

Charged rotating dilaton black holes with Kaluza-Klein asymptoticsChristian Knoll¹ and Petya Nedkova^{1,2,*}¹*Institut für Physik, Universität Oldenburg, D-26111 Oldenburg, Germany*²*Department of Theoretical Physics, Faculty of Physics, Sofia University,
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We construct a class of stationary and axisymmetric solutions to the five-dimensional Einstein-Maxwell-dilaton gravity, which describe configurations of charged rotating black objects with Kaluza-Klein asymptotics. The solutions are constructed by uplifting a vacuum seed solution to six dimensions, performing a boost and a subsequent circle reduction. We investigate the physical properties of the charged solutions and obtain their general relations to the properties of the vacuum seed. We also derive the gyromagnetic ratio and the Smarr-like relations. As particular cases, we study three solutions, which describe a charged rotating black string, a charged rotating black ring on Kaluza-Klein bubbles, and a superposition of two black holes and a Kaluza-Klein bubble.

DOI: [10.1103/PhysRevD.93.064052](https://doi.org/10.1103/PhysRevD.93.064052)**I. INTRODUCTION**

Einstein-Maxwell-dilaton gravity arises as a truncation of the low-energy string theory action and has attracted a lot of attention as a quantum-gravity-motivated gravitational theory. A major topic of interest is the construction of black hole solutions within the theory in various spacetime dimensions and the investigation of their properties. Static exact black hole solutions are known in four dimensions [1,2], due to the integrability of the four-dimensional static axisymmetric Einstein-Maxwell-dilaton equations. The dimensionally reduced action describes a symmetric target space with a $SL(2, R) \times R$ hidden symmetry group which opens a way to the generation of new solutions by algebraic means [3,4]. Thus, a dilaton-Weyl class of solutions can be constructed generalizing the static Einstein-Maxwell solutions in the presence of a dilaton field [4,5]. In the stationary case, no nontrivial hidden symmetries are discovered for general value of the dilaton coupling constant, which hinders the construction of exact solutions describing rotating charged dilaton black holes. Two particular values of the dilaton constant α make an exception when the hidden symmetry group is enhanced: $\alpha = 0$ corresponding to the Einstein-Maxwell case and $\alpha = \sqrt{3}$, which is equivalent to the Kaluza-Klein theory.

The situation is similar in five dimensions, where the static truncation of the Einstein-Maxwell-dilaton theory is highly symmetric and allows for the construction of exact solutions, while in the stationary case the integrability is violated. Even in the Einstein-Maxwell limit $\alpha = 0$, no nontrivial hidden symmetries are discovered when the electromagnetic field is fully exited [6]. An integrable sector was found [7,8], in which the electromagnetic and the twist potentials decouple and give rise to two

independent sigma models. Consequently, some stationary black hole solutions were constructed carrying magnetic (dipole) charge [9,10]. However, electrically charged rotating solutions do not belong to this sector, as well as rotating solutions with dipole charge, for which the electromagnetic vector potential has a component along some of the axes of rotation.

The only case when the dimensionally reduced action for a general stationary and axisymmetric electromagnetic field describes a symmetric space corresponds to the Kaluza-Klein theory. It is contained as a particular case in the five-dimensional Einstein-Maxwell-dilaton gravity for a dilaton coupling constant $\alpha = \sqrt{8/3}$. Thus, Kaluza-Klein solutions provide examples for exact solutions describing charged rotating black holes, which do not have counterparts in the pure Einstein-Maxwell case. Due to the lack of solution-generation techniques, five-dimensional charged rotating black holes for dilaton coupling $\alpha \neq \sqrt{3/8}$ are investigated numerically [11–14] or perturbatively [15,16]. Some exact solutions exist only for extremal charged rotating black holes with Kaluza-Klein asymptotics [17].

In five dimensions, a great variety of exact stationary and axisymmetric black hole solutions in vacuum were constructed. In addition to the higher-dimensional generalization of the Kerr solution [18], asymptotically flat black objects with non-spherical horizon topology were discovered, such as black rings [19], and black lenses [20]. Nonlinear superpositions of such objects were obtained resulting in different multihorizon solutions, including a system of concentric or orthogonal black rings [21,22], and black Saturns [23]. Black hole solutions with Kaluza-Klein asymptotics ($M^4 \times S^1$) were also considered motivated by braneworld scenarios with compact extra dimensions. They include a large class of solutions called black holes on

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Kaluza-Klein bubbles [24–28], which represent superposition between black objects with different horizon topology and Schwarzschild instantons [29]. These systems can be further generalized to solutions which are only locally asymptotically Kaluza-Klein, but globally the spacelike boundary at infinity represents some nontrivial fibration of S^3 . In this way, configurations of black objects and more complicated types of gravitational instantons were obtained [30]. Different versions of such solutions including electromagnetic fields were constructed [31–34]. They possess interesting thermodynamical properties, since certain electromagnetic fluxes also give contributions in their Smarr-like relations and the first law of thermodynamics [35–37].

Charged generalizations of rotating vacuum solutions can be obtained in the Kaluza-Klein sector of Einstein-Maxwell-dilaton gravity by means of a simple procedure. The D -dimensional vacuum solution is trivially embedded in $D + 1$ -dimensional spacetime by adding an extra dimension, a boost is performed with respect to the extra dimension, and the resulting solution is dimensionally reduced again to D dimensions along the additional coordinate. As a result a generalization of the initial solution possessing an electromagnetic and a dilaton field is constructed. Some charged rotating dilaton black objects were obtained in five dimensions in this way.¹ They include the charged counterpart of the Myers-Perry black hole [39], the black ring [40], and the black Saturn [41]. More general black ring solutions with double rotation and carrying also a dipole charge were obtained by a related scheme [42–45]. A suitable vacuum solution with a compact dimension is constructed in six dimensions by means of the inverse scattering method. Performing a Kaluza-Klein reduction along the compact dimension leads to a solution to the five-dimensional Einstein-Maxwell-dilaton gravity which can possess a dipole charge [42,44], and an electric charge [45].

In this paper, we obtain charged rotating solutions to the five-dimensional Einstein-Maxwell-dilaton gravity, which describe configurations of black objects with Kaluza-Klein asymptotics. We consider a class of stationary and axisymmetric solutions which contain at least one horizon and for which the Killing vector associated with the compact dimension is hypersurface orthogonal. Consequently, the solutions possess a single angular momentum, and the possible fixed point sets of the spacelike Killing fields are restricted to Kaluza-Klein bubbles, and the axis of rotation. Such solutions describe configurations of black objects with different topology, which may be also superposed with Kaluza-Klein bubbles. Given a particular vacuum configuration, we apply on it the described Kaluza-Klein transformation by uplifting it to six dimensions, performing a boost, and a subsequent circle reduction. As a result, we

obtain the corresponding charged dilaton solution in the Kaluza-Klein sector of the five-dimensional Einstein-Maxwell-dilaton gravity. Various general relations exist between the properties of the vacuum seed solution and the charged dilaton one. We investigate these relations and show how the physical characteristics of the charged solution can be expressed by means of the characteristics of its vacuum counterpart.

The paper is organized in the following way. In Secs. II and III, we describe the class of solutions we consider, and the solution generation technique. In Sec. IV, we study the physical properties of the charged solutions and their relations to the properties of the vacuum ones. In particular, we consider the conserved charges, the local Komar masses and angular momenta of the black holes, the local electric charges associated with the horizons, the Smarr-like relations, and the gyromagnetic ratio. In Sec. V, we apply our results for the investigation of three particular solutions—the charged rotating black string, the charged rotating black ring on Kaluza-Klein bubbles, and a superposition of two rotating black holes and a Kaluza-Klein bubble. In the paper, we work in geometrical units $G = c = 1$.

II. CLASS OF ROTATING BLACK HOLES WITH KALUZA-KLEIN ASYMPTOTICS

In this work, we consider a class of stationary and axisymmetric solutions with Kaluza-Klein asymptotics, which contain at least one horizon. The solutions possess $R \times U(1)^2$ isometry group generated by three Killing vectors, one of which is asymptotically timelike. We can introduce a coordinate system adapted to their orbits, such that the asymptotically timelike Killing vector is expressed as $\xi = \frac{\partial}{\partial t}$, the Killing vector associated with the compact dimension is given by $k = \frac{\partial}{\partial \psi}$, and the remaining spacelike Killing vector is $\eta = \frac{\partial}{\partial \phi}$. We consider only solutions for which the Killing vector k is hypersurface orthogonal. Such solutions rotate only with respect to the axis of the Killing vector η , and do not contain fixed point sets of nontrivial linear combinations of the spacelike Killing vectors. The general form of the metric for this class is given by

$$ds^2 = g_{tt}(dt + \omega d\phi)^2 + \tilde{g}_{\phi\phi}d\phi^2 + g_{\psi\psi}d\psi^2 + \gamma_{ab}dx^a dx^b, \quad a, b = 1, 2, \quad (1)$$

where x^a , $a = 1, 2$ are the coordinates parametrizing the two-dimensional surfaces orthogonal to the Killing fields, and all the metric functions depend only on them. A useful coordinate system are the canonical Weyl coordinates ρ and z , in which the two-dimensional metric is diagonal, and the determinant of the three-dimensional metric spanned by the Killing fields is equal to $-\rho^2$.

