Relativistic formulation of the Hall-Vinen-Bekarevich-Khalatnikov superfluid hydrodynamics

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The relativistic analogue of the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) hydrodynamics is derived making use of the phenomenological method similar to that used by Bekarevich and Khalatnikov [1] in their derivation of HVBK hydrodynamics. The resulting equations describe a finite-temperature superfluid liquid with the distributed vorticity. The main dissipative effects, including mutual friction, are taken into account. The proposed hydrodynamics is needed for reliable modeling of the dynamical properties of superfluid neutron stars.

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I. INTRODUCTION

Despite the fact that superfluid flow must be irrotational, it is well known [2–5] that in a rotating bucket a superfluid mimics solid body rotation *on average* by creating arrays of topological defects—vortex lines, near which the irrotationality condition breaks down.

Hall and Vinen [6] developed in 1956 a coarse-grained hydrodynamic equations capable of describing a superfluid liquid with the continuously distributed vorticity. Their equations are only valid in situations when a typical length scale of the problem is much larger than the intervortex spacing. Later in 1960-1961 Hall [7] and, independently, Bekarevich and Khalatnikov [1] presented a more elaborated version of these equations which is now called Hall-Vinen-Bekarevich-Khalatnikov (HVBK) hydrodynamics. Note that the most general phenomenological derivation of HVBK-hydrodynamics, based upon conservation laws, were given by the last two authors in the abbreviation (Bekarevich and Khalatnikov). Subsequently, many authors have repeated and analyzed their derivation in order to generalize it and/or make it more transparent (see Donnelly [5] and Sonin [8] for details and, especially, Refs. [9,10]). The main conclusion of their work is that, basically, the structure of the HVBK-hydrodynamics remains unaffected if one is not interested in the oscillation modes related to the elasticity of the vortex lattice [11,12].

The HVBK-hydrodynamics has received a great deal of attention in relation to the interpretation of liquid helium II experiments [5,8] and, somewhat unexpectedly, in relation to the neutron star physics (see, e.g., Refs. [13–17]). Since HVBK-equations are essentially nonrelativistic, the majority of studies of superfluid neutron-star dynamics have been performed in the nonrelativistic framework. This framework is (as a rule) acceptable for a qualitative analysis of the problem but is inadequate for obtaining the quantitative results since neutron stars are essentially relativistic objects.

Clearly, one needs a Lorentz-covariant formulation of HVBK-hydrodynamics. In the literature there were only a few attempts to find such a formulation [18,19] (see also Ref. [20], lectures [21], and references therein). The authors of these works restrict themselves to the case of a vanishing temperature ($T = 0$), when there are no thermal excitations (normal component) in the liquid and hence no dissipative interaction (the so-called "mutual friction") between the superfluid and normal liquid components. The resulting hydrodynamics, generalized subsequently to describe superfluid mixtures [22,23], have then been applied to model oscillations of cold ($T = 0$) superfluid rotating neutron stars in Ref. [24]. Note, however, that in many physically interesting situations the approximation of vanishing stellar temperature is not justified and leads to *qualitatively* wrong results when studying neutron star dynamics (see, e.g., Refs. [25–30] for illustration of principal importance of finite temperature effects in some problems). Moreover, as we argue in Appendix F, the hydrodynamics of Refs. [18,19] is internally inconsistent, which can have important consequences for those problems (see Ref. [31] for an example) for which the contribution of the vortex energy to the total energy density cannot be neglected.[1]

The aim of the present study is to fill the existing gap by deriving the self-consistent relativistic dissipative HVBK-hydrodynamics, valid at arbitrary temperature. Our derivation will closely follow the ideas of the original derivation of Bekarevich and Khalatnikov [1].

[1]The results of Ref. [24] remain unaffected since it (legitimately) ignores a small vortex contribution to the total energy density.

The paper is organized as follows. In Sec. II we present the derivation of the well-known vortex-free superfluid relativistic hydrodynamics. In Sec. III A we formulate the relativistic HVBK-hydrodynamics under the assumption that the contribution of vortices to the total energy density of a superfluid can be neglected. In Sec. III B this assumption is relaxed and the most general relativistic HVBK-equations are found. Finally, we conclude in Sec. IV.

The paper also contains a number of important appendixes. In Appendix A we present the original (nonrelativistic) HVBK-hydrodynamics; in Appendix B we list the full system of equations of relativistic HVBK-hydrodynamics; in Appendix C we analyze the nonrelativistic limit of one of the most important equations of the proposed hydrodynamics—the superfluid equation; in Appendix D we find the vortex contribution to the energy density; in Appendix E we present an alternative microscopic derivation of the vortex contribution to the energy-momentum tensor (more precisely, derivation of its spatial components); finally, in Appendix F we discuss the internal inconsistency of the zero-temperature vortex hydrodynamics of Refs. [18,19].

Unless otherwise stated, in what follows the speed of light c, the Planck constant $\hbar$, and the Boltzmann constant k_B are all set to unity, $c = \hbar = k_\mathrm{B} = 1$.

II. RELATIVISTIC SUPERFLUID HYDRODYNAMICS IN THE ABSENCE OF VORTICES

A. General equations

Neglecting vortices, relativistic superfluid hydrodynamics for a one-component liquid has been studied in many papers and is well known (see, e.g., [18,20,32–41]). Here we present its derivation partly in order to establish notations and partly because, as we believe, it can be of independent interest. Our derivation adopts the same strategy as that used by Khalatnikov [3] to derive equations of *nonrelativistic* superfluid hydrodynamics.

Hydrodynamic equations include the energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0 \tag{1}$$

and particle conservation

$$\partial_\mu j^\mu = 0, \tag{2}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$; $T^{\mu\nu}$ is the energy-momentum tensor (which must be symmetric) and j^μ is the particle four-current density. Here and below, unless otherwise stated, μ, ν, and other Greek letters are space-time indices running over 0, 1, 2, and 3. Generally, $T^{\mu\nu}$ and j^μ can be presented as

$$T^{\mu\nu} = \underline{(P+\varepsilon)u^\mu u^\nu} + \underline{P g^{\mu\nu}} + \Delta T^{\mu\nu}, \tag{3}$$

$$j^\mu = \underline{n u^\mu} + \Delta j^\mu, \tag{4}$$

where P is the pressure given by Eq. (21) below; ε is the energy density; n is the number density; $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the space-time metric.[2] Finally, u^μ is the four-velocity of the normal (nonsuperfluid) liquid component (thermal excitations), normalized by the condition

$$u_\mu u^\mu = -1. \tag{5}$$

The underlined terms in Eqs. (3) and (4) have the familiar form of, respectively, the energy-momentum tensor and particle current density of nonsuperfluid matter (see, e.g., Ref. [2]). Correspondingly, additional "superfluid" terms $\Delta T^{\mu\nu}$ and Δj^μ characterize deviation of superfluid hydrodynamics from the ordinary one. Note that the thermodynamic quantities introduced in Eqs. (3) and (4) do not have any direct physical meaning unless a comoving frame where they are measured (defined) is specified. In what follows we *define* the comoving frame by the condition $u^\mu = (1,0,0,0)$ indicating, that it is the frame where the normal liquid component is at rest. This definition coincides with the definition of the comoving frame in the ordinary relativistic hydrodynamics. It means, in particular, that the components T^{00} and j^0 in this frame are given by the conditions, $T^{00} = \varepsilon$ and $j^0 = n$, which, in an arbitrary frame, translates into

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon, \tag{6}$$

$$u_\mu j^\mu = -n, \tag{7}$$

or, in view of the expressions (3) and (4) to

$$u_\mu u_\nu \Delta T^{\mu\nu} = 0, \tag{8}$$

$$u_\mu \Delta j^\mu = 0. \tag{9}$$

As a matter of fact, we can extract even more information about the form of j^μ in the comoving frame. Since in that frame normal component does not move, spatial components of the current j^i ($i = 1,\ 2,\ 3$) are nonzero only because of the motion of superfluid component. In the nonrelativistic limit the contribution of the superfluid

[2] Throughout the paper we assume that the metric is flat. Generalization of our results to arbitrary $g_{\mu\nu}$ is straightforward provided that all relevant length scales of the problem (e.g., thermal excitation mean free path) are small enough compared with the characteristic gravitational length scale (e.g., neutron star radius) [42]. In the latter case general relativity effects can easily be incorporated into hydrodynamics by replacing ordinary derivatives in all equations with their covariant analogues.

component in this situation would be $\rho_s \boldsymbol{V}_s$, where ρ_s is the superfluid density and $\boldsymbol{V}_s = \boldsymbol{\nabla}\phi/m$ is the superfluid velocity. (Here m is the bare particle mass and ϕ is a scalar proportional to the phase Φ of the condensate wave function; for Bose-Einstein condensate $\phi = \Phi$, for Cooper-pair condensate $\phi = \Phi/2$ [43,44].) By analogy, in the relativistic case it is natural to introduce a superfluid four-velocity

$$V^{\mu}_{(s)} \equiv \frac{\partial^{\mu}\phi}{m}, \tag{10}$$

and assume that j^i can be represented as $j^i = mYV^i_{(s)} = Y\partial^i\phi$, where Y is some coefficient, a relativistic equivalent of the superfluid density ρ_s (it is easily verified that in the nonrelativistic limit $Y = \rho_s/(m^2c^2)$ in dimensional units [44]). Consequently, in the comoving frame one has [see Eqs. (4) and (9)]

$$\Delta j^0 = 0, \tag{11}$$

$$\Delta j^i = j^i = Y\partial^i\phi. \tag{12}$$

In an arbitrary frame this expression can be rewritten by introducing a new four-vector, b^{μ}, as

$$\Delta j^{\mu} = Y(\partial^{\mu}\phi + b^{\mu}). \tag{13}$$

To satisfy Eq. (12), a spatial part of b^{μ} should vanish in the comoving frame, $b^i = 0$. That is, b^{μ} and u^{μ} should be collinear in that frame, hence they must be collinear in all other frames, i.e.,

$$b^{\mu} = -Bu^{\mu}, \tag{14}$$

where B is some scalar to be determined below. In view of Eqs. (9) and (13) B and ϕ are interrelated by the following equation

$$u_{\mu}\partial^{\mu}\phi = -B. \tag{15}$$

Note that, Eq. (11) is then automatically satisfied.

Let us now introduce a new four-vector,

$$w^{\mu} \equiv \partial^{\mu}\phi - Bu^{\mu} \tag{16}$$

instead of $\partial^{\mu}\phi$. Since this vector depends on the four-gradient of the scalar ϕ, it is not arbitrary and is constrained by the condition

$$\partial_{\mu}(w_{\nu} + Bu_{\nu}) = \partial_{\nu}(w_{\mu} + Bu_{\mu}), \tag{17}$$

which is simply the statement that $\partial_{\mu}\partial_{\nu}\phi = \partial_{\nu}\partial_{\mu}\phi$. In what follows Eq. (17) is called the *potentiality* condition or simply the *superfluid* equation. In terms of the new four-vector w^{μ} one has [see Eqs. (4) and (13)]

$$j^{\mu} = nu^{\mu} + Yw^{\mu}, \tag{18}$$

while the condition (9) transforms into

$$u_{\mu}w^{\mu} = 0. \tag{19}$$

Equations (1)–(3), (6), (7), and (17)–(19) are key equations that will be used below. They should be supplemented by the second law of thermodynamics.

In a normal matter the energy density ε of a one-component liquid can generally be presented as a function of the number density n and the entropy density S. In superfluid matter, there is an additional degree of freedom associated with the vector w^{μ}. One can construct two scalars associated with w^{μ}, namely, $u_{\mu}w^{\mu}$ and $w_{\mu}w^{\mu}$. The first scalar vanishes on account of (19), so that $\varepsilon = \varepsilon(n, S, w_{\mu}w^{\mu})$. Consequently, variation of ε can generally be written as

$$d\varepsilon = \mu dn + TdS + \frac{\Lambda}{2}d(w_{\mu}w^{\mu}), \tag{20}$$

where we defined the relativistic chemical potential $\mu \equiv \partial\varepsilon(n, S, w_{\mu}w^{\mu})/\partial n$; temperature $T \equiv \partial\varepsilon(n, S, w_{\mu}w^{\mu})/\partial S$; and $\Lambda \equiv 2\partial\varepsilon(n, S, w_{\mu}w^{\mu})/\partial(w_{\mu}w^{\mu})$. Equation (20) is interpreted as the second law of thermodynamics for a superfluid liquid.

