Galilean anomalies and their effect on hydrodynamics

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We study flavor and gravitational anomalies in Galilean theories coupled to torsional Newton-Cartan backgrounds. We establish that the relativistic anomaly inflow mechanism with an appropriately modified anomaly polynomial can be used to generate these anomalies. Similar to the relativistic case, we find that Galilean anomalies also survive only in even dimensions. Further, these anomalies only effect the flavor and rotational symmetries of a Galilean theory; in particular, the Milne boost symmetry remains nonanomalous. We also extend the transgression machinery used in relativistic fluids to Galilean fluids, and use it to determine how these anomalies affect the constitutive relations of a Galilean fluid. Unrelated to the Galilean fluids, we propose an analogue of the off-shell second law of thermodynamics for relativistic fluids, to include torsion and a conserved spin current in the vielbein formalism. Interestingly, we find that even in the absence of spin current and torsion the entropy currents in the two formalisms are different: while the usual entropy current gets a contribution from the gravitational anomaly, the entropy current in the vielbein formalism does not have any anomaly-induced part.

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I. NULL REDUCTION AND ANOMALIES

For most practical purposes, the world around us can be regarded as nonrelativistic. So it is natural to ask how various exotic results in relativistic theories can be interpreted in the nonrelativistic limit. Taking this limit (sending the speed of light $c \to \infty$) however turns out to be a notoriously nontrivial task. Except in a few special cases, the nonrelativistic limit is either not well defined or is not unique,[1] which forces the analysis to resort to other methods. One such (and historically the first) method is, rather than taking a limit of a relativistic theory, to define nonrelativistic theories in their own right guided by the symmetries. Nonrelativistic theories are known to transform covariantly under the action of "Galilean algebra." This algebra[2] is spanned by the following generators:

$$\begin{aligned} \text{Continuity (mass operator)}&: M, \\ \text{Time Translation}&: H, \\ \text{Translations}&: P_a, \\ \text{Galilean Boosts}&: B_a, \\ \text{Rotations}&: M_{ab}, \end{aligned}$$

with the commutation relations

$$\begin{gathered} [H,P_a]=0, \qquad [H,M_{ab}]=0, \qquad [H,B_a]=-P_a, \\ [P_a,P_b]=0, \qquad [M_{ab},P_c]=\delta_{ac}P_b-\delta_{bc}P_a, \\ [B_a,B_b]=0, \qquad [M_{ab},B_c]=\delta_{ac}B_b-\delta_{bc}B_a, \\ [M_{ab},M_{cd}]=\delta_{ac}M_{bd}-\delta_{ad}M_{bc}-\delta_{bc}M_{ad}+\delta_{bd}M_{ac}, \\ [M,\cdot]=0, \qquad [B_a,P_b]=\delta_{ab}M. \end{gathered} \tag{1.1}$$

In this work we will be interested in studying properties of Galilean theories, defined as theories respecting the Galilean algebra. Note that this definition spans a larger class of theories than just the nonrelativistic theories, as every Galilean theory might not arise as a $c \to \infty$ limit of a relativistic theory.

About a decade after the inception of general relativity it was realized that the spacetimes with the Galilean isometry group (called Galilean spacetimes) can also be packaged into a nice covariant language: Newton-Cartan geometries [2,3]. Since then there has been a huge amount of development in our understanding of how Galilean theories couple to Newton-Cartan backgrounds [4–18].[3] We recommend looking at Sec. 2.1 of Ref. [14] for a short and self-contained review of Newton-Cartan geometries, which will be extensively used throughout this work. References [19–22] contain some more recent work on Galilean physics which will not be touched upon here.

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[1]For example, Maxwell's electromagnetism is known to have more than one nonrelativistic limits [1].

[2]To be more precise, what we call Galilean algebra is generally known as the *Bargmann algebra* which is the *central extension of Galilean algebra* with the mass operator M. Galilean algebra sits inside as a special case with $M = 0$.

[3]It is far from the reach of a mortal being to compile an exhaustive list of the work on nonrelativistic physics; please refer to the mentioned works and references therein.

There is another well-known way to approach nonrelativistic physics: *null reduction* [23–25]. It has been known for a long time that the Galilean group can be embedded into a Poincaré group that is one dimension higher. Correspondingly, one can constrain the Poincaré algebra in a certain way, and reduce it to the Galilean algebra. To be more precise, consider generators of the five-dimensional Poincaré algebra written in null coordinates[4] ($A, B = -, +, 1, 2, 3$),

$$\begin{aligned} \text{Spacetime Translations:}\ & P_A, \\ \text{Lorentz Transformations:}\ & M_{AB}, \end{aligned} \tag{1.2}$$

with the usual commutation relations,

$$\begin{aligned} [P_A, P_B] &= 0, \\ [M_{AB}, P_C] &= \eta_{AC} P_B - \eta_{BC} P_A, \\ [M_{AB}, M_{CD}] &= \eta_{AC} M_{BD} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD} + \eta_{BD} M_{AC}. \end{aligned} \tag{1.3}$$

We can check that a subset of these generators, those that commute with null momenta P_- (a, $b = 1$, 2, 3),

$$P_-,\ \ P_+,\ \ P_a,\ \ M_{a-},\ \ M_{ab}, \tag{1.4}$$

span the Galilean algebra (1.1), with P_- acting as a new Casimir. $M \equiv P_-$ can be interpreted as a continuity operator (with mass as its conserved charge), P_a as translations, $H \equiv P_+$ as time translation, $B_a \equiv M_{a-}$ as Galilean boosts, and finally M_{ab} as rotations (look at Ref. [26] for an extensive review). This is rather convenient as instead of starting from a four-dimensional relativistic theory and taking $c \to \infty$, one can start with a five-dimensional relativistic theory and reduce it over a light cone (introduce a null Killing vector) to get a Galilean theory. This idea (and its generalizations to higher and lower dimensions) have been used readily in the literature to reproduce known results and to get new insights into nonrelativistic physics. Probably the most important of these results, in the current context, was to reproduce (torsional) Newton-Cartan geometries starting from a *Bargmann structure* (relativistic manifold carrying a covariantly constant null Killing vector) in one higher dimension [9,12,27–29]. Also, the authors of Ref. [30] and many following them (e.g., Refs. [31,32]) established that reducing a relativistic fluid on a light cone indeed gives the expected constitutive relations of a Galilean fluid, which was discussed, e.g., in Ref. [33].

The authors of Ref. [32] realized that this mechanism fails to reproduce the most generic Galilean theories. In particular, the thermodynamics of a Galilean fluid gained via null reduction is in a sense more restrictive than the most generic Galilean fluids.[5] Further, the parity-violating sector of the reduced fluid is highly restrictive and survives only in a very special case of "incompressible fluids kept in a constant magnetic field." In Ref. [35], the same authors provided a resolution to this issue, which however is a little different from the usual spirit of null reduction. Rather than performing null reduction of a relativistic fluid, the authors suggested constructing a theory of fluids coupled to Bargmann structures from scratch, henceforth referred to as a *Bargmann fluid* or *null fluid*.[6] In the process it was realized that there are certain aspects of null fluids which arise just by the introduction of null isometry and have no analogue in usual relativistic fluids. Upon null reduction,[7] this null fluid gives rise to the most generic Galilean fluid. In a sense null fluids can be seen as a particular embedding of Galilean fluids into a spacetime of one higher dimension. This approach is more in line with the axiomatic approach to study Galilean theories, but has the benefit that we have all of the well-developed machinery of relativistic physics at our disposal.

The aim of this paper is to address a similar issue, but in a different setting: *anomalies*. Flavor and gravitational anomalies for a nonrelativistic quantum field theory (Lifshitz fermions) were discussed in Ref. [36] using path integral methods. Reference [37], on the other hand, took the conventional null reduction approach to this problem, where the author started with an anomalous relativistic theory and figured out its fate upon reduction. There is however an issue with this approach: relativistic anomalies[8] are known to exist only in even dimensions, and hence this approach will essentially give anomalies *only in odd-dimensional* Galilean theories. This is slightly

[4]We define the transformation to null coordinates as $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^4)$.

[5]See Eq. IV. 121 of Ref. [32] and footnote (7) of Ref. [34] for more details on this issue.

[6]Why A "null" fluid? A fluid is generally called "null" if the corresponding fluid velocity is a null vector. Unlike usual relativistic fluids, one can show that on a Bargmann structure (with null Killing vector V^M), a null fluid ($u^M u_M = 0$) and a unit normalized fluid ($w^M w_M = -1$) are related by merely a field redefinition: $u^M = w^M + \frac{1}{2 w^N V_N} V^M$. The authors of Ref. [35] found that writing a Bargmann fluid in terms of a "null fluid velocity" is more natural from the point of view of a Galilean fluid.

[7]Since the theory already has a null Killing field, null reduction is defined as choosing a foliation transverse to the Killing field and compactifying the null direction. As we shall discuss in Sec. II C, doing this requires introducing a Galilean frame of reference, or in other words, a preferred notion of time.

[8]The author of Ref. [37] considered both flavor/gravitational as well as Weyl anomalies; however, in this work we will only be concerned with the former.

unpleasant, because if one is to look at Galilean theories as a makeshift version of nonrelativistic theories which in turn are the "low-velocity" limit of relativistic theories in the same number of dimensions, one would expect them to be anomalous *only in even dimensions* (see footnote 8). Half of this problem can be solved by noting that all of the anomalies found in Ref. [37] crucially depend on the components of the higher-dimensional gauge field and affine connection along the Killing direction ($A_\sim$, $\Gamma^M_{\sim N}$, where A_M is the gauge field, Γ^R_{MN} is the affine connection, and the null Killing vector is chosen to be $\partial_\sim$). It was noted in Ref. [32] that these components act as sources in the mass conservation Ward identity [look at the discussion around Eq. (2.42)]. Since we do not know of any such mass sources appearing in nature, it would be better to switch these off (one can check that these mass sources $A_\sim$, $\Gamma^M_{\sim N}$ are well-defined gauge-covariant tensors). Doing so will eliminate all the anomalies in odd-dimensional Galilean theories. We call the Bargmann structures with these mass sources set to zero as *compatible Bargmann structures* or *null backgrounds*, following Ref. [34]. The other half of the problem is however more challenging: we need to find a consistent mechanism to introduce anomalies in theories coupled to odd-dimensional null backgrounds.

The basic idea to do this was illustrated in Ref. [34] using U(1) anomalies. To motivate this let us consider the simplest case of a four-dimensional flat relativistic theory with a U(1) anomaly. Conservation of the corresponding (covariant) current J^μ is given as

$$\partial_\mu J^\mu = \frac{3}{4} C^{(4)} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{1.5}$$

where $F_{\mu\nu}$ is the field-strength tensor and $C^{(4)}$ is the anomaly constant. Upon taking a nonrelativistic limit, one would qualitatively expect the conservation law to look like[9]

$$\partial_t q + \partial_i j^i = -6C^{(4)} e_i b^i, \tag{1.6}$$

where e, b are the electric and magnetic fields, respectively. This effect can be reproduced after null reduction of a five-dimensional conservation law,

$$\partial_M J^M = \frac{3}{4} C^{(4)} \epsilon^{MNRST} \overline{V}_M F_{NR} F_{ST}, \tag{1.7}$$

where $\overline{V}^M$ is an arbitrary null vector with $\overline{V}_\sim = -1$. Note that $F_{\sim M} = \partial_\sim A_M - \partial_M A_\sim = 0$ when $A_\sim = 0$. Since one index on ϵ must be "$\sim$," this responsibility lands on $\overline{V}_M$, implying that the mentioned expression does not depend on which $\overline{V}^M$ is chosen (these statements will be made more rigorous in Sec. III B). It was observed in Ref. [34] that this anomaly can indeed be generated by the anomaly inflow mechanism exactly in the same way as it works for usual relativistic anomalies, but with a tweaked anomaly polynomial. The authors there were interested in Abelian anomalies and how they affect the hydrodynamics at the level of constitutive relations. This work will generalize these arguments to non-Abelian and gravitational anomalies, and will give a more rigorous and transparent mechanism to compute their contribution to Galilean hydrodynamics using the transgression machinery of relativistic fluids [38].

However, unlike Ref. [34], we would need to introduce torsion into the game for a clearer analysis of the gravitational sector. In Newton-Cartan geometries it is known (see Ref. [14]) that torsionlessness imposes a constraint $\mathrm{d}\boldsymbol{n} = 0$ on the time metric $\boldsymbol{n} = n_\mu \mathrm{d}x^\mu$. It has been noted in Refs. [12,18,39] that lifting this constraint off shell is necessary to study energy transport in Galilean theories. A similar issue also showed up in the context of Galilean hydrodynamics discussed in Ref. [35], where the authors noted that on torsionless Galilean backgrounds the second law of thermodynamics fails to capture all the constraints obeyed by the transport coefficients of a Galilean fluid. Since we will be interested in off-shell physics to understand anomalies, imposing torsionlessness would only make matters less clear. Nevertheless, at the cost of some added technicalities, it will allow us to explore null reduction for theories with a nonzero spin current, which as far as we can tell has not been attempted.[10] Reference [17] considered the most generic Galilean theories on a torsional Newton-Cartan background (without a conserved spin current), which follows very nicely via null reduction. Notably, the authors of Ref. [17] presented their results in a "frame-independent" manner using an "extended space representation" of the Galilean group; we will show in Appendix B that this representation is nothing but the theory on a null background seen prior to null reduction.

It is worth noting here that the essence of null reduction—whether usual or axiomatic—lies in the fact that the sophisticated machinery of relativistic theories can be used to say something useful about nonrelativistic theories. This method however has its limitations; one

[9]Note that Eq. (1.6) is *not* just Eq. (1.5) expanded into coordinates. When we take the $c \to \infty$ limit of $\partial_\mu J^\mu = \partial_t J^0/c + \partial_i J^i$ we get $\partial_t q + \partial_i j^i$, where $q = \lim_{c\to\infty} J^0/c$ and $j^i = \lim_{c\to\infty} J^i$. For right-hand side we use the definitions $\varepsilon^{ijk} = \epsilon^{0ijk}/c$, $e_i = \lim_{c\to\infty} cF_{i0}$, $b^i = \lim_{c\to\infty} \frac{1}{2}\varepsilon^{ijk} F_{jk}$, and assume $C^{(4)} \sim \mathcal{O}(c^0)$.

[10]Some authors (including those of Ref. [12]) have considered null reduction in the presence of torsion, but have not included a spin connection as an independent background source.

needs to be acquainted with the relativistic side of the story to appreciate the construction. Although we review whatever is required for this work, readers might find it helpful to consult the relativistic results first, or from time to time during the reading. The respective relativistic references will be mentioned on the go.

Unrelated to Galilean fluids, we also make some observations regarding the entropy current for a relativistic fluid. Recently, an off-shell generalization of the second law of thermodynamics was considered in Ref. [40] in the context of torsionless relativistic hydrodynamics. The authors of Refs. [41,42] also proposed a new Abelian $\mathrm{U}(1)_{\mathrm{T}}$ symmetry in hydrodynamics associated with this off-shell statement, with entropy as its conserved charge. We propose a natural generalization of this off-shell statement of the second law in the vielbein formalism, in the presence of torsion and a conserved spin current. More interestingly, even in the absence of torsion we find that the entropy current defined by the off-shell second law in the vielbein formalism is different from what is defined in the metric-like formalism (we call the latter the Belinfante entropy current). The vielbein entropy current does not have any anomaly-induced parts, while the Belinfante entropy current has been shown to get contributions from a gravitational anomaly [42]. A similar distinction between the two formalisms has been known for the energy-momentum tensor as well: while the vielbein formalism deals with an asymmetric canonical energy-momentum (EM) tensor (which is the Noether current of translations), the metric-like formalism deals with a symmetric Belinfante EM tensor (which couples to the metric in general relativity) (see footnote 15 for related comments). Motivated by this, and the fact that the vielbein entropy current does not get contributions from an anomaly, we guess that it should be in some sense more naturally related to the fundamental $\mathrm{U}(1)_{\mathrm{T}}$ symmetry of Refs. [41,42]. In passing we would also like to note that the two entropy currents are found to differ only off shell, and boil down to the same thing upon imposing the equations of motion. Further, for a spinless fluid[11] the difference only survives in the anomalous sector, and is precisely what accounts for the vielbein entropy current being independent of anomalies. Interested readers can jump directly to Appendix D.

This work is broadly categorized in five sections. The remainder of the Introduction contains a summary of our main results in Sec. I A. Section II starts off by extending the null background construction of Ref. [34] to include torsion, which is further used to derive the Ward identities of a Galilean theory with a nontrivial spin current in Sec. II C. A review of the relativistic anomaly inflow mechanism has been provided in Sec. III, which we modify in Sec. III B to account for anomalies in the null/Galilean backgrounds and derive the corresponding anomalous Ward identities. Later, in Sec. IV we discuss how these anomalies affect the constitutive relations of null/Galilean hydrodynamics. Keeping in mind the technicality of this work, a detailed walkthrough example for the simplest case of three-dimensional null theories (two-dimensional Galilean theories) is given in Sec. V A. These results are further generalized to arbitrary higher dimensions in Sec. V B. In Appendix A we present some of our results in the conventional noncovariant basis for the benefit of readers not acquainted with the Newton-Cartan language. Appendix B is devoted to a comparison of null backgrounds to the extended space representation of Ref. [17]. In Appendix C we give some notations and conventions for differential forms used throughout this work. Finally, in Appendix D we comment on the entropy current in relativistic hydrodynamics in the vielbein formalism.

A. Overview and results

Skipping all the technicalities, we start directly with the results, keeping in mind that these results have been obtained by null reduction of anomalies on null backgrounds. In the following we denote indices on a Newton-Cartan (NC) manifold $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ by $\mu\nu\ldots$, and on a flat spatial manifold $\mathbb{R}^{(d)}$ by $a, b\ldots$. The NC structure is defined by a time metric n_μ, a degenerate vielbein $e_a{}^\mu$, and a flat metric δ_{ab}. Further, we define a NC frame velocity v^μ, and by using it an "inverse" vielbein $e^a{}_\mu$ satisfying $v^\mu n_\nu + e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu$ and $e^a{}_\mu e_b{}^\mu = \delta^a{}_b$. Indices on $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ cannot be raised/lowered, while on $\mathbb{R}^{(d)}$ they can be raised/lowered by δ^{ab}, δ_{ab}. $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ indices can be projected down to $\mathbb{R}^{(d)}$ using $e^a{}_\mu$, $e_a{}^\mu$. An NC manifold is also equipped with a connection $\Gamma^\lambda{}_{\mu\nu}$, a spin connection $C^a{}_{\mu b}$, a non-Abelian gauge field A_μ, and a covariant derivative $\tilde{\nabla}_\mu$ associated with all of these. We also define the spacetime dependence of the frame velocity (also known as the boost connection) as $c_\mu{}^a = e^a{}_\nu \tilde{\nabla}_\mu v^\nu$. Differential forms are denoted by bold symbols.

Similar to the relativistic case, we find that (flavor and gravitational) anomalies on an even-dimensional NC background $\mathcal{M}^{\mathrm{NC}}_{(2n)}$ are governed by a $(2n+2)$-dimensional anomaly polynomial $\mathfrak{p}^{(2n+2)}$. However, here the anomaly polynomial is written in terms of the Chern classes of the gauge field strength $\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A}\wedge\boldsymbol{A} = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ and the Pontryagin classes of NC spatial curvature $\boldsymbol{R}^a{}_b = \mathrm{d}\boldsymbol{C}^a{}_b + \boldsymbol{C}^a{}_c\wedge\boldsymbol{C}^c{}_b = \frac{1}{2}R_{\mu\nu}{}^a{}_b\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$. On the other hand, the odd-dimensional Galilean theories are nonanomalous (in the absence of any extra mass sources). In the presence of anomalies, the conservation laws of the theory are given as

[11] By "spinless" we mean that the theory does not contain an independent conserved spin current (coupled to torsion).

$$
\begin{aligned}
\text{Mass Cons (Continuity)}&: \tilde{\underline{\nabla}}_\mu \rho^\mu = 0,\\
\text{Energy Cons (Time Translation)}&: \tilde{\underline{\nabla}}_\mu \epsilon^\mu = [\text{power}] - p^{\mu a} c_{\mu a},\\
\text{Momentum Cons (Translations)}&: \tilde{\underline{\nabla}}_\mu p^\mu{}_a = [\text{force}]_a - \rho^\mu c_{\mu a},\\
\text{Temporal Spin Cons (Galilean Boosts)}&: \tilde{\underline{\nabla}}_\mu \tau^{\mu a} = \frac{1}{2}(\rho^a - p^a),\\
\text{Spatial Spin Cons (Rotations)}&: \tilde{\underline{\nabla}}_\mu \sigma^{\mu ab} = p^{[ba]} + 2\tau^{\mu[a} c^{b]}{}_\mu + \sigma_{\mathrm{H}}^{\perp ab},\\
\text{Charge Cons (Flavor Transformations)}&: \tilde{\underline{\nabla}}_\mu j^\mu = \mathrm{j}_{\mathrm{H}}^{\perp},
\end{aligned}
\tag{1.8}
$$

where $\tilde{\underline{\nabla}}_\mu = \tilde{\nabla}_\mu + v^\nu H_{\nu\mu} - e_a{}^\nu \mathrm{T}^a{}_{\nu\mu}$. Here $H_{\mu\nu}$ is the temporal torsion and $\mathrm{T}^a{}_{\mu\nu}$ is the spatial torsion. Along with the conservation laws, the associated symmetries and conserved quantities have been specified above. We see that the mass current is exactly conserved. The energy/momentum current is sourced by the power/force densities (expressions can be found in Sec. II C) and pseudo-power/force densities due to the spacetime dependence of the frame velocity $c_\mu{}^a$. The temporal spin current is sourced by the difference between the spatial mass current and momentum density (for spinless theories it implies equality of the two). Barring anomalies, the spatial spin current is sourced by the antisymmetric part of the momentum density (causing torque) and pseudotorque density, while the charge current is exactly conserved. In addition to these, the spatial spin and charge currents are also sourced by gravitational[12] $\sigma_{\mathrm{H}}^{\perp ab}$ and flavor $\mathrm{j}_{\mathrm{H}}^{\perp}$ anomalies, respectively. These anomaly sources can be determined from the anomaly polynomial $\mathfrak{p}^{(2n+2)}$ as

$$
\sigma_{\mathrm{H}}^{\perp ab} = -*_\uparrow \left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{R}_{ba}}\right], \quad \mathrm{j}_{\mathrm{H}}^{\perp} = -*_\uparrow \left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{F}}\right]. \tag{1.9}
$$

In the study of Galilean hydrodynamics, we can construct the sector of constitutive relations completely determined by these anomalies, following the transgression machinery developed to do that same job in relativistic fluids [38]. To do this, we first need to define the hydrodynamic shadow gauge field $\hat{\boldsymbol{A}} = \boldsymbol{A} - \mu \boldsymbol{n}$ and spin connection $\hat{\boldsymbol{C}}^a{}_b = \boldsymbol{C}^a{}_b - [\mu_\sigma]^a{}_b \boldsymbol{n}$, where μ is the flavor chemical potential and $[\mu_\sigma]^a{}_b$ is the spatial spin chemical potential. We call the corresponding field strengths $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{R}}^a{}_b$, and the anomaly polynomial made out of these is $\hat{\mathfrak{p}}^{(2n+2)}$. Using these we define the transgression form, $\boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)} = -\frac{\boldsymbol{n}}{\boldsymbol{H}} \wedge (\mathfrak{p}^{(2n+2)} - \hat{\mathfrak{p}}^{(2n+2)})$, where[13] $\boldsymbol{H} = -\mathrm{d}\boldsymbol{n}$. It can be used to generate the anomalous sector of constitutive relations; only nonzero contributions are given as

$$
\begin{aligned}
(\epsilon^\mu)_{\mathrm{A}} &= *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{H}}\right]^\mu,\\
(\sigma^{\mu ab})_{\mathrm{A}} &= *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{R}_{ba}}\right]^\mu,\\
(j^\mu)_{\mathrm{A}} &= *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{F}}\right]^\mu.
\end{aligned}
$$

We leave it for the reader to convince themselves that these formulas are well defined. These constitutive relations follow the second law of thermodynamics and off-shell adiabaticity with a trivially zero entropy current. We would like to caution the reader that these are merely the contributions from anomalies to the constitutive relations; there will be further contributions which are independent of the anomalies and have not been discussed here. We would like to mention that in this derivation of the anomaly-induced constitutive relations, we rely on the existence of an equilibrium partition function which describes the fluid in the equilibrium configuration. These ideas were discussed for a relativistic fluid in Refs. [43–45] and were later adapted to Galilean fluids in Refs. [15,16,34,35].

