

Cardy-Verlinde entropy in Hořava-Lifshitz gravityOrlando Luongo,^{1,2,3,4,*} Giovanni Battista Pisani,⁵ and Hernando Quevedo^{6,5,†}¹*Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa*²*Astrophysics, Cosmology and Gravity Centre (ACGC), University of Cape Town, Rondebosch 7701, Cape Town, South Africa*³*Dipartimento di Fisica, Università di Napoli “Federico II,” Via Cinthia, I-80126 Napoli, Italy*⁴*Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli, Via Cinthia, Napoli I-00185, Italy*⁵*Dipartimento di Fisica, “Sapienza” Università di Roma, Piazzale Aldo Moro 5, I-00185 Rome, Italy*⁶*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, Mexico*

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We investigate homogeneous cosmological models with perfect-fluid sources in the framework of the Hořava-Lifshitz model for quantum gravity. We show that the Hamiltonian constraint of such spacetimes can be rewritten as the Cardy formula for the entropy in conformal field theory. The Cardy entropy is shown to depend explicitly on the value of the Hořava parameter λ so that it can be interpreted as determining the entropy and the gravitational interaction of the theory. Moreover, we show that Verlinde’s Pythagorean representation of the Hamiltonian constraint is also valid in the case of homogeneous Hořava-Lifshitz spacetimes. We interpret these results as a further indication of a deep relationship between gravity, thermodynamics and holography in the quantum regime.

DOI: [10.1103/PhysRevD.93.064057](https://doi.org/10.1103/PhysRevD.93.064057)**I. INTRODUCTION**

All the efforts spent to develop a self-consistent theory of quantum gravity have been so far incomplete [1]. Even though many approaches have been carried forward during the last few decades, the problem of obtaining a covariant scheme for quantum gravity still remains open and, consequently, many phenomenological proposals have been discussed in the literature [2–5]. An appealing technique consists in assuming the violation of the Lorentz invariance in some particular energy regimes [6]. The idea is that Lorentz invariance is somehow broken at ultraviolet scales, providing quantum effects which have consequences on the observable Universe itself. Recently, following this philosophy Hořava proposed an alternative theory which reduces to Einstein’s gravity with a cosmological constant in the infrared regime, leading to complicated modifications in the ultraviolet regime [7]. The underlying philosophy of Hořava’s model is to include Lorentz-breaking terms in the action, which provides a different scaling between space and time at an ultraviolet fixed point. The transformation $x^i \rightarrow lx^i$, $t \rightarrow l^z t$, where z is the scaling exponent, in the case $z = 3$, leads to a theory which is renormalizable by power counting. This idea was first proposed by Lifshitz [8] and, in the case of the Hořava approach, it leads to a model which is nonrelativistic in the ultraviolet regime. However, in the infrared limit, the model manifests an emerging four-dimensional general covariance. Due to these considerations, the Hořava framework is

frequently referred to as the Hořava-Lifshitz (HL) model. Although the model is clearly appealing for the description of quantum effects in gravity, its physical interpretation is still open. For instance, the model depends upon several parameters, each of them having a precise physical role. Nevertheless, the interpretation of these parameters is not completely known. In this work, we will focus on the parameter λ which is usually fixed in the infrared limit in order to obtain Einstein’s gravity.

In this work, we will consider homogeneous spacetimes with a perfect fluid as the gravitational source under the condition that they satisfy the field equations of the Hořava-Lifshitz model. In particular, we will explicitly analyze the isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and the anisotropic Bianchi cosmologies of the types I, V and IX, following the procedure formulated in Ref. [9]. We first derive the field equations and write them in a form which is suitable for our analysis. In particular, we express the Hamiltonian constraint in terms of the Hubble parameters which determine the expansion in different spatial directions. Then, we assume the validity of the Cardy entropy which was originally derived in two-dimensional conformal field theory. Assuming the validity of the Cardy formula implies that we are assuming a holographic treatment of the entropy in the HL model. If this assumption turns out to be true in the corresponding cosmological scenarios, we can conclude that the holographic principle can be used in the corresponding models. In fact, we will show that for all homogeneous spacetimes under consideration in this work it is possible to define the central charge and the eigenvalue of the Virasoro operator in

*luongo@na.infn.it

†quevedo@nucleares.unam.mx

such a way that the Cardy formula is identically satisfied, generating an explicit expression for the Cardy entropy. In doing so, it turns out that the Hořava parameter λ enters explicitly the Cardy entropy, allowing us to interpret it, determining the entropy and the gravitational interaction of the theory. Moreover, we will calculate the Hubble, Bekenstein and Bekenstein-Hawking entropies for the anisotropic Bianchi IX cosmological model, and will show that they can be arranged in such a way that the corresponding Hamiltonian constraint can be cast into the Pythagorean representation proposed recently by Verlinde [10].