We need also to specify the pressure P. According to the standard definition it equals to a partial derivative of the full system energy εV with respect to volume V at constant total number of particles, total entropy, and $w_{\mu}w^{\mu}$ [3,45],

$$P \equiv -\frac{\partial(\varepsilon V)}{\partial V} = -\varepsilon + \mu n + TS. \tag{21}$$

Using (20) and (21) one arrives at the following Gibbs-Duhem equation for a superfluid liquid,

$$dP = nd\mu + SdT - \frac{\Lambda}{2}d(w_{\mu}w^{\mu}). \tag{22}$$

B. Determination of $\Delta T^{\mu\nu}$ and the parameters B and Λ

We discussed above a general structure of the *non-dissipative* hydrodynamics of superfluid liquid, which must conserve entropy of any closed system. This means that the entropy generation equation must take the form of the continuity equation,

$$\partial_{\mu}S^{\mu} = 0, \tag{23}$$

where S^{μ} is the entropy current density (it will be shown below that the entropy flows with the normal liquid component, i.e., $S^{\mu} = Su^{\mu}$).

We will find $\Delta T^{\mu\nu}$, B, and Λ from this requirement. To do this, we should derive the entropy generation equation from the hydrodynamics of the previous section. Let us consider a combination $u_\nu \partial_\mu T^{\mu\nu}$, which vanishes in view of Eq. (1). Using Eqs. (3), (5), (21), and (22) one obtains

$$0 = -u^\mu T\partial_\mu S - ST\partial_\mu u^\mu - \mu\partial_\mu(nu^\mu) - u^\mu \Lambda w_\nu \partial_\mu w^\nu + u_\nu \partial_\mu \Delta T^{\mu\nu}, \quad (24)$$

or, using Eq. (2) with j^μ from Eq. (18),

$$T\partial_\mu(Su^\mu) = \mu\partial_\mu(Yw^\mu) - u^\mu\Lambda w_\nu\partial_\mu w^\nu + u_\nu\partial_\mu \Delta T^{\mu\nu}. \quad (25)$$

This equation can be further transformed to

$$T\partial_\mu(Su^\mu) = \partial_\mu(\mu Y w^\mu) - Yw^\mu\partial_\mu\mu - u^\mu\Lambda w_\nu\partial_\mu w^\nu + \partial_\mu(u_\nu\Delta T^{\mu\nu}) - \Delta T^{\mu\nu}\partial_\mu u_\nu. \quad (26)$$

The derivative $\partial_\mu w^\nu$ in the third term on the right-hand side of Eq. (26) can be expressed by making use of Eq. (17). After substitution of the result a few terms vanish and we left with

$$T\partial_\mu(Su^\mu) = w^\mu(\Lambda\partial_\mu B - Y\partial_\mu\mu) + \partial_\mu(\mu Y w^\mu + u_\nu \Delta T^{\mu\nu}) + \partial_\mu u_\nu(\Lambda w^\mu w^\nu + \Lambda B u^\mu w^\nu - \Delta T^{\mu\nu}). \quad (27)$$

To obtain Eq. (27) we used the equalities

$$u_\mu\partial_\nu u^\mu = 0, \quad (28)$$

$$u_\mu\partial^\nu w^\mu = -w^\mu\partial^\nu u_\mu, \quad (29)$$

following from Eqs. (5) and (19), respectively. The second and third terms in Eq. (27) can be symmetrized by employing Eqs. (5) and (19). As a result, Eq. (27) can be rewritten in its final form as

$$T\partial_\mu(Su^\mu) = w^\mu(\Lambda\partial_\mu B - Y\partial_\mu\mu) + \partial_\mu[u_\nu(\Delta T^{\mu\nu} - \Lambda w^\mu w^\nu - \mu Y w^\mu u^\nu - \mu Y w^\nu u^\mu)] + \partial_\mu u_\nu(\Lambda w^\mu w^\nu + \Lambda B u^\mu w^\nu + \Lambda B u^\nu w^\mu - \Delta T^{\mu\nu}) \quad (30)$$

or

$$\partial_\mu(Su^\mu) = \frac{w^\mu}{T}(\Lambda\partial_\mu B - Y\partial_\mu\mu) + (\mu Y - \Lambda B)\frac{\partial_\mu T}{T^2}w^\mu + \partial_\mu\left[\frac{u_\nu}{T}(\Delta T^{\mu\nu} - \Lambda w^\mu w^\nu - \mu Y w^\mu u^\nu - \mu Y w^\nu u^\mu)\right] + \partial_\mu\left(\frac{u_\nu}{T}\right)(\Lambda w^\mu w^\nu + \Lambda B u^\mu w^\nu + \Lambda B u^\nu w^\mu - \Delta T^{\mu\nu}). \quad (31)$$

The right-hand side of this equation must be a four-divergence for *any* $\partial_\mu u_\nu$, $\partial_\mu T$, and $\partial_\mu \mu$. This requirement, together with the assumption that $\Delta T^{\mu\nu}$ should depend on the four-velocities u^μ and w^μ and various thermodynamic quantities (but not on their gradients), while B and Λ should depend on thermodynamic quantities only, allows us to identify the unknown parameters Λ, B, $\Delta T^{\mu\nu}$, and S^μ as

$$\Lambda = \frac{Y}{k}, \quad (32)$$

$$B = k\mu, \quad (33)$$

$$\Delta T^{\mu\nu} = Y\left(\frac{w^\mu w^\nu}{k} + \mu u^\mu w^\nu + \mu u^\nu w^\mu\right), \quad (34)$$

$$S^\mu = Su^\mu, \quad (35)$$

where k is some constant which should be equal to 1, as follows from the comparison with the nonrelativistic theory.[3] These equalities complete the formulation of relativistic superfluid hydrodynamics in the absence of vortices. One can see that the resulting energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = (P+\varepsilon)u^\mu u^\nu + Pg^{\mu\nu} + Y(w^\mu w^\nu + \mu u^\mu w^\nu + \mu u^\nu w^\mu), \quad (36)$$

is symmetric and satisfies the condition (6) [on account of Eq. (19)].

Equations (20) and (22) now take the form

$$d\varepsilon = \mu\, dn + T\, dS + \frac{Y}{2}d(w_\mu w^\mu), \quad (37)$$

$$dP = n\, d\mu + S\, dT - \frac{Y}{2}d(w_\mu w^\mu), \quad (38)$$

while the potentiality condition (17) becomes

$$\partial_\mu(w_\nu + \mu u_\nu) = \partial_\nu(w_\mu + \mu u_\mu) \Leftrightarrow m[\partial_\mu V_{(s)\nu} - \partial_\nu V_{(s)\mu}] = 0. \quad (39)$$

Remark 1.—It is relatively straightforward to include dissipation into this hydrodynamics. The corresponding corrections (the largest of them) have been first obtained in Refs. [18,20] and have received a great deal of attention in the recent years [34,35,37,38,40]. For the superfluid hydrodynamics in the form discussed above they were formulated in Ref. [35].

[3]Another way to verify that k can be chosen equal to 1 is to note that *both* ϕ and Y are introduced into the theory through the definition (12) of j^i in the comoving frame. They can, therefore, be simultaneously rescaled, $Y \to Y/k$ and $\phi \to k\phi$, without affecting j^i and other observables of the theory. This is equivalent to choosing $k = 1$ in Eqs. (32)–(34).

Dissipation adds a correction $\tau^{\mu\nu}_{\rm diss}$ to the energy-momentum tensor $T^{\mu\nu}$ (36) and also changes the relation between the superfluid velocity $V^{\mu}_{(\rm s)}$ and the four-vector w^{μ}, which becomes [35][4]

$$V^{\mu}_{(\rm s)} = \frac{w^{\mu} + (\mu + \varkappa_{\rm diss})u^{\mu}}{m}, \tag{40}$$

where $\varkappa_{\rm diss}$ is the correction depending on the bulk viscosity coefficients ξ_3 and ξ_4. Both these corrections are briefly discussed in Appendix B, where we present the full system of equations of relativistic superfluid HVBK-hydrodynamics.

III. RELATIVISTIC SUPERFLUID HYDRODYNAMICS IN THE PRESENCE OF VORTICES

A thorough discussion of vortices in the nonrelativistic superfluid hydrodynamics can be found in many references (see, e.g., [3,5,8,46]); a brief summary of results is given in Appendix A. An extension of the concept of vortices to the relativistic case is rather straightforward (see, e.g., Refs. [18–21,47]). When there are no vortices in the system the wave function phase of a superfluid condensate is a well-defined quantity everywhere so that the integral $\oint \partial_{\mu}\phi dx^{\mu}$ over *any* closed loop vanishes. If there are topological defects—vortices—in the system, this integral should not be necessarily zero and can be a multiple of 2π (it cannot be arbitrary in order for the wave function of the condensate to be uniquely defined),

$$\oint V_{(\rm s)\mu} dx^{\mu} = \frac{2\pi N}{sm}, \tag{41}$$

where N is an integer; $s = 1$ for Bose-superfluids and $s = 2$ for Fermi-superfluids, and we introduced the superfluid velocity $V_{(\rm s)\mu}$ instead of $\partial_{\mu}\phi$ (sufficiently far from the vortices, where the "hydrodynamic approach" is justified, they are related by Eq. (10); however, in the immediate vicinity of the vortex cores this equation is violated [43]). It can be shown [3] that in a real superfluid it is energetically favorable to form vortices in the form of thin lines, each carrying exactly one quantum of circulation [i.e., an integral (41) over a closed loop around any given vortex line is $2\pi/(sm)$].

Equation (41) can be rewritten, using the Stokes' theorem, as an integral over the surface encircled by the loop,

$$\int df^{\mu\nu} F_{\mu\nu} = \frac{2\pi N}{s}, \tag{42}$$

where $F_{\mu\nu}$ defines vorticity multiplied by m, $F_{\mu\nu} \equiv m[\partial_{\mu}V_{(\rm s)\nu} - \partial_{\nu}V_{(\rm s)\mu}]$ (for brevity, $F_{\mu\nu}$ is called "vorticity" in what follows). In many physically interesting situations[5] vortices are so densely packed on a typical length-scale of the problem that it makes no sense to follow the evolution of each of them in order to describe dynamics of the system as a whole. Instead, it is more appropriate to use *coarse-grained* dynamical equations which depend on quantities averaged over the volume containing large amount of vortices.

The main parameters of such a theory are the smooth-averaged superfluid velocity and vorticity (to be defined as $V^{\mu}_{(\rm s)}$ and $F_{\mu\nu}$ in what follows); they are analogous to, respectively, the averaged superfluid velocity $\boldsymbol{V}_{\rm s}$ and $m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$ of the nonrelativistic theory. Note that, in view of Eq. (42), the smooth-averaged vorticity $F_{\mu\nu} \neq 0$ (and $V^{\mu}_{(\rm s)}$ is not simply given by a gradient of scalar). In other words, when there are vortices in the system, Eq. (39) should be replaced by a weaker constraint (see below).

A. Hydrodynamic equations under condition that the vortex contribution to the energy density can be neglected

To get an insight into the problem, let us first determine the form of large-scale hydrodynamics in the case when one can neglect contribution of vortices to the second law of thermodynamics and to the energy-momentum tensor.[6] In the nonrelativistic theory this limit corresponds to HVBK-hydrodynamics with $\lambda = 0$ (when $\hbar$ is formally set to 0; see Appendix A and Remark 2 there). In this limit vortices affect only the superfluid equation (39) [Eq. (A4) of the nonrelativistic theory], while other equations of Sec. II remain unchanged. Note, however, that now these equations depend on the smooth-averaged four-velocity $V^{\mu}_{(\rm s)}$ which is not given by simply $\partial^{\mu}\phi/m$. Correspondingly, the smooth-averaged quantity w^{μ} in these equations should now be written as [see Eq. (40) with $\varkappa_{\rm diss} = 0$]

$$w^{\mu} = mV^{\mu}_{(\rm s)} - \mu u^{\mu}. \tag{43}$$

To find an explicit form of the smooth-averaged superfluid equation in the presence of vortices we will again make use of the fact that the entropy of a closed system cannot decrease. Employing the energy-momentum and particle conservation laws (1) and (2) with j^{μ} and $T^{\mu\nu}$ given by, respectively, Eqs. (18) and (36), as well as Eqs. (5), (19), (21), (37), and (38), we arrive at the following entropy generation equation,

[4] In the absence of dissipation $V^{\mu}_{(\rm s)} = (w^{\mu} + \mu u^{\mu})/m$, as follows from Eqs. (10), (16), and (33).

[5] For example, in rotating neutron stars, the mean distance between the neighboring vortices is $\sim 10^{-2}$–10^{-4} cm, while the typical length-scale, the stellar radius, is ~ 10 km.

[6] For clarity, we also ignore in what follows the standard viscous and thermal conduction terms in the expression for $T^{\mu\nu}$ ($\tau^{\mu\nu}_{\rm diss} = 0$) and in the relation (40) between $V^{\mu}_{(\rm s)}$ and w^{μ} ($\varkappa_{\rm diss} = 0$).

$$T\partial_\mu(Su^\mu) = u^\nu Y w^\mu F_{\mu\nu}. \tag{44}$$

This equation can be derived in the same way as in Sec. II B with the only difference that now it is obtained without making use of the potentiality condition (39), which is not valid in the system with the distributed vorticity ($F_{\mu\nu} \neq 0$).