Explicit examples of the above results in case of the U(1) and spin anomalies for two dimensions and a generalization to $2n$ dimensions is given in Sec. V. But probably the most important take-home message of this work is that one can perform a consistent analysis of flavor and spin anomalies for Galilean theories using guidelines laid out by the relativistic construction. This should be taken as yet another point in the favor of (or rather an advertisement for) the axiomatic approach to null reduction: null backgrounds [35].

[12] It should be noted that the gravitational anomaly appears as a "spin anomaly" in Eq. (1.8). This is already familiar from the respective relativistic version, where the gravitational anomaly in a metric-like formulation appears as a Lorentz anomaly in the vielbein formalism.

[13] It is not immediately clear why it is okay to divide by a two-form $\boldsymbol{H}$. One can however check that the numerator always contains at least one power of $\boldsymbol{H}$, which will cancel the $\boldsymbol{H}$ in the denominator.

II. GALILEAN THEORIES WITH SPIN AND TORSION

The aim of this section is to extend the null background construction of Refs. [34,35] to torsional backgrounds, and derive the nonanomalous Ward identities for a Galilean theory with a nonzero spin current. We will introduce anomalies later in Sec. III. The construction is mainly based on the work of Refs. [12,28] on torsional null reductions, with certain modifications. We will be working in the vielbein formalism, which is a more natural choice for a spin system. Hence the language and expressions will be slightly different from what has been seen in the earlier work on null backgrounds [34] where authors focused on the torsionless and spinless case.

A. Einstein-Cartan backgrounds

We start with a short review of the Einstein-Cartan backgrounds, mostly to set up notation for our later discussion on the torsional null backgrounds. A more comprehensive introduction to this formalism can be found in, e.g., Ref. [46]. Consider a manifold $\mathcal{M}_{(d+2)}$, on which theories are invariant under diffeomorphisms and (possibly non-Abelian) flavor gauge group $\mathcal{G}$. We denote the infinitesimal diffeomorphism and flavor variation parameters by

$$\psi_\xi = \{\xi = \xi^M \partial_M, \Lambda_{(\xi)}\} \in \mathrm{T}\mathcal{M}_{(d+2)} \times \mathfrak{g}. \tag{2.1}$$

We have denoted the tangent bundle of $\mathcal{M}_{(d+2)}$ as $\mathrm{T}\mathcal{M}_{(d+2)}$, and the Lie algebra corresponding to $\mathcal{G}$ as $\mathfrak{g}$. Indices on $\mathcal{M}_{(d+2)}$ are denoted by $M, N, R, S\ldots$. $\mathcal{M}_{(d+2)}$ is endowed with a metric $\mathrm{d}s^2 = \mathrm{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N$, a $\mathfrak{g}$-valued gauge field $\boldsymbol{A} = A_M \mathrm{d}x^M$, and a metric compatible affine connection $\Gamma^R{}_{MS}$ which is not necessarily symmetric in its last two indices. In the case of torsional geometries it is more natural to shift to the vielbein formalism, which we describe in the following. The condition of local flatness of a manifold allows us to define a map between $\mathrm{T}\mathcal{M}_{(d+2)}$ and (pseudo-Riemannian) flat space $\mathbb{R}^{(d+1,1)}$, realized in terms of a vielbein $\mathrm{E}^A{}_M$ and its inverse $\mathrm{E}_A{}^M$, restricted by

$$\mathrm{G}_{MN} = \mathrm{E}^A{}_M \mathrm{E}^B{}_N \eta_{AB}, \qquad \mathrm{G}^{MN} = \mathrm{E}_A{}^M \mathrm{E}_B{}^N \eta^{AB}, \tag{2.2}$$

where η_{AB} is the flat Minkowski metric, and $A, B, C, D\ldots$ denote indices on $\mathbb{R}^{(d+1,1)}$. Indices on $\mathcal{M}_{(d+2)}$ can be raised and lowered by G_{MN}, and those on $\mathbb{R}^{(d+1,1)}$ by η_{AB}. Indices on $\mathcal{M}_{(d+2)}$ and $\mathbb{R}^{(d+1,1)}$ can also be interchanged using the $\mathrm{E}^A{}_M$. The vielbein has $(d+2)^2$ components out of which $\frac{1}{2}(d+2)(d+3)$ are taken away by Eq. (2.2). The remaining $\frac{1}{2}(d+1)(d+2)$ components can be fixed by introducing an additional $\mathrm{SO}(d+1,1)$ symmetry in the definition of the vielbein: $\mathrm{E}^A{}_M \sim O^A{}_B \mathrm{E}^B{}_M$. Hence $\mathrm{E}^A{}_M$ modded by diffeomorphisms and $\mathrm{SO}(d+1,1)$ has the same physical information as G_{MN} modded with only diffeomorphisms. We also define a spin connection for fields living in $\mathbb{R}^{(d+1,1)}$,

$$\boldsymbol{C}^A{}_B = C^A{}_{MB}\mathrm{d}x^M = \mathrm{E}_B{}^S(\mathrm{E}^A{}_R \Gamma^R{}_{MS} - \partial_M \mathrm{E}^A{}_S)\mathrm{d}x^M, \tag{2.3}$$

which has the same information as $\Gamma^R{}_{MS}$. So finally our system can be described by the trio $\{\mathrm{E}^A{}_M, C^A{}_{MB}, A_M\}$ modded by diffeomorphisms, flavor transformations, and $\mathrm{SO}(d+1,1)$ rotations denoted by infinitesimal parameters,

$$\psi_\xi = \{\xi^M\partial_M, [\Lambda_{\Sigma(\xi)}]^A{}_B, \Lambda_{(\xi)}\} \in \mathrm{T}\mathcal{M}_{(d+2)} \times \mathfrak{so}(d+1,1) \times \mathfrak{g}. \tag{2.4}$$

Here $\mathfrak{so}(d+1,1)$ denotes the Lie algebra of $\mathrm{SO}(d+1,1)$. ψ_ξ is given a Lie algebra structure by defining a commutator on it,

$$\psi_{[\xi_1,\xi_2]} = [\psi_{\xi_1}, \psi_{\xi_2}] = \delta_{\xi_1}\psi_{\xi_2} = -\delta_{\xi_2}\psi_{\xi_1}, \tag{2.5}$$

where

$$\begin{aligned}
\delta_{\xi_1}\xi_2 &= \pounds_{\xi_1}\xi_2 = -\pounds_{\xi_2}\xi_1 = -\delta_{\xi_2}\xi_1, \\
\delta_{\xi_1}[\Lambda_{\Sigma(\xi_2)}]^A{}_B &= \pounds_{\xi_1}[\Lambda_{\Sigma(\xi_2)}]^A{}_B + [\Lambda_{\Sigma(\xi_2)}]^A{}_C[\Lambda_{\Sigma(\xi_1)}]^C{}_B \\
&\quad - [\Lambda_{\Sigma(\xi_1)}]^A{}_C[\Lambda_{\Sigma(\xi_2)}]^C{}_B - \pounds_{\xi_2}[\Lambda_{\Sigma(\xi_1)}]^A{}_B \\
&= -\delta_{\xi_2}[\Lambda_{\Sigma(\xi_1)}]^A{}_B, \\
\delta_{\xi_1}\Lambda_{(\xi_2)} &= \pounds_{\xi_1}\Lambda_{(\xi_2)} + [\Lambda_{(\xi_2)}, \Lambda_{(\xi_1)}] - \pounds_{\xi_2}\Lambda_{(\xi_1)} \\
&= -\delta_{\xi_2}\Lambda_{(\xi_1)}.
\end{aligned} \tag{2.6}$$

Similarly, the action of ψ_ξ (denoted by δ_ξ) on an arbitrary field φ (all indices suppressed) obeys an algebra: $[\delta_{\xi_1}, \delta_{\xi_2}]\varphi = \delta_{[\xi_1,\xi_2]}\varphi$. Under the action of ψ_ξ, constituent fields vary as

$$\begin{aligned}
\delta_\xi \mathrm{E}^A{}_M &= \pounds_\xi \mathrm{E}^A{}_M - [\Lambda_{\Sigma(\xi)}]^A{}_B \mathrm{E}^B{}_M \\
&= \nabla_M \xi^A + \xi^N \mathrm{T}^A{}_{NM} - [\nu_{\Sigma(\xi)}]^A{}_B \mathrm{E}^B{}_M, \\
\delta_\xi C^A{}_{MB} &= \pounds_\xi C^A{}_{MB} + \nabla_M [\Lambda_{\Sigma(\xi)}]^A{}_B \\
&= \nabla_M [\nu_{\Sigma(\xi)}]^A{}_B + \xi^N R_{NM}{}^A{}_B, \\
\delta_\xi A_M &= \pounds_\xi A_M + \nabla_M \Lambda_{(\xi)} = \nabla_M \nu_{(\xi)} + \xi^N F_{NM},
\end{aligned} \tag{2.7}$$

where $\xi^A = \mathrm{E}^A{}_M \xi^M$ and $\pounds_\xi$ denotes the Lie derivative along ξ^M. The covariant derivative ∇_M is associated with all the connections $\Gamma^R{}_{MS}, C^A{}_{MB}, A_M$, which acts on a general field $\varphi^R{}_S{}^A{}_B$ transforming in the adjoint representation of the flavor group as

$$\begin{aligned}
\nabla_M \varphi^R{}_S{}^A{}_B &= \partial_M \varphi^R{}_S{}^A{}_B + \Gamma^R{}_{MN}\varphi^N{}_S{}^A{}_B - \Gamma^N{}_{MS}\varphi^R{}_N{}^A{}_B \\
&\quad + C^A{}_{MC}\varphi^R{}_S{}^C{}_B - C^C{}_{MB}\varphi^R{}_S{}^A{}_C + [A_M, \varphi^R{}_S{}^A{}_B],
\end{aligned} \tag{2.8}$$

and similarly on higher-rank objects. In Eq. (2.7) we have defined[14]

$$\text{Scaled flavor chemical potential:}\ \nu_{(\xi)} = \Lambda_{(\xi)} + \xi^N A_N,$$
$$\text{Scaled spin chemical potential:}\ [\nu_{\Sigma(\xi)}]^A{}_B = [\Lambda_{\Sigma(\xi)}]^A{}_B + \xi^N C^A{}_{NB}, \tag{2.9}$$

associated with ψ_ξ. One can check that, despite appearing noncovariant, these scaled chemical potentials transform covariantly under the action of ψ_ξ. We have also defined curvatures of all the constituent fields,

$$\text{Gauge Field Strength:}\ \boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A}\wedge\boldsymbol{A} = \frac{1}{2}F_{MN}\mathrm{d}x^M\wedge\mathrm{d}x^N,$$
$$\text{Spacetime Curvature:}\ \boldsymbol{R}^A{}_B = \mathrm{d}\boldsymbol{C}^A{}_B + \boldsymbol{C}^A{}_C\wedge\boldsymbol{C}^C{}_B = \frac{1}{2}R_{MN}{}^A{}_B\mathrm{d}x^M\wedge\mathrm{d}x^N,$$
$$\text{Spacetime Torsion:}\ \mathbf{T}^A = \mathrm{d}\mathbf{E}^A + \boldsymbol{C}^A{}_B\wedge\mathbf{E}^B = \frac{1}{2}\mathrm{T}^A{}_{MN}\mathrm{d}x^M\wedge\mathrm{d}x^N. \tag{2.10}$$

One can check that all these quantities also transform covariantly under the action of ψ_ξ. It is interesting to note that $C^A{}_{MB}$ transforms as a $\mathfrak{so}(d+1,1)$-valued gauge field. In terms of torsion it is possible to give an exact expression for the connections, which we note for completeness:

$$\Gamma^R{}_{MS} = \frac{1}{2}\mathrm{G}^{RN}(\partial_M\mathrm{G}_{NS} + \partial_S\mathrm{G}_{NM} - \partial_N\mathrm{G}_{MS} + \mathrm{T}_{NMS} - \mathrm{T}_{MSN} - \mathrm{T}_{SMN}),$$
$$C^A{}_{MB} = \frac{1}{2}\eta_{BD}\mathrm{E}^{[D|S}[2(2\partial_{[S}\mathrm{E}^{A]}{}_{M]} - \mathrm{T}^{A]}{}_{SM}) + \mathrm{E}_{CM}\mathrm{E}^{A]N}(2\partial_{[S}\mathrm{E}^C{}_{N]} - \mathrm{T}^C{}_{SN})]. \tag{2.11}$$

A physical theory on $\mathcal{M}_{(d+2)}$ can be described by a partition function $W[\mathrm{E}^A{}_M, C^A{}_{MB}, A_M]$ which is a functional of the vielbein and connections. Under an infinitesimal variation of the sources its response is captured by

$$\delta W = \int\{\mathrm{d}x^M\}\sqrt{|\mathrm{G}|}(T^M{}_A\delta\mathrm{E}^A{}_M + \Sigma^{MA}{}_B\delta C^B{}_{MA} + J^M\cdot\delta A_M), \tag{2.12}$$

where $X\cdot Y = \mathrm{Tr}[XY]$ for $X, Y\in\mathfrak{g}$ is the inner product on $\mathfrak{g}$. T^{MA} is the *canonical energy-momentum tensor*, Σ^{MAB} is the *spin current* (antisymmetric in its last two indices), and J^M is the *charge current*. Demanding the partition function to be invariant under the action of ψ_ξ, we can find the Ward identities[15] related to these currents,

$$\underline{\nabla}_M T^M{}_N = \mathrm{T}^B{}_{NM}T^M{}_B + R_{NM}{}^A{}_B\Sigma^{MB}{}_A + F_{NM}\cdot J^M,$$
$$\underline{\nabla}_M\Sigma^{MAB} = T^{[BA]},$$
$$\underline{\nabla}_M J^M = 0. \tag{2.13}$$

Here $\underline{\nabla}_M = \nabla_M - \mathrm{T}^N{}_{NM}$ has been introduced for brevity.

B. Null backgrounds

We are now ready to define null backgrounds. These kinds of backgrounds and their Galilean interpretation goes back to Refs. [9,12,28,48]. The idea of null backgrounds is to somewhat tweak the procedure, so that we not only get the correct symmetries, but also reproduce the required background field content after reduction. As we shall show, this even allows us to add anomalies in odd-dimensional null backgrounds which naively does not look possible.

We will call ψ_ξ a *compatible symmetry data* if the scaled chemical potentials associated with it defined in Eq. (2.9) are identically zero. Now, a manifold $\mathcal{M}_{(d+2)}$ along with fields $\{\mathrm{E}^A{}_M, C^A{}_{MB}, A_M\}$ will be called a *null background* (or more formally a *compatible Bargmann structure*) if it admits a *covariantly constant compatible null isometry* generated by $\psi_V = \{V^M\partial_M, [\Lambda_{\Sigma(V)}]^A{}_B, \Lambda_{(V)}\}$, i.e.,

1. the action of ψ_V is an isometry, $\delta_V\mathrm{E}^A{}_M = \delta_V C^A{}_{MB} = \delta_V A_M = 0$;
2. V is null, $V^M V_M = 0$;
3. V is covariantly constant, $\nabla_M V^N = 0$; and
4. ψ_V is compatible, $\nu_{(V)} = V^M A_M + \Lambda_{(V)} = 0$, $[\nu_{\Sigma(V)}]^A{}_B = V^M C^A{}_{MB} + [\Lambda_{\Sigma(V)}]^A{}_B = 0$.

Although this definition of null backgrounds is a little different from that in Ref. [34], one can check that it boils down to the same thing in the torsionless limit. If we drop condition (4), i.e., compatibility, we would be left with the definition of *Bargmann structures* [9]

[14]By scaled we mean scaled with temperature: $\nu_{(\xi)} = \mu_{(\xi)}/\vartheta_{(\xi)}$, where $\mu_{(\xi)}$ is the chemical potential and $\vartheta_{(\xi)}$ is the temperature. Note that at this point these quantities are just introduced for computational convenience, and they will get a physical meaning only in the presence of preferred symmetry data, e.g., when spacetime admits an isometry.

[15]Note that we can use the spin Ward identity to eliminate the antisymmetric part of the canonical EM tensor in the EM conservation equation. Doing this is particularly helpful in torsionless theories where the new EM conservation becomes $\nabla_M T^{MN}_{(b)} = F^{NM}\cdot J_M$. Here we have defined the symmetric *Belinfante energy-momentum tensor*, $T^{MN}_{(b)} = T^{(MN)} + 2\nabla_R\Sigma^{(MN)R}$. In this work, however, we will mostly talk in terms of the canonical EM tensor as this is the Noether charge corresponding to translations. Also, it is well known that gravitational anomalies do not affect the canonical EM conservation [47].

extended to the vielbein formalism. They have some nice properties:

$$\mathrm{T}^A{}_{MN} V_A = H_{MN} \equiv 2\partial_{[M} V_{N]}, \qquad R_{MN}{}^A{}_B V_A = 0. \tag{2.14}$$

Hence, if we are interested in a torsionless theory, we would have to apply a constraint on V, which can be violated off shell. The requirement of compatibility further imposes

$$V^M \mathrm{T}^A{}_{MN} = V^M F_{MN} = V^M R_{MN}{}^A{}_B = 0, \qquad V^M \nabla_M \varphi = \delta_V \varphi \tag{2.15}$$

for any tensor φ transforming in an appropriate representation of $\mathfrak{g}$ and $\mathfrak{so}(d+1,1)$ (all indices suppressed). These restrictions are in some sense the backbone of the null background construction. First and foremost, they eliminate the unphysical mass sources that would otherwise appear in the mass conservation law after reduction. Hints of it were originally found in Ref. [32] in an attempt at naive null reduction of charged fluids. We will have more to say about it later. As we shall see, these restrictions also allow for anomalies in the odd-dimensional null backgrounds and forbid them in even-dimensional ones. This is an important feature, if we are to reproduce physically realizable anomalies in Galilean theories in one lower dimension.

We demand that physical theories on null backgrounds (referred to as null theories) are not invariant under the action of any arbitrary ψ_ξ but only those which leave ψ_V invariant, i.e., $[\psi_V, \psi_\xi] = 0$. This requirement ensures that there is no dynamics along the isometry even off shell. The new partition function variation can be written following Eq. (2.12) as

$$\delta W = \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} (T^M{}_A \delta \mathrm{E}^A{}_M + \Sigma^{MA}{}_B \delta C^B{}_{MA} + J^M \cdot \delta A_M + \#_A \delta V^A). \tag{2.16}$$

Note the last term in this expression, which is valid since our restriction does not forbid us from varying V^A. The astute reader might note that we could have absorbed that term into T^{MA} owing to the fact that $\delta V^M = 0$, but we have a better setup in mind. The conditions of a null background along with the restrictions we have imposed imply that the null theories are invariant under the following set of current redefinitions:

$$T^{MA} \to T^{MA} + V^M \theta_1^A,$$
$$\Sigma^{MAB} \to \Sigma^{MAB} + V^M \theta_2^{AB},$$
$$J^M \to J^M + \theta_3 V^M, \tag{2.17}$$

$$\#^A \to \#^A - \theta_1^A + \theta_4 V^A, \tag{2.18}$$

where the θ's are arbitrary scalars transforming in appropriate representations of $\mathfrak{g}$ and $\mathfrak{so}(d+1,1)$. The Ward identities on null backgrounds will also be slightly modified compared to Eq. (2.13),[16]

$$\underline{\nabla}_M T^M{}_N = \mathrm{T}^B{}_{NM} T^M{}_B + R_{NM}{}^A{}_B \Sigma^{MB}{}_A + F_{NM} \cdot J^M,$$
$$\underline{\nabla}_M \Sigma^{MAB} = T^{[BA]} + \#^{[A} V^{B]},$$
$$\underline{\nabla}_M J^M = 0. \tag{2.19}$$

One can check that these equations are invariant under the redefinitions (2.17). To interpret the new $\#^A$ term, note that the spin (angular momentum) conservation consists of $\frac{1}{2}(d+1)(d+2)$ equations. However, as was pointed out in the Introduction, after reduction the system only respects $\frac{1}{2}d(d-1)$ equations corresponding to rotations and d equations corresponding to Galilean boosts. The job of $\#^A$ is then to eliminate the remaining $(d+1)$ conservation equations. Practically, it is best to fix an "off-shell gauge" $\delta V^A = 0$, which renders a new invariance in the spin current,

$$\Sigma^{MAB} \to \Sigma^{MAB} + \theta_5^{M[A} V^{B]}, \tag{2.20}$$

and omits the remaining $(d+1)$ components of the spin conservation. Note that it will further restrict ψ_ξ to obey $[\nu_{\Sigma(\xi)}]^A{}_B V^B = 0$. From this point onward we will assume that every symmetry data ψ_ξ satisfies these requirements, and will term them ψ_V *compatible symmetry data*. From this viewpoint, the spin conservation in Eq. (2.19) must be true for some $\#^M$, hence ruling out components involving $\#^M$ as they carry no information.

On null backgrounds, by using ψ_V we can also define some more "thermodynamic" variables associated with ψ_ξ similar to Eq. (2.9),

$$\text{Temperature}: \ \vartheta_{(\xi)} = -\frac{1}{\xi^N V_N},$$
$$\text{Scaled mass chemical potential}: \ \varpi_{(\xi)} = -\frac{\xi^M \xi_M}{2\xi^N V_N}, \tag{2.21}$$

and by using it we can define chemical potentials from scaled chemical potentials,

$$\mu_{(\xi)} = \vartheta_{(\xi)} \nu_{(\xi)},$$
$$[\mu_{\Sigma(\xi)}]^A{}_B = \vartheta_{(\xi)} [\nu_{\Sigma(\xi)}]^A{}_B,$$
$$\mu_{\varpi(\xi)} = \vartheta_{(\xi)} \varpi_{(\xi)}. \tag{2.22}$$

These abstract definitions will be useful later.

[16] Following footnote 15, one might wonder what the respective Belinfante EM conservation law looks like for null theories. Similar to the non-null case, one can use the spin conservation in the EM conservation law, which will give $\nabla_M (T^{MN}_{(b)} - \#^{[N} V^{M]}) = F^{NM} \cdot J_M$. One can show that the $\#^M$ dependence can be removed by using the T^{MA} redefinition (2.17), after which one recovers the standard Belinfante conservation law (given in footnote 15) even for null theories. The Belinfante EM tensor, on the other hand, is left with redefinition freedom, $T^{MN}_{(b)} \to T^{MN}_{(b)} + \theta_1 V^M V^N$. These were derived directly for a spinless null theory in Ref. [35].

C. Null reduction—Newton-Cartan backgrounds

Having obtained the Ward identities in the null background language, it is now time to see what they imply for the Galilean theories. To do this we need to pick up a foliation $\mathcal{M}_{(d+2)} = S^1_V \times \mathcal{M}^{\rm NC}_{(d+1)}$ and compactify along the isometry direction V. Following Ref. [34], we note that since V is null, it is not possible to find a unique such foliation without choosing a set of ψ_V compatible *time data*, $\psi_T = \{T^M\partial_M, [\Lambda_{\Sigma(T)}]^A{}_B, \Lambda_{(T)}\}$. This is tantamount to choosing a preferred Galilean frame of reference.[17] Having chosen ψ_T, we can define such a foliation as $\mathcal{M}_{(d+2)} = S^1_V \times \mathbb{R}_T \times \mathcal{M}^T_{(d)}$, where we identify $\mathcal{M}^{\rm NC}_{(d+1)} = \mathbb{R}_T \times \mathcal{M}^T_{(d)}$ as a degenerate *Newton-Cartan* manifold. We define *null reduction* as this choice of foliation and subsequent compactification.