A convenient way to classify the five-dimensional stationary and axisymmetric solutions is by means of their

¹For a charged dilaton generalization of the Kerr black hole, see [38].

interval structure [46]. The interval structure gives information about the existence and the position of the horizons and the different fixed point sets of spacelike Killing fields. We consider the factor space of the spacetime with respect to the isometry group $\hat{M} = M/R \times U(1)^2$. In Weyl coordinates, it is diffeomorphic to the upper half-plane $\rho > 0$, and all the horizons and the fixed point sets of the spacelike Killing vectors are located on its boundary $\rho = 0$. Consequently, they correspond to finite intervals on the z axis, given by $z_i \leq z \leq z_{i+1}$ for some values z_i and z_{i+1} , or semi-infinite intervals on the z axis. We can associate with each interval a direction vector, which specifies the linear combination of Killing vectors which vanishes on it, or equivalently, the particular kind of fixed point set which it describes. The set of data, containing the position and the type of the intervals for a given solution, and their lengths and directions is called interval structure.

As an example, the interval structure of a solution belonging to the class we consider is presented in Fig. 1. The parameters a_i correspond to the endpoints of the intervals on the z axis. The direction vector is specified over each interval, and it gives the coefficients in the linear combination vanishing on it, with respect to a basis of Killing vectors $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \psi}\}$. The finite intervals correspond either to horizons (with direction $(1, \Omega_i, 0)$), or to fixed point sets of the Killing field associated with the compact dimension (with direction $(0, 0, 1)$). These fixed point sets form regular two-dimensional surfaces, which are called Kaluza-Klein bubbles. Consequently, the presented interval structure describes a sequence of alternating black holes and Kaluza-Klein bubbles. Each horizon rotates with angular velocity Ω_i around the axis of the Killing field $\frac{\partial}{\partial \phi}$. The semi-infinite intervals with direction $(0, 1, 0)$ represent the axis of rotation.

The most general interval structure associated to a solution within the class we consider contains three types of intervals: finite intervals with direction $(1, \Omega_i, 0)$, corresponding to rotating black objects with angular velocity Ω_i , finite intervals with direction $(0, 0, 1)$ corresponding to Kaluza-Klein bubbles, and finite and semi-infinite intervals with direction $(0, 1, 0)$ representing the axis of rotation. Both semi-infinite intervals should be directed along the Killing vector $\eta = \frac{\partial}{\partial \phi}$ in order for the solutions to have Kaluza-Klein asymptotics.

III. CONSTRUCTION OF THE CHARGED DILATON SOLUTIONS

We consider the Einstein-Maxwell-dilaton gravity in five dimensions with the action

$$I = \frac{1}{16\pi} \int d^5x \sqrt{-g} (R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - e^{-2\alpha\varphi} F^{\mu\nu} F_{\mu\nu}). \quad (2)$$

It leads to the following field equations,

$$\begin{aligned} R_{\mu\nu} &= 2\partial_\mu \varphi \partial_\nu \varphi + 2e^{-2\alpha\varphi} \left[F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} F_{\beta\rho} F^{\beta\rho} \right], \\ \nabla_\mu \nabla^\mu \varphi &= -\frac{\alpha}{2} e^{-2\alpha\varphi} F_{\nu\rho} F^{\nu\rho}, \\ \nabla_\mu [e^{-2\alpha\varphi} F^{\mu\nu}] &= 0, \end{aligned} \quad (3)$$

where $R_{\mu\nu}$ is the Ricci tensor for the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu}$ is the Maxwell tensor, and φ is the dilaton field. We assume that the dilaton coupling parameter takes the value $\alpha = \sqrt{8/3}$, which constrains the Kaluza-Klein sector in the theory.

Solutions to the five-dimensional Einstein-Maxwell-dilaton gravity with $\alpha = \sqrt{8/3}$ can be obtained by the following procedure. We consider a known vacuum solution to the five-dimensional Einstein equations, which we call a seed. For our purposes, we assume that the seed solution has the general form

$$\begin{aligned} ds_0^2 &= g_{tt}^{(0)} (dt + \omega^{(0)} d\phi)^2 + \tilde{g}_{\phi\phi}^{(0)} d\phi^2 + g_{\psi\psi}^{(0)} d\psi^2 \\ &+ \gamma_{ab}^{(0)} dx^a dx^b, \quad a, b = 1, 2; \end{aligned} \quad (4)$$

i.e., it belongs to the class of solutions described in Sec. II. We embed it trivially in six dimensions by adding an extra dimension. Thus, we obtain a vacuum solution to the six-dimensional Einstein equations with the metric

$$ds_6^2 = ds_0^2 + dx_6^2, \quad (5)$$

where ds_0^2 is the metric of the seed solution, and the coordinate x_6 parametrizes the extra dimension. Then we perform a hyperbolic rotation in the $t - x_6$ plane with the matrix

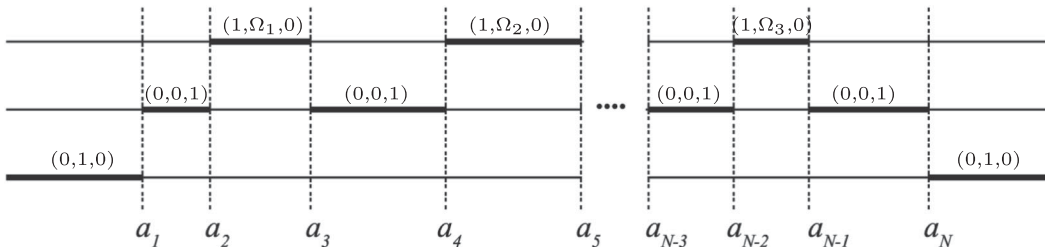


FIG. 1. Interval structure of sequence of rotating black holes and Kaluza-Klein bubbles.

$$L = \begin{pmatrix} \cosh \gamma & \sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{pmatrix} \quad (6)$$

and reduce the resulting solution along the extra dimension using the ansatz

$$ds_6^2 = e^{\sqrt{\frac{2}{3}}\varphi} ds_5^2 + e^{-2\sqrt{\frac{2}{3}}\varphi} (dx_6 + 2A_\mu dx^\mu)^2. \quad (7)$$

The five-dimensional metric ds_5^2 corresponds to a solution in the Kaluza-Klein sector of the five-dimensional Einstein-Maxwell-dilaton gravity with a scalar field φ and an electromagnetic potential A_μ . The metric of the constructed solution is connected to the seed solution (4) as

$$ds_5^2 = \Lambda^{-\frac{2}{3}} g_{tt}^{(0)} (dt + \cosh \gamma \omega^{(0)} d\phi)^2 + \Lambda^{\frac{1}{3}} [\tilde{g}_{\phi\phi}^{(0)} d\phi^2 + g_{\psi\psi}^{(0)} d\psi^2 + \gamma_{ab}^{(0)} dx^a dx^b], \quad (8)$$

where the function Λ is given by

$$\Lambda = \cosh^2 \gamma + \sinh^2 \gamma g_{tt}^{(0)}, \quad (9)$$

while the Maxwell 2-form F and the dilaton field are

$$\begin{aligned} F &= dt \wedge dA_t + d\phi \wedge dA_\phi, \\ A_t &= \frac{1}{2} \Lambda^{-1} \sinh \gamma \cosh \gamma (1 + g_{tt}^{(0)}), \\ A_\phi &= \frac{1}{2} \Lambda^{-1} \sinh \gamma \omega^{(0)} g_{tt}^{(0)}, \\ e^{-2\alpha\varphi} &= e^{-4\sqrt{\frac{2}{3}}\varphi} = \Lambda^{\frac{4}{3}}. \end{aligned} \quad (10)$$

The constructed solution possesses the same interval structure as the seed solution i.e., it contains the same number of disconnected horizons and the same fixed point sets of the spacelike Killing vectors. The only difference is that the angular velocity of each horizon is modified by a factor $\cosh^{-1} \gamma$. Thus, if the i th horizon in the seed solution is rotating with angular velocity $\Omega_i^{(0)}$, the angular velocity of the same horizon in the charged solution will be $\Omega_i = \cosh^{-1} \gamma \Omega_i^{(0)}$.

Solutions with multiple disconnected fixed point sets of a spacelike Killing field may suffer from conical singularities. To avoid conical singularities the orbits of the Killing field in the vicinity of each of its fixed point sets should have the same periodicity, which should be also consistent with the asymptotic structure of the solution. For our class of solutions, it follows that on each of the bubble intervals we should have $\Delta\psi = L$, where L is the length of the Kaluza-Klein circle at infinity, while on the axis of rotation it should be satisfied $\Delta\phi = 2\pi$. In Weyl canonical coordinates, these conditions possess the form

$$\begin{aligned} (\Delta\psi)_{I_{B_j}} &= 2\pi \lim_{\rho \rightarrow 0} \sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\psi\psi}}} = L, \\ (\Delta\phi)_{I_\eta} &= 2\pi \lim_{\rho \rightarrow 0} \sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} = 2\pi, \end{aligned} \quad (11)$$

where I_{B_j} is the j th bubble interval, and I_η is an interval corresponding to the rotation axis. Considering the general form of the metric (8) we see that the charging transformation does not influence the regularity conditions.² Therefore, if the seed solution is free of conical singularities and the length of its Kaluza-Klein circle at infinity is equal to $L^{(0)}$, the charged solution will be also free of conical singularities with the same length of the compact dimension at infinity $L = L^{(0)}$. In the following discussion, we will consider only balanced solutions.