This paper is structured as follows: In Sec. II, we present a brief review of the Hořava action and comment on the free parameters that enter the theory. In Sec. III, we study homogeneous spacetimes with a perfect-fluid source. We consider the FLRW metric and the Bianchi I, V and IX metrics for which we derive the field equations explicitly. Special attention is given to the form of the Hamiltonian constraint which is expressed in a compact form in terms of the scale factors, the Hubble parameters and the parameters entering the HL model. In Sec. IV, we analyze the Hamiltonian constraints of all the homogeneous cosmological spacetimes and establish a unique relationship that allows us to define the Cardy entropy for each spacetime. We also show that the Pythagorean representation of the constraints in terms of entropies is valid in the HL model. Finally, in Sec. V, we discuss our results.

II. THE HOŘAVA-LIFSHITZ PARADIGM

Consider an arbitrary four-dimensional metric in the framework of the ADM (Arnowitt-Deser-Misner) formalism, i.e.,

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (1)$$

where we introduce the lapse function N , the shift vector N^i and the three-dimensional metric g_{ij} . The Einstein-Hilbert covariant action in the ADM decomposition is given by

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N (K_{ij} K^{ij} - K^2 + R - 2\Lambda), \quad (2)$$

where

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad (3)$$

is the extrinsic curvature tensor and K is its trace, where the dot represents the time derivative.

The HL quantum gravity model is based upon a generalization of the Einstein-Hilbert action which breaks covariance in order to obtain in the ultraviolet limit a theory that is renormalizable by power counting. Then, the Hořava action can be expressed as [7]

$$S = \int dt d^3x (\mathcal{L}_0 + \mathcal{L}_1), \quad (4)$$

with

$$\mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R^{(3)} - 3\Lambda_W^2)}{8(1-3\lambda)} \right\} \quad (5)$$

and

$$\mathcal{L}_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} (R^{(3)})^2 - \frac{\kappa^2}{2\nu^4} \left(C_{ij} - \frac{\mu\nu^2}{2} R_{ij}^{(3)} \right) \left(C^{ij} - \frac{\mu\nu^2}{2} R^{(3)ij} \right) \right\}, \quad (6)$$

where $R_{ij}^{(3)}$ and $R^{(3)}$ are the Ricci tensor and the scalar curvature for the three-dimensional metric g_{ij} , respectively, and

$$C^{ij} = \epsilon^{ikl} \nabla_k \left(R_l^{(3)j} - \frac{1}{4} R^{(3)} \delta_l^j \right) \quad (7)$$

is known in literature as the Cotton tensor. The free parameters entering the Hořava action (4) are λ , κ , μ , ν and Λ_W . Soon after the publication of the model, it was found that the Schwarzschild-AdS black hole solution is not recovered in the infrared limit, although Einstein's theory with cosmological model was obtained at the level of the action [11]. This difficulty was solved by introducing an additional parameter which modifies the IR behavior [12,13]. In this work, however, we will use the original Hořava action, since our results do not depend on the value of the additional parameter.

The five constants κ , λ , μ , ν , and Λ_W are parameters which determine the velocity of the light c , the gravitational constant G and the Einstein cosmological constant Λ by means of [14]

$$c^2 = \frac{\kappa^4 \mu^2 |\Lambda_W|}{8(3\lambda - 1)^2}, \quad (8)$$

$$G = \frac{\kappa^2 c^2}{16\pi(3\lambda - 1)} \Lambda_W = \frac{2}{3} \Lambda.$$

These three constraints imply that three constants of the HL model can be fixed by using experimental data. Notice that for $\lambda < 1/3$ the gravitational constant becomes negative, indicating the presence of repulsive gravity. We therefore limit ourselves to the case $\lambda > 1/3$. Consequently, the HL model possesses only two free parameters, which should be chosen in accordance with observations. In this work, we will focus on the investigation of the physical significance of the parameter λ in the context of cosmological models.