Newton-Cartan structure: Null reduction of a generic torsional relativistic spacetime in the metric-like formulation was first performed in Ref. [49,50]. Here we add to their results a non-Abelian gauge field and present them in the vielbein formalism. Using ψ_T we can define a null field orthonormal to V as

$$\overline{V}^M_{(T)} = \vartheta_{(T)}T^M + \mu_{\varpi(T)}V^M, \tag{2.23}$$

such that $\overline{V}^M_{(T)}\overline{V}_{(T)M} = 0$, and $\overline{V}^M_{(T)}V_M = -1$. Here $\vartheta_{(T)}$, $\mu_{\varpi(T)}$ have been defined in Eqs. (2.21) and (2.22). Without loss of generality we choose a basis on $\mathcal{M}_{(d+2)}$, $x^M = \{x^\sim, x^\mu\}$ such that $\psi_V = \{\partial_\sim, 0, 0\}$. On the other hand, on $\mathbb{R}^{(d+1,1)}$ we choose a basis $x^A = \{x^-, x^+, x^a\}$ such that $V = \partial_-$ and $\overline{V}_{(T)} = \partial_+$. At this stage we choose a specific representation of η_{AB}, $\mathrm{E}^A{}_M$, and $\mathrm{E}_A{}^M$ that is compatible with the mentioned basis,

$$\eta_{AB} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ab} \end{pmatrix}, \quad \mathrm{E}^A{}_M = \begin{pmatrix} 1 & -B_\mu \\ 0 & n_\mu \\ 0 & e^a{}_\mu \end{pmatrix},$$
$$\mathrm{E}_A{}^M = \begin{pmatrix} 1 & 0 \\ B_\nu v^\nu & v^\mu \\ B_\nu e_a{}^\nu & e_a{}^\mu \end{pmatrix}, \tag{2.24}$$

such that

$$n_\mu v^\mu = 1, \qquad e_a{}^\mu n_\mu = 0, \qquad e^a{}_\mu v^\mu = 0,$$
$$e^a{}_\mu e_b{}^\mu = \delta^a{}_b, \qquad v^\mu n_\nu + e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu. \tag{2.25}$$

This can be identified as the *Newton-Cartan* structure. We can also define the NC degenerate metric by

$$h_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}, \qquad h^{\mu\nu} = e_a{}^\mu e_b{}^\nu \delta^{ab}. \tag{2.26}$$

Since there is no nondegenerate metric on $\mathcal{M}^{\rm NC}_{(d+1)}$, the raising/lowering of $\mu,\nu\ldots$ indices is not permitted. However, $a,b\ldots$ indices can be raised/lowered using δ_{ab}. The NC vielbein $e^a{}_\mu$ is not a "square matrix" and hence does not furnish an invertible map between tensors on $\mathcal{M}^{\rm NC}_{(d+1)}$ and $\mathbb{R}^{(d)}$. However it can be used to project tensors on $\mathcal{M}^{\rm NC}_{(d+1)}$ to tensors on $\mathbb{R}^{(d)}$, and tensors on $\mathbb{R}^{(d)}$ to "spatial tensors" on $\mathcal{M}^{\rm NC}_{(d+1)}$,

$$e^a{}_\mu X^\mu = X^a, \qquad e_a{}^\mu Y_\mu = Y_a,$$
$$X^a e_a{}^\mu = h^\mu{}_\nu X^\nu, \qquad Y_a e^a{}_\mu = h^\nu{}_\mu Y_\nu, \tag{2.27}$$

where $h^\mu{}_\nu = h^{\mu\rho}h_{\sigma\nu}$. The compatibility of null isometry switches off many components of the connections: $\Gamma^M_{\sim N}$, $\Gamma^M_{\mu\sim}$, $C^A_{\sim B}$, $C^A_{\mu-}$, $C^+_{\mu B}$, and $A_\sim$. The remaining nonzero components can be determined to be

$$C^-{}_{\mu a} = c_{\mu a}, \qquad C^a_{\mu +} = c_\mu{}^a, \qquad \Gamma^\sim{}_{\mu\nu} = c_{\mu\nu} - \tilde\nabla_\mu B_\nu,$$
$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda\partial_\mu n_\nu + \frac{1}{2}h^{\lambda\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})$$
$$+ n_{(\mu}\Omega_{\nu)\sigma}h^{\lambda\sigma} + \frac{1}{2}(e_a{}^\lambda \mathrm{T}^a{}_{\mu\nu} - 2e_{a(\nu}\mathrm{T}^a{}_{\mu)\sigma}h^{\lambda\sigma}),$$
$$C^a{}_{\mu b} = \frac{1}{2}n_\mu\Omega_b{}^a + \frac{1}{2}\eta_{bd}e^{[d|\nu}[2(2\partial_{[\nu}e^{a]}{}_{\mu]} - \mathrm{T}^{a]}{}_{\nu\mu})$$
$$+ e_{c\mu}e^{a]\sigma}(2\partial_{[\nu}e^c{}_{\sigma]} - \mathrm{T}^c{}_{\nu\sigma})]. \tag{2.28}$$

Here we have defined the spacetime dependence of the frame velocity $c_{\mu\nu} = h_{\sigma\nu}\tilde\nabla_\mu v^\sigma$ in terms of which frame vorticity is given by $\Omega_{\mu\nu} = 2c_{[\mu\nu]}$. We say that a time data (reference frame) ψ_T is *globally inertial* if $c_{\mu\nu} = 0$. We choose the connections on $\mathcal{M}^{\rm NC}_{(d+1)}$ to be $\Gamma^\lambda{}_{\mu\nu}$, $C^a{}_{\mu b}$, and A_μ, and denote the associated covariant derivative by $\tilde\nabla_\mu$, acting on a general field $\varphi^\rho{}_\sigma{}^a{}_b$ transforming in the adjoint representation of the flavor group as

$$\tilde\nabla_\mu\varphi^\rho{}_\sigma{}^a{}_b = \partial_\mu\varphi^\rho{}_\sigma{}^a{}_b + \Gamma^\rho{}_{\mu\nu}\varphi^\nu{}_\sigma{}^a{}_b - \Gamma^\nu{}_{\mu\sigma}\varphi^\rho{}_\nu{}^a{}_b + C^a{}_{\mu c}\varphi^\rho{}_\sigma{}^c{}_b$$
$$- C^c{}_{\mu b}\varphi^\rho{}_\sigma{}^a{}_c + [A_\mu, \varphi^\rho{}_\sigma{}^a{}_b], \tag{2.29}$$

and similarly on higher-rank objects. The action of $\tilde\nabla_\mu$ on the NC structure can be found to be

$$\tilde\nabla_\mu n_\nu = 0, \qquad \tilde\nabla_\mu e_a{}^\nu = 0, \qquad \tilde\nabla_\mu h^{\rho\sigma} = 0,$$
$$\tilde\nabla_\mu h_{\nu\rho} = -2c_{\mu(\nu}n_{\rho)}, \qquad \tilde\nabla_\mu e^a{}_\nu = -n_\nu c_\mu{}^a. \tag{2.30}$$

One can check that $\tilde\nabla_\mu, \Gamma^\lambda{}_{\mu\nu}$ agrees with the most generic NC covariant derivative and connection written down in Ref. [17]. One can also perform the reduction of curvatures. The surviving components of the gauge field strength are

[17]The authors of Ref. [17] proposed a formalism for Galilean theories independent of the choice of frame. But on a closer look it would be clear that they just discovered null backgrounds from a different perspective. The Ward identities as described in Ref. [17] are just the null background Ward identities with a slight rearrangement; we give a comparison in Appendix B.

$F_{\mu\nu}$ which act as the NC gauge field strength. Similarly, the surviving components of the torsion are the spatial torsion $\mathrm{T}^a{}_{\mu\nu}$, "mass torsion" $\mathrm{T}_{+\mu\nu} = -\mathrm{T}^-{}_{\mu\nu}$, and temporal torsion $H_{\mu\nu} = -\mathrm{T}^+_{\mu\nu}$. Finally, we have the surviving components of the curvature,

$$R_{\mu\nu}{}^a{}_+ = 2\partial_{[\mu}c_{\nu]}{}^a + 2C^a{}_{[\mu|b}c_{\nu]}{}^b,$$
$$R_{\mu\nu}{}^a{}_b = 2\partial_{[\mu}C^a{}_{\nu]b} + 2C^a{}_{[\mu|c}C^c{}_{\nu]b}, \tag{2.31}$$

which act as the NC temporal and spatial curvatures, respectively. Both curvatures can also be combined into a full NC curvature,

$$R_{\mu\nu}{}^\rho{}_\sigma = e_a{}^\rho(R_{\mu\nu}{}^a{}_+ n_\sigma + R_{\mu\nu}{}^a{}_b e^b{}_\sigma). \tag{2.32}$$

We define the raised NC volume element,

$$\varepsilon_\uparrow^{\mu\nu\cdots} = \overline{V}_M \epsilon^{M\mu\nu\cdots} = -\epsilon^{-\mu\nu\cdots}. \tag{2.33}$$

Again, since the volume element is defined with all indices up, and there is no lowering operation, the corresponding Hodge dual $*_\uparrow$ gives a map from differential forms to completely antisymmetric contravariant tensor fields. It is also possible to define a lowered volume element, but we would not require it for our purposes. More details on NC volume forms and Hodge duals can be found in Appendix C.

Conserved currents and Ward identities: Now we need to decompose the currents in this basis,

$$T^M{}_A = \begin{pmatrix} \times & \times & \times \\ -\rho^\mu & -\epsilon^\mu & p^\mu{}_a \end{pmatrix}, \qquad J^M = \begin{pmatrix} \times \\ j^\mu \end{pmatrix},$$
$$\Sigma^{\sim AB} = \times, \qquad \Sigma^{\mu AB} = \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \tau^{\mu b} \\ \times & -\tau^{\mu a} & \sigma^{\mu ab} \end{pmatrix}. \tag{2.34}$$

Here we have denoted unphysical components by $\times$ which can be eliminated using the redefinitions (2.17) and (2.20). We identify ρ^μ as the mass current, ϵ^μ as the energy current, $p^{\mu a}$ as the momentum current, $\tau^{\mu a}$ as the temporal spin current, $\sigma^{\mu ab}$ as the spatial spin current, and finally j^μ as the charge current. We can also project the μ index in these currents onto $\mathbb{R}^{(d)}$ to get the corresponding "spatial currents." On the other hand, we define various densities as the projection of these currents along n_μ,

$$\rho = n_\mu \rho^\mu, \qquad \epsilon = n_\mu \epsilon^\mu, \qquad p^a = n_\mu p^{\mu a},$$
$$\tau^a = n_\mu \tau^{\mu a}, \qquad \sigma^{ab} = n_\mu \sigma^{\mu ab}, \qquad q = n_\mu j^\mu. \tag{2.35}$$

In terms of these, the physical components of the Ward identities (2.19) can be expressed as

$$\text{Mass Cons (Continuity)}: \underline{\tilde{\nabla}}_\mu \rho^\mu = 0,$$
$$\text{Energy Cons (Time Translation)}: \underline{\tilde{\nabla}}_\mu \epsilon^\mu = [\text{power}] - p^{\mu a} c_{\mu a},$$
$$\text{Momentum Cons (Translations)}: \underline{\tilde{\nabla}}_\mu p^\mu{}_a = [\text{force}]_a - \rho^\mu c_{\mu a},$$
$$\text{Temporal Spin Cons (Galilean Boosts)}: \underline{\tilde{\nabla}}_\mu \tau^\mu{}_a = \frac{1}{2}(\rho_a - p_a),$$
$$\text{Spatial Spin Cons (Rotations)}: \underline{\tilde{\nabla}}_\mu \sigma^{\mu ab} = p^{[ba]} + 2\tau^{\mu[a} c_\mu{}^{b]},$$
$$\text{Charge Cons (Flavor Transformations)}: \underline{\tilde{\nabla}}_\mu j^\mu = 0, \tag{2.36}$$

where $\underline{\tilde{\nabla}}_\mu = \tilde{\nabla}_\mu + v^\nu H_{\nu\mu} - e_a{}^\nu \mathrm{T}^a{}_{\nu\mu}$. These are the (non-anomalous) conservation laws of a Galilean theory with spin current. The conserved quantities have been mentioned above (and the underlying symmetry). The temporal conservation equation, which is slightly less familiar, is akin to the Milne boost Ward identity of the torsionless case, which states that the spatial mass current must be equal to the momentum density (look, e.g., at Ref. [14] and follow references therein). Here [power] and $[\text{force}]_a$ are power and force densities due to background fields,

$$[\text{power}] = -v^\nu(H_{\nu\mu}\epsilon^\mu + \mathrm{T}_{+\nu\mu}\rho^\mu + \mathrm{T}^a{}_{\nu\mu} p^\mu{}_a + R_{\mu\nu}{}^a{}_+ \tau^\mu{}_a$$
$$+ R_{\nu\mu ab}\sigma^{\mu ba} + F_{\nu\mu} \cdot j^\mu),$$
$$[\text{force}]_a = e_a{}^\nu(H_{\nu\mu}\epsilon^\mu + \mathrm{T}_{+\nu\mu}\rho^\mu + \mathrm{T}^a{}_{\nu\mu} p^\mu{}_a + R_{\mu\nu}{}^a{}_+ \tau^\mu{}_a$$
$$+ R_{\nu\mu ab}\sigma^{\mu ba} + F_{\nu\mu} \cdot j^\mu), \tag{2.37}$$

which act as the energy and momentum sources, respectively. The terms coupling to $c_{\mu a}$ in Eq. (2.36) are due to the fact that the chosen Galilean frame (time data) is not globally inertial and hence causes pseudopower, pseudoforce, and pseudotorque.

One could have taken a slightly different approach to get these Ward identities and performed null reduction at the level of the partition function (2.16) itself,

$$\delta W = \int \{\mathrm{d}x^\mu\}\sqrt{|\gamma|}(\rho^\mu \delta B_\mu - \epsilon^\mu \delta n_\mu + p^\mu{}_a \delta e^a{}_\mu + 2\tau^{\mu a}\delta c_{\mu a}$$
$$+ \sigma^{\mu a}{}_b \delta C^b{}_{\mu a} + j^\mu \cdot \delta A_\mu), \tag{2.38}$$

where $\gamma_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu$ and $\gamma = \det \gamma_{\mu\nu} = -\mathrm{G}$. The symmetry data ψ_ξ breaks up in the NC basis as

$$\psi_\xi^{\rm NC} = \{\Lambda_{{\rm M}(\xi)} \equiv -\xi^\sim,\ \ \xi^\mu,\ \ [\Lambda_{\tau(\xi)}]_a \equiv [\Lambda_{\Sigma(\xi)}]^-{}_a,\\ [\Lambda_{\sigma(\xi)}]^a{}_b \equiv [\Lambda_{\Sigma(\xi)}]^a{}_b,\ \ \Lambda_{(\xi)}\}. \tag{2.39}$$

The variation of various constituent fields under the action of $\psi_\xi^{\rm NC}$ (also denoted as δ_ξ) can be obtained via null reduction,[18]

$$\begin{aligned}
\delta_\xi B_\mu &= \pounds_\xi B_\mu + \partial_\mu \Lambda_{{\rm M}(\xi)} + [\Lambda_{\tau(\xi)}]_a e^a{}_\mu \\
&= \partial_\mu \nu_{{\rm M}(\xi)} + \xi^\nu {\rm T}_{+\nu\mu} - \xi^\nu c_{\mu\nu} + [\nu_{\tau(\xi)}]_a e^a{}_\mu, \\
\delta_\xi n_\mu &= \pounds_\xi n_\mu = \partial_\mu \xi^+ - \xi^\nu H_{\nu\mu}, \\
\delta_\xi e^a{}_\mu &= \pounds_\xi e^a{}_\mu - [\Lambda_{\sigma(\xi)}]^a{}_b e^b{}_\mu - [\Lambda_{\tau(\xi)}]^a n_\mu \\
&= \tilde\nabla_\mu \xi^a + \xi^+ c_\mu{}^a + \xi^\nu {\rm T}^a{}_{\nu\mu} \\
&\quad - [\nu_{\sigma(\xi)}]^a{}_b e^b{}_\mu - [\nu_{\tau(\xi)}]^a n_\mu, \\
\delta_\xi c_{\mu a} &= \pounds_\xi c_{\mu a} + (\partial_\mu [\Lambda_{\tau(\xi)}]_a - C^b{}_{\mu a} [\Lambda_{\tau(\xi)}]_b) + [\Lambda_{\sigma(\xi)}]^b{}_a c_{\mu b} \\
&= \tilde\nabla_\mu [\nu_{\tau(\xi)}]_a + [\nu_{\sigma(\xi)}]^b{}_a c_{\mu b} - \xi^\nu R_{\nu\mu + a}, \\
\delta_\xi C^a{}_{\mu b} &= \pounds_\xi C^a{}_{\mu b} + (\partial_\mu [\Lambda_{\sigma(\xi)}]^a{}_b + C^a{}_{\mu c} [\Lambda_{\sigma(\xi)}]^c{}_b \\
&\quad - C^c{}_{\mu b} [\Lambda_{\sigma(\xi)}]^a{}_c) = \tilde\nabla_\mu [\nu_{\sigma(\xi)}]^a_b + \xi^\nu R_{\nu\mu}{}^a{}_b, \\
\delta_\xi A_\mu &= \pounds_\xi A_\mu + \partial_\mu \Lambda_{(\xi)} + [A_\mu, \Lambda_{(\xi)}] \\
&= \tilde\nabla_\mu \nu_{(\xi)} + \xi^\nu F_{\nu\mu}.
\end{aligned} \tag{2.40}$$

Looking at these expressions we can identify $\Lambda_{{\rm M}(\xi)}$ as the continuity parameter, ξ^μ as the spacetime translation parameter, $[\Lambda_{\tau(\xi)}]_a$ as the Galilean boost parameter, $[\Lambda_{\sigma(\xi)}]^a_b$ as the rotation parameter, and $\Lambda_{(\xi)}$ as the flavor parameter. It is further noteworthy that $\xi^+ = n_\mu \xi^\mu$ and $\xi^a = e^a{}_\mu \xi^\mu$ serve as time translation and space translation parameters, respectively. Demanding the invariance of Eq. (2.38) under all of these parameters, one can recover the Ward identities (2.36). One can compare these results to those of Ref. [17].

In the first equation of Eq. (2.40) we have defined the *scaled total mass chemical potential* associated with ψ_ξ as $\nu_{{\rm M}(\xi)} = \Lambda_{{\rm M}(\xi)} + \xi^\mu B_\mu = \xi^M \overline{V}_{(T)M}$. It differs from the scaled mass chemical potential $\varpi_{(\xi)}$ defined in Eq. (2.21) by a "kinetic" part, $\nu_{{\rm M}(\xi)} = \varpi_{(\xi)} - \frac{1}{2\vartheta_{(\xi)}} \overline{V}^a_{(\xi)} \overline{V}_{(\xi)a}$. Following Eq. (2.22) we can also define the total mass chemical potential as $\mu_{{\rm M}(\xi)} = \vartheta_{(\xi)} \nu_{{\rm M}(\xi)} = \mu_{\varpi(\xi)} - \frac{1}{2} \overline{V}^a_{(\xi)} \overline{V}_{(\xi)a}$.[19]

We would like to note that mass, being exactly conserved, is a consequence of compatibility. Otherwise the respective conservation equation would look something like

$$\begin{aligned}
-\underline{\tilde\nabla}_\mu \rho^\mu &= {\rm T}^A{}_{\sim M} T^M{}_A + R_{\sim M}{}^A{}_B \Sigma^{MB}{}_A + F_{\sim M} \cdot J^M \\
&= T^M{}_A ({\rm E}^B{}_M [\nu_{\Sigma(V)}]^A{}_B - {\rm E}^A{}_N \nabla_M V^N) \\
&\quad - \Sigma^{MB}{}_A \nabla_M [\nu_{\Sigma(V)}]^A{}_B - J^M \cdot \tilde\nabla_M \nu_{(V)}.
\end{aligned} \tag{2.42}$$

One can clearly see that $\nabla_M V^N$, $[\nu_{\Sigma(V)}]^A{}_B$, and $\nu_{(V)}$ source this conservation. One of the prime reasons for imposing compatibility is to get rid of these mass sources.

Comparing our analysis to the torsionless case of Ref. [34], one would note that the authors there also imposed a "T-redefinition" invariance in the theory, which leads to a Galilean boost transformation upon reduction. Note that on defining $\overline{\psi}^\mu = [\Lambda_{\tau(\xi)}]^a e_a{}^\mu$, our Galilean boost transformation,

$$\delta_\xi|_{\overline\psi} B_\mu = \overline\psi_\mu, \quad \delta_\xi|_{\overline\psi} v^\mu = \overline\psi^\mu, \quad \delta_\xi|_{\overline\psi} h_{\mu\nu} = -2\overline\psi_{(\mu} n_{\nu)}, \tag{2.43}$$

boils down to the (infinitesimal) T-redefinition transformation of Ref. [34]. Hence for us imposing the T redefinition is redundant. Actually, even in Ref. [34], imposing the T redefinition was redundant, as the authors noted that the corresponding Ward identity is trivially satisfied for theories obtained by null reduction. It was helpful however to have this transformation there, because Galilean currents are not boost invariant and there was no nontrivial inherent symmetry of the partition function to keep track of these transformations. It is worth noting that the Galilean boost transformation (2.43) is same as the "Milne boost" transformation [11] encountered in the metric-like formulation of Newton-Cartan backgrounds.

III. GALILEAN FLAVOR AND SPIN ANOMALIES

In the previous section we used null reduction to obtain Ward identities for a Galilean theory with a nontrivial spin

[18] Note that fixing V^M or V^A is not a "gauge fixing," as transformations shifting these are not part of our symmetries on null backgrounds. On the other hand, fixing $\bar V^A_{(T)}$ is a gauge fixing which can be violated off shell. If we fix this gauge even off shell we would miss the corresponding temporal spin conservation equation.

[19] Although we will not be using it in this work, it is interesting to differentiate two types of mass chemical potentials. Consider that our system has a preferred symmetry data ψ_U. Naively μ_ϖ corresponds to the first law of thermodynamics written in terms of the internal energy E, while $\mu_{\rm M}$ corresponds to the first law in terms of the total energy $E_{\rm tot} = E + \frac{1}{2} R u^a u_a$ (where $u^a = \bar V^a_{(U)}$; subscripts (U) have been dropped),

$$\begin{aligned}
{\rm d}E &= \vartheta {\rm d}S + \mu_\varpi {\rm d}R + [\mu_\Sigma]^B{}_A {\rm d}[Q_\Sigma]^A{}_B + \mu \cdot {\rm d}Q, \\
{\rm d}E_{\rm tot} &= \vartheta {\rm d}S + \mu_{\rm M} {\rm d}R + u^a {\rm d}(R u_a) + [\mu_\Sigma]^B{}_A {\rm d}[Q_\Sigma]^A{}_B + \mu \cdot {\rm d}Q\cdot
\end{aligned} \tag{2.41}$$

When working with the total energy as a thermodynamic variable, the thermodynamics becomes frame dependent and the first law has a term corresponding to the work done due to the momentum density $R u_a$ as well. The notation used here can be found in Ref. [34].

current. Now we would like to take this a step ahead and ask, how are these identities modified in the presence of flavor and gravitational anomalies? We will give away the suspense right away, because the following story is quite technical. As one would expect, the flavor anomaly in null theory translates to the flavor anomaly in Galilean theory as well, while the gravitational anomaly manifests itself purely through the spatial spin conservation. The other four of the six conservation laws in Eq. (2.36) remain nonanomalous. In the formulation of anomalies in Cartan language it is not surprising; it is known that the gravitational anomaly acts as a Lorentz anomaly in this formalism and only violates the spin conservation [47]. What is surprising is that we did not find any anomalies in the temporal spin conservation (or correspondingly the Milne boost Ward identity). We do not claim that this anomaly cannot be introduced by other means or that we are not missing anything, but the fact that the number of anomaly coefficients in our treatment and that of a relativistic theory match exactly (in fact, they both are determined by the same anomaly polynomial) gives us some confidence in our results.

