by Eq. (92). Hence, at $\rho = 0$ the magnetic field strength of the soliton with nonquantized magnetic flux has a positive maximum for $m > 0$ and a negative minimum for $m < 0$. For $m = 1$ the power expansion for the energy density of the soliton with nonquantized magnetic flux is slightly modified compared to Eq. (93):

$$\begin{aligned}\tilde{\mathcal{E}}(\rho) = 1 + 2\frac{\alpha_1^2}{\lambda} + \frac{2}{\kappa}\frac{A_2^2}{\lambda^2} \\ - \frac{\alpha_1^2}{\lambda}\left(2 + 4\frac{\alpha_1^2}{\lambda} + 3\frac{A_2}{\lambda}\right)\rho^2 + O(\rho^4). \qquad (95)\end{aligned}$$

We see that the coefficient of ρ^2 in Eq. (95) is negative as it is for the soliton with quantized magnetic flux. Hence we conclude that at $\rho = 0$ the energy density of the soliton with nonquantized magnetic flux has a minimum for $|m| \geq 2$ and a maximum for $|m| = 1$. We see also from Fig. 5 that in the neighborhood of $\rho = 0$ the behavior of $\tilde{B}(\rho)$ and $\tilde{\mathcal{E}}(\rho)$ corresponds to Eqs. (92) and (95).

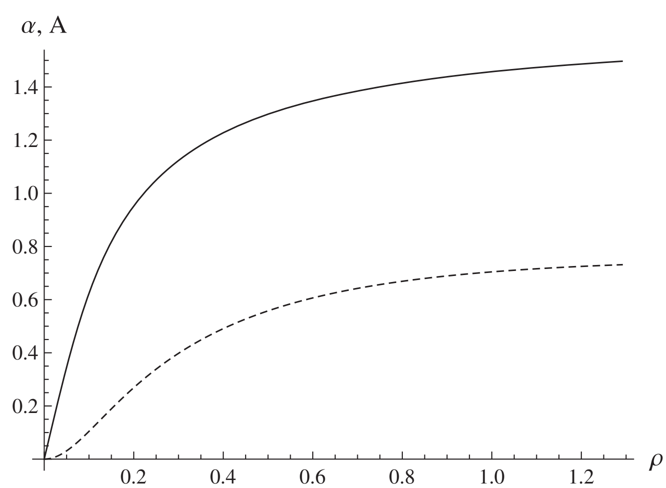

FIG. 4. The numerical solution $\alpha(\rho)$, $A(\rho)$ for the soliton with nonquantized magnetic flux, corresponding to $\kappa = 60$. The solid curve is for $\alpha(\rho)$; the dashed curve is for $A(\rho)$.

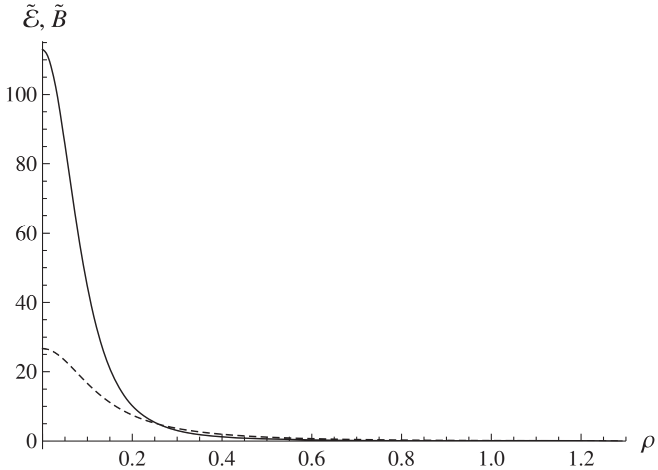

FIG. 5. The dependence of the energy density $\tilde{\mathcal{E}}$ and the magnetic field strength $\tilde{B}$ of the soliton with nonquantized magnetic flux on ρ. The solid curve is for $\tilde{\mathcal{E}}$; the dashed curve is for $\tilde{B}$. The parameter κ is the same as in Fig. 4.