III. COSMOLOGICAL DYNAMICS

The simplest cosmological models assume a high degree of symmetry in order to be able to handle the corresponding field equations. In this section, we will assume that the spacetime is homogeneous so that we are left with only two possible classes of models, namely isotropic and anisotropic cosmological models. In the first case, the geometry of the spacetime is described by the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (9)$$

where $a(t)$ is the scale factor and $k = -1, 0, 1$. The corresponding field equations can be derived from the variation of the Hořava action. We obtain

$$H^2 = \frac{2}{3\lambda - 1} \left[\frac{8\pi G}{3} \rho_{\text{tot}} - \frac{k}{a^2} - \frac{2k^2}{3\lambda - 1} \left(\frac{4\pi G\mu}{a^2} \right)^2 \right], \quad (10)$$

$$\frac{\ddot{a}}{a} = \frac{2}{3\lambda - 1} \left[-\frac{4\pi G}{3} (\rho_{\text{tot}} + 3p_{\text{tot}}) + \frac{2k^2}{3\lambda - 1} \left(\frac{4\pi G\mu}{a^2} \right)^2 \right], \quad (11)$$

where the total density and pressure are defined as $\rho_{\text{tot}} = \rho_{\Lambda_W} + \rho$ and $p_{\text{tot}} = p_{\Lambda_W} + p$, respectively, with $\rho_{\Lambda_W} \equiv \frac{3\Lambda_W}{16\pi G}$ and $p_{\Lambda_W} \equiv -\rho_{\Lambda_W}$. The limiting case of the Λ CDM model is recovered when $\lambda \rightarrow 1$ and $\mu \rightarrow 0$.

For later use, it is convenient to rewrite the Hamiltonian constraint (10) as

$$\frac{3\lambda - 1}{2} H^2 + \frac{k + M}{a^2} = \frac{8\pi G}{3} \rho_{\text{tot}}, \quad (12)$$

where $M = M(a, \lambda, \mu)$ is a function of the scale factor and the Hořava parameters. The explicit value of this function depends on the topology of the cosmological model and can be expressed as given in Table I.

We now consider the case of anisotropic cosmological models. We will restrict ourselves to the study of the Bianchi I, V and IX models, which are described by the metrics

$${}^{(I)} ds^2 = -dt^2 + a_1^2 dx^2 + a_2^2 dy^2 + a_3^2 dz^2, \quad (13)$$

TABLE I. Hamiltonian constraint (12) for isotropic spacetimes.

FLRW universe	k	M
Flat	0	0
Open	-1	$\frac{2}{3\lambda - 1} \left(\frac{4\pi G\mu}{a} \right)^2$
Closed	+1	$\frac{2}{3\lambda - 1} \left(\frac{4\pi G\mu}{a} \right)^2$

$${}^{(V)} ds^2 = -dt^2 + a_1^2 dx^2 + e^{2x} a_2^2 dy^2 + e^{2x} a_3^2 dz^2, \quad (14)$$

$${}^{(IX)} ds^2 = -dt^2 + a_1^2 (\omega_1)^2 + a_2^2 (\omega_2)^2 + a_3^2 (\omega_3)^2, \quad (15)$$

where the scale factors a_i ($i = 1, 2, 3$) depend on time only, and

$$\begin{aligned} \omega_1 &= \frac{1}{2} (-dx \sin z + dy \sin x \cos z), \\ \omega_2 &= \frac{1}{2} (dx \cos z + dy \sin x \sin z), \\ \omega_3 &= \frac{1}{2} (dy \cos x + dz). \end{aligned} \quad (16)$$

In the case of the Bianchi IX metric, it is convenient to introduce the alternative representation

$${}^{(IX)} ds^2 = -dt^2 + e^{-2\Omega} [e^{2X+2Y} (\omega_1)^2 + e^{2X-2Y} (\omega_2)^2 + e^{-4X} (\omega_3)^2], \quad (17)$$

which is useful for concrete calculations in terms of the scale factors

$$a_1 = e^{-\Omega+X+Y}, \quad a_2 = e^{-\Omega+X-Y}, \quad a_3 = e^{-\Omega-2X}. \quad (18)$$

Introducing the directional Hubble parameters $H_i = \dot{a}_i/a_i$, the computation of the corresponding Hamiltonian constraint yields

$$\begin{aligned} &\frac{1}{6} (\lambda - 1) (H_1^2 + H_2^2 + H_3^2) \\ &+ \frac{\lambda}{3} (H_1 H_2 + H_1 H_3 + H_2 H_3) + \frac{k + F}{a_1^2} \\ &= \frac{8\pi G}{3} \rho_{\text{tot}}, \end{aligned} \quad (19)$$

where $F = F(a_1, a_2, a_3, \lambda, \mu)$ depends on the scale factors and the Hořava parameters, and its explicit form is given in Table II, where

$$\begin{aligned} \epsilon^2 &\equiv 1 - \frac{a_3^2}{a_2^2} - 2 \left(1 - \frac{a_1^2}{a_2^2} \right) - \frac{a_2^2}{a_3^2} \left(1 - \frac{a_1^2}{a_2^2} \right)^2, \\ \theta^4 &\equiv 1 - \frac{a_3^4}{a_2^4} - 2 \left(1 - \frac{a_1^4}{a_2^4} \right) - \frac{a_2^4}{a_3^4} \left(1 - \frac{a_1^4}{a_2^4} \right)^2. \end{aligned} \quad (20)$$

As for the dynamical equations, we now have three second-order differential equations which relate the second time derivatives of the scale factors. Although these equations are not strictly necessary for the following analysis, for the sake of completeness and for future use we present them explicitly in the Appendix.