IV. PHYSICAL PROPERTIES

A. Conserved charges

The constructed class of solutions are characterized by a number of conserved charges—the ADM mass M_{ADM} and angular momentum \mathcal{J} , the electric charge \mathcal{Q} , and the dilaton charge \mathcal{D} . They can be computed by the following integrals,

$$M_{\text{ADM}} = -\frac{L}{16\pi} \int_{S_\infty^2} [2i_k \star d\xi - i_\xi \star dk], \quad \mathcal{J} = \frac{L}{16\pi} \int_{S_\infty^2} i_k \star d\eta, \quad (12)$$

$$\mathcal{Q} = \frac{1}{4\pi} \int_{S_\infty^2 \times S^1} e^{-2\alpha\varphi} \star F, \quad \mathcal{D} = -\frac{1}{4\pi} \int_{S_\infty^2} i_k i_\xi \star d\varphi, \quad (13)$$

over the two-dimensional sphere at spacial infinity S_∞^2 . Due to the Kaluza-Klein asymptotics, the solutions are characterized by an additional conserved charge—the gravitational tension [47,48]. It can be calculated by the generalized Komar integral [35,49]

$$\mathcal{T} = -\frac{1}{16\pi} \int_{S_\infty^2} [i_k \star d\xi - 2i_\xi \star dk]. \quad (14)$$

The conserved charges can be extracted from the asymptotic behavior of the metric functions, and the electromagnetic and dilaton potentials. We introduce the asymptotic coordinates r and θ defined as

$$\rho = r \sin \theta, \quad z = r \cos \theta$$

and consider the general expansion of the metric functions at the asymptotic infinity $r \rightarrow \infty$,

²We do not consider solutions which contain closed timelike curves.

$$g_{tt} \approx -1 + \frac{c_t}{r}, \quad g_{\psi\psi} \approx 1 + \frac{c_\psi}{r}. \quad (15)$$

Then the Komar integrals for the ADM mass and the gravitational tension are equal to

$$M_{\text{ADM}} = \frac{L}{4}(2c_t - c_\psi), \quad \mathcal{T}L = \frac{L}{4}(c_t - 2c_\psi). \quad (16)$$

The angular momentum, the electric charge and the dilaton charge are contained as coefficients in the leading order terms in the asymptotic expansion of the metric function $g_{t\phi}$, the electromagnetic potential A_t , and the dilaton potential φ , respectively,

$$g_{t\phi} \approx -\frac{2\mathcal{J} \sin^2\theta}{L} \frac{1}{r}, \quad A_t \approx \frac{\mathcal{Q}}{Lr}, \quad \varphi \approx -\frac{\mathcal{D}}{r} \quad (17)$$

We can also define the magnetic moment \mathcal{M} as the coefficient in the asymptotic expansion of the electromagnetic potential A_ϕ

$$A_\phi \approx -\frac{\mathcal{M} \sin^2\theta}{L} \frac{1}{r}. \quad (18)$$

The conserved charges for our class of charged dilaton solutions can be expressed by the characteristics of the corresponding seed solution in the following way. We obtain the asymptotic expansions for the metric functions and the potentials in the form

$$\begin{aligned} g_{tt} &\approx -1 + \frac{c_t}{r} = -1 + \frac{c_t^{(0)}}{r} + \frac{2}{3} \sinh^2\gamma \frac{c_t^{(0)}}{r}, \\ g_{t\phi} &\approx -\cosh\gamma \frac{2\mathcal{J}^{(0)} \sin^2\theta}{L} \frac{1}{r}, \\ g_{\psi\psi} &\approx 1 + \frac{c_\psi}{r} = 1 + \frac{c_\psi^{(0)}}{r} + \frac{1}{3} \sinh^2\gamma \frac{c_t^{(0)}}{r}, \end{aligned} \quad (19)$$

$$\begin{aligned} A_t &\approx \frac{1}{2} \sinh\gamma \cosh\gamma \frac{c_t^{(0)}}{r}, \\ A_\phi &\approx -\sinh\gamma \frac{\mathcal{J}^{(0)} \sin^2\theta}{L} \frac{1}{r}, \\ \varphi &\approx -\frac{1}{\sqrt{6}} \sinh^2\gamma \frac{c_t^{(0)}}{r}, \end{aligned} \quad (20)$$

where we denote with zero index the corresponding quantities for the seed solution, and L is the length of the Kaluza-Klein circle at infinity. Consequently, the ADM mass, the angular momentum and the tension are related to the conserved charges of the seed solution as

$$\begin{aligned} M_{\text{ADM}} &= M_{\text{ADM}}^{(0)} + \frac{L}{4} \sinh^2\gamma c_t^{(0)}, \\ \mathcal{J} &= \cosh\gamma \mathcal{J}^{(0)}, \quad \mathcal{T} = \mathcal{T}^{(0)}, \end{aligned} \quad (21)$$

while the electric charge, the dilaton charge, and the magnetic moment are

$$\begin{aligned} \mathcal{Q} &= \frac{L}{2} \sinh\gamma \cosh\gamma c_t^{(0)}, \quad \mathcal{D} = \frac{1}{\sqrt{6}} \sinh^2\gamma c_t^{(0)}, \\ \mathcal{M} &= \sinh\gamma \mathcal{J}^{(0)}. \end{aligned} \quad (22)$$

The magnetic moment and the dilaton charge are not independent characteristics of the solutions. They are related to the product of the angular momentum, or electric charge, respectively, and the so-called corotating potential Σ_H defined on the black hole horizons. The corotating potential on the i th black hole horizon is defined as

$$\Sigma_{H_i} = (v^\mu A_\mu)_{H_i}, \quad (23)$$

where A_μ is the electromagnetic potential, and $v = \xi + \Omega_i \eta$ is the Killing vector which becomes null on the corresponding horizon. For the class of solutions which we consider, the corotating potential takes the constant value $\Sigma_H = \frac{1}{2} \tanh\gamma$ on every black hole horizon. Thus, the dilaton charge and the magnetic moment are equal to

$$\mathcal{D} = \sqrt{\frac{8}{3}} \frac{\Sigma_H \mathcal{Q}}{L} = \alpha \frac{\Sigma_H \mathcal{Q}}{L}, \quad \mathcal{M} = 2\Sigma_H \mathcal{J}. \quad (24)$$

B. Twist potentials and local quantities

We consider the twist 1-form $i_\xi i_k \star d\xi$. It satisfies the Ricci identity

$$di_\xi i_k \star d\xi = 2 \star [R(\xi) \wedge k \wedge \xi], \quad (25)$$

where $R(\xi)$ is the Ricci 1-form, and from the field equations we can obtain the identity

$$\star R(\xi) = -2e^{-2\alpha\varphi} \left(-\frac{2}{3} i_\xi F \wedge \star F + \frac{1}{3} F \wedge i_\xi \star F \right). \quad (26)$$

Using the explicit form of the electromagnetic field (10), it is reduced to

$$\star [R(\xi) \wedge k \wedge \xi] = 2dA_t \wedge i_\xi i_k e^{-2\alpha\varphi} \star F. \quad (27)$$

The field equations imply that $di_\xi i_k e^{-2\alpha\varphi} \star F = 0$, consequently we can introduce an electromagnetic potential \mathcal{B}_ξ such that $d\mathcal{B}_\xi = i_\xi i_k e^{-2\alpha\varphi} \star F$. Taking advantage of it, Eq. (25) yields

$$di_{\xi}i_k \star d\xi - 4d(A_t d\mathcal{B}_{\xi}) = 0. \quad (28)$$

The 1-form $i_{\xi}i_k \star d\xi - 4A_t d\mathcal{B}_{\xi}$ is invariant under the Killing fields ξ , k and η and can be viewed as defined on the factor space $\hat{M} = M/R \times U(1)^2$. The factor space is simply connected, therefore there exists a globally defined potential χ , such that

$$d\chi = i_{\xi}i_k \star d\xi - 4A_t d\mathcal{B}_{\xi}. \quad (29)$$

In a similar way, we can define a potential related to the twist 1-form $i_{\eta}i_k \star d\eta$ with respect to the Killing field η . It satisfies the relation

$$di_{\eta}i_k \star d\eta = 2 \star [R(\eta) \wedge k \wedge \eta], \quad (30)$$

where

$$\star [R(\eta) \wedge k \wedge \eta] = 2dA_{\phi} \wedge i_{\eta}i_k e^{-2\alpha\phi} \star F. \quad (31)$$

Introducing an electromagnetic potential B_{η} such that $dB_{\eta} = i_{\eta}i_k e^{-2\alpha\phi} \star F$, we obtain the identity

$$di_{\eta}i_k \star d\eta - 4d(A_{\phi} dB_{\eta}) = 0, \quad (32)$$

which implies that there exists a potential μ such that

$$d\mu = i_{\eta}i_k \star d\eta - 4A_{\phi} dB_{\eta}. \quad (33)$$

We consider the corresponding potentials $\chi^{(0)}$ and $\mu^{(0)}$ for the vacuum seed solution, defined by the relations $d\chi^{(0)} = (i_{\xi}i_k \star d\xi)^{(0)}$ and $d\mu^{(0)} = (i_{\eta}i_k \star d\eta)^{(0)}$ in terms of the corresponding Killing fields. We introduce in addition the potential $\lambda^{(0)}$ satisfying $d\lambda^{(0)} = (i_{\eta}i_k \star d\xi)^{(0)}$. By direct calculation we can prove that the following identities are satisfied,

$$\begin{aligned} i_{\xi}i_k \star d\xi &= \cosh \gamma d\chi^0 - 2A_t \sinh \gamma d\chi^0, \\ i_{\eta}i_k \star d\eta &= \cosh \gamma d\mu^{(0)} - 2A_{\phi} \sinh \gamma \cosh \gamma d\lambda^{(0)}, \\ d\mathcal{B}_{\xi} &= -\frac{1}{2} \sinh \gamma d\chi^{(0)}, \\ d\mathcal{B}_{\eta} &= -\frac{1}{2} \sinh \gamma \cosh \gamma d\lambda^{(0)}. \end{aligned} \quad (34)$$

Consequently, we obtain the relations

$$i_{\xi}i_k \star d\xi = \cosh \gamma d\chi^0 + 4A_t d\mathcal{B}_{\xi}, \quad (35)$$

$$i_{\eta}i_k \star d\eta = \cosh \gamma d\mu^{(0)} + 4A_{\phi} d\mathcal{B}_{\eta}, \quad (36)$$

and the twist potentials are connected as

$$d\chi = \cosh \gamma d\chi^{(0)}, \quad d\mu = \cosh \gamma d\mu^{(0)}. \quad (37)$$