In Fig. 6 we can see the dependence of the dimensionless energy $\tilde{E}$ of the soliton with nonquantized magnetic flux on the logarithm of the dimensionless combination $\kappa = g^2h/\lambda$. The dependence $\tilde{E}$ on $\ln(\kappa)$ is presented in the range from the minimum value of κ to its maximum value that we managed to reach by numerical methods. In Fig. 6 the minimum value $\kappa_{\min} \approx 50.9$. It was found numerically that the mixed boundary value problem does not have solutions as $\kappa < \kappa_{\min}$. We see later that $dA_\infty(\kappa)/d\kappa \to \infty$ as κ tends to $\kappa_{\min}$. This fact also indicates that there are no solutions of the mixed boundary value problem as $\kappa < \kappa_{\min}$. The absence of solutions at small κ can be explained as follows. The limit $\kappa \to 0$ can be achieved as $g \to 0, h = \text{const}, \lambda = \text{const}$. In this limit the electromagnetic energy E_{em} of the soliton with nonquantized magnetic flux must tend to zero. It follows from Eq. (68) that the potential part E_{p} of the soliton energy must also tend to zero in this limit. Hence we conclude that for g sufficiently small the function $A(r)$ becomes small for all r and the variation of the function $\alpha(r)$ from 0 to $\pi/2$ is concentrated in a small neighborhood of $r = 0$. Then in Eqs. (80)–(81) we can neglect the term $A(r)/2$ and approximate $\alpha(r)$ by the first term in its power expansion: $\alpha(r) \approx \alpha_m r^m$. Because the variation of $\alpha(r)$ is concentrated in a small neighborhood of $r = 0$, the coefficient α_m increases indefinitely as $g \to 0, h = \text{const}, \lambda = \text{const}$. But it can be easily shown that Eq. (80) cannot be satisfied for sufficiently large α_m. Consequently, there are no solutions of the mixed boundary value problem for sufficiently small κ. Arguments based on continuity lead us to the conclusion that solutions of the mixed boundary value problem do not exist in some interval $[0, \kappa_{\min})$.

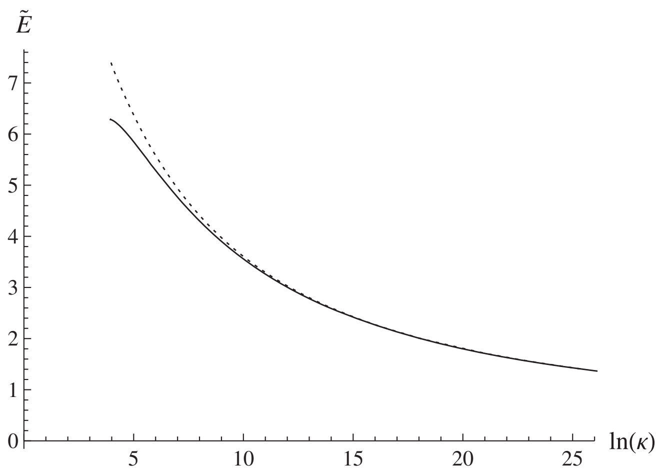

FIG. 6. The energy $\tilde{E}$ of the soliton with nonquantized magnetic flux (solid line) as a function of the logarithm of the dimensionless combination κ. The dotted line corresponds to the energy of the soliton with quantized magnetic flux.