A. Anomaly inflow on Einstein-Cartan backgrounds

In relativistic theories anomaly inflow has been by far the most efficient way to understand flavor and gravitational anomalies [51]. We would like to take a step back and first describe the anomaly inflow mechanism for generic Einstein-Cartan theories. The extension to null theories will then be more transparent and straightforward. A good discussion on anomaly inflow for torsionless relativistic theories can be found in Sec. II of Ref. [52]. We consider that our manifold of interest $\mathcal{M}_{(d+2)}$ lives on the boundary of a *bulk* manifold $\mathcal{B}_{(d+3)}$. Bulk coordinates are denoted with a bar, and we choose a basis $x^{\overline{M}} = \{x^\perp, x^M\}$, where $x^\perp$ corresponds to depth into the bulk. All of the field content $\mathrm{E}^{\overline{A}}{}_{\overline{M}}$, $A_{\overline{M}}$, $C^{\overline{A}}{}_{\overline{M}\,\overline{B}}$ is extended down into the bulk with the requirement that all $\perp$ components vanish at the boundary.

Now we keep our theory of interest on $\mathcal{M}_{(d+2)}$, whose generating functional $W_\mathcal{M}$ is not necessarily invariant under the symmetries of the theory, i.e., it is anomalous. In the bulk we keep some theory with the generating functional $W_\mathcal{B}$, which is invariant under all the symmetries up to some nontrivial boundary terms. The full theory described by $W = W_\mathcal{M} + W_\mathcal{B}$ is assumed to be invariant under all the symmetries. It is actually this nontrivial boundary term in $W_\mathcal{B}$ which induces the anomaly in the boundary theory, hence the name *anomaly inflow*. Note that in the absence of anomalies $W_\mathcal{B} = 0 \Rightarrow W = W_\mathcal{M}$ which was discussed in the last section. Let us assume for now that we have figured out such a $W_\mathcal{B}$, and parametrize its infinitesimal variation as[20]

$$\delta W_\mathcal{B} = \int \{\mathrm{d}x^{\overline{M}}\}\sqrt{|\mathrm{G}_{(d+3)}|}(\mathrm{T}_\mathrm{H}^{\overline{M}\,\overline{A}}\delta\mathrm{E}_{\overline{A}\,\overline{M}} + \Sigma_\mathrm{H}^{\overline{M}\,\overline{A}\,\overline{B}}\delta C_{\overline{B}\,\overline{M}\,\overline{A}} + \mathrm{J}_\mathrm{H}^{\overline{M}}\cdot\delta A_{\overline{M}}) + \int\{\mathrm{d}x^M\}\sqrt{|\mathrm{G}|}(\mathrm{T}_\mathrm{BZ}^{MA}\delta\mathrm{E}_{AM} + \Sigma_\mathrm{BZ}^{MAB}\delta C_{BMA} + \mathrm{J}_\mathrm{BZ}^{M}\cdot\delta A_M). \tag{3.1}$$

It is generally known that $W_\mathcal{B}$ is topological and hence does not depend on the metric/vielbein, but we keep it here just for the sense of generality; we will see that the respective terms vanish when we put in the allowed expression for $W_\mathcal{B}$. The *Hall currents* in the bulk must be manifestly symmetry covariant by the definition of $W_\mathcal{B}$. The boundary *Bardeen-Zumino currents* on the other hand are symmetry noncovariant. The variation of $W_\mathcal{M}$ will generate the *consistent currents* which due to the anomaly are not symmetry covariant either,

$$\delta W_\mathcal{M} = \int\{\mathrm{d}x^M\}\sqrt{|\mathrm{G}|}(T_\mathrm{cons}^{MA}\delta\mathrm{E}_{AM} + \Sigma_\mathrm{cons}^{MAB}\delta C_{BMA} + J_\mathrm{cons}^{M}\cdot\delta A_M). \tag{3.2}$$

Since the full partition function W should be symmetry invariant, we can read off the symmetry-covariant, *covariant currents* in the boundary,

$$T^{MA} = T_\mathrm{cons}^{MA} + \mathrm{T}_\mathrm{BZ}^{MA}, \qquad \Sigma^{MAB} = \Sigma_\mathrm{cons}^{MAB} + \Sigma_\mathrm{BZ}^{MAB}, \qquad J^M = J_\mathrm{cons}^M + \mathrm{J}_\mathrm{BZ}^M. \tag{3.3}$$

By demanding that W is invariant under all symmetries of the theory, we will get the anomalous Ward identities for these currents,

$$\begin{aligned} \underline{\nabla}_M T^M{}_N &= \mathrm{T}^A{}_{NM}T^M{}_A + R_{NM}{}^A{}_B\Sigma^{MB}{}_A + F_{NM}\cdot J^M + \mathrm{T}_\mathrm{H}^\perp{}_N, \\ \underline{\nabla}_M\Sigma^{MAB} &= T^{[BA]} + \Sigma_\mathrm{H}^{\perp AB}, \\ \underline{\nabla}_M J^M &= \mathrm{J}_\mathrm{H}^\perp. \end{aligned} \tag{3.4}$$

We verify that the bulk Hall currents source the anomaly in the boundary theory. Note that the gravitational anomaly purely manifests itself as a Lorentz anomaly in the spin conservation equation. On the other hand, the Hall currents themselves must satisfy the nonanomalous Ward identities (2.13) in the bulk, which will be trivial if $W_\mathcal{B}$ is chosen properly. Now depending on the field content of the theory one will have to construct the most generic allowed $W_\mathcal{B}$ and read off the Hall currents from

[20]Note that $\mathrm{SO}(d+1,1)$ transformations leave the flat metric η_{AB} invariant, and hence it can commute freely through variations.

there. This would determine the most generic anomalies that can occur in the respective theory which can be modeled using the anomaly inflow mechanism. In the notation of differential forms $W_{\mathcal{B}}$ is given by the integration of a full rank form $\boldsymbol{I}^{(d+3)}$,

$$W_{\mathcal{B}} = \int_{\mathcal{B}_{(d+3)}} \boldsymbol{I}^{(d+3)}. \tag{3.5}$$

The requirement that its variation should be symmetry invariant up to a boundary term can be recast into the requirement that $\boldsymbol{\mathcal{P}}^{(d+4)} = \mathrm{d}\boldsymbol{I}^{(d+3)}$ should be symmetry invariant. $\boldsymbol{\mathcal{P}}^{(d+4)}$ is called the anomaly polynomial, which encodes all the nontrivial information about the anomaly. It is evident that $\boldsymbol{\mathcal{P}}^{(d+4)}$ needs to be closed, symmetry invariant, and should not be expressible as the exterior derivative of a symmetry-invariant form. For example, on usual backgrounds (not null), $\boldsymbol{\mathcal{P}}^{(2n+4)}$ is given by the Chern-Simons anomaly polynomial $\boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$ for even-dimensional boundary theories, and no such term is possible in odd dimensions. $\boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$ is a "polynomial" made out of Chern classes of $\boldsymbol{F}$ and Pontryagin classes of $\boldsymbol{R}$. See, e.g., Ref. [52] for more details.

B. Anomaly inflow on null/Newton-Cartan backgrounds

Now we come back to our case of interest: null backgrounds. We follow the above procedure, except that the bulk $\mathcal{B}_{(d+3)}$ is now required to possess a compatible null isometry ψ_V, which translates itself to a compatible null isometry on the boundary $\mathcal{M}_{(d+2)}$ since all the $\perp$ components vanish. The variation of $W_{\mathcal{B}}$ in Eq. (3.1) remains unchanged under a ψ_V compatible variation, except that all the currents now follow the redefinitions specified in Eqs. (2.17) and (2.20). Consequently, we can find the anomalous Ward identities for null backgrounds,

$$\begin{aligned}
\underline{\nabla}_M T^M{}_N &= \mathrm{T}^A{}_{NM} T^M{}_A + R_{NM}{}^A{}_B \Sigma^{MB}{}_A + F_{NM}\cdot J^M + \mathrm{T}^{\perp}_{\mathrm{H}N},\\
\underline{\nabla}_M \Sigma^{MAB} &= T^{[BA]} + \Sigma^{\perp AB}_{\mathrm{H}} + \#^{[A} V^{B]},\\
\underline{\nabla}_M J^M &= \mathrm{J}^{\perp}_{\mathrm{H}},
\end{aligned} \tag{3.6}$$

for some $\#^M$. These are the same as the non-null Ward identities except that, just like the nonanomalous case, some components of the spin current conservation have been discarded using the spin current redefinition (2.20). The physical components of these laws can be expressed after reduction as anomalous Galilean conservation laws,

$$\begin{aligned}
\text{Mass Cons (Continuity)}: \tilde{\underline{\nabla}}_\mu \rho^\mu &= \rho^{\perp}_{\mathrm{H}},\\
\text{Energy Cons (Time Translations)}: \tilde{\underline{\nabla}}_\mu \epsilon^\mu &= [\text{power}] - p^{\mu a} c_{\mu a} + \varepsilon^{\perp}_{\mathrm{H}},\\
\text{Momentum Cons (Translations)}: \tilde{\underline{\nabla}}_\mu p^\mu{}_a &= [\text{force}]_a - \rho^\mu c_{\mu a} + \mathrm{p}^{\perp}_{\mathrm{H}a},\\
\text{Temporal Spin Cons (Galilean Boosts)}: \tilde{\underline{\nabla}}_\mu \tau^{\mu a} &= \frac{1}{2}(\rho^a - p^a) + \tau^{\perp a}_{\mathrm{H}},\\
\text{Spatial Spin Cons (Rotations)}: \tilde{\underline{\nabla}}_\mu \sigma^{\mu ab} &= p^{[ba]} + 2\tau^{\mu[a} c_\mu{}^{b]} + \sigma^{\perp ab}_{\mathrm{H}},\\
\text{Charge Cons (Flavor Transformations)}: \tilde{\underline{\nabla}}_\mu j^\mu &= \mathrm{j}^{\perp}_{\mathrm{H}},
\end{aligned} \tag{3.7}$$

where we have decomposed the Hall currents as

$$\begin{aligned}
\mathrm{T}^{\perp}_{\mathrm{H}A} &= \begin{pmatrix} -\rho^{\perp}_{\mathrm{H}} & -\varepsilon^{\perp}_{\mathrm{H}} & \mathrm{p}^{\perp}_{\mathrm{H}a} \end{pmatrix},\\
\mathrm{J}^{\perp}_{\mathrm{H}} &= \mathrm{j}^{\perp}_{\mathrm{H}},\\
\Sigma^{\perp AB}_{\mathrm{H}} &= \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \tau^{\perp b}_{\mathrm{H}} \\ \times & -\tau^{\perp a}_{\mathrm{H}} & \sigma^{\perp ab}_{\mathrm{H}} \end{pmatrix}.
\end{aligned} \tag{3.8}$$

We hence see that in principle the anomaly inflow can destroy all the conservation laws. It is now the form of $\boldsymbol{\mathcal{P}}^{(d+4)}$ which will determine how many of these anomalies are permissible and in what number of dimensions.

On even-dimensional $(d = 2n)$ null backgrounds the allowed anomaly polynomial takes the usual Chern-Simons structure of relativistic theories $\boldsymbol{\mathcal{P}}^{(2n+4)} = \boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$, which is made up of Chern classes of $\boldsymbol{F}$ and Pontryagin classes of $\boldsymbol{R}$. Note however that neither $\boldsymbol{F}$ nor $\boldsymbol{R}$ have a leg along V, and hence $\boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$ is identically zero. The corresponding $\boldsymbol{I}_{\mathrm{CS}}$ might still have a leg along V since $\iota_V \boldsymbol{A}, \iota_V \boldsymbol{C}^A_B \neq 0$ for a general null theory. But one can check that the corresponding $\perp$ components of the (dual) Hall currents again have no leg along V and hence the Ward identities become nonanomalous. This suggests that we cannot get anomalies in an even-dimensional null theory, and hence odd-dimensional Galilean theories are anomaly free.

At this point we would like to point out some subtle differences from the analysis of Ref. [37]. In the cited reference the author did not impose compatibility of the isometry, and hence $\boldsymbol{F}, \boldsymbol{R}$ do have a leg along V. This results in anomalous conservation laws that crucially depend on

$\nu_{(V)}$, $[\nu_{\Sigma(V)}]^A{}_B$—additional fields which are otherwise switched off by compatibility. As we mentioned in the Introduction, we have chosen to switch off these fields as they serve as "mass sources" in the Galilean theory, and we do not see these sources in the nonrelativistic theories that occur in nature.

Now we shift our attention to the more interesting case of odd-dimensional ($d = 2n - 1$) null backgrounds. One can check that with the field content at hand, it is not possible to naively construct an anomaly polynomial. Following Ref. [34], however, we note that we can remedy this problem by introducing the auxiliary *time data* ψ_T that was used to perform null reduction in Sec. II C. Using the corresponding $\overline{V}_{(T)}$ defined in Eq. (2.23), we can write the only allowed anomaly polynomial,

$$\mathcal{P}^{(2n+3)} = \overline{\boldsymbol{V}}_{(T)} \wedge \mathcal{P}^{(2n+2)}_{\mathrm{CS}}, \tag{3.9}$$

where $\overline{\boldsymbol{V}}_{(T)} = \overline{V}_{(T)M}\mathrm{d}x^M$. Although this expression has an explicit dependence on ψ_T, one can show that it is invariant under any arbitrary redefinition of ψ_T. This follows from the fact that a change in $\overline{V}_{(T)M}$ does not have any leg along V, due to the normalization property $\delta(\overline{V}_{(T)M}V^M) = V^M\delta\overline{V}_{(T)M} = 0$. For this reason we drop the subscript (T) from $\overline{V}_{(T)}$ from this point onward. Readers should convince themselves that there are no more terms which can be written in the anomaly polynomial. However, we have a problem: the anomaly polynomial in Eq. (3.9) is not exact,

$$\mathcal{P}^{(2n+3)} = -\mathrm{d}(\overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}) + \mathrm{d}\overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}. \tag{3.10}$$

Hence for $\boldsymbol{I}^{(2n+2)}$ (and hence $W_{\mathcal{B}}$) to be well defined, the second term must vanish. In general however it does not, as $\boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}$ can have a leg along V. In fact, it can be shown that $\boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}$ does not have a leg along V if and only if ψ_V is in the *transverse gauge*, i.e.,

$$\Lambda_{(V)} = [\Lambda_{\Sigma(V)}]^A{}_B = 0. \tag{3.11}$$

Some comments are in order. Different choices of ψ_V represent different null theories, as we are not allowed to perform transformations which alter these (we demanded that the partition function be invariant only under ψ_V-preserving transformations). Hence this mechanism can only generate anomalies in null theories with null isometry in transverse gauge; otherwise, the last term in Eq. (3.10) will not vanish and we would not be able to define a $W_{\mathcal{B}}$. Note that in conventional null reduction, one generally chooses $\psi_V = \{\partial_\sim, 0, 0\}$ which by choice satisfies the transversality requirement. Modulo this subtlety, we can find

$$\boldsymbol{I}^{(2n+2)} = -\overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}. \tag{3.12}$$

Computing its variation, one can find the *Hall* and *Bardeen-Zumino* currents defined in Eq. (3.1),

$$\mathrm{T}^{\overline{M}\,\overline{A}}_{\mathrm{H}} = 0, \qquad \star_{(2n+2)}\Sigma^{\overline{A}\,\overline{B}}_{\mathrm{H}} = \overline{\boldsymbol{V}} \wedge \frac{\partial \mathcal{P}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{\overline{B}\,\overline{A}}},$$
$$\star_{(2n+2)}\mathbf{J}_{\mathrm{H}} = \overline{\boldsymbol{V}} \wedge \frac{\partial \mathcal{P}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{F}}, \qquad \mathrm{T}^{MA}_{\mathrm{BZ}} = 0,$$
$$\star\Sigma^{AB}_{\mathrm{BZ}} = \overline{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{BA}}, \qquad \star\mathbf{J}_{\mathrm{BZ}} = -\overline{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}}{\partial \boldsymbol{F}}. \tag{3.13}$$

We verify that $T^{\overline{M}\,\overline{A}}_{\mathrm{H}}, T^{MA}_{\mathrm{BZ}}$ vanish. It immediately follows that the mass, energy, and momentum conservations are nonanomalous. Also the (Milne) boost Ward identity stays nonanomalous as the matrix indices of $\Sigma^{MAB}_{\mathrm{H}}$ come from $\boldsymbol{R}^A{}_B$ which have a zero contraction along V. Again this follows from the compatibility of the isometry, and is not true for the considerations of Ref. [37], which is why they found a Milne anomaly. These statements can be recast as

$$\rho^{\perp}_{\mathrm{H}} = \varepsilon^{\perp}_{\mathrm{H}} = p^{\perp a}_{\mathrm{H}} = \tau^{\perp a}_{\mathrm{H}} = 0, \tag{3.14}$$

which follows directly from null reduction. The only laws that become anomalous are hence the spin and charge conservation. Explicit expressions for their Hall currents follow from reduction,

$$\mathrm{j}^{\perp}_{\mathrm{H}} = -*_{\uparrow}\left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{F}}\right], \qquad \sigma^{\perp ab}_{\mathrm{H}} = -*_{\uparrow}\left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{R}_{ba}}\right]. \tag{3.15}$$

Here we have formally denoted $\mathcal{P}^{(2n+2)}_{\mathrm{CS}}$ as $\mathfrak{p}^{(2n+2)}$ after reduction; the distinction is purely notational. $*_{\uparrow}$ is the Hodge dual associated with the raised Newton-Cartan volume element $\varepsilon_{\uparrow}$; refer to Appendix C for more details. Putting Eqs. (3.14) and (3.15) back into Eq. (2.36), we can get the anomalous Ward identities for Galilean theories.

Before closing this section, we would like to make some comments on the even-dimensional case. One might worry that we can use ψ_T to define anomalies in even dimensions as well. However one can check that the only possible symmetry-covariant anomaly polynomial we can write involving ψ_T is

$$\mathcal{P}^{(2n+4)} = \boldsymbol{V} \wedge \overline{\boldsymbol{V}} \wedge \mathcal{P}^{(2n+2)}_{\mathrm{CS}}, \tag{3.16}$$

where $\boldsymbol{V} = V_M\mathrm{d}x^M$. This anomaly polynomial is however not an exact form,

$$\mathcal{P}^{(2n+4)} = \mathrm{d}(\boldsymbol{V} \wedge \overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}) - \boldsymbol{H} \wedge \overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}} + \boldsymbol{V} \wedge \mathrm{d}\overline{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}. \tag{3.17}$$

The last term can be removed just like before by going to the transverse gauge, but the second to last term cannot. We hence see that the current formalism does not allow for anomalous even-dimensional null theories. From this point onward we will assume that our null background is odd-dimensional, and hence set $d = 2n - 1$.

With this we conclude our discussion of generic anomalous Galilean theories. Using the construction of null backgrounds, we have found a set of conservation laws which determine the dynamics of these theories in terms of a set of currents. These laws have already been well explored in the literature, but the fact that they follow by trivially choosing a basis in a higher-dimensional null theory is to be appreciated. Going along the lines of Ref. [17], it appears to us that null backgrounds are the true "covariant" and "frame-independent" formalism of Galilean physics, which appear pretty naturally from a five-dimensional perspective. We refer the reader to Appendix B for more comments on these issues.

All of the results presented here are in the Newton-Cartan notation, which is the natural covariant prescription for Galilean physics. In Appendix A we present some of our results in the conventional noncovariant notation, for the benefit of readers who are not comfortable with the Newton-Cartan language. In addition, seeing the results in noncovariant form might help us relate it better to everyday physics, where we are used to viewing time and space separately.

IV. ANOMALOUS GALILEAN HYDRODYNAMICS

In previous sections we have obtained the anomalous conservation laws for a null/Galilean theory with a nonzero spin current. Here we want to study these theories in the hydrodynamic limit—the near-equilibrium effective description of any quantum system. Before going to that, let us make some general comments about the hydrodynamics on Einstein-Cartan backgrounds. We start by picking up a collection of *hydrodynamic fields* which can be exactly solved for by using the equations of motion of the theory. Since there is an equation of motion for each symmetry data, we choose the hydrodynamic fields to be a set of symmetry data[21] $\psi_U = \{U^M, [\Lambda_\Sigma]^A{}_B, \Lambda\}$. The *fluid* (hydrodynamic system) is characterized by conserved currents T^{MA}, Σ^{MAB}, J^M written as the most generic tensors made out of these hydrodynamic fields ψ_U and background sources E^A_M, C^A_{MB}, A_M, arranged in a derivative expansion. These are known as the *constitutive relations* of the fluid. The near equilibrium assumption of hydrodynamics implies that derivatives of the quantities are small compared to the quantities themselves, which allows for a proper truncation of the derivative expansion. The dynamics of these constitutive relations in turn is governed by the conservation laws (3.4). These constitutive relations are further subjected to the second law of thermodynamics, i.e., the requirement of an entropy current S^M such that $\underline{\nabla}_M S^M \geq 0$, whenever equations of motion are satisfied. This requirement imposes various constraints on the constitutive relations, and the job of hydrodynamics is to monitor these constraints. Having done so, one can in principle plug these constitutive relations back into the equations of motion and solve for exact "configurations" of the hydrodynamic fields, which is not in the scope of hydrodynamics. A nice and modern review of relativistic hydrodynamics can be found in Sec. I of Ref. [42].

Another notion which is inherent to any statistical system is *equilibrium*. Equilibrium is the steady state of hydrodynamics, when the fluid has came in terms with the background and has aligned itself accordingly, i.e. hydrodynamic variables ψ_U are completely determined in terms of the background fields. In this state, the fluid can be described by a partition function W^{eqb} written purely in terms of the background data, and the equations of motion are trivially satisfied [43–45]. Equilibrium is generally defined by a collection of symmetry data $\psi_K = \{K^M, [\Lambda_{\Sigma(K)}]^A{}_B, \Lambda_{(K)}\}$ which acts as an isometry on the background. For our constitutive relations to be physical, we will need to ensure that on introducing ψ_K they trivially satisfy the equations of motion (3.4).

Please note that ψ_U is a set of variables we have picked up to solve the system; like in any field theory, we could do an arbitrary field redefinition of ψ_U without changing the physics. This is known as the *hydrodynamic redefinition freedom*. By convention ψ_U is defined to agree with ψ_K in equilibrium at zero derivative order (this goes into the definition of the fluid velocity, temperature, and chemical potential in equilibrium), which fixes a huge amount of this freedom. Further fixing of this freedom can be dealt with in various different ways, which takes the name of hydrodynamic frames (a more thorough discussion on these frames for null fluids can be found in Ref. [34]). Here we will work in the so-called *equilibrium frame* where $\psi_U = \psi_K$ exactly in equilibrium, not just at zero derivative order. Note that this does not fix the freedom completely; we can still perturb this relation with anything that vanishes in equilibrium. For now we conclude that on setting $\psi_U = \psi_K$, i.e., on promoting ψ_U to an isometry, the constitutive relations should identically satisfy the equations of motion.

It was noted in Ref. [40] for relativistic fluids that it is helpful to remove the clause "whenever equations of motion are satisfied" from the second law requirement and upgrade it to an off-shell statement [53], which for us will read

$$\begin{aligned}\underline{\nabla}_M S^M &+ U^N(\underline{\nabla}_M T^M{}_N - \mathrm{T}^A{}_{NM} T^M{}_A - R_{NM}{}^A{}_B \Sigma^{MB}{}_A \\ &- F_{NM} \cdot J^M) + [\nu_\Sigma]_{BA}(\underline{\nabla}_M \Sigma^{MAB} - T^{[BA]} - \Sigma^{\perp AB}_{\mathrm{H}}) \\ &+ \nu \cdot (\underline{\nabla}_M J^M - \mathrm{J}^{\perp}_{\mathrm{H}}) \geq 0. \end{aligned} \tag{4.1}$$

[21]We drop the subscript (U) for ψ_U and hope that it will be clear from the context.