The identities (34) lead to relations between some local characteristics of the charged dilaton solution and the corresponding vacuum seed solution. We can define an intrinsic mass M_{H_i} and angular momentum J_{H_i} of each black hole by evaluating the integrals (12) on its horizon. Using that the Killing field $v = \xi + \Omega_i \eta = \xi - \omega_i^{-1} \eta$ vanishes on the horizon H_i and that the corotating potential is constant on each horizon, i.e. $\Sigma_H = \frac{1}{2} \tanh \gamma$, we obtain the following relations:

$$\begin{aligned} M_{H_i} &= -\frac{L}{16\pi} \int_{H_i} [2i_k \star d\xi - i_{\xi} \star dk] \\ &= \frac{L}{4} \int_{I_{H_i}} i_{\eta}i_k \star d\xi \\ &= \frac{L}{4} \omega_i \int_{I_{H_i}} i_{\xi}i_k \star d\xi \\ &= \frac{L}{4} \omega_i^{(0)} \int_{I_{H_i}} \cosh^2 \gamma (i_{\xi}i_k \star d\xi)^{(0)} + L\omega_i \int_{I_{H_i}} A_t d\mathcal{B}_{\xi} \\ &= \frac{L}{4} \omega_i^{(0)} \int_{I_{H_i}} (i_{\xi}i_k \star d\xi)^{(0)} - L\omega_i \int_{I_{H_i}} \Sigma_H d\mathcal{B}_{\xi} \\ &\quad + L\omega_i \int_{I_{H_i}} A_t d\mathcal{B}_{\xi} \\ &= M_{H_i}^{(0)} + L \int_{I_{H_i}} A_{\phi} d\mathcal{B}_{\xi}, \end{aligned} \quad (38)$$

$$\begin{aligned} J_{H_i} &= \frac{L}{16\pi} \int_{H_i} i_k \star d\eta \\ &= -\frac{L}{8} \int_{I_{H_i}} i_{\eta}i_k \star d\eta \\ &= -\frac{L}{8} \int_{I_{H_i}} \cosh \gamma (i_{\eta}i_k \star d\eta)^{(0)} - \frac{L}{2} \int_{I_{H_i}} A_{\phi} d\mathcal{B}_{\eta} \\ &= \cosh \gamma J_{H_i}^{(0)} - \frac{L}{2} \int_{I_{H_i}} A_{\phi} d\mathcal{B}_{\eta}. \end{aligned} \quad (39)$$

The integration is performed over the interval I_{H_i} associated with the i -th horizon in the interval structure, and we denote by $M_{H_i}^{(0)}$ and $J_{H_i}^{(0)}$ the intrinsic Komar mass and angular momentum of the corresponding black hole in the seed solution. We see that these quantities are modified due to the interaction with the electromagnetic field. Similar behavior is observed, for example, for the Kerr-Newman black hole, or magnetized black holes when the magnetic field is aligned with the axis of rotation (e.g. [50]).

We can also define an intrinsic mass M_{B_j} of each bubble by considering the Komar integral (12) on its surface

$$M_{B_j} = \frac{L}{16\pi} \int_{B_j} i_\xi \star dk = -\frac{L}{8} \int_{I_{B_j}} i_\eta i_\xi \star dk. \quad (40)$$

The intrinsic mass is related to the length of the interval Δl_{B_j} associated with the bubble in the interval structure

$$M_{B_j} = \frac{L}{4} \Delta l_{B_j} = M_{B_j}^{(0)}. \quad (41)$$

Consequently, it coincides with the intrinsic mass $M_{B_j}^{(0)}$ of the corresponding bubble in the seed solution, if the balance conditions (11) are satisfied.

We can further assign a local charge \mathcal{Q}_{H_i} of each black hole by evaluating the integral (13) on its horizon. Performing the calculation

$$\begin{aligned} \mathcal{Q}_{H_i} &= \frac{1}{4\pi} \int_{H_i} e^{-2\alpha\varphi} \star F \\ &= -\frac{L}{2} \int_{I_{H_i}} i_\eta i_k e^{-2\alpha\varphi} \star F \\ &= -\frac{L}{2} \int_{H_i} d\mathcal{B}_\eta \\ &= \frac{L}{4} \sinh \gamma \cosh \gamma \int_{H_i} (i_\eta i_k \star d\xi)^{(0)} \\ &= \sinh \gamma \cosh \gamma M_{H_i}^{(0)}, \end{aligned} \quad (42)$$

we obtain that the local charge on each horizon is proportional to the intrinsic mass of the corresponding black hole in the seed solution. Then, the total electric charge of the solution can be expressed as

$$\mathcal{Q} = \sum_i \mathcal{Q}_{H_i} = \sinh \gamma \cosh \gamma \sum_i M_{H_i}^{(0)}. \quad (43)$$

Other quantities, which are of physical interest, are the surface gravity κ_{H_i} and the area A_{H_i} of each horizon

$$\kappa_{H_i} = \sqrt{-\frac{1}{2} v_{\mu;\nu} v^{\mu;\nu}}|_{H_i}, \quad A_{H_i} = \int_{H_i} \sqrt{g_{H_i}} dz d\phi d\psi,$$

where $v = \xi + \Omega_i \eta$. Explicit calculation shows that they are related to the corresponding quantities for the seed solution as

$$\kappa_{H_i} = \cosh \gamma^{-1} \kappa_{H_i}^{(0)}, \quad A_{H_i} = \cosh \gamma A_{H_i}^{(0)}, \quad (44)$$

so that the charging transformation preserves their product. Taking into account the Smarr relation on each horizon,

$$M_{H_i} = \frac{1}{4\pi} \kappa_{H_i} A_{H_i} + 2\Omega_i J_{H_i}, \quad (45)$$

the horizon Komar masses and angular momenta should satisfy

$$M_{H_i} - 2\Omega_i J_{H_i} = M_{H_i}^{(0)} - 2\Omega_i^{(0)} J_{H_i}^{(0)}, \quad (46)$$

which is consistent with the derived expressions (38)–(39).

C. Smarr-like relations

Let us consider the generalized Komar integral for the ADM mass (12). It is convenient to reduce it to the factor space $\hat{M} = M/R \times U(1)^2$ by acting with the Killing field η associated with the azimuthal symmetry of the two-dimensional sphere at infinity:

$$M_{\text{ADM}} = \frac{L}{8} \int_{\text{Arc}(\infty)} [2i_\eta i_k \star d\xi - i_\eta i_\xi \star dk]. \quad (47)$$

The integration is now performed over the semicircle representing the boundary of the two-dimensional factor space at infinity. Using Stokes's theorem, the integral can be further expanded into a bulk term over \hat{M} and an integral over the rest of the boundary of the factor space, which is represented by the interval structure I_n

$$\begin{aligned} M_{\text{ADM}} &= \frac{L}{8} \int_{\hat{M}} [2di_\eta i_k \star d\xi - di_\eta i_\xi \star dk] \\ &\quad + \frac{L}{8} \sum_n \int_{I_n} [2i_\eta i_k \star d\xi - i_\eta i_\xi \star dk]. \end{aligned} \quad (48)$$

If we take into account the definitions of the intrinsic masses of the black holes M_{H_i} and the bubbles M_{B_j} , and the fact that the integral vanishes on the axis of rotation, we obtain

$$\begin{aligned} M_{\text{ADM}} &= \sum_i M_{H_i} + \sum_j M_{B_j} \\ &\quad + \frac{L}{8} \int_{\hat{M}} [2di_\eta i_k \star d\xi - di_\eta i_\xi \star dk], \end{aligned} \quad (49)$$

where the index i enumerates the horizons, and the index j runs over the Kaluza-Klein bubbles. Let us consider the bulk integral and use the Ricci identity $d \star d\zeta = 2 \star R(\zeta)$, which applies for any Killing field ζ

$$\begin{aligned} &\frac{L}{8} \int_{\hat{M}} [2di_\eta i_k \star d\xi - di_\eta i_\xi \star dk] \\ &= \frac{L}{4} \int_{\hat{M}} [2i_\eta i_k \star R(\xi) - i_\eta i_\xi \star R(k)]. \end{aligned} \quad (50)$$

We further apply the relation

$$\star R(\zeta) = -2e^{-2\alpha\varphi} \left(-\frac{2}{3} i_\zeta F \wedge \star F + \frac{1}{3} F \wedge i_\zeta \star F \right), \quad (51)$$

which is also valid for any Killing field ζ , for the Killing fields ξ and k . Considering the explicit form of the electromagnetic field, we obtain

$$\begin{aligned} & 2i_\eta i_k \star R(\xi) - i_\eta i_\xi \star R(k) \\ &= 2(i_\xi F \wedge i_\eta i_k e^{-2\alpha\varphi} \star F + i_\eta F \wedge i_\xi i_k e^{-2\alpha\varphi} \star F) \\ &= 2(dA_t \wedge dB_\eta + dA_\phi \wedge dB_\xi). \end{aligned} \quad (52)$$

The bulk term can be simplified using the Stokes' theorem and taking into account that the integrals at infinity vanish, as well as the integrals over the bubble intervals and over the axis of the Killing field η . Thus, we obtain

$$\begin{aligned} & \frac{L}{2} \int_{\hat{M}} [dA_t \wedge dB_\eta + dA_\phi \wedge dB_\xi] \\ &= -\frac{L}{2} \sum_i \int_{I_{H_i}} [A_t dB_\eta + A_\phi dB_\xi]. \end{aligned} \quad (53)$$