Because entropy does not decrease, one should have

$$u^\nu Y w^\mu F_{\mu\nu} \geq 0. \tag{45}$$

Let us now introduce a new four-vector,

$$f_\mu \equiv \frac{u^\nu F_{\mu\nu}}{\mu n}. \tag{46}$$

In terms of f_μ Eq. (45) can be rewritten as

$$W^\mu f_\mu \geq 0, \tag{47}$$

where we also defined

$$W^\mu \equiv \frac{Y w^\mu}{n}. \tag{48}$$

In the comoving frame [where $u^\mu = (1,0,0,0)$], $f_0 = F_{00} = 0$ and Eq. (47) transforms into

$$W^i f_i \geq 0, \tag{49}$$

where $i = 1, 2, 3$ is the spatial index.[7] In order for the inequality (49) to hold true the vector $\boldsymbol{f} \equiv (f^1, f^2, f^3)$ should satisfy a number of conditions (forget for a moment about its definition (46): (i) it must be polar; (ii) must vanish at $F_{\mu\nu} = 0$ (because the potentiality condition (39) is valid in that case); and (iii) should depend on $\boldsymbol{W} \equiv (W^1, W^2, W^3)$ in order to satisfy Eq. (49) at arbitrary $\boldsymbol{W}$ [note that $\boldsymbol{V}_{\rm s} \equiv [V^1_{(\rm s)}, V^2_{(\rm s)}, V^3_{(\rm s)}] = n\boldsymbol{W}/(Ym)$ and thus is not an independent variable; see Eqs. (43) and (48)]. These conditions are clearly insufficient to determine the most general form of $\boldsymbol{f}$. However, it seems reasonable to further require that (iv) $\boldsymbol{f}$ may only depend on $\boldsymbol{W}$ and $F_{\mu\nu}$ (as noted by Clark [9], in the nonrelativistic theory a similar assumption was implicitly made in Ref. [1]; see Ref. [9] for a detailed critical analysis of HVBK-hydrodynamics).

In analogy with electrodynamics, instead of the antisymmetric tensor $F_{\mu\nu} = m[\partial_\mu V_{(\rm s)\nu} - \partial_\nu V_{(\rm s)\mu}]$ it is convenient to introduce an axial vector $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$ and a polar vector $\boldsymbol{E} \equiv m[\partial \boldsymbol{V}_{\rm s}/\partial t + \boldsymbol{\nabla} V^0_{(\rm s)}]$. Then the most general form of $\boldsymbol{f}$, satisfying the conditions (i)–(iv), can, in principle, be found. The resulting expression will contain many more kinetic coefficients (and additional terms) in comparison to the original HVBK-expression (A13), because now we allow $\boldsymbol{f}$ to depend not only on $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$, like in the nonrelativistic theory, but also on the vector $\boldsymbol{E}$.[8] The physical meaning of these additional terms is not clear and deserves a further study. However, in the nonrelativistic limit these terms are presumably suppressed in comparison to the $\boldsymbol{H}$-dependent terms presented in Eq. (50) below [because $\boldsymbol{E} \sim 1/c \to 0$ at $c \to \infty$, see Appendix C and Eqs. (C5) and (C7) there]. Since here we are mainly interested in the straightforward generalization of HVBK-equations to the relativistic case, below we only present the terms which have direct counterparts in the nonrelativistic theory. They exclusively depend on the vector $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$, namely,

$$\boldsymbol{f} = -\alpha[\boldsymbol{H} \times \boldsymbol{W}] - \beta \mathbf{e} \times [\boldsymbol{H} \times \boldsymbol{W}] + \gamma \mathbf{e}(\boldsymbol{W}\boldsymbol{H}), \tag{50}$$

where $\boldsymbol{e} \equiv \boldsymbol{H}/H$ is the unit vector in the direction of $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$; α, β, and γ are some scalars (kinetic coefficients), which can generally depend on invariants of $\boldsymbol{W}$ and $\boldsymbol{H}$. Note that the first term in the right-hand side of Eq. (50), depending on α, is dissipationless. In contrast, the other terms there are dissipative and to satisfy (47) the coefficients β and γ should be positive, $\beta, \gamma \geq 0$. In appendix C it is shown that these coefficients indeed coincide with the coefficients α, β, and γ of HVBK-hydrodynamics.

We found the form of the four-vector f^μ in the comoving frame, $f^\mu = (0, \boldsymbol{f})$, where $\boldsymbol{f}$ is given by Eq. (50). Now our aim will be to rewrite f^μ in an arbitrary frame. To do this let us introduce a four-vector H^μ, given by (in the orthonormal basis)

$$H^\mu \equiv \epsilon^{\mu\nu\lambda\eta} u_\nu m \partial_\lambda V_{(\rm s)\eta} = \frac{1}{2}\epsilon^{\mu\nu\lambda\eta} u_\nu F_{\lambda\eta}, \tag{51}$$

where $\epsilon^{\mu\nu\lambda\eta}$ is the four-dimensional Levi-Civita tensor and we use the antisymmetry property of the tensor $F_{\mu\nu}$ in the second equality. In the comoving frame this vector equals $H^\mu = (0, \boldsymbol{H}) = (0, m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s})$. Also, assume that we have two four-vectors, say, B^μ and C^μ, whose spatial components $\boldsymbol{B}$ and $\boldsymbol{C}$ form a 3D-vector $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{C}$ in the comoving frame. Then we *define* the four-vector A^μ in an arbitrary frame according to

$$A^\mu \equiv \epsilon^{\mu\nu\lambda\eta} u_\nu B_\lambda C_\eta. \tag{52}$$

The definitions (51) and (52) are trivial extensions of the curl operator and cross product, defined in the comoving frame [$u^\mu = (1,0,0,0)$], to an arbitrary frame (see also Refs. [48,49] for similar definitions). Using these definitions, one can immediately write out a general Lorentz-covariant expression for f^μ,

[7] It is worth noting that, in view of Eq. (19), $W^0 = Yw^0/n$ also vanishes in the comoving frame, $W^0 = 0$.

[8] Among the $\boldsymbol{E}$-dependent terms which can enter the expression for $\boldsymbol{f}$ there should be terms of the form $[\boldsymbol{E} \times \boldsymbol{W}] \times \boldsymbol{W}$, $(\boldsymbol{E}\boldsymbol{W})\boldsymbol{E}$, $[\boldsymbol{E} \times \boldsymbol{W}] \times \boldsymbol{E}$ and a number of "mixed" terms depending on both $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$ and $\boldsymbol{E}$, e.g., $[\boldsymbol{W} \times \boldsymbol{E}](\boldsymbol{E}\boldsymbol{H})$ and $\boldsymbol{W} \times [\boldsymbol{E} \times \boldsymbol{H}](\boldsymbol{E}\boldsymbol{H})$.

$$f^{\mu} = -\alpha X^{\mu} - \beta \epsilon^{\mu\nu\lambda\eta} u_{\nu} \mathrm{e}_{\lambda} X_{\eta} + \gamma \mathrm{e}^{\mu} (W^{\lambda} H_{\lambda}), \qquad (53)$$

where $\mathrm{e}^{\mu} = H^{\mu}/H$ with $H = (H_{\mu} H^{\mu})^{1/2}$ and $X^{\mu} \equiv \epsilon^{\mu\nu\lambda\eta} u_{\nu} H_{\lambda} W_{\eta}$.[9] The same expression can be reformulated without making use of the Levi-Civita tensor,[10]

$$f^{\mu} = \alpha \perp^{\mu\nu} F_{\nu\lambda} W_{\delta} \perp^{\lambda\delta} + \frac{\beta - \gamma}{H} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\eta\sigma} F_{\lambda\nu} W_{\delta} \perp^{\lambda\delta} + \gamma H W_{\delta} \perp^{\mu\delta}, \qquad (54)$$

where $\perp^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$ is the projection operator and

$$H = \sqrt{\frac{1}{2} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\mu\nu} F_{\eta\sigma}}. \qquad (55)$$

Because f^{μ} is now specified, Eq. (46) can now be treated as a *new superfluid equation* which replaces the potentiality condition (39) and generalizes it to the case of a superfluid liquid with distributed vorticity. It can be rewritten as

$$u^{\nu} F_{\mu\nu} = \mu n f_{\mu}. \qquad (56)$$

Note that it is valid as long as one can neglect the contribution of vortices to the energy density (i.e., $\lambda = 0$, see Appendix D). Otherwise, the definition of the vector W^{μ} should be modified [see Eq. (82) in Sec. III B]. In Appendix C we demonstrate that, in the nonrelativistic limit, Eq. (56) reduces to Eq. (A4) with $\lambda = 0$.

Remark 1.—To derive the superfluid equation (56) we first introduced the vector $f^{\mu} = u^{\nu} F_{\mu\nu}/(\mu n)$ and then deduced its possible form from the condition $f_{\mu} W^{\mu} \geq 0$. This is not the only way of obtaining this equation. In fact, Eq. (56) can also be derived by introducing a vector $g_{\nu} \equiv W^{\mu} F_{\mu\nu}$ and then requiring it to satisfy a condition $g_{\nu} u^{\nu} \geq 0$, which follows from the constraint (45).

Remark 2.—Equation (56) imposes certain restrictions on the possible form of the tensor $F_{\mu\nu}$. Assume that $F_{\mu\nu}$ satisfies this equation. Then it can be shown by direct calculation that, if the coefficient γ in Eq. (54) vanishes, then a four-vector $V^{\mu}_{(\mathrm{L})}$ exists, given by,

$$V^{\mu}_{(\mathrm{L})} = u^{\mu} - \mu n \alpha W_{\nu} \perp^{\mu\nu} + \frac{\mu n \beta}{H} \perp^{\mu\alpha} \perp^{\nu\beta} F_{\alpha\beta} W_{\nu}, \qquad (57)$$

such that the combination $V^{\nu}_{(\mathrm{L})} F_{\mu\nu}$ is identically zero,

$$V^{\nu}_{(\mathrm{L})} F_{\mu\nu} = 0. \qquad (58)$$

Equation (58) is analogous to the vorticity conservation equation (A17) of the nonrelativistic HVBK-hydrodynamics (see Appendix A).

Remark 3.—In Appendix A we consider the strong and weak-drag limits for superfluid equation (A4) (or (A17)) of the nonrelativistic HVBK-hydrodynamics. Similar limits can also be considered in relativistic hydrodynamics. In particular, strong-drag limit corresponds to $\alpha = \beta = \gamma = 0$ in Eq. (54) so that Eq. (58) reduces to

$$u^{\nu} F_{\mu\nu} = 0. \qquad (59)$$

This equation describes vortex motion (vorticity transfer) with the velocity u^{μ} of normal liquid component. (In ordinary nonsuperfluid hydrodynamics a similar equation takes place, but vorticity $F_{\mu\nu}$ there is expressed through the same velocity u^{μ}, with which it is transferred, $F_{\mu\nu} = \partial_{\mu}(\mu u_{\nu}) - \partial_{\nu}(\mu u_{\mu})$, see, e.g., Ref. [50].) Weak-drag limit is described by the equation

$$V^{\nu}_{(\mathrm{s})} F_{\mu\nu} = 0 \qquad (60)$$

and follows from Eq. (58) when $\alpha = -1/(\mu^2 Y)$ and $\beta = \gamma = 0$ (cf. the corresponding limit in the nonrelativistic HVBK-hydrodynamics). It corresponds to a vortex motion with the superfluid velocity $V^{\mu}_{(\mathrm{s})}$. Note that both these limits were analyzed in Ref. [23] in application to zero-temperature superfluid neutron stars.[11]

B. Accounting for the vortex energy

In this section we formulate the relativistic generalization of the HVBK-hydrodynamics taking into account contribution of vortices to the energy-density ε and the energy-momentum tensor $T^{\mu\nu}$. It is convenient to formulate this hydrodynamics in terms of the four-vectors u^{μ} and w^{μ} as primary degrees of freedom. For that it is necessary to define more rigorously what we actually mean by w^{μ}. In what follows we *define* w^{μ} by the formula

$$j^{\mu} = n u^{\mu} + Y w^{\mu}, \qquad (61)$$

where the quantities j^{μ}, u^{μ}, n have the same meaning as in the previous sections [in particular, n is the number density measured in the comoving frame where $u^{\mu} = (1, 0, 0, 0)$],

[9] Possible $\boldsymbol{E}$-dependent terms in the expression for f^{μ} (see the footnote 8) can be obtained in a similar way by introducing a four-vector $E^{\nu} \equiv u_{\mu} F^{\mu\nu}$, which reduces to $(0, \boldsymbol{E})$ in the comoving frame.