This statement is slightly different from what was considered for the torsionless case in Ref. [40], but we verify its equivalence with theirs in Appendix D.

Now we come back to null fluids—fluids on null backgrounds. On null backgrounds, hydrodynamic data ψ_U needs to be compatible with ψ_V, i.e., $[\psi_V, \psi_U] = 0$ and $[\nu_\Sigma]^A{}_B V^B = 0$. This makes sense because 1) the resulting constitutive relations must follow the null isometry, and 2) not all components of the spin conservation in Eq. (3.6) are physical. Further, the constitutive relations are allowed to depend on ψ_V as well. One can check that upon making these tweaks, the off-shell second law (4.1) remains unchanged. We can now go back and study the most generic constitutive relations for null fluids, which have been thoroughly considered in Ref. [34] for a charged spinless torsionless null fluid with U(1) anomalies up to leading order in derivatives. In this work, however, we are only interested in the sector of hydrodynamics that is governed and is completely determined by the anomalies.[22] To accomplish this task in relativistic fluids, the authors of Ref. [38] (see also Ref. [54]) proposed a mechanism based on transgression forms, which allows us to "integrate" the anomalous equations of motion (3.4) and directly figure out the anomalous contribution to the constitutive relations. We will attempt to extend this construction to null fluids.

A. Anomalous null fluids

We start by defining the hydrodynamic shadow gauge field and spin connection,

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu \boldsymbol{V}, \qquad \hat{\boldsymbol{C}}^A{}_B = \boldsymbol{C}^A{}_B + [\mu_\Sigma]^A{}_B \boldsymbol{V}, \quad (4.2)$$

where μ, $[\mu_\Sigma]^A{}_B$ are flavor and spin chemical potentials associated with ψ_U defined in Eq. (2.22). One can check that both ψ_U, ψ_V are compatible with this new gauge field and spin connection, i.e.,

$$\begin{aligned} \hat{\nu} &= U^M \hat{A}_M + \Lambda = 0, \\ [\hat{\nu}_\Sigma]^A{}_B &= U^M \hat{C}^A{}_{MB} + [\Lambda_\Sigma]^A{}_B = 0, \\ \hat{\nu}_{(V)} &= V^M \hat{A}_M = 0, \\ [\hat{\nu}_{\Sigma(V)}]^A{}_B &= V^M \hat{C}^A{}_{MB} = 0. \end{aligned} \quad (4.3)$$

Recall that we have chosen $\Lambda_{(V)} = [\Lambda_{\Sigma(V)}]^A{}_B = 0$ to be able to define anomalies. We define the operation $(\hat{\ })$ as $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}(\boldsymbol{A} \to \hat{\boldsymbol{A}}, \boldsymbol{C}^A{}_B \to \hat{\boldsymbol{C}}^A{}_B)$. One can check that the hatted field strengths also follow the null background conditions (2.14) and (2.15). We would like to import one result from the transgression machinery without proof (see Sec. 11 of Ref. [55] for more details), which implies that

$$\boldsymbol{I}^{(2n+1)}_{\text{CS}} - \hat{\boldsymbol{I}}^{(2n+1)}_{\text{CS}} = \boldsymbol{\mathcal{V}}^{(2n+1)}_{\mathcal{P}_{\text{CS}}} + \mathrm{d}\boldsymbol{\mathcal{V}}^{(2n)}_{I_{\text{CS}}}, \quad (4.4)$$

where[23]

$$\begin{aligned} \boldsymbol{\mathcal{V}}^{(2n+1)}_{\mathcal{P}_{\text{CS}}} &= \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\boldsymbol{\mathcal{P}}^{(2n+2)}_{\text{CS}} - \hat{\boldsymbol{\mathcal{P}}}^{(2n+2)}_{\text{CS}}), \\ \boldsymbol{\mathcal{V}}^{(2n)}_{I_{\text{CS}}} &= \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\boldsymbol{I}^{(2n+1)}_{\text{CS}} - \hat{\boldsymbol{I}}^{(2n+1)}_{\text{CS}}). \end{aligned} \quad (4.5)$$

One can check that these quantities are well defined. We argue that the fluid in the equilibrium configuration can be described by a (bulk + boundary) partition function $W^{\text{eqb}} = W^{\text{eqb}}_{\mathcal{B}} + W^{\text{eqb}}_{\mathcal{M}}$ which has been discussed in the preceding sections.[24] Away from equilibrium, however, the system is described by an effective action $S = S_{\mathcal{B}} + S_{\mathcal{M}}$ which, in equilibrium, boils down to W^{eqb}. It is important as W^{eqb} is only defined in equilibrium. We claim that the appropriate $S_{\mathcal{B}}$ to generate the anomalous sector of the null hydrodynamics is[25]

$$\begin{aligned} S_{\mathcal{B}} &= W_{\mathcal{B}} + \int_{\mathcal{B}_{(2n+2)}} \overline{\boldsymbol{V}} \wedge \hat{\boldsymbol{I}}^{(2n+1)}_{\text{CS}} \\ &= -\int_{\mathcal{B}_{(2n+2)}} \overline{\boldsymbol{V}} \wedge (\boldsymbol{I}^{(2n+1)}_{\text{CS}} - \hat{\boldsymbol{I}}^{(2n+1)}_{\text{CS}}). \end{aligned} \quad (4.6)$$

In equilibrium ($\psi_U = \psi_K$) and on choosing transverse gauge for ψ_K (i.e., $\Lambda_{(K)} = [\Lambda_{\Sigma(K)}]^A{}_B = 0$) the added piece vanishes, as it does not have any leg along V, and we recover the equilibrium partition function. Using Eq. (4.4) we can decompose $S_{\mathcal{B}}$ as

$$S_{\mathcal{B}} = \int_{\mathcal{B}_{(2n+2)}} \boldsymbol{\mathcal{V}}^{(2n+2)}_{\mathcal{P}} + \int_{\mathcal{M}_{(2n+2)}} \boldsymbol{\mathcal{V}}^{(2n+1)}_{I}, \quad (4.7)$$

where we have identified

[22] In relativistic hydrodynamics it is known [52] that there are certain coefficients which appear as independent constants in the naive derivative expansion, but can be fixed in terms of anomaly coefficients appearing at higher derivative orders by demanding consistency of the Euclidean vacuum. Similar constants have also showed up for Galilean fluids in Refs. [15,34], but their connection to the anomaly is not yet clear. Here however we do not consider these contributions.

[23] See footnote 13.

[24] In making this statement, we are implicitly relying on the existence of an equilibrium partition function which describes the fluid in the equilibrium configuration. These ideas were discussed for a relativistic fluid in Refs. [43–45] and were later adopted to Galilean fluids in Refs. [15,16,34,35].

[25] It was argued in Ref. [56] that while this effective action is appropriate to give solutions to the off-shell second law of thermodynamics, the minimization of this action with respect to dynamic fields does not give the correct dynamics. To get the correct dynamics we need to further modify this action in the Schwinger-Keldysh formalism, which we do not touch upon here.

$$\begin{aligned}\mathcal{V}_{\mathcal{P}}^{(2n+2)} &= \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\mathcal{P}^{(2n+3)} - \hat{\mathcal{P}}^{(2n+3)}) \\ &= -\overline{\boldsymbol{V}} \wedge \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\mathcal{P}_{\mathrm{CS}}^{(2n+2)} - \hat{\mathcal{P}}_{\mathrm{CS}}^{(2n+2)}), \\ \mathcal{V}_{\boldsymbol{I}}^{(2n+1)} &= \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\boldsymbol{I}^{(2n+2)} - \hat{\boldsymbol{I}}^{(2n+2)}) \\ &= \overline{\boldsymbol{V}} \wedge \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} - \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)}). \end{aligned} \tag{4.8}$$

The bulk term in Eq. (4.7) is manifestly symmetry invariant, and the full S is symmetry invariant by definition; hence, if we decompose $S_{\mathcal{M}} = S_{\text{n-a}} + S_{\mathcal{M},\text{anom}}$ with the first piece being totally symmetry invariant we can infer

$$S_{\mathcal{M},\text{anom}} = -\int_{\mathcal{B}_{(2n+2)}} \overline{\boldsymbol{V}} \wedge \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)}. \tag{4.9}$$

$S_{\mathcal{M},\text{anom}}$ will generate the anomalous sector of the consistent currents. On the other hand, for the full effective action we will be left with $S = S_{\text{anom}} + S_{\text{n-a}}$ where

$$S_{\text{anom}} = \int_{\mathcal{B}_{(2n+2)}} \mathcal{V}_{\mathcal{P}}^{(2n+2)}. \tag{4.10}$$

S_{anom} will generate the anomalous sector of the covariant currents.

Constitutive relations: In light of our discussion above, we should be able to generate the anomalous sector of covariant currents by varying S_{anom}. We will get

$$\begin{aligned}\delta S_{\text{anom}} = &\int_{\mathcal{B}_{(2n+2)}} (\delta\boldsymbol{A} \wedge \cdot\star_{(2n+2)}\mathbf{J}_{\mathrm{H}} - \delta\hat{\boldsymbol{A}} \wedge \cdot\star_{(2n+2)}\hat{\mathbf{J}}_{\mathrm{H}} \\ &+ \delta\boldsymbol{C}^{\overline{A}}{}_{\overline{B}} \wedge \star_{(2n+2)}\Sigma_{\mathrm{H}}{}^{\overline{B}}{}_{\overline{A}} - \delta\hat{\boldsymbol{C}}^{\overline{A}}{}_{\overline{B}} \wedge \star_{(2n+2)}\hat{\Sigma}_{\mathrm{H}}{}^{\overline{B}}{}_{\overline{A}}) \\ &+ \int_{\mathcal{M}_{(2n+1)}} (\delta\boldsymbol{A} \wedge \cdot\star\boldsymbol{J}_{\mathcal{P}} + \delta\boldsymbol{C}^{A}{}_{B} \wedge \star\Sigma_{\mathcal{P}}{}^{B}{}_{A} \\ &+ \delta\boldsymbol{V} \wedge \star\boldsymbol{E}_{\mathcal{P}}), \end{aligned} \tag{4.11}$$

where we have defined

$$\begin{aligned}\star\boldsymbol{E}_{\mathcal{P}} &= \frac{\partial\mathcal{V}_{\mathcal{P}}^{(2n+2)}}{\partial\boldsymbol{H}} \\ &= \frac{\boldsymbol{V}}{\boldsymbol{H}^{\wedge 2}} \wedge [\hat{\mathcal{P}}^{(2n+3)} - \mathcal{P}^{(2n+3)} \\ &\quad - \boldsymbol{H} \wedge \star_{(2n+2)}(\mu\cdot\hat{\mathbf{J}}_{\mathrm{H}} + [\mu_\Sigma]^{A}{}_{B}\hat{\Sigma}_{\mathrm{H}}{}^{B}{}_{A})], \\ \star\Sigma_{\mathcal{P}}{}^{A}{}_{B} &= \frac{\partial\mathcal{V}_{\mathcal{P}}^{(2n+2)}}{\partial\boldsymbol{R}^{B}{}_{A}} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \star_{(2n+2)}(\Sigma_{\mathrm{H}}{}^{A}{}_{B} - \hat{\Sigma}_{\mathrm{H}}{}^{A}{}_{B}), \\ \star\boldsymbol{J}_{\mathcal{P}} &= \frac{\partial\mathcal{V}_{\mathcal{P}}^{(2n+2)}}{\partial\boldsymbol{F}} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \star_{(2n+2)}(\mathbf{J}_{\mathrm{H}} - \hat{\mathbf{J}}_{\mathrm{H}}), \end{aligned} \tag{4.12}$$

and the Hall currents have been defined in Eq. (3.13). Since S_{anom} is invariant under the symmetries by construction, we can find a set of Bianchi identities that these currents must follow,

$$\begin{aligned}\underline{\nabla}_M(T^M{}_N)_{\mathrm{A}} &= \mathrm{T}^A{}_{NM}(T^M{}_A)_{\mathrm{A}} + R_{NK}{}^{A}{}_{B}(\Sigma^{MB}{}_A)_{\mathrm{A}} \\ &\quad + F_{NM}(J^M)_{\mathrm{A}} - V_N(\mu\cdot\hat{\mathrm{J}}_{\mathrm{H}}^{\perp} + [\mu_\Sigma]_{AB}\hat{\Sigma}_{\mathrm{H}}^{\perp BA}), \\ \underline{\nabla}_M(\Sigma^{MAB})_{\mathrm{A}} &= (T^{[BA]})_{\mathrm{A}} + \Sigma_{\mathrm{H}}^{\perp AB} - \hat{\Sigma}_{\mathrm{H}}^{\perp AB} + \#^{[A}V^{B]}, \\ \underline{\nabla}_M(J^M)_{\mathrm{A}} &= \mathrm{J}_{\mathrm{H}}^{\perp} - \hat{\mathrm{J}}_{\mathrm{H}}^{\perp}, \end{aligned} \tag{4.13}$$

where we have defined the anomalous "class" of constitutive relations,

$$(T^{MA})_{\mathrm{A}} = E_{\mathcal{P}}^{M}V^{A}, \quad (\Sigma^{MAB})_{\mathrm{A}} = \Sigma_{\mathcal{P}}^{MAB}, \quad (J^M)_{\mathrm{A}} = J_{\mathcal{P}}^{M}. \tag{4.14}$$

One can check that on plugging in $\psi_U = \psi_K$, the hatted Hall currents vanish as they do not have any leg along K and V simultaneously. Consequently the Bianchi identities (4.13) reduce to the equations of motion (3.6). In other words the currents $(T^{MA})_{\mathrm{A}}$, $(\Sigma^{MAB})_{\mathrm{A}}$, $(J^M)_{\mathrm{A}}$ identically satisfy the equations of motion in the equilibrium configuration, as required.

We would like to remind the reader that $\overline{V}$ was added as an arbitrary choice of frame and the anomaly polynomial was invariant under a ψ_T redefinition which shifts $\overline{V}$. The currents we have constructed should then also be invariant under a ψ_T redefinition. One can check that under a ψ_T redefinition the currents in Eq. (4.12) shift by a closed form. By definition the currents always have this ambiguity, and hence we do not change the physics. In hydrodynamics the most natural choice of ψ_T to define anomalies is to set $\psi_T = \psi_U$.

Adiabaticity and entropy current: To claim that the currents we have constructed are physical, we must find a $(S^M)_{\mathrm{A}}$ which satisfies the off-shell second law (4.1). The anomalous sector is bound to be parity violating, implying that no scalar expression can be guaranteed to be positive definite. This turns Eq. (4.1) into a more stringent condition,

$$\begin{aligned}&\underline{\nabla}_M(S^M)_{\mathrm{A}} + U^N[\underline{\nabla}_M(T^M{}_N)_{\mathrm{A}} - \mathrm{T}^A{}_{NM}(T^M{}_A)_{\mathrm{A}} \\ &\quad - R_{NM}{}^{A}{}_{B}(\Sigma^{MB}{}_A)_{\mathrm{A}} - F_{NM}\cdot(J^M)_{\mathrm{A}}] \\ &\quad + [\nu_\Sigma]_{BA}[\underline{\nabla}_M(\Sigma^{MAB})_{\mathrm{A}} - (T^{[BA]})_{\mathrm{A}} - \Sigma_{\mathrm{H}}^{\perp AB}] \\ &\quad + \nu\cdot(\underline{\nabla}_M(J^M)_{\mathrm{A}} - \mathrm{J}_{\mathrm{H}}^{\perp}) = 0, \end{aligned} \tag{4.15}$$

known as the *adiabaticity equation* [41]. By putting the constitutive relations directly into this expression we can get

$$\nabla_M(S^M)_{\mathrm{A}} = 0. \tag{4.16}$$

Hence it suffices to choose an identically zero anomaly-induced entropy current $(S^M)_{\mathrm{A}} = 0$ to satisfy the adiabaticity equation. We would like to comment here that the vanishing of the anomaly-induced entropy current does not rely on the background being null; it is equally true for the usual Einstein-Cartan backgrounds as well. See Appendix D for more comments on the relativistic entropy current.

Equilibrium partition function: In the beginning of this section we argued that at equilibrium, fluid can be described by a partition function written purely in terms of the background data. We will now attempt to find such an equilibrium partition function. We start by computing the variation of the boundary effective action $S_{\mathcal{M},\mathrm{anom}}$ given in Eq. (4.9),

$$\delta S_{\mathcal{M},\mathrm{anom}} = \int \{\mathrm{d}x^M\}\sqrt{|G|}[(T^{MA})_{\mathrm{A}}\delta E_{AM} + \{(\Sigma^{MAB})_{\mathrm{A}} - \Sigma^{MAB}_{\mathrm{BZ}}\}\delta C_{BMA} + \{(J^M)_{\mathrm{A}} - \mathrm{J}^M_{\mathrm{BZ}}\}\cdot\delta A_M] + \int\{\mathrm{d}x^M\}\sqrt{|G|}[\hat{\Sigma}^{MAB}_{\mathrm{BZ}}\delta\hat{C}_{BMA} + \hat{\mathrm{J}}^M_{\mathrm{BZ}}\cdot\delta\hat{A}_M]. \tag{4.17}$$

In equilibrium and choosing transverse gauge for ψ_K, i.e., $\Lambda_{(K)} = [\Lambda_{\Sigma(K)}]^A{}_B = 0$, the terms in the last line vanish. Hence we can define the equilibrium boundary partition function as

$$W^{\mathrm{eqb}}_{\mathcal{M},\mathrm{anom}} = S_{\mathcal{M},\mathrm{anom}}|_{\psi_U=\psi_K} = -\int_{\mathcal{M}_{(2n+1)}}\frac{\boldsymbol{V}}{\boldsymbol{H}}\wedge(\boldsymbol{I}^{(2n+2)} - \hat{\boldsymbol{I}}^{(2n+2)})|_{\psi_U=\psi_K}. \tag{4.18}$$

Putting it together with $W_{\mathcal{B}}$, we can get the equilibrium partition function for the full theory. In practice, however, if one knows the expressions for the Bardeen-Zumino currents, it suffices to have the boundary partition function to generate the covariant currents.

B. Null reduction—anomalous Galilean fluids

Having obtained the constitutive relations for anomalous null fluids, it is now time to perform null reduction and extract the Galilean results. To see this we can directly break up the anomaly-induced constitutive relations $(T^{MA})_{\mathrm{A}}$, $(\Sigma^{MAB})_{\mathrm{A}}$, $(J^M)_{\mathrm{A}}$ into the basis given in Eq. (2.34). A straightforward computation will yield trivial identifications,

$$(\rho^\mu)_{\mathrm{A}} = 0, \quad (p^{\mu a})_{\mathrm{A}} = 0, \quad (\tau^{\mu a})_{\mathrm{A}} = 0, \quad (s^\mu)_{\mathrm{A}} = 0,$$
$$(\epsilon^\mu)_{\mathrm{A}} = E^\mu_{\mathcal{P}}, \quad (\sigma^{\mu ab})_{\mathrm{A}} = \Sigma^{\mu ab}_{\mathcal{P}}, \quad (j^\mu)_{\mathrm{A}} = J^\mu_{\mathcal{P}}. \tag{4.19}$$

We have also included an entropy current $(s^\mu)_{\mathrm{A}} = (S^\mu)_{\mathrm{A}}$ here which of course is trivially zero. For the record we write down the off-shell second law of thermodynamics for Galilean fluids,

$$\underline{\tilde{\nabla}}_\mu s^\mu + \nu_{\mathrm{M}}\underline{\tilde{\nabla}}_\mu\rho^\mu - \frac{1}{\vartheta}(\underline{\tilde{\nabla}}_\mu\epsilon^\mu - [\mathrm{power}] + p^{\mu a}c_{\mu a}) + \frac{1}{\vartheta}u^a(\underline{\tilde{\nabla}}_\mu p^\mu{}_a - [\mathrm{force}]_a + \rho^\mu c_{\mu a}) + [\nu_\tau]_a\left(\underline{\tilde{\nabla}}_\mu\tau^{\mu a} - \frac{1}{2}(\rho^a - p^a)\right) + [\nu_\sigma]_{ba}(\underline{\tilde{\nabla}}_\mu\sigma^{\mu ab} - p^{[ba]} - 2\tau^{\mu[a}c_\mu{}^{b]} - \sigma^{\perp ab}_{\mathrm{H}}) + \nu\cdot(\underline{\tilde{\nabla}}_\mu j^\mu - \mathrm{j}^\perp_{\mathrm{H}}) \geq 0, \tag{4.20}$$

where $u^\mu = \overline{V}^\mu_{(U)}$ [defined in Eq. (2.23)] and $u^a = e^a{}_\mu u^\mu$ is the spatial velocity of the fluid. ϑ is the temperature, $\nu_{\mathrm{M}} = \varpi - \frac{1}{2\vartheta}u^a u_a$ is the total mass chemical potential, ϖ is the mass chemical potential, $[\nu_\tau]_a = [\nu_\Sigma]^-{}_a$ is the boost chemical potential, $[\nu_\sigma]^a{}_b = [\nu_\Sigma]^a{}_b$ is the spatial spin chemical potential, and ν is the flavor chemical potential associated with the fluid data ψ_U [the respective definitions can be found in Eqs. (2.9) and (2.21)]. A version of this off-shell second law of thermodynamics for Galilean fluids in a metric-like formalism was first written down in Ref. [15]. This expression will be greatly simplified if we choose $\psi_T = \psi_U$, i.e., choose to describe the fluid in its local rest frame, because then $u^a = 0$,

$$\underline{\tilde{\nabla}}_\mu s^\mu + \varpi\underline{\tilde{\nabla}}_\mu\rho^\mu - \frac{1}{\vartheta}(\underline{\tilde{\nabla}}_\mu\epsilon^\mu - [\mathrm{power}] + p^{\mu a}c_{\mu a}) + [\nu_\tau]_a\left(\underline{\tilde{\nabla}}_\mu\tau^{\mu a} - \frac{1}{2}(\rho^a - p^a)\right) + [\nu_\sigma]_{ba}(\underline{\tilde{\nabla}}_\mu\sigma^{\mu ab} - p^{[ba]} - 2\tau^{\mu[a}c_\mu{}^{b]} - \sigma^{\perp ab}_{\mathrm{H}}) + \nu\cdot(\underline{\tilde{\nabla}}_\mu j^\mu - j^\perp_{\mathrm{H}}) \geq 0. \tag{4.21}$$

It should be apparent that on putting in the equations of motion it simply gives the second law of thermodynamics, $\underline{\tilde{\nabla}}_\mu s^\mu \geq 0$. If one does not prefer to perform reduction to get $(\epsilon^\mu)_{\mathrm{A}}$, $(j^\mu)_{\mathrm{A}}$, $(\sigma^{\mu ab})_{\mathrm{A}}$, these can be generated directly from the Newton-Cartan transgression form,

$$\boldsymbol{\mathcal{V}}^{(2n+1)}_{\mathfrak{p}} = -\frac{\boldsymbol{n}}{\boldsymbol{H}}\wedge(\mathfrak{p}^{(2n+2)} - \hat{\mathfrak{p}}^{(2n+2)}), \tag{4.22}$$

where $\mathfrak{p}^{(2n+2)}$ is the NC anomaly polynomial defined at the end of Sec. III B, and the hatted fields are

$$\hat{\boldsymbol{A}} = \boldsymbol{A} - \mu\boldsymbol{n}, \qquad \hat{\boldsymbol{C}}^a{}_b = \boldsymbol{C}^a{}_b - [\mu_\sigma]^a{}_b\boldsymbol{n}. \tag{4.23}$$

In terms of these, the anomaly-induced constitutive relations can be generated as

$$(j^\mu)_{\rm A} = *_\uparrow\left[\frac{\partial\mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial\boldsymbol{F}}\right]^\mu, \qquad (\sigma^{\mu ab})_{\rm A} = *_\uparrow\left[\frac{\partial\mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial\boldsymbol{R}_{ba}}\right]^\mu,$$
$$(\epsilon^\mu)_{\rm A} = *_\uparrow\left[\frac{\partial\mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial\boldsymbol{H}}\right]^\mu. \tag{4.24}$$

To write the equilibrium partition function in Newton-Cartan language we can use the natural time data in equilibrium $\psi_T = \psi_K = \psi_U$. Hence, using Eq. (4.18) we can find

$$W^{\rm eqb}_{\rm anom} = -\int_{\mathcal{M}^K_{(2n)}} \frac{\boldsymbol{n}}{\boldsymbol{H}}\wedge(\mathbf{i}^{(2n+1)} - \hat{\mathbf{i}}^{(2n+1)})|_{\psi_U=\psi_K}, \tag{4.25}$$

where $d\mathbf{i}^{(2n+1)} = \mathfrak{p}^{(2n+2)}$; $\mathbf{i}^{(2n+1)}$ is just $\boldsymbol{I}^{(2n+1)}_{\rm CS}$ after reduction. Please refer to Appendix C for conventions on reducing the integral.