TABLE II. Hamiltonian constraint (19) for anisotropic cosmological models.

Bianchi type	k	F
I	0	0
V	-1	$\frac{2}{3\lambda-1} \left(\frac{4\pi G\mu}{a_1}\right)^2$
IX	+1	$\frac{c^2}{3} + \frac{2}{3\lambda-1} \left(\frac{4\pi G\mu}{a_1}\right)^2 \left[3(3-8\lambda) \left(1 + \frac{c^2}{3}\right)^2 - 8(1-3\lambda) \left(1 + \frac{\theta^4}{3}\right)\right]$

IV. THE CARDY-VERLINDE ENTROPY

There are several ways to treat a cosmological model as a thermodynamic system to which a particular value of entropy can be associated. One can, for instance, demand that the cosmological model satisfy the first law of thermodynamics $dE = TdS - pdV$, which in the case of homogeneous cosmological models implies that [9]

$$T\dot{S} = V \left[\dot{\rho} + (p + \rho) \frac{\dot{V}}{V} \right], \quad (21)$$

where we assume that $E = \rho V$. Then, using the conservation law, an equation of state and Euler's identity, it is possible to obtain explicit expressions for the entropy and the temperature in terms of the volume V or, equivalently, the cosmic time t . Although this procedure gives reasonable results in the case of homogeneous Bianchi models [9,15], under the presence of inhomogeneities the physical predictions are incompatible with observations [16,17].

An alternative approach consists in using the holographic principle and the results of conformal field theory (CFT) [18]. Indeed, in two-dimensional CFT it is possible to count the microscopic states of a physical system in a simple manner by applying the Cardy formula [19]

$$S_C = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)}, \quad (22)$$

where c is the central charge and L_0 is the eigenvalue of the Virasoro operator. Recently, Verlinde proposed the universal validity of Cardy's entropy and found an interesting link with Friedmann's equations in general relativity [10]. Indeed, the Hamiltonian constraint in close FLRW spacetimes turns out to coincide with the formula for the Cardy entropy if the relationships

$$S_C = \frac{VH}{2G}, L_0 = \frac{aE}{3}, c = \frac{3V}{a\pi G} \quad (23)$$

are satisfied with $V = a^3$. This formal analogy was confirmed also in the case of more general FLRW spacetimes [20] and of Bianchi models [9,15], for which the generalized Cardy formula reads

$$S_C = 2\pi \sqrt{\frac{c}{6} \left(L_0 - k \frac{c}{24} \right)}. \quad (24)$$

The validity of this formula in so many different cosmological scenarios can be interpreted as indicating the existence of a deeper relationship between general relativity, thermodynamics and holography.

From the results presented in the previous section, one can easily see that in the HL quantum gravity model, the Hamiltonian constraints of all FLRW cosmological models satisfy the generalized Cardy formula if the corresponding entropy is identified as

$$S_C = \frac{VH}{2G} \sqrt{\frac{3\lambda-1}{2}}, \quad (25)$$

while the central charge and the eigenvalue of the Virasoro operator must be chosen as given in Table III. In the limiting case $\lambda \rightarrow 1$ and $\mu \rightarrow 0$, our results reduce to those of Einstein gravity.

We see that the parameter λ enters explicitly the expression for the Cardy entropy (25), allowing us to investigate its physical significance. Indeed, the constant factor $\sqrt{(3\lambda-1)/2}$ represents the only difference between Einstein classical gravity and the HL model for quantum gravity. Thus, the parameter λ can be considered as a measure of the classical and quantum gravitational interaction. For the limiting value $\lambda = 1/3$, the Cardy entropy vanishes, indicating that no microscopic states can be associated with the corresponding physical system. This is in accordance with the physical interpretation of the HL model, since this case corresponds to a vanishing value for the gravitational constant, i.e. no gravitational interaction is present. In fact, the value $\lambda = 1/3$ corresponds to the transition point between repulsive gravity ($\lambda < 1/3$) and attractive gravity ($\lambda > 1/3$). For $\lambda < 1/3$, the Cardy entropy becomes imaginary, which can be interpreted as a manifestation of the absence of physical microscopic states in a system dominated by repulsive gravity. We interpret this result as an indication that repulsive gravity is not a physically meaningful theory within the HL quantum

TABLE III. Cardy entropy for FLRW spacetimes in HL gravity.