[10] Equation (54) is the most general expression for f^{μ} valid for arbitrary W^{μ}. However, the four-vector W^{μ}, introduced in this section (cf. the definition of W^{μ} in Sec. III B), satisfies a condition $u_{\mu} W^{\mu} = 0$ [see Eqs. (19) and (48)], which allows one to simplify Eq. (54) in this particular case and write

$$f^{\mu} = \alpha \perp^{\mu\nu} F_{\nu\lambda} W^{\lambda} + \frac{\beta - \gamma}{H} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\eta\sigma} F_{\lambda\nu} W^{\lambda} + \gamma H W^{\mu}.$$

[11] In the "weak-drag" equation (33) of Ref. [23] one finds the total neutron current density instead of the superfluid velocity $V^{\mu}_{(\mathrm{s})}$. This is not surprising since the authors of Ref. [23] work in the limit $T = 0$, when all particles (neutrons) are paired and move with one and the same superfluid velocity $V^{\mu}_{(\mathrm{s})}$.

while the parameter Y is defined by the second law of thermodynamics [see Eq. (64) below], $Y = 2\partial\varepsilon/\partial(w_\mu w^\mu)$. [It is straightforward to show that it is always possible to define w^μ by Eq. (61) such that the coefficients Y in Eq. (61) and Y in Eq. (64) will indeed coincide.] A definition (61) implies that the conditions (9) and (19) must be satisfied automatically.

After defining w^μ, the superfluid velocity $V^\mu_{(\mathrm{s})}$ of our smooth-averaged hydrodynamics can be defined by Eq. (43), $V^\mu_{(\mathrm{s})} = (w^\mu + \mu u^\mu)/m$.[12] (We again ignore here a viscous dissipative correction $\varkappa_{\mathrm{diss}}$, which has the same form [35] as in the vortex-free case and does not affect our derivation; it can easily be included in the final equations, see Appendix B.)

Next, we present the energy-momentum tensor in the form

$$T^{\mu\nu} = (P+\varepsilon)u^\mu u^\nu + P g^{\mu\nu} + Y(w^\mu w^\nu + \mu u^\mu w^\nu + \mu u^\nu w^\mu) + \tau^{\mu\nu}, \tag{62}$$

where P is defined by Eq. (21) and $\tau^{\mu\nu}(=\tau^{\nu\mu})$ is the symmetric vortex contribution to $T^{\mu\nu}$, which will be determined below [without this contribution Eq. (62) coincides with (36)]. Because ε is the total energy density in the comoving frame (including the contribution of vortices), $T^{\mu\nu}$ should satisfy condition (6) which, in view of Eq. (19), translates into

$$u_\mu u_\nu \tau^{\mu\nu} = 0. \tag{63}$$

Finally, the most important step in building up the relativistic HVBK-hydrodynamics is to postulate the form of the second law of thermodynamics in the presence of vortices. Obviously, one can write

$$d\varepsilon = \mu dn + T dS + \frac{Y}{2} d(w_\mu w^\mu) + d\varepsilon_{\mathrm{vortex}}, \tag{64}$$

where $d\varepsilon_{\mathrm{vortex}}$ is the term responsible for the vortex contribution to $d\varepsilon$, while other terms are the same as in the vortex-free superfluid hydrodynamics [see Eq. (37)].

Before guessing a possible form of $d\varepsilon_{\mathrm{vortex}}$ let us derive the entropy generation equation. Using equations of this section together with Eqs. (1), (2), (5), (19), and (21), one gets

$$T\partial_\mu(Su^\mu) = u^\nu Y w^\mu F_{\mu\nu} - u^\mu \partial_\mu \varepsilon_{\mathrm{vortex}} + u_\nu \partial_\mu \tau^{\mu\nu}, \tag{65}$$

where[13]

$$F_{\mu\nu} \equiv m[\partial_\mu V_{(\mathrm{s})\nu} - \partial_\nu V_{(\mathrm{s})\mu}] = \partial_\mu(w_\nu + \mu u_\nu) - \partial_\nu(w_\mu + \mu u_\mu). \tag{66}$$

The first term in the right-hand side of Eq. (65) is the same as in Eq. (44), the second and third terms are induced by the vortex-related terms in Eqs. (62) and (64).

Now let us specify $d\varepsilon_{\mathrm{vortex}}$. In the absence of vortices the energy density ε depends on three scalars, n, S, and $w_\mu w^\mu$. When vortices are present a new dynamical quantity $F_{\mu\nu} \neq 0$ appears and the (smooth-averaged) energy density ε can depend on its various invariants. In fact, it is possible to compose many different scalars from the quantities $F_{\mu\nu}$, u^μ, w^μ, and their derivatives. One can single out one or few of them on the basis of physical arguments or intuition. As it is argued in Appendix D, it is a good approximation to treat $\varepsilon_{\mathrm{vortex}}$ as a function of only one additional invariant $H = (H_\mu H^\mu)^{1/2}$, where H^μ is given by Eq. (51) and equals $(0, m\,\mathrm{curl}\,\boldsymbol{V}_{\mathrm{s}})$ in the comoving frame. Correspondingly, $H = m|\mathrm{curl}\,\boldsymbol{V}_{\mathrm{s}}|$ is analogous to the invariant $\omega = |\mathrm{curl}\,\boldsymbol{V}_{\mathrm{s}}|$ of the nonrelativistic theory (see Appendix A). If ε depends on H, one can write

$$d\varepsilon_{\mathrm{vortex}} = \frac{\partial\varepsilon}{\partial H} dH = \frac{\lambda}{2mH} d(H_\mu H^\mu), \tag{67}$$

where the partial derivative is taken at constant n, S, and $w_\mu w^\mu$; $\lambda \equiv m\partial\varepsilon/\partial H$ is the relativistic analogue of the parameter λ of the nonrelativistic theory (see Appendix D); both parameters coincide in the nonrelativistic limit.

Equation (67) can be rewritten as

$$d\varepsilon_{\mathrm{vortex}} = \frac{\Gamma}{2}(O^{\alpha\beta} dF_{\alpha\beta} + 2F_{\alpha\beta}F^{\alpha\gamma}u^\beta du_\gamma), \tag{68}$$

where we used Eq. (55) together with the identity $u_\mu u_\nu F^{\mu\nu} = 0$, and defined[14]

$$\Gamma \equiv \frac{\lambda}{mH}, \tag{69}$$

$$O^{\alpha\beta} \equiv \perp^{\alpha\gamma}\perp^{\beta\delta} F_{\gamma\delta}. \tag{70}$$

In what follows we will be interested in the quantity $-u^\mu\partial_\mu\varepsilon_{\mathrm{vortex}}$, which appears in the entropy generation equation (65). Using Eq. (68), it is given by

[12] This way of reasoning is similar to that of Bekarevich & Khalatnikov [1]. In a purely phenomenological approach it is not obvious, however, that the superfluid velocity $V^\mu_{(\mathrm{s})}$ defined in this manner will coincide with the velocity, whose vorticity is directly related to the area density of vortex lines and satisfies, for example, the "continuity equation" for vortices [see Eq. (58)]. The fact that both definitions coincide follows from the self-consistency of the resulting hydrodynamics [in particular, Eq. (58) remains to be satisfied, see below]. This conclusion can also be verified by a microscopic consideration similar to that presented in Appendixes D and E.

[13] If $\varkappa_{\mathrm{diss}}$ were nonzero, one would have a combination $F_{\mu\nu} - \partial_\mu(\varkappa_{\mathrm{diss}}u_\nu) + \partial_\nu(\varkappa_{\mathrm{diss}}u_\mu)$ instead of $F_{\mu\nu}$ in Eq. (65).

[14] Note that $O^{\alpha\beta}$ can also be presented in the form, $O^{\alpha\beta} = \frac{1}{2}\epsilon^{\delta\eta\alpha\beta}u_\eta\epsilon_{\delta abc}u^a F^{bc}$ (a, b, and c are the space-time indices).

$$-u^{\mu}\partial_{\mu}\varepsilon_{\text{vortex}} = -\frac{\Gamma}{2}u^{\mu}O^{\alpha\beta}\partial_{\mu}F_{\alpha\beta} - \Gamma u^{\mu}u^{\delta}F_{\alpha\delta}F^{\alpha\nu}\partial_{\mu}u_{\nu}. \tag{71}$$

The first term in the right-hand side of Eq. (71) can be transformed as

$$-\frac{\Gamma}{2}u^{\mu}O^{\alpha\beta}\partial_{\mu}F_{\alpha\beta} = u^{\nu}F_{\mu\nu}\partial_{\alpha}(\Gamma O^{\mu\alpha}) - \partial_{\mu}(u^{\nu}\Gamma O^{\mu\alpha}F_{\nu\alpha}) + \partial_{\mu}u^{\nu}(\Gamma O^{\mu\alpha}F_{\nu\alpha}). \tag{72}$$

To obtain this expression we used the identity [see Eq. (66)]

$$\partial_{\mu}F_{\alpha\beta} = \partial_{\alpha}F_{\mu\beta} + \partial_{\beta}F_{\alpha\mu} \Leftrightarrow \epsilon^{iklm}\partial_{k}F_{lm} = 0, \tag{73}$$

and the fact that both tensors $F^{\mu\nu}$ and $O^{\mu\nu}$ are antisymmetric.

In turn, the second term in the right-hand side of Eq. (71) can be rewritten as

$$\begin{aligned}-\Gamma u^{\mu}u^{\delta}F_{\alpha\delta}F^{\alpha\nu}\partial_{\mu}u_{\nu} &= -\Gamma[u^{\mu}u^{\delta}F_{\alpha\delta}F^{\alpha\nu} \\ &\quad + \underline{u^{\mu}u^{\nu}u^{\beta}u_{\gamma}F_{\alpha\beta}F^{\alpha\gamma}}]\partial_{\mu}u_{\nu} \\ &= -\Gamma u^{\mu}u^{\gamma}\perp_{\nu\beta}F^{\alpha\beta}F_{\alpha\gamma}\partial_{\mu}u^{\nu} \\ &= -\Gamma u^{\mu}u^{\gamma}\perp_{\nu\beta}F^{\alpha\beta}F_{\alpha\gamma}\partial_{\mu}u^{\nu} \\ &\quad + \underline{\partial_{\mu}(\Gamma u^{\nu}u^{\mu}u^{\gamma}\perp_{\nu\beta}F^{\alpha\beta}F_{\alpha\gamma})},\end{aligned} \tag{74}$$

where the underlined terms equal zero [because of Eq. (28) and the equality $u^{\nu}\perp_{\nu\beta} = 0$] and are added here in order to symmetrize the tensor $\tau^{\mu\nu}$ and to satisfy the condition (63), see below. Using Eqs. (72) and (74), one obtains

$$\begin{aligned}-u^{\mu}\partial_{\mu}\varepsilon_{\text{vortex}} &= u^{\nu}F_{\mu\nu}\partial_{\alpha}(\Gamma O^{\mu\alpha}) \\ &\quad - \partial_{\mu}[u^{\nu}(\Gamma O^{\mu\alpha}F_{\nu\alpha} - \Gamma u^{\mu}u^{\gamma}\perp_{\nu\beta}F^{\alpha\beta}F_{\alpha\gamma})] \\ &\quad + \partial_{\mu}u^{\nu}(\Gamma O^{\mu\alpha}F_{\nu\alpha} - \Gamma u^{\mu}u^{\gamma}\perp_{\nu\beta}F^{\alpha\beta}F_{\alpha\gamma}) \\ &= u^{\nu}F_{\mu\nu}\partial_{\alpha}(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta}) \\ &\quad - \partial_{\mu}[u_{\nu}(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma})] \\ &\quad + \partial_{\mu}u_{\nu}(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma}),\end{aligned} \tag{75}$$

where in the second equality we make use of the definition (70) for $O^{\alpha\beta}$. Returning now to the entropy generation equation (65), one can present it in the form

$$\begin{aligned}&T\partial_{\mu}(Su^{\mu}) \\ &= u^{\nu}F_{\mu\nu}[Yw^{\mu} + \partial_{\alpha}(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})] \\ &\quad - \partial_{\mu}[u_{\nu}(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma} - \tau^{\mu\nu})] \\ &\quad + \partial_{\mu}u_{\nu}(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma} - \tau^{\mu\nu})\end{aligned} \tag{76}$$

or

$$\begin{aligned}&\partial_{\mu}(Su^{\mu}) \\ &= \frac{u^{\nu}F_{\mu\nu}}{T}[Yw^{\mu} + \partial_{\alpha}(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})] \\ &\quad - \partial_{\mu}\left[\frac{u_{\nu}}{T}(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma} - \tau^{\mu\nu})\right] \\ &\quad + \partial_{\mu}\left(\frac{u_{\nu}}{T}\right)(\Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma} - \tau^{\mu\nu}).\end{aligned} \tag{77}$$

Neglecting dissipation, the right-hand side of this equation should be a four-divergence at arbitrary $\partial_{\mu}u_{\nu}$, $\partial_{\mu}T$, w^{μ}, Γ, etc. This allows us to find[15]

$$u^{\nu}F_{\mu\nu}[Yw^{\mu} + \partial_{\alpha}(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})] = 0 \quad \text{and} \tag{78}$$

$$\tau^{\mu\nu} = \tau^{\mu\nu}_{\text{vortex}} = \Gamma\perp_{\delta\alpha}F^{\mu\delta}F^{\nu\alpha} - \Gamma u^{\mu}u^{\nu}u^{\gamma}u_{\beta}F^{\alpha\beta}F_{\alpha\gamma}. \tag{79}$$

The first of these equations is similar to a nondissipative version, $u^{\nu}Yw^{\mu}F_{\mu\nu} = 0$, of the condition (45), analyzed in the previous section. It will clearly give us a (nondissipative) superfluid equation generalized to the case when the terms depending on $\Gamma = \lambda/(mH)$ [see Eq. (69)] cannot be neglected. A more general form of this equation will be discussed a little bit later.