This concludes the main abstract results of this work. We have been able to construct flavor and gravitational anomalies in the Galilean theories, and find their effect on the Galilean hydrodynamics. We explicitly constructed the sector of fluid constitutive relations that is totally determined in terms of anomalies. These constitutive relations obey the second law of thermodynamics with a trivially zero entropy current. We also found the equilibrium partition function which generates these constitutive relations in the equilibrium configuration.

V. EXAMPLES

The entire discussion of this work until now has been very abstract. We will now try to illustrate it with a few examples. In the following we will only discuss the case of an Abelian gauge field for simplicity. In Sec. V A we start with a thorough walkthrough example for three-dimensional null theories (two-dimensional Galilean theories), where we perform each and every step as was done in the main work. We hope it will help the reader to understand the procedure more clearly. Later in Sec. V B we present the results for the arbitrary dimensional case up to next-to-leading order in derivatives.

A. Walkthrough: One spatial dimension

Let us go step by step for the case of three-dimensional null backgrounds. The corresponding five-dimensional anomaly polynomial contains squared $\boldsymbol{F}$ and $\boldsymbol{R}$,

$$\mathcal{P}^{(5)} = \overline{\boldsymbol{V}}\wedge(C^{(2)}\boldsymbol{F}^{\wedge 2} + C^{(2)}_g \boldsymbol{R}^{\overline{A}}{}_{\overline{B}}\wedge\boldsymbol{R}^{\overline{B}}{}_{\overline{A}}), \tag{5.1}$$

from which we can read off the expression for $\boldsymbol{I}^{(4)}$,

$$\boldsymbol{I}^{(4)} = -\overline{\boldsymbol{V}}\wedge\Big[C^{(2)}\boldsymbol{A}\wedge\boldsymbol{F} + C^{(2)}_g\Big(\boldsymbol{C}^{\overline{A}}{}_{\overline{B}}\wedge\boldsymbol{R}^{\overline{B}}{}_{\overline{A}} - \frac{1}{3}\boldsymbol{C}^{\overline{A}}{}_{\overline{B}}\wedge\boldsymbol{C}^{\overline{B}}{}_{\overline{C}}\wedge\boldsymbol{C}^{\overline{C}}{}_{\overline{A}}\Big)\Big]. \tag{5.2}$$

From here we can define the bulk partition function $W_{\mathcal{B}} = \int_{\mathcal{B}_{(4)}}\boldsymbol{I}^{(4)}$, and compute its variation [see Eq. (3.5)],

$$\delta W_{\mathcal{B}} = \int_{\mathcal{B}_{(4)}} 2(C^{(2)}\delta\boldsymbol{A}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F} + C^{(2)}_g\delta\boldsymbol{C}^{\overline{A}}{}_{\overline{B}}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{R}^{\overline{B}}{}_{\overline{A}}) - \int_{\mathcal{M}_{(3)}}(C^{(2)}\delta\boldsymbol{A}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{A} + C^{(2)}_g\delta\boldsymbol{C}^A{}_B\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{C}^B{}_A). \tag{5.3}$$

Now using Eq. (3.1) or Eq. (3.13), we can find the Hall and Bardeen-Zumino currents,

$$\begin{aligned}
\star_{(4)}\mathbf{J}_{\rm H} &= \frac{\partial\mathcal{P}^{(5)}}{\partial\boldsymbol{F}} = 2C^{(2)}\overline{\boldsymbol{V}}\wedge\boldsymbol{F}\\
\Rightarrow \mathrm{J}^{\overline{M}}_{\rm H} &= C^{(2)}\epsilon^{\overline{N}\,\overline{R}\,\overline{S}\,\overline{M}}\overline{V}_{\overline{N}}F_{\overline{R}\,\overline{S}},\\
\star_{(4)}\Sigma^{\overline{A}\,\overline{B}}_{\rm H} &= \frac{\partial\mathcal{P}^{(5)}}{\partial\boldsymbol{R}_{\overline{B}\,\overline{A}}} = 2C^{(2)}_g\overline{\boldsymbol{V}}\wedge\boldsymbol{R}^{\overline{A}\,\overline{B}}\\
\Rightarrow \Sigma^{\overline{M}\,\overline{A}\,\overline{B}}_{\rm H} &= C^{(2)}_g\epsilon^{\overline{N}\,\overline{R}\,\overline{S}\,\overline{M}}\overline{V}_{\overline{N}}R_{\overline{R}\,\overline{S}}{}^{\overline{A}\,\overline{B}},\\
\star\mathbf{J}_{\rm BZ} &= \frac{\partial\boldsymbol{I}^{(4)}}{\partial\boldsymbol{F}} = -C^{(2)}\overline{\boldsymbol{V}}\wedge\boldsymbol{A}\\
\Rightarrow \mathrm{J}^M_{\rm BZ} &= C^{(2)}\epsilon^{NRM}\overline{V}_N A_R,\\
\star\Sigma^{AB}_{\rm BZ} &= \frac{\partial\boldsymbol{I}^{(4)}}{\partial\boldsymbol{R}_{BA}} = -C^{(2)}_g\overline{\boldsymbol{V}}\wedge\boldsymbol{C}^{AB}\\
\Rightarrow \Sigma^{MAB}_{\rm BZ} &= C^{(2)}_g\epsilon^{NRM}\overline{V}_N C^A{}_R{}^B.
\end{aligned} \tag{5.4}$$

The anomalous sources in Eq. (3.6) are hence given as

$$\begin{aligned}
\Sigma^{\perp AB}_{\rm H} &= -C^{(2)}_g\epsilon^{MRS}\overline{V}_M R_{RS}{}^{AB},\\
\mathrm{J}^\perp_{\rm H} &= -C^{(2)}\epsilon^{MNR}\overline{V}_M F_{\rm NR}.
\end{aligned} \tag{5.5}$$

Here we have defined the volume element of the boundary manifold as $\epsilon^{\perp MNR} = \epsilon^{MNR}$. After null reduction we can trivially read off the anomalous sources for the NC conservation laws (3.7),

$$\sigma^{\perp ab}_{\rm H} = -C^{(2)}_g\varepsilon^{\mu\nu}_\uparrow R_{\mu\nu}{}^{ab}, \qquad \mathrm{j}^\perp_{\rm H} = -2C^{(2)}\varepsilon^{\mu\nu}_\uparrow F_{\mu\nu}. \tag{5.6}$$

Hydrodynamics: We want to generate the fluid constitutive relations which are compatible with the anomalies described above. As described in the main text, it can be done by using a transgression form [Eq. (4.8)],

$$\mathcal{V}_{\mathcal{P}}^{(4)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge (\boldsymbol{\mathcal{P}}^{(5)} - \hat{\boldsymbol{\mathcal{P}}}^{(5)}) = -\boldsymbol{V} \wedge \overline{\boldsymbol{V}} \wedge [2C^{(2)}\mu \boldsymbol{F} + 2C_g^{(2)}[\mu_\Sigma]^{\overline{A}}{}_{\overline{B}} \boldsymbol{R}^{\overline{B}}{}_{\overline{A}} + (C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]^{\overline{A}}{}_{\overline{B}}[\mu_\Sigma]^{\overline{B}}{}_{\overline{A}})\boldsymbol{H}]. \quad (5.7)$$

From its derivatives we can find the various currents defined in Eq. (4.12),

$$\begin{aligned}
\star \boldsymbol{J}_{\mathcal{P}} &= -2C^{(2)}\mu \boldsymbol{V}\wedge\overline{\boldsymbol{V}} \\
\Rightarrow J^M_{\mathcal{P}} &= 2C^{(2)}\mu \epsilon^{RSM} V_R \overline{V}_S, \\
\star \Sigma^{AB}_{\mathcal{P}} &= -2C_g^{(2)}[\mu_\Sigma]^{AB}\boldsymbol{V}\wedge\overline{\boldsymbol{V}} \\
\Rightarrow \Sigma^{MAB}_{\mathcal{P}} &= 2C_g^{(2)}[\mu_\Sigma]^{AB}\epsilon^{RSM}V_R\overline{V}_S, \\
\star \boldsymbol{E}_{\mathcal{P}} &= -(C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]^A{}_B[\mu_\Sigma]^B{}_A)\boldsymbol{V}\wedge\overline{\boldsymbol{V}} \\
\Rightarrow E^M_{\mathcal{P}} &= (C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]^A{}_B[\mu_\Sigma]^B{}_A)\epsilon^{RSM}V_R\overline{V}_S. \quad (5.8)
\end{aligned}$$

Using Eq. (4.14) we can trivially get the anomalous sector of the constitutive relations from here. These constitutive relations satisfy the adiabaticity equation (4.15) with zero entropy current, and at equilibrium also satisfy the anomalous equations of motion (3.6). Upon null reduction we can get the anomalous contribution to the Galilean constitutive relations from here; the only surviving quantities are

$$(\epsilon^\mu)_{\rm A} = (C^{(2)}\mu^2 + C_g^{(2)}[\mu_\sigma]^a{}_b[\mu_\sigma]^b{}_a)\varepsilon_\uparrow^{\nu\mu}n_\nu,$$
$$(\sigma^{\mu ab})_{\rm A} = 2C_g^{(2)}[\mu_\sigma]^{ab}\varepsilon_\uparrow^{\nu\mu}n_\nu, \qquad (j^\mu)_{\rm A} = 2C^{(2)}\mu\varepsilon_\uparrow^{\nu\mu}n_\nu. \quad (5.9)$$

Finally we can write an equilibrium partition function $W^{\rm eqb}_{\rm anom}$ which generates these currents in the equilibrium configuration. Using Eq. (4.18) we can directly find

$$\begin{aligned}
W^{\rm eqb}_{\rm anom} &= -\int_{\mathcal{M}_{(3)}} \frac{\boldsymbol{V}}{\boldsymbol{H}}\wedge(\boldsymbol{I}^{(4)} - \hat{\boldsymbol{I}}^{(4)}) \\
&= -\int_{\mathcal{M}_{(3)}} \boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge(C^{(2)}\mu\boldsymbol{A} + C_g^{(2)}[\mu_\Sigma]^A{}_B\boldsymbol{C}^B{}_A) \\
&= \int {\rm d}^3x\sqrt{|{\rm G}|}\epsilon^{MNR}V_M\overline{V}_N(C^{(2)}\mu A_R + C_g^{(2)}[\mu_\Sigma]^A{}_B C^B{}_{RA}). \quad (5.10)
\end{aligned}$$

This can be written in the NC language as

$$\begin{aligned}
W^{\rm eqb}_{\rm anom} &= \int_{\mathcal{M}_{(2)}} \boldsymbol{n}\wedge(C^{(2)}\mu\boldsymbol{A} + C_g^{(2)}[\mu_\sigma]^a{}_b\boldsymbol{C}^b{}_a) \\
&= \int {\rm d}^3x\sqrt{|\gamma|}\varepsilon_\uparrow^{\mu\nu}n_\mu(C^{(2)}\mu A_\nu + C_g^{(2)}[\mu_\sigma]^a{}_bC^b{}_{\nu a}). \quad (5.11)
\end{aligned}$$

B. Arbitrary odd spatial dimensions up to subsubleading order

Before proceeding with this example we should clarify the usage of "subsubleading" or "second nontrivial" derivative order for null/Galilean fluids derived from relativistic fluids in Ref. [57]. One can check that in the partition function or constitutive relations of a $(2n+1)$-dimensional null fluid, the first nontrivial contribution from the parity-odd sector comes at $(n-1)$ derivatives, which is generally known as the "leading parity-odd derivative order." Correspondingly n derivatives are called subleading while $(n+1)$ derivatives are called subsubleading. It is also trivial to check that the anomaly polynomial always has two more derivatives than the partition function or constitutive relations. In the anomalous sector one can check that the first nontrivial contribution comes at the leading order (flavor anomaly) while no contribution comes at the subleading order. Hence the "second nontrivial correction" comes at the "subsubleading order."

Coming back to our example, one can check that up to subsubleading order $\boldsymbol{\mathcal{P}}^{(2n+3)}$ and $\boldsymbol{I}^{(2n+2)}$ (for $n>1$) are given as

$$\boldsymbol{\mathcal{P}}^{(2n+3)} = \overline{\boldsymbol{V}}\wedge(C^{(2n)}\boldsymbol{F}^{\wedge(n+1)} + C_g^{(2n)}\boldsymbol{F}^{\wedge(n-1)}\wedge\boldsymbol{R}^{\overline{A}}{}_{\overline{B}}\wedge\boldsymbol{R}^{\overline{B}}{}_{\overline{A}}),$$
$$\boldsymbol{I}^{(2n+2)} = -\overline{\boldsymbol{V}}\wedge\boldsymbol{A}\wedge(C^{(2n)}\boldsymbol{F}^{\wedge n} + C_g^{(2n)}\boldsymbol{F}^{\wedge(n-2)}\wedge\boldsymbol{R}^{\overline{A}}{}_{\overline{B}}\wedge\boldsymbol{R}^{\overline{B}}{}_{\overline{A}}). \quad (5.12)$$

It is worth noting that the contribution from anomalies terminates at subsubleading order in three spatial dimensions ($d=3$, $n=2$), and hence these expressions are exact for $n=2$. From here we can get the Hall currents

$$\begin{aligned}
\Sigma^{\perp AB}_{\rm H} &= -2C_g^{(2n)}\star[\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1)}\wedge\boldsymbol{R}^{AB}], \\
J^\perp_{\rm H} &= -(n+1)C^{(2n)}\star[\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge n}] - (n-1)C_g^{(2n)}\star[\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-2)}\wedge\boldsymbol{R}^A{}_B\wedge\boldsymbol{R}^B{}_A] \quad (5.13)
\end{aligned}$$

that provide anomalies in Eq. (3.6). The results can be trivially transformed to Newton-Cartan language,

$$\begin{aligned}
\sigma^{\perp ab}_{\rm H} &= -2C_g^{(2n)} *_\uparrow [\boldsymbol{F}^{\wedge(n-1)}\wedge\boldsymbol{R}^{ab}], \\
{\rm j}^\perp_{\rm H} &= -(n+1)C^{(2n)} *_\uparrow [\boldsymbol{F}^{\wedge n}] - (n-1)C_g^{(2n)} *_\uparrow [\boldsymbol{F}^{\wedge(n-2)}\wedge\boldsymbol{R}^a{}_b\wedge\boldsymbol{R}^b{}_a], \quad (5.14)
\end{aligned}$$

which provide anomalies in Eq. (3.7).

Hydrodynamics: Using the anomaly polynomial one can find the constitutive relations for the anomalous sector of hydrodynamics,

$$
\begin{aligned}
J^M_{\mathcal{P}} &= (n+1)C^{(2n)}\sum_{m=1}^{n} {}^{n}\boldsymbol{C}_m\mu^m\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-m)}\wedge\boldsymbol{H}^{\wedge(m-1)}]^M \\
&\quad + (n-1)C_g^{(2n)}\left\{\sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_m\mu^m\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{R}^A{}_B\wedge\boldsymbol{R}^B{}_A\wedge\boldsymbol{H}^{\wedge(m-1)}]^M\right. \\
&\quad \left. + \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_m\mu^m[\mu_\Sigma]^A{}_B\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-2-m)}\wedge(2\boldsymbol{R}^B{}_A + [\mu_\Sigma]^B{}_A\boldsymbol{H})\wedge\boldsymbol{H}^{\wedge m}]^M\right\}, \\
\Sigma^{MAB}_{\mathcal{P}} &= 2C_g^{(2n)}\left\{\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^{AB}\wedge\boldsymbol{H}^{\wedge(m-1)}]^M\right. \\
&\quad \left. + \sum_{m=0}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m[\mu_\Sigma]^{AB}\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{H}^{\wedge m}]^M\right\}, \\
E^M_{\mathcal{P}} &= \sum_{m=0}^{n-1}\mu^m\left({}^{n+1}\boldsymbol{C}_{m+2}C^{(2n)}\mu^2 + {}^{n-1}\boldsymbol{C}_m C_g^{(2n)}[\mu_\Sigma]^A{}_B[\mu_\Sigma]^B{}_A\right)\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{H}^{\wedge m}]^M \\
&\quad + C_g^{(2n)}\left\{\sum_{m=2}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^A{}_B\wedge\boldsymbol{R}^B{}_A\wedge\boldsymbol{H}^{\wedge(m-2)}]^M\right. \\
&\quad \left. +2\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m[\mu_\Sigma]^A{}_B\star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^B{}_A\wedge\boldsymbol{H}^{\wedge(m-1)}]^M\right\}.
\end{aligned}
\tag{5.15}
$$

The anomalous sector of constitutive relations in terms of these are given by Eq. (4.14), while the entropy current is zero. Again, by a trivial choice of basis these results can be transformed to the Newton-Cartan basis; the only nonzero constitutive relations are

$$
\begin{aligned}
(j^\mu)_{\rm A} &= (n+1)C^{(2n)}\sum_{m=1}^{n} {}^{n}\boldsymbol{C}_m\mu^m *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-m)}\wedge\boldsymbol{H}^{\wedge(m-1)}]^\mu \\
&\quad + (n-1)C_g^{(2n)}\left\{\sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_m\mu^m *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{R}^a{}_b\wedge\boldsymbol{R}^b{}_a\wedge\boldsymbol{H}^{\wedge(m-1)}]^\mu\right. \\
&\quad \left. + \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_m\mu^m[\mu_\sigma]^a{}_b *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-2-m)}\wedge(2\boldsymbol{R}^b{}_a + [\mu_\sigma]^b{}_a\boldsymbol{H})\wedge\boldsymbol{H}^{\wedge m}]^\mu\right\}, \\
(\sigma^{\mu ab})_{\rm A} &= 2C_g^{(2n)}\left\{\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^{ab}\wedge\boldsymbol{H}^{\wedge(m-1)}]^\mu\right. \\
&\quad \left. + \sum_{m=0}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m[\mu_\sigma]^{ab} *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{H}^{\wedge m}]^\mu\right\}, \\
(\epsilon^\mu)_{\rm A} &= \sum_{m=0}^{n-1}\mu^m\left({}^{n+1}\boldsymbol{C}_{m+2}C^{(2n)}\mu^2 + {}^{n-1}\boldsymbol{C}_m C_g^{(2n)}[\mu_\sigma]^a{}_b[\mu_\sigma]^b{}_a\right) *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{H}^{\wedge m}]^\mu \\
&\quad + C_g^{(2n)}\left\{\sum_{m=2}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^a{}_b\wedge\boldsymbol{R}^b{}_a\wedge\boldsymbol{H}^{\wedge(m-2)}]^\mu\right. \\
&\quad \left. +2\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_m\mu^m[\mu_\sigma]^a{}_b *_\uparrow[\boldsymbol{n}\wedge\boldsymbol{F}^{\wedge(n-1-m)}\wedge\boldsymbol{R}^b{}_a\wedge\boldsymbol{H}^{\wedge(m-1)}]^\mu\right\}.
\end{aligned}
\tag{5.16}
$$

Finally we can write an equilibrium partition function $W^{\rm eqb}_{\rm anom}$ which generates these currents in the equilibrium configuration; for null fluids,

$$W_{\rm anom}^{\rm eqb} = -\int_{\mathcal{M}_{(2n+1)}} \boldsymbol{V}\wedge\overline{\boldsymbol{V}}\wedge\boldsymbol{A}\wedge\Bigg\{\sum_{m=1}^{n} {}^{n}\boldsymbol{C}_{m} C^{(2n)}\mu^{m}\boldsymbol{F}^{\wedge(n-m)}\wedge\boldsymbol{H}^{\wedge(m-1)} + C_{g}^{(2n)}\sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{H}^{\wedge(m-1)}\wedge\boldsymbol{R}^{A}{}_{B}\wedge\boldsymbol{R}^{B}{}_{A} + C_{g}^{(2n)}\sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}[\mu_{\Sigma}]^{A}{}_{B}\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{H}^{\wedge m}\wedge(2\boldsymbol{R}^{B}{}_{A}+[\mu_{\Sigma}]^{B}{}_{A}\boldsymbol{H})\Bigg\}, \tag{5.17}$$

and for Galilean fluids,

$$W_{\rm anom}^{\rm eqb} = -\int_{\mathcal{M}_{(2n)}} \boldsymbol{n}\wedge\boldsymbol{A}\wedge\Bigg\{\sum_{m=1}^{n} {}^{n}\boldsymbol{C}_{m} C^{(2n)}\mu^{m}\boldsymbol{F}^{\wedge(n-m)}\wedge\boldsymbol{H}^{\wedge(m-1)} + C_{g}^{(2n)}\sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{H}^{\wedge(m-1)}\wedge\boldsymbol{R}^{a}{}_{b}\wedge\boldsymbol{R}^{b}{}_{a} + C_{g}^{(2n)}\sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}[\mu_{\sigma}]^{a}{}_{b}\boldsymbol{F}^{\wedge(n-2-m)}\wedge\boldsymbol{H}^{\wedge m}\wedge(2\boldsymbol{R}^{b}{}_{a}+[\mu_{\sigma}]^{b}{}_{a}\boldsymbol{H})\Bigg\}. \tag{5.18}$$

This finishes our discussion about anomalies in generic even-dimensional Galilean fluids up to subsubleading order in the derivative expansion. The one spatial dimensional case was discussed separately in Sec. VA for illustrative purposes. The one-dimensional case is also qualitatively different from higher dimensions, because only in this special case do we get a pure gravitational anomaly term in the anomaly polynomial up to subsubleading order. Three and higher spatial dimensional cases are qualitatively similar as we illustrated above. For physically interesting results one might want to put $n = 2$ and recover three spatial dimensional results, which are found to be in agreement with the path integral calculation of Ref. [36].

VI. CONCLUSIONS AND FURTHER DIRECTIONS

In this work we examined the effect of flavor and gravitational anomalies on Galilean theories with a spin current, coupled to a torsional Newton-Cartan background. In particular it is to be noted that we primarily studied anomalous theories on torsional null backgrounds, from where the aforementioned system is just a choice of basis (null reduction) away. It strengthens our belief that null theories are just an embedding of Galilean theories into a higher-dimensional spacetime, which are closer to their relativistic cousins, are frame independent, and are easier to handle compared to Newton-Cartan backgrounds. The transition from null to Galilean (Newton-Cartan) theories is essentially trivial.

We used the anomaly inflow mechanism prevalent in relativistic theories, with slight modifications, to construct these anomalies. We found that after null reduction the anomalies only contribute to the spatial spin and charge conservation equations, and only in even dimensions. In other words only the rotational and flavor symmetry of the Galilean theory becomes anomalous. This is in contrast with the results of Ref. [37] where Galilean boost symmetry was also seen to be anomalous. As we mentioned in the Introduction and in the main work, the discrepancy can be attributed to the presence of extra background fields in Ref. [37] which have been explicitly switched off in our null background construction. It is interesting to note that the Galilean anomaly polynomial $\mathfrak{p}^{(2n+2)}$ is structurally the same as the relativistic anomaly polynomial $\mathcal{P}_{\rm CS}^{(2n+2)}$, and hence the number of anomaly coefficients on both sides match. Owing to this, the structure of the Hall currents that enter the conservation laws is also quite similar in both cases. Hence the results we have obtained promise to be genuine nonrelativistic anomalies and not just the manifestation of (stronger) Galilean invariance.