It follows from Fig. 6 that the energy $\tilde{E}$ decreases monotonically with the increase of κ. It was checked numerically that the dependence $\tilde{E}(\kappa)$ satisfies Eq. (76). From $\ln(\kappa) \approx 10$ the dependence $\tilde{E}(\kappa)$ for the soliton with nonquantized magnetic flux does not differ visually from that shown in Fig. 3 for the soliton with quantized magnetic flux. This fact can be explained as follows. The limit $\kappa \to \infty$ can be achieved as $g \to \infty, h = \text{const}, \lambda = \text{const}$. In this case differential equations (29) and (47) will contain the large factor g^2 in the terms $g^2h\sin^2(\alpha(r))(m - A(r)/2)$ and $(g^2h/4)\sin^2(2\alpha(r))(m - A(r)/2)$, respectively. In order to compensate this factor, the function $A(r)$ must reach the values in a small neighborhood of $2m$ at small r. Note that the function $A(r)$ of the soliton with nonquantized magnetic flux also reaches the values close to the quantized value $2m$. It is easily seen that for small r, when $\alpha(r) \ll 1$, systems of differential equations (28)–(29) and (46)–(47) are close to each other. As r increases, the function $A(r)$ quickly reaches the values close to $2m$ and the differing terms are suppressed in Eqs. (28) and (46) [and in Eqs. (29) and (47)]. Thus systems of differential equations (28)–(29) and (46)–(47) are close to each other for all r. Therefore, the solution $\alpha(r)$, $A(r)$ for the soliton with nonquantized magnetic flux tends to that of the soliton with quantized magnetic flux as $g \to \infty, h = \text{const}, \lambda = \text{const}$. This fact was checked numerically. Expressions (33) and (49) for the energy densities are also close to each other for all r as $g \to \infty, h = \text{const}, \lambda = \text{const}$. Therefore the energies of the two solitons tend to each other as $\kappa \to \infty$. Note that the energy of the soliton with nonquantized magnetic flux is smaller than that of the soliton with quantized magnetic flux for any given κ. Nevertheless, the soliton with quantized magnetic flux is stable to the transition into the soliton with nonquantized magnetic flux for any given parameters g, h, and λ. This is because these solitons are separated by a kinetic barrier of infinite energy (see Sec. III C).

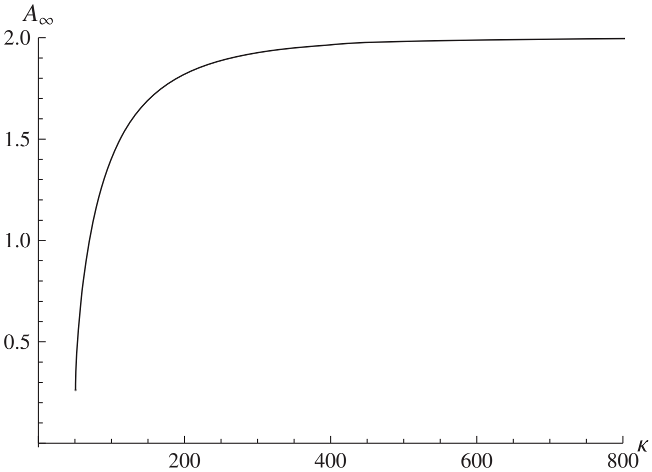

FIG. 7. The dependence of the limiting value A_∞ of the soliton with nonquantized magnetic flux on the dimensionless combination κ.

Figure 7 shows the dependence of the limiting value A_∞ on κ for the soliton with nonquantized magnetic flux in the topological sector with $m = 1$. The dependence is presented in the range of κ from the minimum value 50.9 (as in Fig. 6) to 800. We see that $dA_\infty(\kappa)/d\kappa \to \infty$ as κ tends to its minimum value κ_{min}. We see also that as κ is increased, A_∞ increases monotonically to its quantized value 2. In particular, from $\kappa \approx 500$ the value of A_∞ is virtually indistinguishable from 2. Thus the increase of the gauge coupling constant g (or the decrease of the self-interaction constant λ) leads to an effective quantization of the magnetic flux of the soliton. The analogous effect was described in [11] for the Skyrme-Maxwell soliton in $2 + 1$ dimensions. Note that we chose the value of κ equal to 60 in Fig. 4, because the corresponding value of A_∞ differs significantly from the quantized value 2.