The second of these equations, Eq. (79), is the vortex energy-momentum tensor $\tau^{\mu\nu}_{\text{vortex}}$. As it should be, it is symmetric and satisfies the condition (63). Moreover, in the nonrelativistic limit (when $u^0 \approx 1$ and $u^i \ll 1$) its time components τ^{i0} coincide with the energy-density current $\boldsymbol{q}$ (see equation 16.35 in the monograph by Khalatnikov [3]), while its spatial components coincide with the nonrelativistic vortex stress tensor [the last term in the right-hand side of Eq. (A12)]. To demonstrate the latter property it is instructive to rewrite Eq. (79) in terms of the vector H^{μ}. One can verify that

$$\begin{aligned}\tau^{\mu\nu}_{\text{vortex}} &= \Gamma H^2 g^{\mu\nu} - \Gamma H^{\mu}H^{\nu} \\ &\quad + \Gamma H_{\delta}(\mathfrak{F}^{\nu\delta}u^{\mu} + \mathfrak{F}^{\mu\delta}u^{\nu} - H^{\delta}u^{\mu}u^{\nu}),\end{aligned} \tag{80}$$

where $H = (H_{\mu}H^{\mu})^{1/2}$ [see also Eq. (55)] and $\mathfrak{F}^{\mu\nu} = 1/2\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta}$ is the tensor dual to the vorticity tensor $F^{\mu\nu}$. In the nonrelativistic limit the spatial part of this tensor equals $\tau^{ik}_{\text{vortex}} \approx \Gamma(H^2\delta^{ik} - H^iH^k)$ $[i,\ k = 1,\ 2,\ 3]$ and

[15] Unfortunately, detailed analysis shows that Eqs. (78) and (79) do not follow unambiguously from Eq. (77). To obtain them unambiguously one needs to require, in addition, that the spatial components τ^{ij} of the tensor $\tau^{\mu\nu}$ are independent of the components F_{0i} of the vorticity tensor *in the comoving frame* (i.e., τ^{ij} there depend on $\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_s$ only). This additional assumption is confirmed by the results of independent microscopic consideration (see Appendix E and Remark 1 below in this section).

indeed reduces to the nonrelativistic expression [see Eq. (A12)], because in this limit $\boldsymbol{H} \approx m\,\mathrm{curl}\,\boldsymbol{V}_\mathrm{s} = m\boldsymbol{\omega}$ in the laboratory frame and $\Gamma = \lambda/(mH) = \lambda/(m^2\omega)$.

Now, if we allow for the dissipation in the system, $\tau^{\mu\nu}$ will acquire a dissipative correction $\tau^{\mu\nu}_\mathrm{diss}$, so that $\tau^{\mu\nu} = \tau^{\mu\nu}_\mathrm{vortex} + \tau^{\mu\nu}_\mathrm{diss}$ and Eq. (77) can be rewritten as

$$\partial_\mu S^\mu = \frac{u^\nu F_{\mu\nu}}{T}[Yw^\mu + \partial_\alpha(\Gamma \perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})] - \partial_\mu\left(\frac{u_\nu}{T}\right)\tau^{\mu\nu}_\mathrm{diss}, \tag{81}$$

where $S^\mu \equiv Su^\mu - u_\nu \tau^{\mu\nu}_\mathrm{diss}/T$ is the entropy current density. Following the consideration of Sec. III A, let us introduce the four-vectors $f^\mu \equiv u^\nu F_{\mu\nu}/(\mu n)$ [cf. Eq. (46)] and W^μ,[16]

$$W^\mu \equiv \frac{1}{n}[Yw^\mu + \partial_\alpha(\Gamma \perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})], \tag{82}$$

and assume that, in the comoving frame [$u^\mu = (1,0,0,0)$], the vector f^μ depends only on $\boldsymbol{W}$ and $F_{\mu\nu}$ [see the corresponding discussion after Eq. (48) in Sec. III A]. Then, positive definiteness of the right-hand side of Eq. (81) means independent satisfaction of two conditions,

$$W^\mu f_\mu \geq 0, \quad \text{and} \tag{83}$$

$$-\frac{1}{T}\partial_\mu u_\nu \tau^{\mu\nu}_\mathrm{diss} + \frac{1}{T^2}\partial_\mu T u_\nu \tau^{\mu\nu}_\mathrm{diss} \geq 0. \tag{84}$$

The first condition allows us to determine f^μ, which has the same form as in Eq. (54), but with W^μ given by Eq. (82).[17] With this new f^μ, the superfluid equation acquires the same form (56) as in the previous section [note also that Remark 2 of Sec. III A remains fully applicable as well]. The second condition allows us to specify the dissipative correction $\tau^{\mu\nu}_\mathrm{diss}$. This correction can be found in the same way as it was done in Ref. [35]; it includes standard viscous and thermal conduction terms, and is presented (together with the viscous correction $\varkappa_\mathrm{diss}$) in Appendix B, where a complete set of relativistic HVBK-equations is given.

The hydrodynamic equations obtained here fully describe dynamics of superfluid liquid in the system with vortices and are equivalent, in the nonrelativistic limit, to the ordinary HVBK-hydrodynamics (see Appendix C).

Remark 1.—There is another, less general, way of deriving the tensor $\tau^{\mu\nu}_\mathrm{vortex}$ by direct averaging of the "microscopic" tensor $T^{\mu\nu}$ [see Eq. (36)] over a volume containing large amount of vortices. It can be shown that the results of both approaches coincide (see, in particular, Appendix E, where the spatial part of the tensor $\tau^{\mu\nu}_\mathrm{vortex}$ is obtained in this way).

Remark 2.—Zero-temperature limit of the hydrodynamics described above can be obtained if we put $T = 0$, $S = 0$, and $Y = n/\mu$ (the latter condition is the relativistic analogue of the condition $\rho_\mathrm{s} = \rho$ valid at $T = 0$). Since there are no thermal excitations at $T = 0$ (except in the vortex cores), we also need to specify what we mean by "the normal-liquid velocity" u^μ, which does not have a direct physical meaning in this limit. In the nonrelativistic theory the correct superfluid equation valid at $T = 0$ will be obtained if we put $\boldsymbol{V}_\mathrm{n} = \boldsymbol{V}_\mathrm{s} + (1/\rho)\mathrm{curl}\,\lambda\mathbf{e}$ (see Appendix A, where the same notations are used). This velocity coincides with the vortex velocity $\boldsymbol{V}_\mathrm{L}$ [see Eq. (A18)]. The relativistic generalization of this expression can be written as

$$u^\mu = \frac{m}{\mu}V^\mu_{(\mathrm{s})} + \frac{1}{n}\perp^\mu_\nu \partial_\alpha(\Gamma\perp^{\nu\gamma}\perp^{\alpha\delta}F_{\gamma\delta}), \tag{85}$$

which should be considered as an *implicit*[18] definition of u^μ. It satisfies the three conditions:

(i) First, it is easily checked that with this definition u^μ is correctly normalized, $u_\mu u^\mu = -1$.

(ii) Second, one can demonstrate that, with the definition (85) one has $u^\nu W^\mu F_{\mu\nu} = 0$ [see Eq. (82) where W^μ is defined], i.e., the system entropy remains constant [see Eq. (77) with $\tau^{\mu\nu} = \tau^{\mu\nu}_\mathrm{vortex}$].

(iii) Finally, one can verify that the right-hand side of the superfluid equation (56) vanishes in view of the expression (85), which implies

$$u^\nu F_{\mu\nu} = 0. \tag{86}$$

One sees (see Remark 3 in Sec. III A) that, as in the nonrelativistic case, the vortex velocity coincides with u^μ at $T = 0$. Formula (86) is the new superfluid equation valid at $T = 0$; u^μ in this equation can (in principle) be found by solving Eq. (85) and should be considered as a function of $V^\mu_{(\mathrm{s})}$.

Remark 3.—It can be shown, that the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$, which is a superfluous equation in the system with the only one independent velocity field $V^\mu_{(\mathrm{s})}$, is automatically satisfied provided that (86) holds true. The resulting system of zero-temperature relativistic HVBK-equations is thus self-consistent.

Remark 4.—It would be interesting to compare the zero-temperature version of the relativistic HVBK-hydrodynamics discussed here with the results available in the literature. However, as it is argued in Appendix F,

[16]It is interesting to note that the "current density" $\tilde{j}^\mu$ defined as $\tilde{j}^\mu \equiv nu^\mu + nW^\mu = j^\mu + \partial_\alpha(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta})$ is conserved, $\partial_\mu \tilde{j}^\mu = 0$, because $\partial_\mu\partial_\alpha(\Gamma\perp^{\mu\gamma}\perp^{\alpha\delta}F_{\gamma\delta}) = 0$ due to antisymmetry of $F_{\gamma\delta}$.

[17]Clark's analysis [9] of the nonrelativistic HVBK-hydrodynamics shows that, generally, there can be six independent kinetic coefficients instead of three coefficients α, β, and γ, introduced in Ref. [9]. The same consideration applies also to our expression for f^μ, which is not the most general one (but equivalent to that of Ref. [3], see Appendix C).

[18]It is implicit because the right-hand side of Eq. (85) also depends on u^μ.

we have strong concerns about self-consistency/validity of the existing formulations [18,19] of such hydrodynamics. Thus, no such comparison will be made in the present paper.

IV. CONCLUSIONS

We have generalized the nonrelativistic Hall-Vinen-Bekarevich-Khalatnikov (HVBK) hydrodynamics [1,6] to the relativistic case. The corresponding equations are summarized in Appendix B. The main difference of the proposed hydrodynamics from the formulations of Refs. [18–20] is that it accounts for the presence of thermal excitations (i.e., is valid at $T \neq 0$) and allows for the interaction between the normal and superfluid liquid components (mutual friction).

As a by-product of our work we demonstrate that the previous zero-temperature formulations of the relativistic vortex hydrodynamics [18,19] are internally inconsistent (see Appendix F) and should be modified.

The most natural application of the relativistic HVBK-hydrodynamics formulated here is to neutron stars, which are relativistic objects whose cores are composed of various baryon species (neutrons, protons, etc.) that can be in superfluid/superconducting state. However, to directly apply this hydrodynamics to neutron stars one should first generalize it to the case of superfluid mixtures as well as to allow for the possible presence of the magnetic field and the related topological defects—Abrikosov vortices. These issues were successively addressed, in the nonrelativistic framework, in Refs. [51–54]. The relativistic generalization of the corresponding equations to the superfluid/superconducting mixtures without an external magnetic field is straightforward; the formulation of the full system of magneto-hydrodynamic equations is more complicated. We continue to work in this direction and hope to present the first results soon.