Unrelated to the Galilean theories, we found that in the Cartan formulation of relativistic fluids there exists a more natural definition of the entropy current which does not get any anomalous contributions. On the other hand the Belinfante (usual) entropy current used, e.g., in Ref. [42] gets contributions from the gravitational anomaly. See Appendix D for more comments on this issue.

We also studied the anomalous sector of null/Galilean hydrodynamics, in which we explicitly wrote down the constitutive relations which are completely determined in terms of anomalies. For this we used the transgression machinery developed to do the same task in relativistic hydrodynamics. There have been no surprises in this computation; everything went more or less smoothly for null theories, as it did for relativistic theories. The entropy current in Galilean theories is independent of anomalies as well. From a different perspective, it illustrates that the null background construction allows us to use rather sophisticated and developed relativistic machinery directly in nonrelativistic physics, which is encouraging.

It opens up an arena in which to introduce set results from relativistic theories into null theories and see if we can say something new and useful about the Galilean theories from there. An immediate question that comes to mind is regarding the transcendental contribution to hydrodynamics from anomalies. In relativistic hydrodynamics the authors of Ref. [52] showed that there are certain constants in the fluid constitutive relations that are left undetermined by the second law of thermodynamics, but can be related to the anomaly coefficients by requiring the consistency of yjr Euclidean vacuum. Similar constants have also been found for Galilean theories in Refs. [15,34]. It would be nice to see if these constants can be associated with the Galilean anomalies found in this work. Being a little more ambitious, one can hope for a complete classification of Galilean hydrodynamic transport following its relativistic counterpart suggested recently in Refs. [41,42]. It will also be interesting to see if the Weyl anomaly analysis of Ref. [37] remains unchanged when the additional mass sources have been switched off.

For now we will leave the reader with these questions and possibilities, in the hope that we will be able to unravel new and interesting nonrelativistic physics using null backgrounds. If there is one thing the reader should take away from this work, we would recommend the following approach: if we are interested in a problem pertaining to Galilean physics which we know how to solve in the relativistic case, a good way ahead would be to formulate the problem in terms of null theories, do the computation there, and perform a trivial null reduction to get the Galilean results.

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APPENDIX A: RESULTS IN A NONCOVARIANT BASIS

In this appendix we express some of the results discussed in the main text in a conventional noncovariant notation. We pick up a basis $x^M = \{x^\sim, t, x^i\}$ on $\mathcal{M}_{(d+2)}$ such that $\psi_V = \{\partial_\sim, 0, 0\}$ and $\psi_T = \{\partial_t, 0, 0\}$. $\vec{x} = \{x^i\}$ spans the spatial slice $\mathcal{M}^T_{(d)}$. This is equivalent to choosing the Newton-Cartan decomposition but with $v^i = \Lambda_{(T)} = [\Lambda_{\Sigma(T)}]^A{}_B = 0$. On $\mathbb{R}^{(d+1,1)}$ on the other hand we choose the same basis as before, $x^A = \{x^-, x^+, x^a\}$, such that $V = \partial_-$, $\overline{V}_{(T)} = \partial_+$. In this basis various NC background fields can be decomposed as

$$n_\mu = \begin{pmatrix} \mathrm{e}^{-\Phi} \\ \mathrm{e}^{-\Phi} a_i \end{pmatrix}, \qquad v^\mu = \begin{pmatrix} \mathrm{e}^{\Phi} \\ 0 \end{pmatrix}, \qquad B_\mu = \begin{pmatrix} B_t \\ B_i \end{pmatrix},$$
$$e^a{}_\mu = \begin{pmatrix} 0 \\ e^a{}_i \end{pmatrix}, \qquad e_a{}^\mu = \begin{pmatrix} -a_a \\ e_a{}^i \end{pmatrix},$$
$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix}, \qquad h^{\mu\nu} = \begin{pmatrix} a^k a_k & -a^j \\ -a^i & g^{ij} \end{pmatrix}. \tag{A1}$$

Here the spatial metric has been defined as

$$g_{ij} = \delta_{ab} e^a{}_i e^b{}_j, \qquad g^{ij} = \delta^{ab} e_a{}^i e_b{}^j. \tag{A2}$$

Spatial indices can be raised and lowered by g_{ij} and can be swapped using the spatial vielbein $e^a{}_i$. However in the following we will explicitly work in the $i, j\ldots$ indices. One can check that after this choice of basis we are only allowed to perform $\vec{x}$-dependent transformations, except boosts which are completely fixed. On trivially decomposing the Newton-Cartan expressions into $x^\mu = \{t, x^i\}$, our theory will be manifestly covariant against all these transformations except time translations $t \to t + \xi^t(\vec{x})$, sometimes referred to as Kaluza-Klein (KK) gauge transformations. These transformations act on the background fields as

$$\delta_{KK} a_i = \xi^t \partial_t a_i + \partial_i \xi^t, \quad \delta_{KK} B_i = \xi^t \partial_t B_i + B_t \partial_i \xi^t, \tag{A3}$$

whereas they act on general contra-covariant tensors as

$$\delta_{KK} X^t = \xi^t \partial_t X^t - X^i \partial_i \xi^t, \qquad \delta_{KK} X_t = \xi^t \partial_t X_t,$$
$$\delta_{KK} X^i = \xi^t \partial_t X^i, \qquad \delta_{KK} X_i = \xi^t \partial_t X_i + X_t \partial_i \xi^t, \tag{A4}$$

and similarly for higher-rank tensors. The theory can be made manifestly covariant under KK transformations as well by working with corrected tensors,

$$\acute{X}^t = \mathrm{e}^{-\Phi}(X^t + a_i X^i), \qquad \acute{X}_t = \mathrm{e}^{\Phi} X_t,$$
$$\acute{X}^i = X^i, \qquad \acute{X}_i = X_i - a_i X_t, \tag{A5}$$

and similarly for higher-rank tensors. These are the well-known Kaluza-Klein covariant fields.[26] Under the flat time approximation, i.e., $\Phi = a_i = 0$ this correction becomes trivial. One can check that the NC contraction can be expanded in this format as

$$A^\mu B_\mu = \acute{A}^t \acute{B}_t + \acute{A}^i \acute{B}_i, \tag{A6}$$

which will be helpful later. Now we can decompose various components of connections in this basis as

[26]The original Kaluza-Klein transformation only involves the KK gauge field a_i. The factors of e^Φ can be thought of as redshift factors due to the time component of the time metric n_μ.

$$c_{\mu t} = 0, \qquad \acute{c}_{ij} = \frac{1}{2}\acute{\partial}_t g_{ji} + \frac{1}{2}\acute{\Omega}_{ij} + \acute{\mathrm{T}}_{(ij)t},$$
$$\acute{\alpha}_i = \acute{c}_{ti} = \acute{\Omega}_{ti}, \qquad \acute{\Gamma}^t{}_{tt} = -\acute{\partial}_t \Phi, \qquad \acute{\Gamma}^t{}_{tj} = \mathrm{e}^{-\Phi}\acute{\partial}_t a_j,$$
$$\acute{\Gamma}^t{}_{it} = -\acute{\partial}_i \Phi, \qquad \acute{\Gamma}^t{}_{ij} = \mathrm{e}^{-\Phi}\acute{\partial}_i a_j.$$
$$\acute{\Gamma}^k{}_{tt} = \acute{\Omega}_t{}^k, \qquad \acute{\Gamma}^k{}_{it} = \acute{c}_i{}^k, \qquad \acute{\Gamma}^k{}_{tj} = \acute{c}_j{}^k - \acute{\mathrm{T}}^k{}_{jt}.$$
$$\acute{\Gamma}^k{}_{ij} = \frac{1}{2}g^{kl}(\acute{\partial}_i g_{lj} + \acute{\partial}_j g_{li} - \acute{\partial}_l g_{ij}) + \frac{1}{2}(\acute{\mathrm{T}}^k{}_{ij} - 2\acute{\mathrm{T}}_{(ij)}{}^k). \tag{A7}$$

Here we have also defined the corrected coordinate derivatives $\acute{\partial}_i = \partial_i - a_i\partial_t$, $\acute{\partial}_t = \mathrm{e}^{\Phi}\partial_t$. In *equilibrium* (i.e., $\partial_t\varphi = 0\,\forall\,\phi$) or when the time is flat, we can recover $\acute{\partial}_i = \partial_i$, $\acute{\partial}_t = \partial_t$. We define the covariant derivative $\acute{\nabla}_i$ associated with the corrected derivative[27] $\acute{\partial}_i$ and connections $\acute{\Gamma}^j{}_{ik}$ and $\acute{A}_i$, which act on a general tensor $\varphi^i{}_j$ transforming in the adjoint representation of the flavor group, as

$$\acute{\nabla}_i\varphi^j{}_k = \acute{\partial}_i\varphi^j{}_k + \acute{\Gamma}^j{}_{il}\varphi^l{}_k - \acute{\Gamma}^l{}_{ik}\varphi^j{}_l + [\acute{A}_i, \varphi^j{}_k], \tag{A9}$$

and similarly on higher-rank objects. We also define a "time covariant derivative" $\acute{\nabla}_t$ associated with $\acute{\partial}_t$ and connections $\acute{\Gamma}^j{}_{tk}$ and $\acute{A}_t$, acting on $\varphi^i{}_j$ naturally,

$$\acute{\nabla}_t\varphi^i{}_j = \acute{\partial}_t\varphi^i{}_j + \acute{\Gamma}^i{}_{tk}\varphi^k{}_j - \acute{\Gamma}^k{}_{tj}\varphi^i{}_k + [\acute{A}_t, \varphi^i{}_j], \tag{A10}$$

and similarly on higher-rank objects. One can check that both of these derivatives behave tensorially on the spatial slice and are KK gauge invariant. More importantly both of these preserve the spatial metric g_{ij}. There is no essential need to work with these corrected quantities, but we do so because the statements are manifestly KK gauge invariant and look nicer.

Using a similar decomposition and the KK-corrected expressions for various currents, we can reduce the conservation equations (3.7) into the noncovariant basis as

$$\begin{aligned}
\text{Mass Cons}: & \underline{\acute{\nabla}}_t\rho + \underline{\acute{\nabla}}_i\acute{\rho}^i = 0,\\
\text{Energy Cons}: & \underline{\acute{\nabla}}_t\epsilon + \underline{\acute{\nabla}}_i\acute{\epsilon}^i = [\text{power}] - \acute{p}^i\acute{\alpha}_i - \acute{p}^{ij}\acute{c}_{ij},\\
\text{Momentum Cons}: & \underline{\acute{\nabla}}_t\acute{p}_i + \underline{\acute{\nabla}}_j\acute{p}^j{}_i = [\text{force}]_i - \rho\acute{\alpha}_i - \acute{\rho}^j\acute{c}_{ji},\\
\text{Temporal Spin Cons}: & \underline{\acute{\nabla}}_t\acute{\tau}^i + \underline{\acute{\nabla}}_j\acute{\tau}^{ji} = \frac{1}{2}(\acute{\rho}^i - \acute{p}^i),\\
\text{Spatial Spin Cons}: & \underline{\acute{\nabla}}_t\acute{\sigma}^{ij} + \underline{\acute{\nabla}}_k\acute{\sigma}^{kij} = \acute{p}^{[ji]} + 2\acute{\tau}^{[i}\acute{\alpha}^{j]} + 2\acute{\tau}^{k[i}\acute{c}_k{}^{j]} + \acute{\sigma}_{\mathrm{H}}^{\perp ij},\\
\text{Charge Cons}: & \underline{\acute{\nabla}}_t q + \underline{\acute{\nabla}}_i\acute{j}^i = \mathrm{j}_{\mathrm{H}}^{\perp},
\end{aligned} \tag{A11}$$

where $\underline{\acute{\nabla}}_i = \acute{\nabla}_i - \acute{\mathrm{T}}^j{}_{ji} + \acute{H}_{ti}$ and $\underline{\acute{\nabla}}_t = \acute{\nabla}_t + \acute{\Gamma}^i{}_{ti}$. It is worth noting that the corrected time component of the mass current $\acute{\rho}^t$ is just the mass density ρ, and similarly for all other currents. If we are to expand the covariant derivatives in these equations, the nice-looking expressions will turn notoriously bad, so we do not attempt that here. Rather, we invite the readers to qualitatively access the form of these equations and convince themselves that these are what we expect for a Galilean system. Similarly the [power] and [force] densities can also be decomposed as

$$[\text{power}] = \acute{e}_i\cdot\acute{j}^i + \cdots, \qquad [\text{force}]^i = \acute{e}_i\cdot q + \acute{\beta}_{ij}\cdot\acute{j}^j + \cdots, \tag{A12}$$

where $\acute{e}_i = \acute{F}_{it}$ is the electric field, $\acute{\beta}_{ij} = \acute{F}_{ij}$ is the dual magnetic field, and $\cdots$ corresponds to similar terms coming from all other field-current pairs.

On the other hand, noncovariant expressions for the anomalous sector of the hydrodynamic constitutive relations follow trivially from Eq. (4.19). The only nonzero contributions are given as[28]

[27]One might be lured (e.g., in Ref. [34]) to define the covariant derivative with respect to the original derivative ∂_i and the more conventional affine connection,

$$\begin{aligned}\gamma^k{}_{ij} &= \acute{\Gamma}^k{}_{ij} - \frac{1}{2}a^k\partial_t g_{ij} + g^{kl}a_{(i}\partial_t g_{j)l}\\ &= \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) + \frac{1}{2}(\acute{\mathrm{T}}^k{}_{ij} - 2\acute{\mathrm{T}}_{(ij)}{}^k),\end{aligned} \tag{A8}$$

which however will not be KK gauge invariant. The results hence will be messy and will carry extra time derivatives of the metric. Therefore we will refrain from doing so. Obviously both of these covariant derivatives are same in the flat time case or in equilibrium.

[28]We have assumed that the same ψ_T is being used for reduction and to describe the anomaly polynomial. Had they been different, the currents would shift by a total derivative.

$$(\acute{\epsilon}^i)_{\rm A} = E^i_{\mathcal{P}}, \qquad (\acute{\sigma}^{ijk})_{\rm A} = \Sigma^{ijk}_{\mathcal{P}}, \qquad (\acute{j}^i)_{\rm A} = J^i_{\mathcal{P}}. \quad \text{(A13)}$$

The expressions for the rhs can be obtained from Eq. (4.24).

APPENDIX B: COMPARISON WITH GERACIE *et al.* [17]

The authors of Ref. [17] have prescribed a nice covariant frame-independent description of Galilean physics in terms of an "extended space representation." The extended space is basically a one-dimensional-higher flat space which allows for a nice frame-independent embedding of the Galilean group. On a closer inspection, however, it would be clear that the extended space is nothing but the vielbein space of null theories. To demonstrate this we pick up a basis on $\mathcal{M}_{(d+2)}$ (but do not perform null reduction, which would otherwise require us to choose time data ψ_T and hence will introduce frame dependence), $x^M = \{x^\sim, x^\mu\}$ such that $V = \partial_\sim$. We can then express the anomalous null conservation laws as

$$\begin{aligned}
(\tilde{\nabla}_\mu - \mathrm{T}^\nu{}_{\nu\mu}) j^\mu_\rho &= 0, \\
\mathrm{E}^A{}_\nu (\tilde{\nabla}_\mu - \mathrm{T}^\nu{}_{\nu\mu}) T^\mu{}_A - \mathrm{T}^A{}_{\nu\mu} T^\mu{}_A &= R_{\nu\mu}{}^A{}_B \Sigma^{\mu B}{}_A + F_{\nu\mu} \cdot J^\mu, \\
(\tilde{\nabla}_\mu - \mathrm{T}^\nu{}_{\nu\mu}) \Sigma^{\mu AB} &= -T^{[AB]} + \Sigma^{\perp AB}_{\mathrm{H}} + \#^{[A} V^{B]}, \\
(\tilde{\nabla}_\mu - \mathrm{T}^\nu{}_{\nu\mu}) j^\mu_q &= \mathrm{J}^\perp_{\mathrm{H}}.
\end{aligned} \qquad \text{(B1)}$$

In this and only this section $\tilde{\nabla}_\mu$ is associated with $\Gamma^\rho{}_{\mu\sigma}$, $C^A{}_{\mu B}$, A_μ and the vielbein has been used to transform indices. The results are presented to make them look as close as possible to Eqs. (5.8)–(5.10) of ref. [17]. The authors however did not consider anomalies, and did not report the full spin conservation. Only the boost part of the spin conservation is reported in Eq. (5.13) of Ref. [17] which is identical to our corresponding conservation in Eq. (3.7).

If one looks at these equations and at the currents appearing in them, one would realize that all the unphysical degrees of freedom have been eliminated (except the spin conservation equations). Therefore the EM tensor and charge current as they appear in Ref. [17] only carry physical information. At the cost of some unphysical degrees of freedom (and a consistent prescription to eliminate them) we have been able to transform this set of equations into a nice covariant higher-dimensional null theory.

We would like to note that the authors of Ref. [17] have also used their construction to study $(2+1)$-dimensional Galilean fluids. The same results (for the torsionless case) were gained from "null fluids" in Ref. [34] and a detailed comparison can be found in their last appendix.

APPENDIX C: CONVENTIONS OF DIFFERENTIAL FORMS

In this appendix we will recollect some results about differential forms, and will set notations and conventions used throughout this work. An m-rank differential form $\boldsymbol{\mu}^{(m)}$ on a $(d+2)$-dimensional manifold $\mathcal{M}_{(d+2)}$ can be written in a coordinate basis as

$$\boldsymbol{\mu}^{(m)} = \frac{1}{m!} \mu_{M_1 M_2 \ldots M_m} \mathrm{d}x^{M_1} \wedge \mathrm{d}x^{M_2} \wedge \ldots \wedge \mathrm{d}x^{M_m}, \qquad \text{(C1)}$$

where μ is a completely antisymmetric tensor. On $\mathcal{M}_{(d+2)}$, the volume element is given by a full rank form,

$$\boldsymbol{\epsilon}^{(d+2)} = \frac{1}{(d+2)!} \epsilon_{M_1 M_2 \ldots M_{d+2}} \mathrm{d}x^{M_1} \wedge \mathrm{d}x^{M_2} \wedge \ldots \wedge \mathrm{d}x^{M_{d+2}}, \qquad \text{(C2)}$$

where ϵ is the totally antisymmetric Levi-Civita symbol with value $\epsilon_{0,1,2,\ldots,d+1} = \sqrt{|\mathrm{G}|}$ and $\mathrm{G} = \det \mathrm{G}_{MN}$. Using it, the Hodge dual is defined to be a map from m-rank differential forms to $(d+2-m)$-rank differential forms,

$$\star[\boldsymbol{\mu}^{(m)}] = \frac{1}{(d+2-m)!} \left(\frac{1}{m!} \mu^{M_1 \ldots M_m} \epsilon_{M_1 \ldots M_m N_1 \ldots N_{d+2-m}} \right) \times \mathrm{d}x^{N_1} \wedge \ldots \wedge \mathrm{d}x^{N_{d+2-m}}. \qquad \text{(C3)}$$

One can check that $\star\star\boldsymbol{\mu}^{(m)} = \mathrm{sgn}(\mathrm{G})(-)^{m(d-m)}$, and

$$\boldsymbol{\mu}^{(m)} \wedge \star[\boldsymbol{\nu}^{(m)}] = \frac{1}{m!} \mu^{M_1 \ldots M_m} \nu_{M_1 \ldots M_m} \boldsymbol{\epsilon}^{(d+2)}. \qquad \text{(C4)}$$

We define the $\wedge$ product of two differential forms as

$$\boldsymbol{\mu}^{(m)} \wedge \boldsymbol{\nu}^{(r)} = \frac{1}{(m+r)!} \left(\frac{(m+n)!}{m! r!} \mu_{[M_1 \ldots M_m} \nu_{N_1 \ldots N_r]} \right) \times \mathrm{d}x^{M_1} \wedge \ldots \wedge \mathrm{d}x^{N_1} \wedge \ldots. \qquad \text{(C5)}$$

For multiple differential forms we can find

$$\begin{aligned}
&\boldsymbol{\mu}^{(m)} \wedge \boldsymbol{\nu}^{(r)} \wedge \ldots \wedge \boldsymbol{\rho}^{(s)} \\
&\quad = \frac{1}{(m+r+\cdots+s)!} \\
&\qquad \times \left(\frac{(m+r+\cdots+s)!}{m! r! \ldots s!} \mu_{[M_1 \ldots M_m} \nu_{N_1 \ldots N_r} \rho_{R_1 \ldots R_s]} \right) \\
&\qquad \times \mathrm{d}x^{M_1} \wedge \ldots \wedge \mathrm{d}x^{N_1} \wedge \ldots \wedge \mathrm{d}x^{R_1} \wedge \ldots,
\end{aligned} \qquad \text{(C6)}$$

$$\star[\boldsymbol{\mu}^{(m)} \wedge \boldsymbol{\nu}^{(r)} \ldots \wedge \boldsymbol{\rho}^{(s)}] = \frac{1}{(d+2-m-r\ldots-s)!} \times \left(\frac{1}{m!r!\ldots s!}\mu^{M_1\cdots}\nu^{N_1\cdots}\ldots\rho^{R_1\cdots}\epsilon_{M_1\ldots N_1\ldots R_1\ldots\cdots S_1\ldots}\right) \times \mathrm{d}x^{S_1} \wedge \ldots. \tag{C7}$$

We define the interior product with respect to a vector field X of a differential form as

$$\iota_X \boldsymbol{\mu}^{(m)} = \frac{1}{(m-1)!}(X^M \mu_{[MN_1\ldots N_{m-1}]})\mathrm{d}x^{N_1} \wedge \ldots \wedge \mathrm{d}x^{N_{m-1}}. \tag{C8}$$

One can check the following two useful identities:

$$\iota_X \star[\boldsymbol{\mu}^{(m)}] = \star[\boldsymbol{\mu}^{(m)} \wedge \boldsymbol{X}], \quad \star[\iota_X \boldsymbol{\mu}^{(m)}] = (-)^{m-1}\boldsymbol{X} \wedge \star[\boldsymbol{\mu}^{(m)}]. \tag{C9}$$

Given a one-form $\boldsymbol{Y}^{(1)}$ and a vector field X such that $\iota_X \boldsymbol{Y}^{(1)} = 1$, any differential form $\boldsymbol{\mu}^{(m)}$ can be decomposed as

$$\boldsymbol{\mu}^{(m)} = \iota_X(\boldsymbol{Y}^{(1)} \wedge \boldsymbol{\mu}^{(m)}) + \boldsymbol{Y}^{(1)} \wedge \iota_X \boldsymbol{\mu}^{(m)}. \tag{C10}$$

This is in particular helpful when $\boldsymbol{\mu}^{(d+2)}$ is a full rank form,

$$\boldsymbol{\mu}^{(d+2)} = \boldsymbol{Y}^{(1)} \wedge \iota_X \boldsymbol{\mu}^{(d+2)}. \tag{C11}$$

The exterior derivative of a differential form is defined to be

$$\mathrm{d}\boldsymbol{\mu}^{(m)} = \frac{1}{(m+1)!} \times [(m+1)\partial_{[M_1}\mu_{M_2\ldots M_{m+1}]}]\mathrm{d}x^{M_1} \wedge \ldots \wedge \mathrm{d}x^{M_{p+1}}. \tag{C12}$$

One can check the useful relation

$$\star\mathrm{d}\boldsymbol{\mu}^{(d+1)} = (-)^{d+1}\underline{\nabla}_M \star[\boldsymbol{\mu}^{(d+1)}]^M, \quad \mathrm{d}\star[\boldsymbol{\mu}^{(1)}] = \star\underline{\nabla}_M \mu^M. \tag{C13}$$

The Lie derivative of a differential form satisfies

$$\pounds_X \boldsymbol{\mu}^{(m)} = \iota_X \mathrm{d}\boldsymbol{\mu}^{(m)} + \mathrm{d}(\iota_X \boldsymbol{\mu}^{(m)}). \tag{C14}$$

Integration of a full rank form is defined as

$$\int_{\mathcal{M}_{(d+2)}} \boldsymbol{\mu}^{(d+2)} = \mathrm{sgn(G)} \int \{\mathrm{d}x^M\}\sqrt{|\mathrm{G}|}\star[\boldsymbol{\mu}^{(d+2)}] = \mathrm{sgn(G)} \int \{\mathrm{d}x^M\}\sqrt{|\mathrm{G}|}\frac{1}{(d+2)!} \times \epsilon^{M_1\ldots M_{d+2}}\mu_{M_1\ldots M_{d+2}}. \tag{C15}$$

Here the raised Levi-Civita symbol has the value $\epsilon^{0,1,2,\ldots,d+1} = \mathrm{sgn}(G)/\sqrt{|G|}$. Integration of an exact full rank form is given by an integration on the boundary,

$$\int_{\mathcal{M}_{(d+2)}} \mathrm{d}\boldsymbol{\mu}^{(d+1)} = \int_{\partial\mathcal{M}_{(d+2)}} \boldsymbol{\mu}^{(d+1)}, \tag{C16}$$

where given a unit vector N normal to the boundary, the volume element on the boundary is defined as $\iota_N \boldsymbol{\epsilon}^{(d+2)} = \star\boldsymbol{N}$.