FLRW universe	k	S_C	L_0	c
Flat	0	$2\pi \sqrt{\frac{c}{6} L_0}$	$\frac{aE}{3}$	$\frac{3V}{a\pi G}$
Open	-1	$2\pi \sqrt{\frac{c}{6} \left(L_0 + \frac{c}{24} \right)}$	$\frac{aE}{3\sqrt{-1-M}}$	$\frac{3V\sqrt{1-M}}{a\pi G}$
Closed	+1	$2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)}$	$\frac{aE}{3\sqrt{1+M}}$	$\frac{3V\sqrt{1+M}}{a\pi G}$

gravity model. Starting at $\lambda = 1/3$, the attractive gravitational interaction increases as λ increases, and for $\lambda = 1$ it reaches the classical value of general relativity. As λ increases, the Cardy entropy increases as $\lambda^{1/2}$, and at some point the theory will reach a quantum regime whose gravitational interaction will depend on the value of λ . In particular, λ is related to measurements of Lorentz invariance deviations and can be constrained at different

cosmological regimes to get limits over quantum gravity effects [21].

We now consider Bianchi cosmologies in HL gravity. As before, the idea is to identify S_C , L_0 and c such that the Cardy formula (24) coincides with the Hamiltonian constraint (19). One can show that there is a unique identification which satisfies this condition, namely

$$S_C = \frac{V}{2G} \sqrt{\frac{1}{6}(\lambda - 1)(H_1^2 + H_2^2 + H_3^2) + \frac{\lambda}{3}(H_1H_2 + H_1H_3 + H_2H_3)}, \quad (26)$$

while the values of L_0 and c depend on the model as given in Table IV. In the limiting isotropic case ($H_1 = H_2 = H_3$), we obtain the values for the FLRW spacetime given above. This proves that the Cardy formula is also valid for homogeneous spacetimes in HL gravity.

Notice that the validity of the Cardy entropy imposes certain conditions on the Hubble parameters. In fact, for values of λ within the interval $[1, \infty)$ the Hubble parameters H_1 , H_2 and H_3 can take any positive real values. This means that the velocity of expansion can be arbitrarily large in any spatial direction. In the interval $\lambda \in (1/3, 1)$, however, the reality condition of the Cardy entropy implies that the expansion velocity is limited by the value of λ . This behavior is illustrated in Fig. 1, where we plot the (normalized) Cardy entropy in terms of H_3 for different values of λ and fixed values of H_1 and H_2 . We see that in order for the Cardy entropy to be a real quantity, the value of the Hubble parameter H_3 must be inside a finite interval

whose maximum depends on the value of λ . This result implies that the expansion velocity in the spatial directions cannot be arbitrarily large. The parameter λ acts as a physical barrier which limits the expansion velocity of an anisotropic universe.

In the case of FLRW spacetimes in Einstein gravity, Verlinde noticed that the Hamiltonian constraint can be written in a Pythagorean form

$$S_H^2 - 2S_{BH}S_B + S_{BH}^2 = 0, \quad (27)$$

where S_H , S_{BH} and S_B are the Hubble, Bekenstein-Hawking, and Bekenstein entropies, respectively. A lengthy but straightforward calculation shows that the Hamiltonian constraints (12) and (19) in HL gravity can also be put in the Pythagorean form. Indeed, in the case of the Bianchi IX model, the entropies entering Verlinde's formula (27) can be expressed as

$$S_H = \frac{V}{2G} \sqrt{\frac{\lambda}{3}(H_1H_2 + H_1H_3 + H_2H_3) + \frac{\lambda - 1}{6}(H_1^2 + H_2^2 + H_3^2)}, \quad (28)$$

$$S_{BH} = \frac{V}{2Ga_1} \sqrt{1 + \frac{\epsilon^2}{3} + \left(\frac{2}{3\lambda - 1}\right) \left(\frac{4\pi G\mu}{a_1}\right)^2 \left[8(3\lambda - 1) \left(1 + \frac{\theta^4}{3}\right) - 3(8\lambda - 3) \left(1 + \frac{\epsilon^2}{3}\right)^2\right]}, \quad (29)$$

$$S_B = \frac{2\pi}{3} \frac{Ea_1}{\sqrt{1 + \frac{\epsilon^2}{3} + \left(\frac{2}{3\lambda - 1}\right) \left(\frac{4\pi G\mu}{a_1}\right)^2 \left[8(3\lambda - 1) \left(1 + \frac{\theta^4}{3}\right) - 3(8\lambda - 3) \left(1 + \frac{\epsilon^2}{3}\right)^2\right]}}. \quad (30)$$

TABLE IV. Cardy entropy for Bianchi cosmologies in HL gravity.