We substitute this expression in equation (49) and apply the identity for the horizon Komar mass (38), which leads to

$$\begin{aligned} M_{\text{ADM}} &= \sum_i M_{H_i} + \sum_j M_{B_j} - \frac{L}{2} \sum_i \int_{I_{H_i}} [A_t dB_\eta + A_\phi dB_\xi] \\ &= \sum_i M_{H_i}^{(0)} + \sum_j M_{B_j}^{(0)} - \frac{L}{2} \sum_i \int_{I_{H_i}} [A_t dB_\eta + \Omega_i A_\phi dB_\xi] \\ &= \sum_i M_{H_i}^{(0)} + \sum_j M_{B_j}^{(0)} + \sum_i \Sigma_{H_i} \mathcal{Q}_{H_i} \\ &= \sum_i \left(1 + \frac{1}{2} \sinh^2 \gamma\right) M_{H_i}^{(0)} + \sum_j M_{B_j}^{(0)}. \end{aligned} \quad (54)$$

We can also consider the relation of the total angular momentum to some local quantities by expanding the corresponding Komar integral over the factor space \hat{M} . Thus, we obtain

$$\begin{aligned} \mathcal{J} &= -\frac{L}{8} \int_{\text{Arc}(\infty)} i_\eta i_k \star d\eta \\ &= -\frac{L}{8} \sum_i \int_{I_{H_i}} i_\eta i_k \star d\eta - \frac{L}{4} \int_{\hat{M}} i_\eta i_k \star R(\eta) \\ &= \sum_i J_{H_i} - \frac{L}{2} \int_{\hat{M}} dA_\phi \wedge dB_\eta \\ &= \sum_i \left[J_{H_i} + \frac{L}{2} \int_{I_{H_i}} A_\phi dB_\eta \right] \\ &= \cosh \gamma \sum_i J_{H_i}^{(0)}, \end{aligned} \quad (55)$$

where we have applied the identity (39) for the horizon angular momenta, and the index i again runs over the horizons. In the case when a single horizon is present, we

can express the Smarr relation for the mass in the following form,

$$M_{\text{ADM}} = \frac{1}{4\pi} \kappa_H A_H + 2\Omega \mathcal{J} + \sum_j M_{B_j} + \Sigma_H \mathcal{Q}_H, \quad (56)$$

using this relation.

In a similar way, we can derive a Smarr relation for the tension. Considering the corresponding Komar integral

$$\begin{aligned} \mathcal{T}L &= \frac{L}{8} \int_{\text{Arc}(\infty)} [i_\eta i_k \star d\xi - 2i_\eta i_\xi \star dk] \\ &= \frac{1}{2} \sum_i M_{H_i} + 2 \sum_j M_{B_j} \\ &\quad + \frac{L}{4} \int_{\hat{M}} [i_\eta i_k \star R(\xi) - 2i_\eta i_\xi \star R(k)], \end{aligned} \quad (57)$$

and using the identity

$$\begin{aligned} i_\eta i_k \star R(\xi) - 2i_\eta i_\xi \star R(k) &= 2i_\eta F \wedge i_\xi i_k e^{-2\alpha\varphi} \star F \\ &= 2dA_\phi \wedge dB_\xi, \end{aligned} \quad (58)$$

as well as the expression for the Komar mass of each horizon, we obtain

$$\begin{aligned} \mathcal{T}L &= \frac{1}{2} \sum_i M_{H_i} + 2 \sum_j M_{B_j}^{(0)} - \frac{L}{2} \sum_i \int_{I_{H_i}} A_\phi dB_\xi \\ &= \frac{1}{2} \sum_i M_{H_i}^{(0)} + 2 \sum_j M_{B_j}^{(0)}. \end{aligned} \quad (59)$$

Consequently, the Smarr relation for the tension coincides with that for the seed solution.

D. Gyromagnetic ratio

We can investigate the gyromagnetic ratio of the constructed solutions. It is defined as the constant of proportionality g in the relation

$$\mathcal{M} = g \frac{\mathcal{Q}\mathcal{J}}{2M_{\text{ADM}}}, \quad (60)$$

between the magnetic moment, the angular momentum, the ADM mass and the electric charge of the solution. Since the magnetic moment and the angular momentum are not independent, but related as $\mathcal{M} = \tanh \gamma \mathcal{J}$, the gyromagnetic ratio does not depend on the value of the angular momentum, but only on the ratio between the ADM mass and the electric charge. We use the Smarr-like relation for the mass,

$$M_{\text{ADM}} = \sum_i \left(1 + \frac{1}{2} \sinh^2 \gamma\right) M_{H_i}^{(0)} + \sum_j M_{B_j}^{(0)}, \quad (61)$$

and the expression for the local electric charge on each horizon,

$$Q_{H_i} = \sinh \gamma \cosh \gamma M_{H_i}^{(0)}, \quad (62)$$

and also take into account that the total electric charge of the solution is a sum of the local charges on all of the horizons. Then, we can express the gyromagnetic ratio in the form

$$g = 2 \cosh^{-2} \gamma \left(1 + \frac{1}{2} \sinh^2 \gamma + \frac{\sum_j M_{B_j}^{(0)}}{\sum_i M_{H_i}^{(0)}} \right), \quad (63)$$

where $M_{H_i}^{(0)}$ and $M_{B_j}^{(0)}$ are the Komar masses of the black holes and the bubbles in the seed solution. For large electric charges $\gamma \rightarrow \infty$, the gyromagnetic ratio approaches the value $g = 1$.

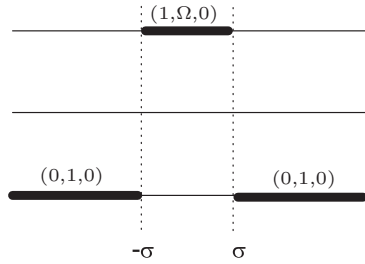
V. PARTICULAR SOLUTIONS

A. Charged rotating dilaton black string

The most simple solution which belongs to the class we consider is the black string. It contains a single horizon and can be described by the interval structure represented in Fig. 2 (left). The rotating black string in vacuum is constructed by trivially embedding the Kerr black hole in five-dimensional spacetime by adding a compact extra dimension. By performing the Kaluza-Klein transformation on it we obtain its charged dilaton generalization. The solution is represented in the most simple form in the Boyer-Lindquist coordinates, in which the metric is given by

$$\begin{aligned} ds^2 &= -\Lambda^{-\frac{2}{3}} \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt + \omega d\phi)^2 \\ &+ \Lambda^{\frac{1}{3}} \left[\frac{\Sigma \Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + d\psi^2 \right], \\ \omega &= \frac{2mr a \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} \cosh \gamma, \\ \Lambda &= 1 + \frac{2mr}{\Sigma} \sinh^2 \gamma, \\ \Delta &= r^2 - 2mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \end{aligned} \quad (64)$$

and the electromagnetic potentials are



$$A_t = \Lambda^{-1} \sinh \gamma \cosh \gamma \frac{mr}{\Sigma},$$

$$A_\phi = -\Lambda^{-1} \sinh \gamma \frac{mr}{\Sigma} a \sin^2 \theta. \quad (65)$$

It is characterized by the real parameters $a \in (0, 1)$, $m > 0$, which correspond to the spin parameter of the vacuum seed solution $a = J^{(0)}/M^{(0)}$, and its mass normalized to the length of the compact dimension $m = M^{(0)}/L$. The length of the horizon interval in the interval structure $\Delta l = 2\sigma$ is connected to them by the relation $\sigma^2 = m^2 - a^2$. The horizon is located at $r = r_H = m + \sqrt{m^2 - a^2}$, and rotates with angular velocity

$$\Omega = \frac{a \cosh^{-1} \gamma}{2m(\sigma + m)}. \quad (66)$$

The ADM mass, angular momentum, and electric charge of the solution are given by

$$\begin{aligned} M_{\text{ADM}} &= Lm \left(1 + \frac{1}{2} \sinh^2 \gamma \right), \\ \mathcal{J} &= L \cosh \gamma a m, \\ \mathcal{Q} &= L \sinh \gamma \cosh \gamma m, \end{aligned} \quad (67)$$

and the gyromagnetic ratio is equal to $g = 1 + \cosh^{-2} \gamma$.

We can obtain the horizon Komar mass and angular momentum by calculating the integral

$$I = L \int_{I_H} A_\phi d\mathcal{B}_\xi = -\frac{L}{2} \sinh \gamma \int_{I_H} A_\phi d\chi^{(0)}. \quad (68)$$

The twist 1-form for the vacuum seed solution on the horizon takes the form

$$d\chi^{(0)} = -\frac{\alpha(1 + \alpha^2)(1 - \alpha^2 \cos^2 \theta) \sin \theta}{(1 + \alpha^2 \cos^2 \theta)^2} d\theta, \quad (69)$$

where $\alpha = a/r_H$, and the restriction of the electromagnetic potential A_ϕ on the horizon is given by

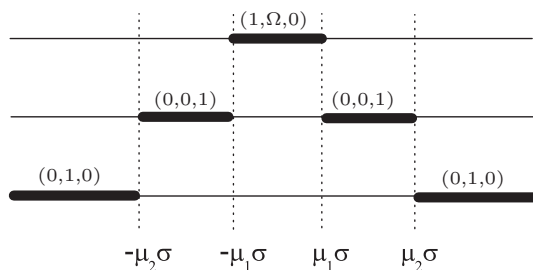


FIG. 2. Interval structure of the rotating black string (left), and the rotating black ring on Kaluza-Klein bubbles (right).