1. Newton-Cartan differential forms

We decompose a vector and a one-form on $\mathcal{M}_{(d+2)}$ in the NC basis,

$$\mathcal{X}^M\partial_M = (\mathcal{X}^\sim - B_\mu\mathcal{X}^\mu)\partial_\sim + \mathcal{X}^\mu(\partial_\mu + B_\mu\partial_\sim), \quad \mathcal{Y}_M\mathrm{d}x^M = \mathcal{Y}_\sim(\mathrm{d}x^\sim - B_\mu\mathrm{d}x^\mu) + (\mathcal{Y}_\mu + B_\mu\mathcal{Y}_\sim)\mathrm{d}x^\mu. \tag{C17}$$

One can check that these results are written in a basis that transforms "nicely" from the NC perspective, which tells us that

$$V^M\mathcal{Y}_M = \mathcal{Y}_\sim, \quad \mathcal{Y}_\mu + B_\mu\mathcal{Y}_\sim, \quad \overline{V}_M\mathcal{X}^M = \mathcal{X}^\sim - X^\mu B_\mu, \quad \mathcal{X}^\mu \tag{C18}$$

are quantities that transform nicely. As is quite apparent, the first and last quantities do not depend on the explicit choice of ψ_T but the middle ones do. A similar analysis can be done for all tensor fields in the theory. Note that if $\mathcal{Y}_M$ satisfies $\iota_V\boldsymbol{\mathcal{Y}} = V^M\mathcal{Y}_M = 0$, the one-form becomes purely NC. On the other hand, if $\overline{V}_M\mathcal{X}^M = 0$ the vector field becomes purely NC. This motivates us to define a NC differential form to be a differential form on $\mathcal{M}_{(d+2)}$ which does not have a leg along V, i.e., $\iota_V\boldsymbol{\mu}^{(m)}$. Such a differential form can be expanded as

$$\boldsymbol{\mu}^{(m)} = \frac{1}{m!}\mu_{\mu_1\mu_2\ldots\mu_m}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_m}. \tag{C19}$$

On the other hand we define a NC "differential contra-form" as a totally antisymmetric contravariant tensor in $\mathcal{M}_{(d+2)}$ which has zero contraction with $\overline{V}_M$. In the basis $\partial'_\mu = \partial_\mu + B_\mu\partial_\sim$ it can be expanded as

$$\boldsymbol{\mu}^{[m]} = \frac{1}{m!}\mu^{\mu_1\mu_2\ldots\mu_m}\partial'_{\mu_1}\wedge\partial'_{\mu_2}\wedge\ldots\wedge\partial'_{\mu_m}. \tag{C20}$$

It is clear that though the basis depends on the choice of ψ_T, the components of the contra-form are independent of it. On a manifold with a nondegenerate metric there exists a map between these two quantities, but for us these two shall be distinct. We can also define a spatial differential form/contra-form with the requirement that it should not have any leg along V and $\overline{V}$. In this case there exists a map between these two quantities realized by $p^{\mu\nu}$ and $p_{\mu\nu}$.

Correspondingly there are three volume elements,

$$\begin{aligned}\boldsymbol{\varepsilon}_{\uparrow}^{[d+1]} &= [\star\overline{\boldsymbol{V}}]^{\sharp} = \frac{1}{(d+1)!}(\overline{V}_M\epsilon^{M\mu_1\ldots\mu_{d+1}})\partial'_{\mu_1}\wedge\ldots\wedge\partial'_{\mu_{d+1}},\\ \boldsymbol{\varepsilon}_{\downarrow}^{(d+1)} &= \star\boldsymbol{V} = \frac{1}{(d+1)!}(V^M\epsilon_{M\mu_1\ldots\mu_{d+1}})\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d+1}},\\ \boldsymbol{\varepsilon}^{(d)} &= \star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}] = \frac{1}{d!}(V^M\overline{V}^N\epsilon_{MN\mu_1\ldots\mu_d})\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_d}.\end{aligned} \tag{C21}$$

In the main text we have primarily used the first one. Correspondingly, there are three Hodge duals that provide maps from forms to contra-forms, contra-forms to forms, and a self-inverse map between spatial forms, respectively,

$$\begin{aligned}*_{\uparrow}[\boldsymbol{\mu}^{(m)}] &= \star[\overline{\boldsymbol{V}}\wedge\boldsymbol{\mu}^{(m)}]^{\sharp} = \frac{1}{(d+1-m)!}\left(\frac{1}{m!}\mu_{\mu_1\ldots\mu_m}\varepsilon_{\uparrow}^{\mu_1\ldots\mu_m\nu_1\ldots\nu_{d+1-m}}\right)\partial'_{\nu_1}\wedge\ldots\wedge\partial'_{\nu_{d+1-m}},\\ *^{\downarrow}[\boldsymbol{\mu}^{[m]}] &= \star[\boldsymbol{V}\wedge\boldsymbol{\mu}^{\flat(m)}] = \frac{1}{(d+1-m)!}\left(\frac{1}{m!}\mu^{\mu_1\ldots\mu_m}\varepsilon^{\downarrow}_{\mu_1\ldots\mu_m\nu_1\ldots\nu_{d+1-m}}\right)\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{d+1-m}},\\ *[\boldsymbol{\mu}^{(m)}] &= \star[\boldsymbol{V}\wedge\boldsymbol{u}\wedge\boldsymbol{\mu}^{(m)}] = \frac{1}{(d-m)!}\left(\frac{1}{m!}\mu^{\mu_1\ldots\mu_m}\varepsilon_{\mu_1\ldots\mu_m\nu_1\ldots\nu_{d-m}}\right)\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{d-m}}.\end{aligned} \tag{C22}$$

One can check that $** = -\mathrm{sgn}(G)(-)^{m(d-m)}$ and $*^{\downarrow}*_{\uparrow} = *_{\uparrow}*^{\downarrow} = -\mathrm{sgn}(G)(-)^{m(d+1-m)}$. Finally we need to define an integration for NC full rank forms and contra-forms,

$$\begin{aligned}\int_{\mathcal{M}_{(d+1)}}\boldsymbol{\mu}^{(d+1)} &= \mathrm{sgn}(G)\int_{\mathcal{M}_{(d+2)}}\overline{\boldsymbol{V}}\wedge\boldsymbol{\mu}^{(d+1)} = \mathrm{sgn}(\gamma)\int\{\mathrm{d}x^{\mu}\}\sqrt{|\gamma|}*_{\uparrow}[\boldsymbol{\mu}^{(d+1)}],\\ \int_{\mathcal{M}_{(d+1)}}\boldsymbol{\mu}^{[d+1]} &= \mathrm{sgn}(G)\int_{\mathcal{M}_{(d+2)}}\boldsymbol{V}\wedge\boldsymbol{\mu}^{\flat(d+1)} = \mathrm{sgn}(\gamma)\int\{\mathrm{d}x^{\mu}\}\sqrt{|\gamma|}*^{\downarrow}[\boldsymbol{\mu}^{[d+1]}],\end{aligned} \tag{C23}$$

where $\gamma_{\mu\nu} = p_{\mu\nu} + n_\mu n_\nu$ and $\gamma = \det\gamma_{\mu\nu} = -\mathrm{G}$. Obviously a full rank spatial form would be zero. The rest of the notations and conventions follow from our relativistic discussion.

2. Noncovariant differential forms

Choosing a noncovariant basis given in Appendix A, a vector and a one-form can be decomposed as

$$\begin{aligned}\mathcal{X}^M\partial_M &= -\mathrm{e}^{\Phi}(\mathcal{X}_t + B_t\mathcal{X}_{\sim})\partial_{\sim} - \mathrm{e}^{\Phi}\mathcal{X}_{\sim}(B_t\partial_{\sim} + \partial_t) + \mathcal{X}^i(\partial_i - a_i\partial_t + (B_i - a_iB_t)\partial_{\sim}),\\ \mathcal{Y}_M\mathrm{d}x^M &= \mathcal{Y}_{\sim}(\mathrm{d}x^{\sim} - B_\mu\mathrm{d}x^\mu) + (\mathcal{Y}_{\sim}B_t + \mathcal{Y}_t)(\mathrm{d}t + a_i\mathrm{d}x^i) + g_{ij}\mathcal{Y}^j\mathrm{d}x^i.\end{aligned} \tag{C24}$$

It immediately follows that a spatial differential form ($\mathcal{Y}_t = \mathcal{Y}_{\sim} = 0$) is indeed a pure differential form on the spatial slice. Such a form can be expanded in the coordinate basis as

$$\boldsymbol{\mu}^{(m)} = \frac{1}{m!}\mu_{i_1i_2\ldots i_m}\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_m}. \tag{C25}$$

Since there exists an invertible metric g_{ij} on this slice, there is a map between forms and contra-forms. One can check that the volume element $\boldsymbol{\varepsilon}^{(d)}$ defined before is indeed a full rank form on the spatial slice and can be written in this setting as

$$\boldsymbol{\varepsilon}^{(d)} = \star[\boldsymbol{V}\wedge\overline{\boldsymbol{V}}] = \frac{1}{d!}(V^M\overline{V}^N\epsilon_{MNi_1\ldots i_d})\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_d}. \tag{C26}$$

The Hodge dual $*$ associated with it serves as the Hodge dual operation on the spatial slice. Finally a full rank spatial form can be integrated on a spatial slice,

$$\int_{\mathcal{M}_{(d)}} \boldsymbol{\mu}^{(d)} = \mathrm{sgn}(G) \int_{\mathcal{M}_{(d+2)}} \mathrm{e}^{\Phi} \boldsymbol{V} \wedge \overline{\boldsymbol{V}} \wedge \boldsymbol{\mu}^{(d)} = \mathrm{sgn}(g) \int \{\mathrm{d}x^\mu\} \sqrt{|g|} * [\boldsymbol{\mu}^{(d)}]. \tag{C27}$$

Here $g = \det g_{ij} = \mathrm{e}^{2\Phi}\gamma = -\mathrm{e}^{2\Phi}\mathrm{G}$. Other conventions and notations are the same as in the relativistic case.

APPENDIX D: COMMENTS ON THE RELATIVISTIC ENTROPY CURRENT

In this appendix we wish to make some comments on the entropy current for a relativistic fluid. To be notationally consistent with some recent works in this direction (e.g., Ref. [42]), in this section we consider the relativistic manifold $\mathcal{M}_{(2n)}$ to be $2n$ dimensional, and denote indices on it by $\mu, \nu \ldots$. On the local flat space $\mathbb{R}^{(2n-1,1)}$, however, we denote the indices by $\alpha, \beta \ldots$. This setup is equipped with a vielbein $e^\alpha{}_\mu$, an affine connection $\Gamma^\lambda{}_{\mu\nu}$, a spin connection $C^\alpha{}_{\mu\beta} = e_\beta{}^\rho(\Gamma^\nu{}_{\mu\rho} e^\alpha{}_\nu - \partial_\mu e^a{}_\rho)$, and a non-Abelian gauge field A_μ. Correspondingly we have a torsion tensor $\mathrm{T}^\alpha{}_{\mu\nu}$, a Riemann curvature tensor $R_{\mu\nu}{}^\alpha{}_\beta$, and a gauge field strength $F_{\mu\nu}$. $\mu, \nu \ldots$ indices can be raised/lowered by the metric $g_{\mu\nu}$, $\alpha, \beta \ldots$ indices can be raised/lowered by the flat metric $\eta_{\alpha\beta}$, while both type of indices can be interchanged by the vielbein. The covariant derivative on the other hand is given by ∇_μ which is associated with all the connections. We take the fluid data to be $\psi_\beta = \{\beta^\mu, [\Lambda_{\Sigma(\beta)}]^\mu{}_\nu, \Lambda_{(\beta)}\}$. In terms of it we define the fluid temperature $T = (-\beta^\mu\beta_\mu)^{-1/2}$, fluid velocity $u^\mu = T\beta^\mu$, scaled chemical potential $\nu = \Lambda_\beta + \beta^\mu A_\mu$, chemical potential $\mu = T\nu$, scaled spin chemical potential $[\nu_\Sigma]^\alpha{}_\beta = [\Lambda_\Sigma]^\alpha{}_\beta + \beta^\mu C^\alpha{}_{\mu\beta}$, and spin chemical potential $[\mu_\Sigma]^\alpha{}_\beta = T[\nu_\Sigma]^\alpha{}_\beta$. Finally we have a canonical EM tensor $T^\mu{}_\alpha$, a spin current $\Sigma^{\mu\alpha}{}_\beta$, a charge current J^μ, an entropy current J^μ_S, a Belinfante EM tensor $T^{\mu\nu}_{(b)}$, and a Belinfante (usual) entropy current $J^\mu_{S(b)}$.

We wrote an off-shell generalization for the second law of thermodynamics in Cartan formalism in Sec. IV, which in the aforementioned notation will become

$$\underline{\nabla}_\mu J^\mu_S + \beta^\nu(\underline{\nabla}_\mu T^\mu{}_\nu - \mathrm{T}^\alpha{}_{\nu\mu} T^\mu{}_\alpha - R_{\nu\mu}{}^\alpha{}_\beta \Sigma^{\mu\beta}{}_\alpha - F_{\nu\mu} \cdot J^\mu) + [\nu_\Sigma]_{\beta\alpha}(\underline{\nabla}_\mu \Sigma^{\mu\alpha\beta} - T^{[\beta\alpha]} - \Sigma_{\mathrm{H}}^{\perp\alpha\beta}) + \nu \cdot (\underline{\nabla}_\mu J^\mu - \mathrm{J}^\perp_{\mathrm{H}}) \geq 0, \tag{D1}$$

where $\underline{\nabla}_\mu = \nabla_\mu - \mathrm{T}^\nu{}_{\nu\mu}$. $\Sigma_{\mathrm{H}}^{\perp\alpha\beta}$, $\mathrm{J}^\perp_{\mathrm{H}}$ are the anomalous Hall currents, which are determined in terms of an anomaly polynomial $\boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}$ as

$$\star_{(2n+1)}\Sigma_{\mathrm{H}}{}^{\alpha\beta} = \frac{\partial \boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{\beta\alpha}}, \quad \star_{(2n+1)}\mathbf{J}_{\mathrm{H}} = \frac{\partial \boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{F}}. \tag{D2}$$

On imposing the equations of motion (3.4) (after the appropriate change of notation) this will boil down to the second law of thermodynamics $\underline{\nabla}_\mu J^\mu_S \geq 0$. To compare this statement with that of Ref. [40] we make a field redefinition,

$$[\nu_\Sigma]_{\mu\nu} \to [\nu'_\Sigma]_{\mu\nu} = [\nu_\Sigma]_{\mu\nu} + e_{\alpha[\mu}\delta_\beta e^\alpha{}_{\nu]} = \nabla_{[\nu}\beta_{\mu]} + \mathrm{T}_{[\mu\nu]\rho}\beta^\rho, \tag{D3}$$

where δ_β is the diffeomorphism, spin, and flavor transformation associated with ψ_β. This field redefinition does not spoil our equilibrium frame as the perturbation vanishes on promoting ψ_β to an isometry. Further, by setting the torsion to zero this statement boils down to the statement appearing in Ref. [40],

$$\nabla_\mu J^\mu_{S(b)} + \beta_\nu(\nabla_\mu T^{\mu\nu}_{(b)} - F^{\nu\mu} \cdot J_\nu - \nabla_\mu \Sigma_{\mathrm{H}}^{\perp\nu\mu}) + \nu \cdot (\nabla_\mu J^\mu - \mathrm{J}^\perp_{\mathrm{H}}) \geq 0, \tag{D4}$$

where we have defined the *Belinfante EM tensor*,

$$T^{\mu\nu}_{(b)} = T^{(\mu\nu)} + 2\nabla_\rho \Sigma^{(\mu\nu)\rho}. \tag{D5}$$

We have also defined the *Belinfante entropy current*,[29]

$$J^\mu_{S(b)} = J^\mu_S + \beta_\nu(\nabla_\rho \Sigma^{\rho\mu\nu} + T^{[\mu\nu]} - \Sigma_{\mathrm{H}}^{\perp\mu\nu}), \tag{D6}$$

which is a more natural quantity to use when working with the Belinfante EM tensor $T^{\mu\nu}_{(b)}$. Note that the two entropy currents differ only off shell and boil down to the same thing when the spin equation of motion,

$$\nabla_\rho \Sigma^{\rho\mu\nu} = T^{[\nu\mu]} + \Sigma_{\mathrm{H}}^{\perp\mu\nu}, \tag{D7}$$

is imposed. For comparison with Ref. [42] we will be interested in relativistic fluids without a spin current. In the absence of anomalies we could achieve this by setting $\Sigma^{\rho\mu\nu} = T^{[\mu\nu]} = 0$, but anomalies would not allow us to make this simple choice. Nevertheless, we can define spinless fluids as configurations for which $\Sigma^{\rho\mu\nu}$, $T^{[\mu\nu]}$ are totally determined in terms of the anomalies.

The transgression form business does not change much in the vielbein formalism. The end result is that we can define certain quantities in terms of the anomaly polynomial $\boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}$ and hatted connections $\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu\boldsymbol{u}$, $\hat{\boldsymbol{C}}^\alpha{}_\beta = \boldsymbol{C}^\alpha{}_\beta + [\mu_\Sigma]^\alpha{}_\beta \boldsymbol{u}$ [refer to the discussion around Eq. (12.25) of Ref. [42] for more details],

[29]The motive for calling $J^M_{S(b)}$ the Belinfante entropy current is primarily to distinguish it from J^M_S, and second to relate it more closely to the Belinfante EM tensor $T^{MN}_{(b)}$. We could not find any existing name in the literature for this quantity.

$$\star\Sigma_{\mathcal{P}}^{\alpha\beta} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \left(\frac{\partial \mathcal{P}^{(2n+2)}}{\partial \boldsymbol{R}_{\beta\alpha}} - \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{R}}_{\beta\alpha}}\right),$$
$$\star\boldsymbol{J}_{\mathcal{P}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \left(\frac{\partial \mathcal{P}^{(2n+2)}}{\partial \boldsymbol{F}} - \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{F}}}\right),$$
$$\star\boldsymbol{q}_{\mathcal{P}} = -\frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \left[\mathcal{P}_{\mathrm{CS}}^{(2n+2)} - \hat{\mathcal{P}}_{\mathrm{CS}} + \mathrm{d}\boldsymbol{u} \wedge \left([\mu_\Sigma]^\alpha{}_\beta \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{R}}^\alpha{}_\beta} + \mu \cdot \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{F}}}\right)\right]. \quad \text{(D8)}$$

In terms of these, the anomalous sector of the constitutive relations is given as

$$(T^{\mu\alpha})_{\mathrm{A}} = q_{\mathcal{P}}^\mu u^\alpha + q_{\mathcal{P}}^\alpha u^\mu,$$
$$(\Sigma^{\mu\alpha\beta})_{\mathrm{A}} = \Sigma_{\mathcal{P}}^{\mu\alpha\beta},$$
$$(J^\mu)_{\mathrm{A}} = J_{\mathcal{P}}^\mu. \quad \text{(D9)}$$

These currents follow the Bianchi identities,[30]

$$\underline{\nabla}_\mu (T^\mu{}_\nu)_{\mathrm{A}} = \mathrm{T}^\alpha{}_{\nu\mu} (T^\mu{}_\alpha)_{\mathrm{A}} + R_{\nu\mu}{}^\alpha{}_\beta (\Sigma^{\mu\beta}{}_\alpha)_{\mathrm{A}} + F_{\nu\mu} \cdot (J^\mu)_{\mathrm{A}} - u_\nu (\mu \cdot \hat{\mathrm{J}}_{\mathrm{H}}^\perp + [\mu_\Sigma]_{\alpha\beta} \hat{\Sigma}_{\mathrm{H}}^{\perp\beta\alpha}) + \frac{1}{\sqrt{-g}} \delta_\beta (\sqrt{-g} T q_{\mathcal{P}\nu}),$$
$$\underline{\nabla}_\mu (\Sigma^{\mu\alpha\beta})_{\mathrm{A}} = \Sigma_{\mathrm{H}}^{\perp\alpha\beta} - \hat{\Sigma}_{\mathrm{H}}^{\perp\alpha\beta},$$
$$\underline{\nabla}_\mu (J^\mu)_{\mathrm{A}} = \mathrm{J}_{\mathrm{H}}^\perp - \hat{\mathrm{J}}_{\mathrm{H}}^\perp. \quad \text{(D10)}$$

By plugging these constitutive relations into the off-shell adiabaticity equation we can get a relation for the entropy current,

$$\underline{\nabla}_\mu (J_S^\mu)_{\mathrm{A}} \geq 0. \quad \text{(D11)}$$

Hence the off-shell second law can be satisfied with a trivially zero entropy current,

$$J_S^\mu = 0. \quad \text{(D12)}$$

In other words, the entropy current J_S^μ does not get any contribution from anomalies. On the other hand, using Bianchi identities in Eq. (D6), we can read off the anomalous Belinfante entropy current,

$$(J_{S(b)}^\mu)_{\mathrm{A}} = \beta_\nu \hat{\Sigma}_{\mathrm{H}}^{\perp\nu\mu}, \quad \text{(D13)}$$

which is what was found in Refs. [42,54]. Note that $\hat{\Sigma}_{\mathrm{H}}^{\perp\nu\mu}$ is by definition antisymmetric in its last two indices, and differs from Ref. [42] by a factor of 2. Hence we have established that the entropy current in the vielbein formalism J_S^μ does not get a contribution from anomalies, while the Belinfante entropy current does. Recall that a similar situation appears for the EM tensor as well; while the canonical EM tensor $T^{\mu\alpha}$ that appears in the vielbein formalism is the Noether current of translations, the symmetric Belinfante EM tensor $T_{(b)}^{\mu\nu}$ that appears in the metric-like formalism couples to the metric in general relativity but does not correspond to any Noether current. Hence from the point of view of symmetries, the canonical EM tensor is a more natural quantity. On the same lines we guess that the vielbein entropy current will be more naturally associated with the fundamental $\mathrm{U}(1)_{\mathrm{T}}$ symmetry introduced in Ref. [42], as opposed to the Belinfante entropy current. The former being independent of anomalies seems to strengthen this natural guess. However one will have to do the explicit computation of $\mathrm{U}(1)_{\mathrm{T}}$ transformations in the presence of torsion to give any weight to this claim.

[30] Upon using the definition of the Belinfante EM tensor from Eq. (D5), and setting the torsion to zero, these Bianchi identities reproduce the ones given in Ref. [42].

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