Bianchi type	k	S_C	L_0	c
I	0	$2\pi\sqrt{\frac{c}{6}L_0}$	$\frac{a_1E}{3}$	$\frac{3V}{a_1\pi G}$
V	-1	$2\pi\sqrt{\frac{c}{6}\left(L_0 + \frac{c}{24}\right)}$	$\frac{a_1E}{3\sqrt{-F-G}}$	$\frac{3V\sqrt{1-F}}{a_1\pi G}$
IX	+1	$2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}$	$\frac{a_1E}{3\sqrt{F+G}}$	$\frac{3V\sqrt{1+F}}{a_1\pi G}$

In the isotropic limit, the results are consistent with those obtained for FLRW spacetimes. This form of the Hamiltonian constraint allows us to represent the dynamical evolution of the entropies as being inside a circle of radius S_B . In the case of the FLRW spacetimes, this radius is constant so that the dynamical evolution of the Hubble and Bekenstein-Hawking entropies are restricted to evolve inside the circle. In the anisotropic case, however, S_B

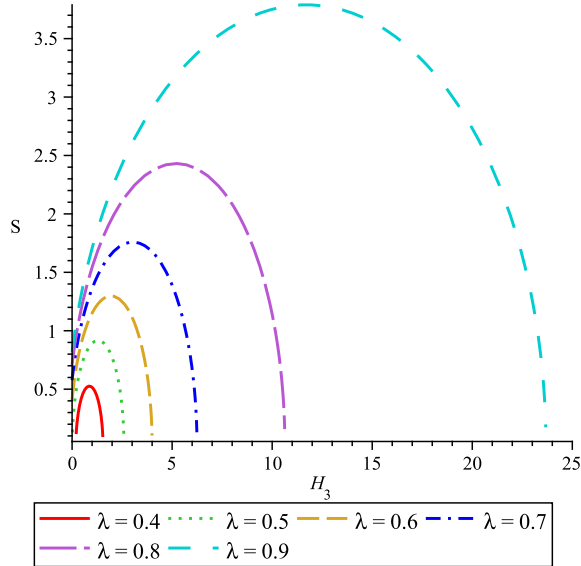


FIG. 1. A normalized Cardy entropy as a function of the Hubble parameter H_3 for different values of the Hořava parameter λ . For concreteness we fix the remaining Hubble parameters as $H_1 = 0.7$ and $H_2 = 0.6$.

depends explicitly on time so that the evolution of the remaining entropies is no longer restricted to be inside a fixed circle, leading to a more complex pictorial representation which physically implies that the evolution depends on the equation of state of the fluid [15]. The radius S_B also depends explicitly on the value of the parameter λ . This means that for a given value of λ , the radius of the circle cannot be arbitrarily small or arbitrarily large as time passes, but it depends explicitly on the value of λ . In other words, the Hořava parameter λ influences the evolution of the entropies in the Verlinde representation.

V. FINAL OUTLOOKS AND PERSPECTIVES

In this work, we investigated the structure of the Hamiltonian constraint for homogeneous cosmological models with a perfect-fluid source in the Hořava-Lifshitz quantum gravity model. We calculated explicitly the field equations for the isotropic FLRW metric and for the anisotropic Bianchi I, V and IX metrics. The metrics were chosen in such a way that in each case the resulting Hamiltonian constraint has a particular compact and simple form, which allowed us to perform a comparative analysis. We assumed the validity of the Cardy entropy formula, which was used in two-dimensional conformal field theory to carry out a counting of the microscopic states of physical systems.

It was shown that the Hamiltonian constraint of homogeneous cosmological spacetimes can be identified with the Cardy entropy, by choosing in a unique way the central charge and the eigenvalue of the Virasoro operator which enter the Cardy formula. As a result, we obtain explicit expressions for the Cardy entropy of the FLRW and Bianchi spacetimes. This proves the validity of the

Cardy entropy in the HL quantum gravity model, implying in turn that the corresponding gravitational fields can be analyzed by using the holographic approach of the Cardy entropy. Moreover, the fact that a particular entropy can be associated with homogeneous spacetimes indicates that a thermodynamic analysis can also be performed. This, however, was not analyzed in the present work.