$$A_\phi = -\sinh\gamma \frac{m\alpha \sin^2\theta}{\beta^2 + \alpha^2 \cos^2\theta}, \quad (70)$$

where we define the parameter $\beta^2 = 1 + \sinh^2\gamma(1 + \alpha^2)$. Consequently, the integral is equal to³

$$I = -L\sigma \frac{(1 + \alpha^2)}{1 - \alpha^2} + L\sigma \cosh^2\gamma \frac{(1 + \alpha^2)^2}{\alpha\beta(1 - \alpha^2)(\beta^2 - 1)} \\ \times \left(2\beta \arctan\alpha - (1 + \beta^2) \arctan\frac{\alpha}{\beta} \right).$$

Taking into account the expression for the horizon Komar mass of the seed solution

$$M_H^{(0)} = Lm = L\sigma \frac{1 + \alpha^2}{1 - \alpha^2}, \quad (71)$$

we obtain for the horizon Komar mass

$$M_H = L\sigma \cosh^2\gamma \frac{(1 + \alpha^2)^2}{\alpha\beta(1 - \alpha^2)(\beta^2 - 1)} \\ \times \left(2\beta \arctan\alpha - (1 + \beta^2) \arctan\frac{\alpha}{\beta} \right). \quad (72)$$

The horizon angular momentum is calculated from (39) using this result and the fact that on the horizon it is satisfied that $d\mathcal{B}_\eta = -\Omega_H^{-1} d\mathcal{B}_\xi$. We obtain the following expression:

$$J_H = -\frac{L\sigma^2(1 + \alpha^2)}{\alpha(1 - \alpha^2)} \cosh\gamma \left[1 - \cosh^2\gamma \frac{(1 + \alpha^2)^2}{\alpha\beta(1 - \alpha^2)(\beta^2 - 1)} \right. \\ \left. \times \left(2\beta \arctan\alpha - (1 + \beta^2) \arctan\frac{\alpha}{\beta} \right) \right].$$

B. Charged rotating black ring on Kaluza-Klein bubbles

The charged rotating black ring on Kaluza-Klein bubbles is an example of a solution which contains a single horizon, but in addition also possesses fixed point sets of the Killing field associated with the compact dimension. The rotating black ring on Kaluza-Klein bubbles in vacuum was constructed recently by means of solitonic techniques [28]. It is characterized by the interval structure represented in Fig. 2 (right). If the solution is balanced, it should contain bubbles with equal size. Then, it is described by three real parameters $\sigma > 0$, $\mu_1 \in (0, 1)$ and $\mu_2 > 1$, which are associated with the lengths of the horizon and bubble intervals, and the angular momentum. Due to the balance conditions

³Instead of the mass and spin parameters m and a , we can parametrize the black string solution with the parameters σ and α . They are related as $m = \frac{\sigma(1+\alpha^2)}{1-\alpha^2}$ and $a = \frac{2\sigma\alpha}{1-\alpha^2}$.

the length of the compact dimension at infinity is not an independent parameter, but is defined by the relation

$$L = 8\pi\sigma \frac{\mu_1\mu_2(\mu_2 - \mu_1)}{(\mu_1\mu_2 - 1)(\mu_2 + \mu_1)} \sqrt{\mu_2^2 - 1}. \quad (73)$$

Applying the Kaluza-Klein transformation we obtain the metric of the charged dilaton generalization in the form

$$ds^2 = -\Lambda^{-\frac{2}{3}} \frac{W_1}{W_2} (dt + \cosh\gamma \omega d\phi)^2 \\ + \Lambda^{\frac{1}{3}} \left[\frac{W_2}{W_1} \rho^2 \frac{e^{2\tilde{U}_{-\mu_2\sigma}} e^{2\tilde{U}_{\mu_1\sigma}}}{e^{2\tilde{U}_{-\mu_1\sigma}} e^{2\tilde{U}_{\mu_2\sigma}}} d\phi^2 + \frac{e^{2\tilde{U}_{-\mu_1\sigma}} e^{2\tilde{U}_{\mu_2\sigma}}}{e^{2\tilde{U}_{-\mu_2\sigma}} e^{2\tilde{U}_{\mu_1\sigma}}} dy^2 \right. \\ \left. + Y(d\rho^2 + dz^2) \right], \\ \Lambda = 1 + \sinh^2\gamma \frac{W_2 - W_1}{W_2},$$

$$\tilde{U}_c = \frac{1}{2} \ln \left[\sqrt{\rho^2 + (z - c)^2} + (z - c) \right],$$

where ρ and z are Weyl coordinates, and the metric functions W_1 , W_2 , ω , and Y depend only on them. Due to their complexity, we provide their explicit form in the Appendix.

The electromagnetic potentials are

$$A_t = \frac{1}{2} \Lambda^{-1} \sinh\gamma \cosh\gamma \frac{W_2 - W_1}{W_2}, \\ A_\phi = -\frac{1}{2} \Lambda^{-1} \sinh\gamma \frac{W_1}{W_2} \omega. \quad (74)$$

We obtain for the ADM mass, the angular momentum, and the electric charge of the solution the expressions

$$M_{\text{ADM}} = \frac{L}{2} \sigma(\mu_2 - \mu_1) \frac{(\mu_1\mu_2 + 1)}{(\mu_1\mu_2 - 1)} + \frac{L}{2} \sinh^2\gamma \sigma \frac{(\mu_2 - \mu_1)}{(\mu_1\mu_2 - 1)}, \\ \mathcal{J} = L \cosh\gamma \sigma^2 \frac{\mu_1\mu_2(\mu_2 - \mu_1)}{(\mu_1\mu_2 - 1)^2} \sqrt{(1 - \mu_1^2)(\mu_2^2 - 1)}, \\ \mathcal{Q} = L \sinh\gamma \cosh\gamma \sigma \frac{(\mu_2 - \mu_1)}{(\mu_1\mu_2 - 1)}. \quad (75)$$

The intrinsic masses of the black hole and the bubbles for the vacuum seed solution are given by

$$M_H^{(0)} = L\sigma \frac{(\mu_2 - \mu_1)}{(\mu_1\mu_2 - 1)}, \\ M_{B_j}^{(0)} = \frac{L}{4} \sigma(\mu_2 - \mu_1), \quad j = 1, 2. \quad (76)$$

Using these expressions, we can calculate the gyromagnetic ratio of the charged solution

$$g = 1 + \cosh^{-2}\gamma \mu_1\mu_2. \quad (77)$$

The horizon angular velocity and area are

$$\Omega = \frac{1}{2\sigma \cosh \gamma} \frac{\mu_1 \mu_2 - 1}{\mu_1(\mu_2 - \mu_1)} \sqrt{\frac{(1 - \mu_1^2)}{(\mu_2^2 - 1)}},$$

$$A_H = 32\pi L \cosh \gamma \sigma^2 \frac{\mu_1^3 \mu_2 (\mu_2 - \mu_1) (\mu_2^2 - 1)}{(\mu_1 \mu_2 - 1)^2 (\mu_1 + \mu_2)^2}. \quad (78)$$

The horizon Komar mass and angular momentum are obtained by calculating the integral

$$I = L \int_{I_H} A_\phi d\mathcal{B}_\xi = -\frac{L}{2} \sinh \gamma \int_{I_H} A_\phi d\chi^{(0)} \quad (79)$$

and using the relation (38). The twist potential of the seed solution possesses the following form on the horizon,

$$\chi^{(0)} = \beta \frac{y(1 + \mu_1)(1 + b^2)}{(1 + \mu_2)(1 + b^2 y^2)}, \quad (80)$$

where $y = z/\sigma$, and

$$b^2 = \frac{(1 - \mu_1^2)(\mu_2^2 - y^2)}{(\mu_2^2 - 1)(\mu_1^2 - y^2)},$$

$$\beta = -\sqrt{\frac{(1 - \mu_1)(1 + \mu_2)}{(\mu_2 - 1)(1 + \mu_1)}}. \quad (81)$$

The restriction of the electromagnetic potential A_ϕ on the horizon is given by

$$A_\phi = -\frac{1}{2} \frac{\sinh \gamma}{\Omega^{(0)}} \frac{(1 - y^2)b^2}{1 + b^2 y^2 + \sinh^2 \gamma (1 + b^2)}. \quad (82)$$

Consequently, the integral (79) is equal to

$$I = -L\sigma \frac{(\mu_2 - \mu_1)}{\mu_1 \mu_2 - 1}$$

$$\times \left[1 - \frac{\mu_1^2}{\tanh^2 \gamma} \left(\frac{2}{Z} \arctan Z - \frac{F^2 + Z^2}{FZ^2} \arctan F \right) \right]$$

$$= -M_H^{(0)} + M_H,$$

$$F^2 = \frac{\mu_1^2(1 - \mu_1^2)}{\mu_1^2(\mu_2^2 - 1) + \sinh^2 \gamma (\mu_2^2 - \mu_1^2)},$$

$$Z^2 = \frac{1 - \mu_1^2}{\mu_2^2 - 1}, \quad (83)$$

and we obtain for the horizon Komar mass

$$M_H = L\sigma \frac{\mu_1^2(\mu_2 - \mu_1)}{\tanh^2 \gamma (\mu_1 \mu_2 - 1)} \left(\frac{2}{Z} \arctan Z - \frac{F^2 + Z^2}{FZ^2} \arctan F \right). \quad (84)$$

The horizon angular momentum is calculated in a similar way using the relation (39), and the evaluated integral (79)

$$J_H = -L \cosh \gamma \mu_1^2 \sigma^2 \frac{(\mu_2 - \mu_1)}{\mu_1 \mu_2 - 1} \sqrt{\frac{\mu_2^2 - 1}{1 - \mu_1^2}}$$

$$\times \left[1 + \frac{\mu_1(\mu_2 - \mu_1)}{\tanh^2 \gamma (\mu_1 \mu_2 - 1)} \left(\frac{F^2 + Z^2}{FZ^2} \arctan F - \frac{2}{Z} \arctan Z \right) \right].$$