Let us consider the value $\tilde{\Delta}(\kappa, m) = \tilde{E}(\kappa, m) - m\tilde{E}(\kappa, 1)$. If $\tilde{\Delta}(\kappa, m) > 0$ ($\tilde{\Delta}(\kappa, m) < 0$), then the soliton with given m and κ is unstable (stable) to the decay $m \to m_1 + m_2 + \ldots, \sum_i m_i = m$. Figure 8 presents the dependence $\tilde{\Delta}(\kappa, m)$ on m for the solitons with quantized magnetic flux. The dependence is presented for several values of κ. It follows from Fig. 8 that the solitons are unstable to the decay if $\kappa < 2$ and stable to it if $\kappa > 2$. If $\kappa = \kappa_{\mathrm{BPS}} = 2$, then we see from Fig. 8 that $\tilde{E}(2, m) = m\tilde{E}(2, 1)$ in accordance with Eq. (90). In this case there are no intersoliton forces between the separated solitons. Note that all these properties of the $CP(2)$ soliton with quantized magnetic flux are exactly analogous to those of the classical vortex [8,9]. Figure 9 presents the dependence $\tilde{\Delta}(\kappa, m)$ on m for the solitons with nonquantized magnetic flux. It was found that these solitons are stable to the decay $m \to m_1 + m_2 + \ldots, \sum_i m_i = m$ for all κ that have been achieved numerically. Figure 9 illustrates this fact for several values of κ.

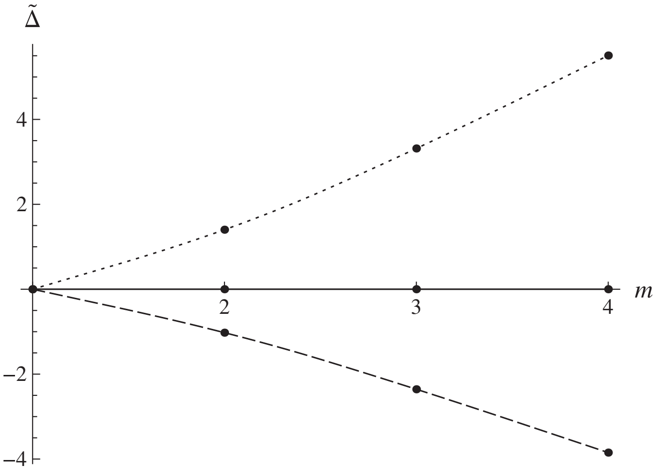

FIG. 8. The dependence of $\tilde{\Delta}$ on m for the soliton with quantized magnetic flux. The dotted, solid, and dashed lines correspond to $\kappa = 1, 2, 4$, respectively.

In conclusion of this section we discuss the possibility of the existence of excited soliton solutions in the topological sector with a given m. Self-interaction potential (15) vanishes at $\phi_3 = 0$. This value of ϕ_3 corresponds to $\alpha = \pi/2 + \pi k$, $k \in \mathbb{Z}$ in ansatz (16) for the normalized $CP(2)$ field ϕ. Boundary condition (19) and the topological solitons considered here correspond to $k = 0$. Note that systems of differential equations (28)–(29) and (46)–(47) are invariant under the discrete transformation $\alpha(r) \to -\alpha(r)$. The expressions (32)–(33) and (48)–(49) for the Noether current density and the energy density are also invariant under this transformation. We can therefore be sure that there are topological solitons corresponding to $k = -1$ in the model. The energy, magnetic flux, and other properties of the solitons with $k = -1$ are completely analogous to those of the solitons with $k = 0$. The following question naturally arises: Are there solutions corresponding to

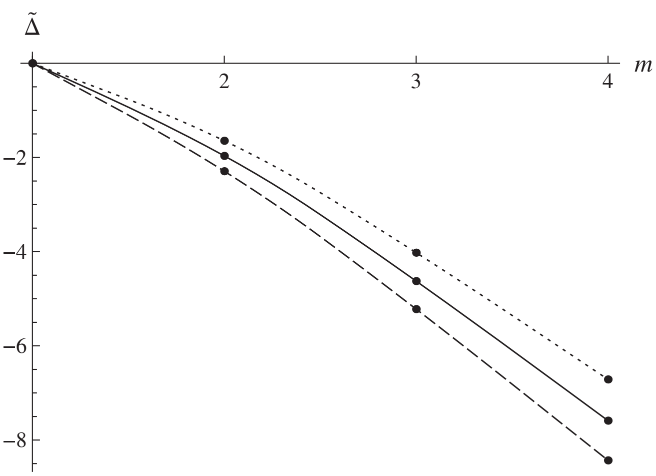

FIG. 9. The dependence of $\tilde{\Delta}$ on m for the soliton with nonquantized magnetic flux. The dotted, solid, and dashed lines correspond to $\kappa = 67, 100, 200$, respectively.