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APPENDIX A: HVBK-HYDRODYNAMICS

We present here the main equations of dissipative Hall-Vinen-Bekarevich-Khalatnikov hydrodynamics. We refer to Refs. [1,3,5–7,10] for more details. Hydrodynamic equations in the presence of vortices take the form (i, $k = 1$, 2, 3)

$$\partial_t \rho + \mathrm{div}\boldsymbol{j} = 0, \tag{A1}$$

$$\partial_t j^i + \partial_k \Pi^{ik} = 0, \tag{A2}$$

$$\partial_t S + \mathrm{div} S\boldsymbol{V}_\mathrm{n} = \frac{R}{T}, \tag{A3}$$

$$\partial_t \boldsymbol{V}_\mathrm{s} + (\boldsymbol{V}_\mathrm{s}\boldsymbol{\nabla})\boldsymbol{V}_\mathrm{s} + \boldsymbol{\nabla}\left(\breve{\mu} - \frac{1}{2}|\boldsymbol{V}_\mathrm{s} - \boldsymbol{V}_\mathrm{n}|^2\right) = \boldsymbol{F}, \tag{A4}$$

$$dE_0 = \breve{\mu} d\rho + TdS + \rho_\mathrm{s}(\boldsymbol{V}_\mathrm{s} - \boldsymbol{V}_\mathrm{n})d(\boldsymbol{V}_\mathrm{s} - \boldsymbol{V}_\mathrm{n}) + dE_\mathrm{vortex}, \tag{A5}$$

and consist of, respectively, continuity equation, momentum conservation, entropy generation equation, superfluid equation, and the second law of thermodynamics. Here $\rho = mn$ is the density; m is the particle mass; ρ_s is the superfluid density; $\boldsymbol{j}$ is the mass current density; Π^{ik} is the stress tensor; $\boldsymbol{V}_\mathrm{n}$ and V_s are the normal and superfluid velocities, respectively; R is the dissipative function; and $\boldsymbol{F}$ is a force to be specified below. Further, $\breve{\mu}$ is the nonrelativistic chemical potential; in the nonrelativistic limit the chemical potential μ, introduced in Sec. II, is related to $\breve{\mu}$ by the formula $\breve{\mu} = (\mu - mc^2)/m$; E_0 is the nonrelativistic energy density as measured in the inertial frame moving with the velocity $\boldsymbol{V}_\mathrm{n}$. Finally, the last term in Eq. (A5) is responsible for the vortex contribution to the energy density and is approximately given by [3,10]

$$dE_\mathrm{vortex} = \hat{E}_\mathrm{vortex} dN_\mathrm{vortex}, \tag{A6}$$

where

$$\hat{E}_\mathrm{vortex} = \rho_\mathrm{s}\frac{\varkappa^2}{4\pi}\ln\frac{b}{a} \tag{A7}$$

is the vortex (kinetic) energy per unit length and

$$N_\mathrm{vortex} = \frac{\omega}{\varkappa} \tag{A8}$$

is the area density of vortices. In Eqs. (A7) and (A8) $\boldsymbol{\omega} \equiv \mathrm{curl}\, \boldsymbol{V}_\mathrm{s}$; $\varkappa = 2\pi\hbar/(sm)$ ($s = 1$ for Bose- and $s = 2$ for Fermi-superfluids); a is the radius of a vortex core; $b = 1/(\pi N_\mathrm{vortex})^{1/2} = \varkappa^{1/2}/(\pi\omega)^{1/2}$ is the quantity of the order of the intervortex distance. Taking into account these definitions, Eq. (A6) can be rewritten as

$$dE_\mathrm{vortex} = \lambda d\omega, \tag{A9}$$

where

$$\lambda \equiv \rho_\mathrm{s}\frac{\varkappa}{4\pi}\ln\frac{b}{a} = \rho_\mathrm{s}\frac{\varkappa}{4\pi}\ln\frac{\varkappa^{1/2}}{a\pi^{1/2}\omega^{1/2}}. \tag{A10}$$

Hydrodynamic Eqs. (A1)–(A5) should be supplemented by the expressions for $\boldsymbol{j}$, Π^{ik}, $\boldsymbol{F}$, and R. Ignoring the thermal diffusivity and viscosity terms (which have the standard form, like in the vortex-free case [1,3]), one has

$$\boldsymbol{j} = \rho_{\rm s}\boldsymbol{V}_{\rm s} + \rho_{\rm n}\boldsymbol{V}_{\rm n}, \tag{A11}$$

$$\Pi^{ik} = P\delta^{ik} + \rho_{\rm s}V_{\rm s}^iV_{\rm s}^k + \rho_{\rm n}V_{\rm n}^iV_{\rm n}^k + \left(\lambda\omega\delta^{ik} - \lambda\frac{\omega^i\omega^k}{\omega}\right), \tag{A12}$$

$$\begin{aligned}\boldsymbol{F} = &-\boldsymbol{\omega}\times(\boldsymbol{V}_{\rm n}-\boldsymbol{V}_{\rm s}) + \alpha\boldsymbol{\omega}\times(\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e})\\ &+\beta\mathbf{e}\times[\boldsymbol{\omega}\times(\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e})]\\ &-\gamma\mathbf{e}[\boldsymbol{\omega}(\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e})],\end{aligned} \tag{A13}$$

$$R = -[\boldsymbol{F} + \boldsymbol{\omega}\times(\boldsymbol{V}_{\rm n}-\boldsymbol{V}_{\rm s})](\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e}). \tag{A14}$$

Here $\rho_{\rm n} = \rho - \rho_{\rm s}$ is the normal density; $\mathbf{e} \equiv \boldsymbol{\omega}/\omega$ is the unit vector along $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{V}_{\rm s}$; $P = -E_0 + \breve{\mu}\rho + TS$ is the pressure; α, β, and γ are kinetic coefficients describing interaction of vortices with the normal liquid component (mutual friction). The term in Eq. (A13), depending on α, is nondissipative, as opposed to the terms proportional to β and γ. The coefficients β and γ should be positive in order for the dissipative function R to be positive-definite, $\beta > 0$ and $\gamma > 0$. HVBK-equations, described above, deserve a few remarks.

Remark 1.—Following [4,10,45], the second law of thermodynamics (A5) is written in a reference frame where the normal liquid component is at rest, $\boldsymbol{V}_{\rm n} = 0$. This is in contrast with Refs. [1,3] where it is written in a reference frame of a superfluid component, $\boldsymbol{V}_{\rm s} = 0$ (see, e.g., Ref. [10] for a more detailed discussion). As a result, definitions of chemical potential and energy density are slightly different in Refs. [1,3]. Namely, it can be shown that their chemical potential $\breve{\mu}_{\rm Kh}$ and the energy density $E_{0\rm Kh}$ are related to our $\breve{\mu}$ and E_0 by the formulas [10]

$$\breve{\mu}_{\rm Kh} = \breve{\mu} - \frac{1}{2}(\boldsymbol{V}_{\rm s}-\boldsymbol{V}_{\rm n})^2, \tag{A15}$$

$$E_{0\rm Kh} = E_0 + \frac{1}{2}\rho(\boldsymbol{V}_{\rm s}-\boldsymbol{V}_{\rm n})^2 - \rho_{\rm s}(\boldsymbol{V}_{\rm s}-\boldsymbol{V}_{\rm n})^2. \tag{A16}$$

At the same time, it is easy to verify that the pressure in both approaches is the same, $P_{\rm Kh} = P$. Because the superfluid equation (A4) depends on $\breve{\mu}$, it (formally) differs from the corresponding equation of Refs. [1,3], which is expressed through $\breve{\mu}_{\rm Kh}$.

Remark 2.—As follows from Eqs. (A7) and (A10), $\hat{E}_{\rm vortex} \to 0$ and $\lambda \to 0$ at $\hbar \to 0$. In this limit, corresponding to a continuously distributed vorticity (like in the ordinary nonsuperfluid hydrodynamics), contribution of vortices to the total energy and momentum of the liquid can be neglected [but the "mutual friction" terms in Eq. (A4), depending on α, β, and γ, will generally survive]. The situation when one can set $\lambda = 0$ in all equations described above is common; the hydrodynamic equations in this limit are often used, e.g., in modeling superfluid dynamics of rotating neutron stars [13,14].

Remark 3.—In the absence of a (generally weak) longitudinal force, $\gamma = 0$, superfluid equation (A4) can be rewritten in an elegant way [3]. Taking the curl of this equation, one obtains

$$\partial_t\boldsymbol{\omega} = \mathrm{curl}(\boldsymbol{V}_{\rm L}\times\boldsymbol{\omega}), \tag{A17}$$

where

$$\begin{aligned}\boldsymbol{V}_{\rm L} = \boldsymbol{V}_{\rm n} &- \alpha(\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e})\\ &-\beta\mathbf{e}\times(\boldsymbol{j}-\rho\boldsymbol{V}_{\rm n}+\mathrm{curl}\,\lambda\mathbf{e}).\end{aligned} \tag{A18}$$

Equation (A17) describes translation of the vector $\boldsymbol{\omega}$ with the velocity of the vortex lines $\boldsymbol{V}_{\rm L}$.

Two extreme regimes of motion of the vortex lines are of interest. Assume that it is possible to neglect the terms depending on λ and β in Eqs. (A17) and (A18). Then, in the *strong-drag regime* $\boldsymbol{V}_{\rm L} = \boldsymbol{V}_{\rm n}$ and vortices are completely entrained by the motion of the normal liquid component. This regime corresponds to $\alpha = 0$. In the *weak-drag regime* the situation is opposite. Interaction with the normal excitations is so weak that vortices move with the superfluid component, $\boldsymbol{V}_{\rm L} = \boldsymbol{V}_{\rm s}$. Equation (A17) then takes the form of a standard vorticity equation of ordinary hydrodynamics,

$$\partial_t\boldsymbol{\omega} = \mathrm{curl}(\boldsymbol{V}_{\rm s}\times\boldsymbol{\omega}). \tag{A19}$$

As follows from Eq. (A18), the weak-drag limit is realized if $\alpha = -1/\rho_{\rm s}$.

APPENDIX B: RELATIVISTIC HVBK-HYDRODYNAMICS: SUMMARY OF RESULTS

Here we present the full system of hydrodynamic equations which reduces to HVBK-hydrodynamics in the nonrelativistic limit. For the reader's convenience, this appendix is made self-contained.

The main ingredients of the relativistic superfluid HVBK-hydrodynamics are the four-velocity of thermal excitations u^μ, normalized by the condition $u_\mu u^\mu = -1$, and the four-vector w^μ, which is defined by Eq. (B11) (see below). This four-vector is orthogonal to u^μ,

$$u_\mu w^\mu = 0, \tag{B1}$$

and is related to the superfluid velocity $V^\mu_{(\rm s)}$ by the formula

$$V^\mu_{(\rm s)} = \frac{w^\mu + (\mu + \varkappa_{\rm diss})u^\mu}{m}, \tag{B2}$$

where m is the bare particle mass; μ is the relativistic chemical potential; and $\varkappa_{\mathrm{diss}}$ is the viscous dissipative correction to be specified below [see Eq. (B14)]. Another important parameter of this hydrodynamics is the vorticity tensor,

$$F_{\mu\nu} = m[\partial_\mu V_{(\mathrm{s})\nu} - \partial_\nu V_{(\mathrm{s})\mu}]. \tag{B3}$$

The relativistic HVBK-hydrodynamics consists of the particle and energy-momentum conservations,

$$\partial_\mu j^\mu = 0, \tag{B4}$$

$$\partial_\mu T^{\mu\nu} = 0, \tag{B5}$$

the second law of thermodynamics [note that all the thermodynamic quantities are measured in the comoving frame, where $u^\mu = (1, 0, 0, 0)$],

$$d\varepsilon = \mu dn + TdS + \frac{Y}{2} d(w_\mu w^\mu) + \frac{\lambda}{m} dH, \tag{B6}$$

and the superfluid equation

$$u^\nu F_{\mu\nu} = \mu n f_\mu. \tag{B7}$$

In Eqs. (B4)–(B7) n, T, and S are the number density, temperature, and entropy density, respectively; Y is the relativistic analogue of the superfluid density [35,44]; λ has the same meaning as the corresponding quantity of the nonrelativistic HVBK-hydrodynamics (see Appendixes A and D); and

$$H = \sqrt{\frac{1}{2} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\mu\nu} F_{\eta\sigma}}, \tag{B8}$$

where $\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. Further, f^μ equals

$$\begin{aligned} f^\mu = {}& \alpha \perp^{\mu\nu} F_{\nu\lambda} W_\delta \perp^{\lambda\delta} + \frac{\beta - \gamma}{H} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\eta\sigma} F_{\lambda\nu} W_\delta \perp^{\lambda\delta} \\ & + \gamma H W_\delta \perp^{\mu\delta}, \end{aligned} \tag{B9}$$

where α, β, and γ are the mutual friction parameters (the same as in the nonrelativistic HVBK-hydrodynamics, see Appendix A); and

$$W^\mu \equiv \frac{1}{n} [Y w^\mu + \partial_\alpha (\Gamma \perp^{\mu\gamma} \perp^{\alpha\delta} F_{\gamma\delta})] \tag{B10}$$

with $\Gamma \equiv \lambda/(mH)$.