Our results show that the Cardy entropy for homogeneous HL cosmologies depends explicitly on the Hořava parameter λ . We use this byproduct of our analysis to investigate the physical significance of this parameter. In the case of FLRW spacetimes, we obtained that if λ takes values within the interval $[1/3, \infty)$, the Cardy entropy is a real quantity. For $\lambda < 1/3$, the entropy becomes imaginary, corresponding to a theory in which only repulsive gravity is present. We interpret this limiting theory as unphysical. The point $\lambda = 1/3$ corresponds to the transition between repulsive and attractive gravity. At this particular point, the Cardy entropy vanishes. Then, for increasing values of λ the entropy increases as $S_C \propto \sqrt{\lambda}$. The particular value $\lambda = 1$ corresponds to the limit of general relativity at which covariance is recovered. Thus, we can interpret λ as the parameter responsible for the intensity of the gravitational interaction in different gravity models. As λ increases and tends to infinity, the HL quantum gravity model describes situations in which quantum effects become more important.

We also calculated the explicit expression for the Cardy entropy of anisotropic spacetimes. The entropy depends explicitly on λ also, as $S_C \propto \sqrt{\lambda}$. We investigated in detail the results for the Bianchi IX model, which is characterized by three different scale factors. Within the interval $\lambda \in [1, \infty)$, the Hubble parameters can be arbitrarily large, indicating that the velocity expansion in the spatial directions is not limited by the reality condition of the Cardy entropy. However, in the interval $\lambda \in (1/3, 1)$, the situation is completely different. If we fix two of the Hubble parameters of the Bianchi IX model, the remaining third Hubble parameter cannot take an arbitrary value, but is limited within the interval $[0, H_{\max}]$ where H_{\max} depends explicitly on the value of λ . This means that an anisotropic spacetime cannot expand with arbitrary velocity along the spatial dimensions. The parameter λ acts as a physical barrier that limits the expansion velocity.

We also proved that the Hamiltonian constraints of homogeneous HL spacetimes can be written in a Pythagorean form, leading to explicit expressions for the Hubble, Bekenstein and Bekenstein-Hawking entropies. This particular form of the Hamiltonian constraint allows us to represent the dynamical evolution of the entropies as being inside a circle of radius S_B . In contrast to the case of the FLRW spacetime in general relativity, where the radius of the circle is constant, in the anisotropic case it depends explicitly on time, so that the evolution of the remaining entropies is no longer restricted to be inside a fixed circle. The radius S_B also depends on the parameter λ . This means that for a given value of λ , the radius

of the circle cannot be arbitrarily small or arbitrarily large as time passes, but it is limited by the value of λ . We conclude that the Hořava parameter λ drastically changes the evolution of the entropies in the Verlinde representation.

Our results show that the holographic approach as expressed in the Cardy entropy can be used in the framework of the HL quantum gravity model to investigate the thermodynamic properties of homogeneous spacetimes. Here, we have only investigated the resulting expressions for the Cardy entropy in order to analyze the physical significance of the Hořava parameter λ . A more detailed thermodynamic analysis is beyond the scope of the present work. Nevertheless, we consider that our results are a further indication of the possible existence of a not-yet-discovered deep relationship between gravity, thermodynamics and holography.

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APPENDIX: DYNAMICAL EQUATIONS FOR THE BIANCHI SPACETIMES

The three dynamical equations for each Bianchi model can be obtained directly from the HL field equations, and can be written as follows:

$$\begin{aligned}
 &(\lambda - 1) \left[-\frac{1}{2}(H_i^2 + H_j^2 + H_l^2) + \frac{\ddot{a}_i}{a_i} + H_i H_j + H_i H_l \right] \\
 &+ \lambda \left[H_j H_l + \frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_l}{a_l} \right] + \frac{k + D}{a_i^2} \\
 &= 8\pi G p_{\text{tot}}, \tag{A1}
 \end{aligned}$$

TABLE V. Dynamical Bianchi equations.

Bianchi type	k	D
I	0	0
V	-1	$-\frac{2}{3\lambda-1} \left(\frac{4\pi G\mu}{a_i}\right)^2$
IX	+1	$\epsilon_i^2 - 2\xi_i^2 - \frac{2}{3\lambda-1} \left(\frac{4\pi G\mu}{a_i}\right)^2 \{4(1-3\lambda)[-9(1+\frac{\epsilon_i^2}{3})^2 + 6(1+\frac{\theta_i^2}{3}) + 4(1+\xi_i^2)^2] - 3(1-4\lambda)[4(1+\frac{\epsilon_i^2}{3})(1+\xi_i^2) - 3(1+\frac{\epsilon_i^2}{3})^2]\}$