$k \neq -1$, 0 If such solutions exist, they must correspond to excited solitons in the topological sector with a given m. Let us consider the soliton with quantized magnetic flux at $\kappa = \kappa_{\rm BPS} = 2$. In this case the soliton solution $\alpha(r)$, $A(r)$ must satisfy Bogomolny equations (88)–(89) as well as field equations (28)–(29). Suppose that there is a BPS soliton solution that satisfies the boundary conditions, $\alpha(0) = 0$, $\alpha(\infty) = \pi/2 + \pi k$, where k is an integer other than -1 and 0. Let $\bar{r}$ be the value of the radial variable r such that $\alpha(\bar{r}) = \pi/2$. Then it follows from Eq. (89) that at $r = \bar{r}$ the function $A(r)$ must have a maximum for $m > 0$ and a minimum for $m < 0$. However, it is shown in Sec. III C that $A(r)$ monotonically increases for $m > 0$ and monotonically decreases for $m < 0$. Thus, our assumption that $\alpha(r)$ can reach the value $\pm\pi/2$ at finite r leads us to the contradiction. We conclude that for $\kappa = 2$ there are no soliton solutions with quantized magnetic flux such that $\alpha(\infty) \neq \pm\pi/2$. Arguments based on continuity lead us to the conclusion that at least in the neighborhood of $\kappa = 2$ these solutions also do not exist. We did not find soliton solutions with quantized magnetic flux or those with nonquantized magnetic flux such that $\alpha(\infty) \neq \pm\pi/2$ by numerical methods for any value of κ.

V. CONCLUSION

In the present paper the topological solitons of the $(2+1)$-dimensional gauged $CP(2)$ model have been investigated. The Lagrangian of the gauged $CP(N-1)$ model is invariant under two local Abelian transformations (6) and (9). The characteristic feature of the $CP(N-1)$ model is the fact that there is no physical gauge field corresponding to local Abelian transformations (6). Instead, the invariance of the Lagrangian (7) under local Abelian transformations (6) is provided by the Hermitian projection operator (3). As a result, the finiteness of the soliton energy leads us to the modified boundary condition: $\mathcal{P}_{ab}(D_i\phi_b) \to 0$ as $r \to \infty$. This modified boundary condition is less restrictive than the traditional one: $D_i\phi_b \to 0$ as $r \to \infty$. As a consequence, there are two types of topological solitons in the spectrum of the gauged $CP(2)$ model with the charge matrix $Q = \lambda_3/2$. For the topological soliton with quantized magnetic flux the covariant derivatives $D_i\phi_b$ tend to zero as $r \to \infty$. For the topological soliton with nonquantized magnetic flux the transverse parts of the covariant derivatives $\mathcal{P}_{ab}(D_i\phi_b)$ tend to zero as $r \to \infty$. At the same time, the covariant derivatives $D_i\phi_b$ of the topological soliton with nonquantized magnetic flux can be different from zero as $r \to \infty$.

In the present paper we found the two different topological solitons of the gauged $CP(2)$ model with the Maxwell term. It should be noted, however, that there are other field models that have in their spectra soliton solutions of different types. It was shown in Ref. [35] that the gauged Higgs model with the Chern-Simons term has two self-dual soliton solutions. The first solution is a topological soliton with quantized magnetic flux; the second solution is a nontopological soliton with nonquantized magnetic flux. Two self-dual soliton solutions also exist in the gauged O(3) σ-model with the Chern-Simons term [17,18]. The first solution is a topological soliton; the second solution is a nontopological soliton. Both these soliton solutions possess a nonquantized magnetic flux.