It remains to specify the particle current density j^μ and the energy-momentum tensor $T^{\mu\nu}$ in Eqs. (B4) and (B5),

$$j^\mu = n u^\mu + Y w^\mu, \tag{B11}$$

$$\begin{aligned} T^{\mu\nu} = {}& (P + \varepsilon) u^\mu u^\nu + P g^{\mu\nu} + Y(w^\mu w^\nu + \mu u^\mu w^\nu + \mu u^\nu w^\mu) \\ & + \tau^{\mu\nu}_{\mathrm{vortex}} + \tau^{\mu\nu}_{\mathrm{diss}}. \end{aligned} \tag{B12}$$

Here $P = -\varepsilon + \mu n + TS$ is the pressure; and $\tau^{\mu\nu}_{\mathrm{vortex}}$ is the vortex contribution to $T^{\mu\nu}$,

$$\tau^{\mu\nu}_{\mathrm{vortex}} = \Gamma \perp_{\delta\alpha} F^{\mu\delta} F^{\nu\alpha} - \Gamma u^\mu u^\nu u^\gamma u_\beta F^{\alpha\beta} F_{\alpha\gamma}. \tag{B13}$$

Finally, the dissipative corrections $\varkappa_{\mathrm{diss}}$ and $\tau^{\mu\nu}_{\mathrm{diss}}$ in Eqs. (B2) and (B12) are given by [35]

$$\varkappa_{\mathrm{diss}} = -\xi_3 \partial_\mu (Y w^\mu) - \xi_4 \partial_\mu u^\mu, \tag{B14}$$

$$\begin{aligned} \tau^{\mu\nu}_{\mathrm{diss}} = {}& -\kappa(\perp^{\mu\gamma} u^\nu + \perp^{\nu\gamma} u^\mu)(\partial_\gamma T + T u^\delta \partial_\delta u_\gamma) \\ & - \eta \perp^{\mu\gamma} \perp^{\nu\delta} \left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} g_{\gamma\delta} \partial_\varepsilon u^\varepsilon \right) \\ & - \xi_1 \perp^{\mu\nu} \partial_\gamma (Y w^\gamma) - \xi_2 \perp^{\mu\nu} \partial_\gamma u^\gamma. \end{aligned} \tag{B15}$$

In these equations κ and η are, respectively, the thermal conductivity and shear viscosity coefficients; $\xi_1, \ldots, \xi_4$ are the bulk viscosity coefficients ($\xi_1 = \xi_4$; $\xi_1^2 \leq \xi_2 \xi_3$; κ, η, ξ_2, $\xi_3 \geq 0$).

APPENDIX C: SUPERFLUID EQUATION IN THE NONRELATIVISTIC LIMIT

Equations of the relativistic HVBK-hydrodynamics are summarized in Appendix B. Our aim here will be to demonstrate that the "superfluid" equation [Eq. (B7)] of this hydrodynamics reduces to its nonrelativistic counterpart (A4) in the nonrelativistic limit. In what follows we use dimensional units. In these units Eq. (B7) becomes

$$u^\nu F_{\mu\nu} = \frac{\mu n}{c^3} f_\mu. \tag{C1}$$

Spatial components of Eq. (C1) can be rewritten as (i, $j = 1, 2, 3$)

$$u^0 F_{i0} + u^j F_{ij} = \frac{\mu n}{c^3} f_i, \tag{C2}$$

or, in view of (B3),

$$m u^0 [\partial_i V_{(\mathrm{s})0} - \partial_0 V_{(\mathrm{s})i}] + m u^j [\partial_i V_{(\mathrm{s})j} - \partial_j V_{(\mathrm{s})i}] = \frac{\mu n}{c^3} f_i, \quad \text{or} \tag{C3}$$

$$\partial_i V_{(\mathrm{s})0} = \partial_0 V_{(\mathrm{s})i} - \frac{u^j}{u^0} [\partial_i V_{(\mathrm{s})j} - \partial_j V_{(\mathrm{s})i}] + \frac{\mu n}{m c^3 u^0} f_i. \tag{C4}$$

On the other hand, it follows from the orthogonality condition (B1), that

$$u^\mu V_{(\mathrm{s})\mu} = -\frac{\mu}{mc} \Rightarrow V_{(\mathrm{s})0} = -\frac{\mu}{m c u^0} - \frac{u^j}{u^0} V_{(\mathrm{s})j}, \tag{C5}$$

where we made use of the definition (B2), which takes the form (in dimensional units and neglecting the dissipative correction $\varkappa_{\rm diss}$)

$$V^{\mu}_{(\rm s)} = \frac{w^{\mu} + \mu u^{\mu}}{mc}. \tag{C6}$$

Substituting Eq. (C5) into (C4), one obtains

$$\partial_0 V_{({\rm s})i} - \frac{u^j}{u^0}[\partial_i V_{({\rm s})j} - \partial_j V_{({\rm s})i}] + \frac{\mu n}{mc^3 u^0} f_i = -\partial_i\left(\frac{\mu}{mcu^0}\right) - \partial_i\left(\frac{u^j}{u^0} V_{({\rm s})j}\right). \tag{C7}$$

Now, introducing the nonrelativistic chemical potential $\breve{\mu} = (\mu - mc^2)/m$ (see Appendix A) and taking into account that u^{μ} is expressed through the velocity $\boldsymbol{V}_{\rm n}$ of the normal component as

$$u^{\mu} = \left(\frac{1}{\sqrt{1-\boldsymbol{V}_{\rm n}^2/c^2}}, \frac{\boldsymbol{V}_{\rm n}}{c\sqrt{1-\boldsymbol{V}_{\rm n}^2/c^2}}\right), \tag{C8}$$

one arrives at the following equation, valid at $|\boldsymbol{V}_{\rm n}|, |\boldsymbol{V}_s| \ll c$ [we recall that $\boldsymbol{V}_{\rm s} \equiv (V^1_{(\rm s)}, V^2_{(\rm s)}, V^3_{(\rm s)})$],

$$\partial_t \boldsymbol{V}_{\rm s} + {\rm curl}\,\boldsymbol{V}_{\rm s} \times \boldsymbol{V}_{\rm n} + \boldsymbol{\nabla}\left[\breve{\mu} - \frac{1}{2}\boldsymbol{V}_{\rm n}^2 + \boldsymbol{V}_{\rm n}\boldsymbol{V}_{\rm s}\right] = -\frac{\mu n}{mc^2} f^i, \tag{C9}$$

or, taking into account that $\boldsymbol{\nabla}(\boldsymbol{V}_{\rm s}^2)/2 = (\boldsymbol{V}_{\rm s}\boldsymbol{\nabla})\boldsymbol{V}_{\rm s} - {\rm curl}\,\boldsymbol{V}_{\rm s} \times \boldsymbol{V}_{\rm s}$ and $\mu \approx mc^2$,

$$\partial_t \boldsymbol{V}_{\rm s} + (\boldsymbol{V}_{\rm s}\boldsymbol{\nabla})\boldsymbol{V}_{\rm s} + \boldsymbol{\nabla}\left[\breve{\mu} - \frac{1}{2}|\boldsymbol{V}_{\rm s} - \boldsymbol{V}_{\rm n}|^2\right] = -{\rm curl}\,\boldsymbol{V}_{\rm s} \times (\boldsymbol{V}_{\rm n} - \boldsymbol{V}_{\rm s}) - n f^i. \tag{C10}$$

This equation is very similar to Eq. (A4), but to draw a final conclusion we need also to analyze the spatial part f^i of the four-vector f^{μ} [see Eq. (B9)]. In the nonrelativistic limit it is given by Eq. (50),

$$\boldsymbol{f} = -\alpha m[{\rm curl}\,\boldsymbol{V}_{\rm s} \times \boldsymbol{W}] - \beta m \mathbf{e} \times [{\rm curl}\,\boldsymbol{V}_{\rm s} \times \boldsymbol{W}] + \gamma m \mathbf{e}(\boldsymbol{W}\,{\rm curl}\,\boldsymbol{V}_{\rm s}), \tag{C11}$$

where $\boldsymbol{W}$ is the spatial part of the four-vector W^{μ} [see Eq. (B10)], which is, in the dimensional form,

$$W^{\mu} = \frac{1}{n}[cYw^{\mu} + \partial_{\alpha}(\Gamma \perp^{\mu\gamma} \perp^{\alpha\delta} F_{\gamma\delta})]. \tag{C12}$$

The spatial component of the first term here equals, in the nonrelativistic limit, $cYw^{\mu} = \rho_{\rm s}(\boldsymbol{V}_{\rm s} - \boldsymbol{V}_{\rm n})/m$ (see Eq. (C6) and note that $Y = \rho_{\rm s}/(m^2c^2)$ at $c \to \infty$ [35,44]).

The last term in Eq. (C12), which equals $\partial_{\alpha}(\Gamma O^{\mu\alpha})$, can be rewritten as [see Eq. (70) for the definition of $O^{\mu\alpha}$ and the footnote 14]

$$\partial_{\alpha}(\Gamma O^{\mu\alpha}) = \partial_{\alpha}\left(\frac{\Gamma}{2}\epsilon^{\delta\eta\mu\alpha} u_{\eta} \epsilon_{\delta abc} u^a F^{bc}\right). \tag{C13}$$

In the nonrelativistic limit the only terms here that survive are those with $\eta = 0$ and $a = 0$. Because both $\partial_{\alpha} u_0$ and $\partial_{\alpha} u^0$ are of the order of $1/c^2$ (see Eq. (28), one may treat u_{η} and u^a in Eq. (C13) as constants ($u^0 \approx 1$ and $u_0 \approx -1$). In this way one finds ($i = 1, 2, 3$),

$$\partial_{\alpha}(\Gamma O^{i\alpha}) = \epsilon^{i\alpha\delta}\partial_{\alpha}\left(\Gamma \frac{1}{2}\epsilon^{\delta bc} F^{bc}\right) = m\,{\rm curl}(\Gamma\,{\rm curl}\,\boldsymbol{V}_{\rm s}) = \frac{{\rm curl}(\lambda \mathbf{e})}{m}, \tag{C14}$$

where we employed Eq. (69), and used the fact that $H = m|{\rm curl}\,\boldsymbol{V}_{\rm s}|$ [see Eq. (51)]. Returning then to the vector $\boldsymbol{W}$, one can write

$$\boldsymbol{W} = \frac{1}{mn}[\rho_{\rm s}(\boldsymbol{V}_{\rm s} - \boldsymbol{V}_{\rm n}) + {\rm curl}(\lambda \mathbf{e})]. \tag{C15}$$

Substituting now Eqs. (C11) and (C15) into Eq. (C10), one verifies that it coincides with the superfluid equation (A4) of nonrelativistic HVBK-hydrodynamics.

APPENDIX D: ENERGY OF A RELATIVISTIC VORTEX AND THE EXPRESSION FOR $d\varepsilon_{\rm vortex}$

Let us consider a homogeneous system without vortices. We assume that all the thermodynamic parameters, as well as the velocities of normal u^{μ} and superfluid $V^{\mu}_{({\rm s})0} = \partial^{\mu}\phi_0/m$ components are constants in time and space. All the quantities related to this (unperturbed) system will be denoted by the subscript "0".

In what follows we shall work in the coordinate frame in which $u^{\mu} = (1,0,0,0)$ [hereafter, the normal-liquid coordinate frame]. In that frame the quantity $w^{\mu}_{(0)} = \partial^{\mu}\phi_0 - \mu_0 u^{\mu}$ can be represented as $w^{\mu}_{(0)} = (0, \partial^i\phi_0)$ on account of Eq. (19) [$i = 1, 2, 3$ is the spatial index]. Correspondingly, the energy density $T^{00}_{(0)}$ is given simply by $\varepsilon_0 = \varepsilon_0(S_0, n_0, w_{\mu(0)} w^{\mu}_{(0)})$ [see Eq. (36)], which is generally a function of the entropy density S_0, the number density n_0, and the scalar $w_{\mu(0)} w^{\mu}_{(0)}$.