where the subindices $\{i, j, l\}$ take the values $\{1, 2, 3\}$, respectively, and the remaining two equations can be obtained by permutation. The function $D = D(a_i, a_j, a_l, \lambda, \mu)$ is different for each Bianchi model, and its explicit value is given in Table V, where

$$\xi_i^2 \equiv 1 - \frac{a_l^2}{a_i^2} - \frac{a_j^2}{a_l^2} \left(1 - \frac{a_i^4}{a_l^4}\right), \tag{A2}$$

$$\epsilon_i^2 \equiv 1 - \frac{a_l^2}{a_i^2} - 2 \left(1 - \frac{a_i^2}{a_j^2}\right) - \frac{a_j^2}{a_l^2} \left(1 - \frac{a_i^2}{a_j^2}\right)^2, \tag{A3}$$

$$\theta_i^4 \equiv 1 - \frac{a_l^4}{a_i^4} - 2 \left(1 - \frac{a_i^4}{a_l^4}\right) - \frac{a_j^4}{a_l^4} \left(1 - \frac{a_i^4}{a_l^4}\right)^2. \tag{A4}$$

[1] S. Carlip, *Rep. Prog. Phys.* **64**, 885 (2001); C. Kiefer, *Ann. Phys. (Berlin)* **15**, 129 (2006).
 [2] D. Colladay and V. A. Kostelecky, *Phys. Rev. D* **58**, 116002 (1998).
 [3] G. Amelino-Camelia, *Living Rev. Relativity* **16**, 5 (2013); C. J. Isham, arXiv:gr-qc/9210011; T. Kifune, *Astrophys. J. Lett.* **518**, L21 (1999); G. Amelino-Camelia and T. Piran, *Phys. Lett. B* **497**, 265 (2001); J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, *Phys. Rev. D* **63**, 124025 (2001).
 [4] D. Mattingly, *Living Rev. Relativity* **8**, 5 (2005); G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, *Nature (London)* **393**, 763 (1998); G. Amelino-Camelia and T. Piran, *Phys. Rev. D* **64**, 036005 (2001); G. Amelino-Camelia, *Phys. Lett. B* **528**, 181 (2002).
 [5] F. Cianfrani, O. M. Lecian, and G. Montani, arXiv:0805.2503; S. Coleman and S. L. Glashow, *Phys. Rev. D* **59**, 116008 (1999); J. Collins, A. Perez, D. Sudarsky, L. Urrutia, and H. Vucetich, *Phys. Rev. Lett.* **93**, 191301 (2004).
 [6] U. Jacob and T. Piran, *J. Cosmol. Astropart. Phys.* 01 (2008) 031; M. Rodriguez Martinez and T. Piran, *J. Cosmol. Astropart. Phys.* 04 (2006) 006; J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, A. S. Sakharov, and E. K. G. Sarkisyan, *Astropart. Phys.* **25**, 402 (2006).
 [7] P. Hořava, *Phys. Rev. D* **79**, 084008 (2009); *Phys. Rev. Lett.* **102**, 161301 (2009).

- [8] E. M. Lifschitz, *Zh. Eksp. Teor. Fiz.* **11**, 255 (1941).
- [9] F. J. Hernandez and H. Quevedo, *Gen. Relativ. Gravit.* **39**, 1297 (2007).
- [10] E. Verlinde, [arXiv:hep-th/0008140](https://arxiv.org/abs/hep-th/0008140).
- [11] M. I. Park, *J. High Energy Phys.* **09** (2009) 123.
- [12] H. Lu, J. Mei, and C. N. Pope, *Phys. Rev. D* **89**, 084008 (2009); H. Nastase, [arXiv:0904.3604](https://arxiv.org/abs/0904.3604).
- [13] R. G. Cai and N. Ohta, *Phys. Rev. D* **81**, 084061 (2010).
- [14] M. I. Park, *J. Cosmol. Astropart. Phys.* **01** (2010) 001.
- [15] O. Obregón, L. Patiño, and H. Quevedo, *Phys. Rev. D* **68**, 026002 (2003).
- [16] H. Quevedo and R. Sussman, *J. Math. Phys.* **36**, 1365 (1995).
- [17] A. Krasinsky, H. Quevedo, and R. Sussman, *J. Math. Phys.* **38**, 2602 (1997).
- [18] L. Susskind, *J. Math. Phys.* **36**, 6377 (1995).
- [19] J. L. Cardy, *Nucl. Phys.* **B270**, 186 (1986).
- [20] D. Youm, *Phys. Rep.* **316**, 1 (1999).
- [21] T. P. Sotiriou, *J. Phys. Conf. Ser.* **283**, 012034 (2011).