C. Charged rotating black holes on a Kaluza-Klein bubble

As a next example, we consider a multihorizon solution, which describes a superposition of two rotating black holes and a Kaluza-Klein bubble. The vacuum solution was constructed in [27] using the inverse scattering method, and it is described by the interval structure presented in Fig. 3. The solution is characterized by 4 parameters $\eta_1 < \eta_2 < -1 < \lambda < 1$, and $\sigma > 0$, which are related to the lengths of the horizon and bubble intervals in the interval structure, and the angular momentum of the solution. The solution is free of conical singularities, if the length of the Kaluza-Klein circle at infinity is related to the solution parameters as

$$L = 4\pi\sigma \frac{(\eta_1 + 1)((\eta_1 - 1)^2 + (\lambda + 1)(\eta_2 - 1))}{(\eta_1 - 1)((\eta_1 + 1)^2 + (\lambda + 1)(\eta_2 + 1))}$$

$$\times \sqrt{\frac{(\lambda - \eta_1)(\lambda - \eta_2)(\eta_2 + 1)}{\eta_2 - 1}}. \quad (85)$$

We apply the Kaluza-Klein transformation on this solution and obtain a charged dilaton solution with metric

$$ds^2 = -\Lambda^{-\frac{2}{3}} \frac{W_1 e^{2\tilde{U}_{\eta_2\sigma}}}{W_2 e^{2\tilde{U}_{\eta_1\sigma}}} (dt + \cosh \gamma \omega d\phi)^2$$

$$+ \Lambda^{\frac{1}{3}} \left[\frac{W_2}{W_1} \rho^2 \frac{e^{2\tilde{U}_{\eta_1\sigma}}}{e^{2\tilde{U}_{\lambda\sigma}}} d\phi^2 + \frac{e^{2\tilde{U}_{\lambda\sigma}}}{e^{2\tilde{U}_{\eta_2\sigma}}} d\psi^2 + Y(d\rho^2 + dz^2) \right],$$

$$\Lambda = 1 + \sinh^2 \gamma \frac{W_2 - W_1}{W_2},$$

$$\tilde{U}_c = \frac{1}{2} \ln \left[\sqrt{\rho^2 + (z - c)^2} + (z - c) \right],$$

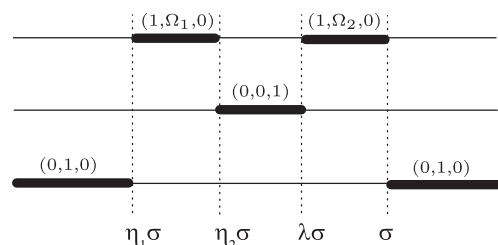


FIG. 3. Interval structure of the rotating black holes on a Kaluza-Klein bubble.

where ρ and z are Weyl coordinates, and the metric functions W_1 , W_2 , ω and Y depend only on them. Similar to the previous solution, the metric functions are given by complicated expressions. Therefore, we provide their explicit form in the Appendix.

The electromagnetic potentials are

$$\begin{aligned} A_t &= \frac{1}{2} \Lambda^{-1} \sinh \gamma \cosh \gamma \frac{W_2 - W_1}{W_2}, \\ A_\phi &= -\frac{1}{2} \Lambda^{-1} \sinh \gamma \frac{W_1}{W_2} \omega. \end{aligned} \quad (86)$$

The black hole horizons rotate with angular velocities

$$\begin{aligned} \Omega_1 &= \frac{-\beta(1-\beta^2)\cosh^{-1}\gamma}{[1-\eta_1+\beta^2(1+\eta_1)]^2\sigma}, \\ \beta^2 &= -\frac{(1+\lambda)(1+\eta_2)}{(1+\eta_1)^2}, \\ \Omega_2 &= \frac{-\beta(1-\beta^2)((1-\eta_2)(1-\lambda)+(1-\eta_1)^2)\cosh^{-1}\gamma}{4[(1-\eta_1)^2+\beta^2(1-\eta_2)(1-\lambda)]\sigma}. \end{aligned} \quad (87)$$

The ADM mass, the angular momentum and the electric charge of the solution possess the following form:

$$\begin{aligned} M_{\text{ADM}} &= \frac{L(4-2\eta_1+\eta_2+\lambda+\beta^2(4+2\eta_1-\eta_2-\lambda))\sigma}{4(1-\beta^2)} \\ &\quad + \frac{L}{4} \sinh^2 \gamma \frac{(\eta_2-\eta_1+2-\beta^2(\eta_2-\eta_1-2))\sigma}{1-\beta^2}, \\ \mathcal{J} &= -\cosh \gamma \\ &\quad \times \frac{2L\beta\sigma^2(2-2\eta_1+\eta_2+\lambda+\beta^2(2+2\eta_1-\eta_2-\lambda))}{(1-\beta^2)^2}, \\ \mathcal{Q} &= \frac{L}{2} \sinh \gamma \cosh \gamma \frac{(\eta_2-\eta_1+2-\beta^2(\eta_2-\eta_1-2))\sigma}{1-\beta^2}. \end{aligned} \quad (88)$$

We calculate the horizon and bubble Komar masses for the vacuum seed solution, and obtain

$$\begin{aligned} M_{H_1}^{(0)} &= \frac{L\sigma(\eta_2-\eta_1)}{2(1-\beta^2)} \frac{[1+\beta^2\frac{(1+\eta_1)}{(1-\eta_1)}][1-\beta^2\frac{(1-\eta_2)(1+\eta_1)}{(1-\eta_1)(1+\eta_2)}]}{[1-\frac{(1-\eta_2)(1+\lambda)}{(1-\eta_1)^2}]}, \\ M_{H_2}^{(0)} &= \frac{L\sigma}{(1-\beta^2)} \frac{[1+\beta^2\frac{(1-\eta_2)(1-\lambda)}{(1-\eta_1)^2}]}{[1-\frac{(1-\eta_2)(1+\lambda)}{(1-\eta_1)^2}]}, \\ M_B^{(0)} &= \frac{L}{4}(\lambda-\eta_2)\sigma, \end{aligned} \quad (89)$$

where we denote by $M_{H_1}^{(0)}$ the Komar mass of the horizon located at $\eta_1\sigma < z < \eta_2\sigma$, and by $M_{H_2}^{(0)}$ the Komar mass of

the horizon located at $\lambda\sigma < z < \sigma$. Using these expressions and the relation (13), we can calculate the local charges on the horizons:

$$\begin{aligned} \mathcal{Q}_{H_1} &= \frac{L}{2} \sinh \gamma \cosh \gamma \frac{\sigma(\eta_2-\eta_1)}{(1-\beta^2)} \\ &\quad \times \frac{[1+\beta^2\frac{(1+\eta_1)}{(1-\eta_1)}][1-\beta^2\frac{(1-\eta_2)(1+\eta_1)}{(1-\eta_1)(1+\eta_2)}]}{[1-\frac{(1-\eta_2)(1+\lambda)}{(1-\eta_1)^2}]}, \\ \mathcal{Q}_{H_2} &= \sinh \gamma \cosh \gamma \frac{L\sigma}{(1-\beta^2)} \frac{[1+\beta^2\frac{(1-\eta_2)(1-\lambda)}{(1-\eta_1)^2}]}{[1-\frac{(1-\eta_2)(1+\lambda)}{(1-\eta_1)^2}]}. \end{aligned} \quad (90)$$

VI. CONCLUSION

We construct a class of stationary and axisymmetric solutions to the five-dimensional Einstein-Maxwell-dilaton equations, which describe sequences of charged rotating black objects and Kaluza-Klein bubbles. The solutions are obtained for the special value of the dilaton coupling constant $\alpha = \sqrt{8/3}$ by uplifting a five-dimensional vacuum solution with the same interval structure to six dimensions, performing a boost and a subsequent circle reduction. We investigate the general relation of their physical properties to the properties of the corresponding vacuum solutions. Certain relations are obtained between the twist potentials and the electromagnetic potentials for the charged and the vacuum cases, which lead to relations between some local properties of the solutions. The horizon Komar masses and angular momenta are modified with respect to the vacuum ones by a term reflecting the interaction with the electromagnetic field. On the other hand, the electric charge associated with each horizon is proportional to the corresponding horizon Komar mass in the vacuum solution. Although our class of solutions possess Kaluza-Klein asymptotics, these relations are also valid for asymptotically flat solutions obtained by the same procedure, up to certain normalization factors. We further derive the Smarr-like relations and obtain an expression for the gyromagnetic ratio.

We investigate three exact solutions as particular cases. The first one represents the charged rotating dilaton black string. It is the most simple solution belonging to the class we consider, since it contains only a single horizon and no fixed point sets of the Killing vector associated with the compact dimension. As more complicated examples we investigate the charged rotating black ring sitting on Kaluza-Klein bubbles, and a multihorizon solution describing a superposition of two rotating charged black holes and a Kaluza-Klein bubble.