Generalizations of the considered topological solitons are possible. Note that the Lagrangian (7) contains only the Maxwell term. It is known [11] that electrically charged solutions of $(2+1)$-dimensional Maxwell gauged models must have an infinite energy. Therefore the topological solitons considered in this paper are electrically neutral. However, the addition of the Chern-Simons term to the Lagrangian (7) (or the replacement of the Maxwell term by the Chern-Simons term) changes this situation [15,17,18,26,27,32,35–37]. In this case we can expect electrically charged solitons in the spectrum of the gauged $CP(2)$ model. The presence of electric and magnetic fields leads to a nonvanishing angular momentum of such solitons. We can also expect that there is the rotating generalization [19] of these solitons in the spectrum of the gauged $CP(2)$ model with the Chern-Simons term.

Axisymmetric ansatz (16) is for the normalized $CP(2)$ field ϕ. Its obvious generalization for the normalized $CP(N-1)$ field is

$$\phi_i = h^{\frac{1}{2}} \exp(im_i\theta)\cos(\alpha_{i+1}(r))\prod_{j=1}^{i}\sin(\alpha_j(r))$$
for $i = 1, \ldots, N-2$,
$$\phi_{N-1} = h^{\frac{1}{2}} \exp(im_{N-1}\theta)\prod_{j=1}^{N-1}\sin(\alpha_j(r)),$$
$$\phi_N = h^{\frac{1}{2}} \exp(im_N\theta)\cos(\alpha_1(r)). \quad (96)$$

We see that the ansatz for the normalized $CP(N-1)$ field ϕ depends on $N-1$ radial functions $\alpha_i(r)$. In addition to Eq. (16) ansatz (96) is axisymmetric under the combined action of spatial SO(2) rotations, diagonal SU$(N+1)$ transformations (5), and U(1) transformations (6). It can be shown that the topological solitons with quantized and nonquantized magnetic fluxes are presented in the gauged $CP(1)$ models with the Lagrangian (7). However, unlike the $CP(2)$ case, there is no gauged $CP(1)$ model with a given potential term that contains simultaneously the topological solitons of both types in its spectrum. At the same time, we expect that for $N \geq 4$ there are gauged $CP(N-1)$ models with both types of topological solitons in their spectra.

We have considered the two combinations of the charge matrices and winding numbers: $Q = \lambda_3/2$, $m_1 = -m_2 = m$, and $Q = \lambda_8/2$, $m_1 = m_2 = m$. All other combinations ($Q = \lambda_3/2$, $m_1 = m_2 = m$, $Q = \lambda_8/2$,

$m_1 = -m_2 = m$, $Q = \lambda_3/2$, $m_1 \neq m_2$, and $Q = \lambda_8/2$, $m_1 \neq m_2$) do not contain anything new compared to those considered. In particular, it was found that the topological solitons with quantized magnetic flux and those with nonquantized magnetic flux simultaneously exist only in the case $Q = \lambda_3/2$, $m_1 = -m_2 = m$. In all other cases, the spectrum of the gauged $CP(2)$ model with potential term (15) contains only the topological solitons with nonquantized magnetic flux.

The topological solitons of the gauged $CP(2)$ model can be quantized by several alternative methods [38–41]. All these methods in one way or another require knowledge of the spectrum of the quadratic fluctuation operator in the functional neighborhood of the soliton. The spectrum can be found only numerically for the specific values of the model's parameters g, h, λ, and m. This, however, is a rather complicated task, and lies beyond the scope of this paper. We remark only that for $g^2 \sim \sqrt{\lambda}$, $h/g^2 \sim h/\sqrt{\lambda} \gg 1$ the topological solitons of both types are classical objects, because their Compton wavelengths are much less than their classical linear sizes. Conversely, for $g^2 \sim \sqrt{\lambda}$, $h/g^2 \sim h/\sqrt{\lambda} \ll 1$ they are quantum objects, because their Compton wavelengths become greater than their classical linear sizes.

ACKNOWLEDGMENTS

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