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APPENDIX: BLACK HOLES ON KALUZA-KLEIN BUBBLES IN VACUUM

1. Rotating black ring on Kaluza-Klein bubbles

The metric of the rotating black ring on Kaluza-Klein bubbles in vacuum is given by

$$ds^2 = -\frac{W_1}{W_2}(dt + \omega d\phi)^2 + \frac{W_2}{W_1}\rho^2 \frac{e^{2\tilde{U}_{-2\sigma}} e^{2\tilde{U}_{\mu_1\sigma}}}{e^{2\tilde{U}_{-1\sigma}} e^{2\tilde{U}_{\mu_2\sigma}}} d\phi^2 + \frac{e^{2\tilde{U}_{-1\sigma}} e^{2\tilde{U}_{\mu_2\sigma}}}{e^{2\tilde{U}_{-2\sigma}} e^{2\tilde{U}_{\mu_1\sigma}}} d\psi^2 + Y(d\rho^2 + dz^2).$$

The metric functions W_1 , W_2 and ω have the form

$$\begin{aligned} W_1 &= [(R_{-\sigma} + R_{\sigma})^2 - 4\sigma^2](1 + ab)^2 \\ &\quad + [(R_{-\sigma} - R_{\sigma})^2 - 4\sigma^2](a - b)^2, \\ W_2 &= [(R_{-\sigma} + R_{\sigma} + 2\sigma) + (R_{-\sigma} + R_{\sigma} - 2\sigma)ab]^2 \\ &\quad + [(R_{-\sigma} - R_{\sigma} - 2\sigma)a - (R_{-\sigma} - R_{\sigma} + 2\sigma)b]^2, \\ \omega &= -\frac{e^{\tilde{U}_{-2\sigma}} e^{\tilde{U}_{\mu_1\sigma}} \hat{\omega}}{e^{\tilde{U}_{-1\sigma}} e^{\tilde{U}_{\mu_2\sigma}} W_1} - \frac{4\sigma\alpha}{1 - \alpha^2}, \\ \hat{\omega} &= [(R_{-\sigma} + R_{\sigma})^2 - 4\sigma^2](1 + ab) \\ &\quad \times [(R_{-\sigma} - R_{\sigma} + 2\sigma)b + (R_{-\sigma} - R_{\sigma} - 2\sigma)a] \\ &\quad - [(R_{-\sigma} - R_{\sigma})^2 - 4\sigma^2](b - a) \\ &\quad \times [(R_{-\sigma} + R_{\sigma} + 2\sigma) - (R_{-\sigma} + R_{\sigma} - 2\sigma)ab], \end{aligned} \quad (A1)$$

where the following functions are used,

$$\begin{aligned} a &= \alpha \frac{e^{2U_{\sigma}} + e^{2\tilde{U}_{-1\sigma}}}{e^{\tilde{U}_{-1\sigma}}} \frac{e^{\tilde{U}_{-2\sigma}}}{e^{2U_{\sigma}} + e^{2\tilde{U}_{-2\sigma}}} \\ &\quad \times \frac{e^{2U_{\sigma}} + e^{2\tilde{U}_{\mu_2\sigma}}}{e^{\tilde{U}_{\mu_2\sigma}}} \frac{e^{\tilde{U}_{\mu_1\sigma}}}{e^{2U_{\sigma}} + e^{2\tilde{U}_{\mu_1\sigma}}}, \\ b &= \beta \frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{-2\sigma}}}{e^{\tilde{U}_{-2\sigma}}} \frac{e^{\tilde{U}_{-1\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{-1\sigma}}} \frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\mu_1\sigma}}}{e^{\tilde{U}_{\mu_1\sigma}}} \\ &\quad \times \frac{e^{\tilde{U}_{\mu_2\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\mu_2\sigma}}}, \\ R_c &= \sqrt{\rho^2 + (z - c)^2}, \\ \tilde{U}_c &= \frac{1}{2} \ln [R_c + (z - c)], \\ U_c &= \frac{1}{2} \ln [R_c - (z - c)]. \end{aligned} \quad (A2)$$

The constants α and β are connected with the solution parameters as

$$\alpha = -\beta = \sqrt{\frac{(1 - \mu_1)(1 + \mu_2)}{(\mu_2 - 1)(1 + \mu_1)}}. \quad (A3)$$

The remaining metric function Y is given by

$$Y = \frac{W_2}{16(1 - \alpha^2)^2 R_{\sigma} R_{-\sigma}} \sqrt{\frac{Y_{\sigma, -\mu_2\sigma} Y_{\sigma, \mu_1\sigma} Y_{-\sigma, -\mu_1\sigma} Y_{-\sigma, \mu_2\sigma}}{Y_{\sigma, -\mu_1\sigma} Y_{\sigma, \mu_2\sigma} Y_{-\sigma, -\mu_2\sigma} Y_{-\sigma, \mu_1\sigma}}} \times \frac{Y_{-\mu_2\sigma, -\mu_1\sigma} Y_{-\mu_2\sigma, \mu_2\sigma} Y_{-\mu_1\sigma, \mu_1\sigma} Y_{\mu_1\sigma, \mu_2\sigma}}{R_{-\mu_2\sigma} R_{-\mu_1\sigma} R_{\mu_1\sigma} R_{\mu_2\sigma} Y_{-\mu_2\sigma, \mu_1\sigma} Y_{-\mu_1\sigma, \mu_2\sigma}} \frac{e^{2\tilde{U}_{-2\sigma}} e^{2\tilde{U}_{\mu_1\sigma}}}{e^{2\tilde{U}_{-1\sigma}} e^{2\tilde{U}_{\mu_2\sigma}}}, \quad (A4)$$

where we define the notation

$$Y_{cd} = R_c R_d + (z - c)(z - d) + \rho^2. \quad (A5)$$

We consider only balanced solutions, i.e. solutions which are free of conical singularities. If conical singularities are allowed, the metric possesses more general form (see [28]).

2. Rotating black holes on a Kaluza-Klein bubble

The metric of the rotating black holes on Kaluza-Klein bubble in vacuum is given by

$$ds^2 = -\frac{W_1}{W_2} \frac{e^{2\tilde{U}_{\eta_2\sigma}}}{e^{2\tilde{U}_{\eta_1\sigma}}} (dt + \omega d\phi)^2 + \frac{W_2}{W_1} \rho^2 \frac{e^{2\tilde{U}_{\eta_1\sigma}}}{e^{2\tilde{U}_{\lambda\sigma}}} d\phi^2 + \frac{e^{2\tilde{U}_{\lambda\sigma}}}{e^{2\tilde{U}_{\eta_2\sigma}}} d\psi^2 + Y(d\rho^2 + dz^2).$$

The metric functions W_1 , W_2 , and ω have the form

$$\begin{aligned} W_1 &= [(R_{-\sigma} + R_{\sigma})^2 - 4\sigma^2](1 + ab)^2 \\ &\quad + [(R_{-\sigma} - R_{\sigma})^2 - 4\sigma^2](a - b)^2, \\ W_2 &= [(R_{-\sigma} + R_{\sigma} + 2\sigma) + (R_{-\sigma} + R_{\sigma} - 2\sigma)ab]^2 \\ &\quad + [(R_{-\sigma} - R_{\sigma} - 2\sigma)a - (R_{-\sigma} - R_{\sigma} + 2\sigma)b]^2, \\ \omega &= -\frac{e^{2\tilde{U}_{\eta_1\sigma}} \hat{\omega}}{e^{\tilde{U}_{\lambda\sigma}} e^{\tilde{U}_{\eta_2\sigma}} W_1} - \frac{4\sigma\alpha}{1 - \alpha^2}, \\ \hat{\omega} &= [(R_{-\sigma} + R_{\sigma})^2 - 4\sigma^2](1 + ab) \\ &\quad \times [(R_{-\sigma} - R_{\sigma} + 2\sigma)b + (R_{-\sigma} - R_{\sigma} - 2\sigma)a] \\ &\quad - [(R_{-\sigma} - R_{\sigma})^2 - 4\sigma^2](b - a) \\ &\quad \times [(R_{-\sigma} + R_{\sigma} + 2\sigma) - (R_{-\sigma} + R_{\sigma} - 2\sigma)ab], \end{aligned} \quad (A6)$$

where the following functions are used:

$$\begin{aligned}
a &= \alpha \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda\sigma}} e^{2U_\sigma} + e^{2\tilde{U}_{\eta_2\sigma}} \left(\frac{e^{\tilde{U}_{\eta_1\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_1\sigma}}} \right)^2}{e^{\tilde{U}_{\lambda\sigma}} e^{\tilde{U}_{\eta_2\sigma}}}, \\
b &= \beta \frac{e^{\tilde{U}_{\lambda\sigma}} e^{\tilde{U}_{\eta_2\sigma}} \left(\frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_1\sigma}}}{e^{\tilde{U}_{\eta_1\sigma}}} \right)^2}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\lambda\sigma}} e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_2\sigma}}}, \\
R_c &= \sqrt{\rho^2 + (z-c)^2}, \\
\tilde{U}_c &= \frac{1}{2} \ln [R_c + (z-c)], \quad U_c = \frac{1}{2} \ln [R_c - (z-c)].
\end{aligned} \tag{A7}$$

The constants α and β are connected with the solution parameters as

$$\alpha = -\beta = \sqrt{-\frac{(1+\lambda)(1+\eta_2)}{(1+\eta_1)^2}}. \tag{A8}$$

The remaining metric function Y is given by

$$\begin{aligned}
Y &= \frac{W_2}{(1-\alpha^2)^2 R_\sigma R_{-\sigma} R_{\lambda\sigma} R_{\eta_1\sigma} R_{\eta_2\sigma}} \\
&\times \sqrt{\frac{Y_{-\sigma,\lambda\sigma} Y_{-\sigma,\eta_2\sigma} Y_{\lambda\sigma,\eta_1\sigma} Y_{\lambda\sigma,\eta_2\sigma} Y_{\eta_1\sigma,\eta_2\sigma}}{Y_{\sigma,\lambda\sigma} Y_{\sigma,\eta_2\sigma}}} \\
&\times \frac{Y_{\sigma,\eta_1\sigma} e^{2\tilde{U}_{\eta_1\sigma}}}{Y_{-\sigma,\eta_1\sigma} e^{2\tilde{U}_{\lambda\sigma}}},
\end{aligned} \tag{A9}$$

where we define the notation

$$Y_{cd} = R_c R_d + (z-c)(z-d) + \rho^2. \tag{A10}$$

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