Now let us *adiabatically* perturb the system by creating a straight vortex, assuming that the total number of particles remains unchanged. Denoting the correction due to the vortex as $\phi_{\rm V}$, one finds for the perturbed system (in the normal-liquid frame): $w^{\mu} = (0, \partial^i\phi_0 + \partial^i\phi_{\rm V})$ and $T^{00} = \varepsilon(n, S, w_{\mu}w^{\mu})$. The vortex energy can be defined as the difference between the energies of the perturbed and unperturbed systems,

$$E_{\rm V} = \int dV(T^{00} - T^{00}_{(0)}) = \int dV[\varepsilon(S, n, w_\mu w^\mu) - \varepsilon_0(S_0, n_0, w_{\mu(0)} w^\mu_{(0)})], \quad \text{(D1)}$$

where the integration is performed over the system volume V. As it will be clear from the subsequent consideration, the main contribution to $E_{\rm V}$ comes from the region far from the vortex, where $S(\boldsymbol{r})$, $n(\boldsymbol{r})$, and $w_\mu(\boldsymbol{r}) w^\mu(\boldsymbol{r})$ only weakly deviate from, respectively, S_0, n_0, and $w_{\mu(0)} w^\mu_{(0)}$. Consequently, one can expand the function under the integral in Eq. (D1) and present $E_{\rm V}$ as

$$\begin{aligned} E_{\rm V} &\approx \int dV \frac{\partial\varepsilon}{\partial S}[S(\boldsymbol{r}) - S_0] + \int dV \frac{\partial\varepsilon}{\partial n}[n(\boldsymbol{r}) - n_0] \\ &\quad + \int dV \frac{\partial\varepsilon}{\partial(w_\mu w^\mu)}[w^\mu w_\mu - w^\mu_{(0)} w_{\mu(0)}] \\ &\approx T_0 \int dV[S(\boldsymbol{r}) - S_0] + \mu_0 \int dV[n(\boldsymbol{r}) - n_0] \\ &\quad + \frac{Y_0}{2}\int dV[w^\mu w_\mu - w^\mu_{(0)} w_{\mu(0)}], \end{aligned} \quad \text{(D2)}$$

where in the second equality use has been made of the second law of thermodynamics (37). Since the total entropy and particle number in the perturbed and unperturbed systems are the same by construction,[19] the first two integrals vanish, so that

$$\begin{aligned} E_{\rm V} &\approx \frac{Y_0}{2}\int dV[w^\mu w_\mu - w^\mu_{(0)} w_{\mu(0)}] \\ &= \frac{Y_0}{2}\int dV[\partial^i\phi_{\rm V}\partial_i\phi_{\rm V} + 2\partial^i\phi_{\rm V}\partial_i\phi_0]. \end{aligned} \quad \text{(D3)}$$

Because $\partial_i\phi_0$ is constant and $\partial^i\phi_{\rm V}$ is symmetric [see Eq. (D6) below], the contribution into the integral from the second term in Eq. (D3) vanishes and we finally arrive at the following formula for $E_{\rm V}$,

$$E_{\rm V} \approx \frac{Y_0}{2}\int dV \partial^i\phi_{\rm V}\partial_i\phi_{\rm V}. \quad \text{(D4)}$$

In the nonrelativistic limit (D4) reduces to the standard expression for the vortex energy,

$$E_{\rm V} = \frac{\rho_{s0}}{2}\int dV \boldsymbol{V}^2_{s\rm V}, \quad \text{(D5)}$$

[19]This is not strictly true because formula (D2) does not include integration over the volume in the immediate vicinity of the vortex core, where the hydrodynamic approach is not applicable. However, the entropy and the number of particles contained in that volume are small (proportional to the radius a of the vortex core squared), hence their contribution to the total entropy and particle number can be neglected.

if we note that the superfluid velocity induced by the vortex, $\boldsymbol{V}_{s\rm V}$, is related to the scalar $\phi_{\rm V}$ by the condition $\boldsymbol{V}_{s\rm V} = \boldsymbol{\nabla}\phi_{\rm V}/m$ and that in the nonrelativistic limit the superfluid density $\rho_{s0} = m^2 Y_0$.

To take an integral in Eq. (D4) one needs to specify $\partial^i\phi_{\rm V}$. If the straight vortex is at rest in the normal-liquid frame and $\partial^i\phi_0 = 0$ then, as follows from the symmetry arguments [see Eq. (41)], it induces the velocity field given simply by[20]

$$\partial^i\phi_{\rm V} = \frac{\boldsymbol{e}_\varphi}{sr}, \quad \text{(D6)}$$

where $\boldsymbol{e}_\varphi$ is the unit vector in the azimuthal direction (φ is the polar angle) and $s = 1$ or 2 is the quantity defined after Eq. (41). In reality, however, we deal with a nonstationary problem: the vortex can move with some velocity in the normal-liquid frame and $\partial^i\phi_0$ does not necessary vanish. In the nonrelativistic theory it is argued (e.g., Ref. [3]) that Eq. (D6) remains a good approximation for $\partial^i\phi_{\rm V}$ even in this case. The latter result can be extended to the fully relativistic case if we assume that the background superfluid velocity $\partial^i\phi_0$ (and hence the vortex velocity) are much smaller than the speed of light c in *the normal-liquid frame*. In an arbitrary frame this requirement means that the difference between the spatial components of normal and background superfluid velocities should be much smaller than the speed of light c. This condition is not very restrictive and, for example, is satisfied in the superfluid matter of neutron stars, where superfluidity is destroyed long before the velocity difference becomes comparable to c [56].

Substituting (D6) into (D4) and performing an integration, one arrives at the following expression for the vortex energy per unit length (we suppress the subscript “0” from here on),

$$\hat{E}_{\rm V} = \frac{\pi Y}{s^2}\ln\frac{b}{a}, \quad \text{(D7)}$$

where a is the radius of the vortex core and b is an “external” radius of the order of the intervortex spacing (as in the nonrelativistic theory). The radius b is related to the number of vortices $N_{\rm V}$ per unit area by the standard formula (cf. Ref. [3]),

$$\pi b^2 = \frac{1}{N_{\rm V}}. \quad \text{(D8)}$$

On the other hand, as follows from Eq. (41) and the Stokes' theorem [see also Eq. (42)], $N_{\rm V}$ is related to the smooth-averaged vorticity (defined in the normal-liquid frame) by the expression

[20]To obtain Eq. (D6) we choose a circle in 3D centered at the vortex line as the integration contour in Eq. (41).

$$N_{\rm V} = \frac{sm|\varepsilon^{ijk}\partial_j V_{({\rm s})k}|}{2\pi}. \tag{D9}$$

To obtain this formula an integration is performed over the surface whose boundary is the contour specified in the footnote 20.

Using Eqs. (D8) and (D9), one obtains the following expression for the vortex energy density $\varepsilon_{\rm vortex}$,

$$\varepsilon_{\rm vortex} = \frac{\hat{E}_{\rm V}}{\pi b^2} = \frac{mY}{2s}\ln\left(\frac{b}{a}\right)|\varepsilon^{ijk}\partial_j V_{({\rm s})k}| \equiv \lambda|\varepsilon^{ijk}\partial_j V_{({\rm s})k}|, \tag{D10}$$

where we introduced the parameter $\lambda \equiv [mY/(2s)]\ln(b/a)$, which only weakly (logarithmically) depends on b (and, as a consequence, weakly depends on $|\varepsilon^{ijk}\partial_j V_{({\rm s})k}|$). Note that $\varepsilon_{\rm vortex}$ is the quantity which is determined, by definition, in the normal-liquid frame. It is thus a Lorentz-invariant quantity and it is useful to rewrite it in an explicitly Lorentz-invariant form. One can do this with the help of the four-vector H^μ [see Eq. (51)],

$$\varepsilon_{\rm vortex} = \frac{\lambda}{m}\sqrt{H^\mu H_\mu}. \tag{D11}$$

Consequently, the differential of this energy density due to a variation of H^μ is given by

$$d\varepsilon_{\rm vortex} = \frac{\lambda}{2mH}d(H_\mu H^\mu), \tag{D12}$$

where $H = \sqrt{H^\mu H_\mu}$ and we neglected, as in the nonrelativistic theory, the dependence of λ on H^μ. This formula coincides with the expression (67) for $d\varepsilon_{\rm vortex}$ used in the text.

Remark 1.—Presence of vortices not only adds an additional term (D12) to the second law of thermodynamics (64) but also renormalizes the particle chemical potential μ, which is now approximately given by [see Eq. (D11)]

$$\mu = \mu_{\rm old} + \frac{\partial\varepsilon_{\rm vortex}}{\partial n} = \mu_{\rm old} + \frac{H}{m}\frac{\partial\lambda}{\partial n}, \tag{D13}$$

where $\mu_{\rm old}$ is the chemical potential in the absence of vortices.

APPENDIX E: SPATIAL PART OF THE TENSOR $\tau^{\mu\nu}_{\rm vortex}$ FROM THE MICROSCOPIC AVERAGING PROCEDURE

Here we briefly demonstrate how to obtain the spatial part of the tensor $\tau^{\mu\nu}_{\rm vortex}$ from the "microscopic" tensor $T^{\mu\nu}$ [see Eq. (36)]. In the comoving frame [i.e., in the frame in which $u^\mu = (1, 0, 0, 0)$] the spatial components of the tensor $T^{\mu\nu}$ equal

$$T^{ij} = Pg^{ij} + Yw^i w^j = Pg^{ij} + Y\partial^i\phi\partial^j\phi. \tag{E1}$$

In the system with vortices $\phi = \phi_0 + \phi_{\rm V}$ (the notations are the same as in Appendix D). Assume that we have a bunch of vortices with locally constant density, which are directed along the axis z. Let us introduce a notion of the "Wigner-Seitz cell"—a cylinder of radius b surrounding each vortex line. We then average the tensor T^{ij} out over one such Wigner-Seitz cell. Since we neglect interaction between vortices, the neighboring vortices "do not interfere" when averaging Eq. (E1). The result can be written as

$$\langle T^{ij}\rangle = \frac{1}{\pi b^2\Delta z}\int dV T^{ij} = \langle P\rangle g^{ij} + \langle Y\rangle\partial^i\phi_0\partial^j\phi_0 + \langle Y\partial^i\phi_{\rm V}\partial^j\phi_{\rm V}\rangle, \tag{E2}$$

where angle brackets mean averaging over the Wigner-Seitz cell; $\Delta z \sim b$ is a height of cylinder (the actual value of Δz is not important); and dV is the volume element. Note that the main contribution to $\langle Y\rangle$ comes from the region far from the vortex core, where Y can be considered as constant (the dependence of Y on $\partial^i\phi_{\rm V}$ is weak). This means that $\langle Y\rangle \approx Y_0$, where Y_0 is the value of Y at a distance $\sim b$ from the vortex (or, equivalently, the value of Y in the system without vortices; see Appendix D). Similarly, one can also replace Y with Y_0 when taking other averages.

The cross-terms $\langle Y\partial^i\phi_0\partial^j\phi_{\rm V}\rangle$ and $\langle Y\partial^j\phi_0\partial^i\phi_{\rm V}\rangle$ in Eq. (E2) vanish on account of the symmetry of the problem. Clearly, the only "interesting" (nonstandard) contribution to $\langle T^{\mu\nu}\rangle$ comes from the last term in Eq. (E2), which can be identified as the vortex tensor, i.e., $\tau^{ij}_{\rm vortex} = \langle Y\partial^i\phi_{\rm V}\partial^j\phi_{\rm V}\rangle$. To find this tensor, let us write $\partial^i\phi_{\rm V}$ in Cartesian coordinates (x, y, z) using Eq. (D6),

$$\partial^x\phi_{\rm V} = -\frac{\sin\varphi}{sr}, \tag{E3}$$

$$\partial^y\phi_{\rm V} = \frac{\cos\varphi}{sr}, \tag{E4}$$

$$\partial^z\phi_{\rm V} = 0, \tag{E5}$$

where the axes x and y are located in the plane perpendicular to the axis z (and all the three axes cross at the vortex line). Using Eqs. (E3)–(E5) it is easily verified that the only nonzero components of the vortex tensor are $\tau^{xx}_{\rm vortex}$ and $\tau^{yy}_{\rm vortex}$; they are given by

$$\tau^{xx}_{\rm vortex} = \tau^{yy}_{\rm vortex} = \langle Y\partial^x\phi_{\rm V}\partial^x\phi_{\rm V}\rangle = \frac{1}{\pi b^2\Delta z}\int dz\, rdr\, d\varphi Y\frac{\sin^2\varphi}{s^2r^2} \approx \frac{Y_0}{s^2b^2}\ln\frac{b}{a} = \lambda|\epsilon^{ijk}\partial_j V_{({\rm s})k}|, \tag{E6}$$

where λ is defined by the same formula as in Appendix D and we used Eqs. (D8) and (D9) to obtain the last equality. Making 3D rotation, $\tau^{ij}_{\rm vortex}$ can generally be presented as ($\boldsymbol{H} = m\,\mathrm{curl}\,\boldsymbol{V}_{\rm s}$)

$$\tau^{ij}_{\rm vortex} = \frac{\lambda}{m} H - \frac{\lambda}{m} \frac{H^i H^j}{H}. \quad \text{(E7)}$$

This tensor exactly coincides with the spatial part of the tensor $\tau^{\mu\nu}_{\rm vortex}$, written in the comoving frame and presented in Sec. III B [see Eq. (80) there]. Interestingly, the same tensor $\tau^{ij}_{\rm vortex}$ can be determined from the purely thermodynamic arguments following the method of Ref. [57].

APPENDIX F: INCONSISTENCY OF THE ZERO-TEMPERATURE VORTEX HYDRODYNAMICS OF REFS. [18,19]

Here we shall demonstrate that the vortex hydrodynamics of Ref. [18] is internally inconsistent and hence the energy-momentum tensor $T^{\mu\nu}$ of that hydrodynamics should be modified. Since Ref. [19] obtained the same expression[21] for $T^{\mu\nu}$ (although some of its other equations are different due to some unexplained reason), it suffers from the same inconsistency problem. Thus, we shall not discuss Ref. [19] in what follows. The notations used in this section differ from those adopted in other parts of the paper and coincide with the notations of Ref. [18].

Let us consider the formula (77) of Ref. [18]. It gives the energy-momentum tensor $T^{\mu\nu}$ for the superfluid liquid with the distributed vorticity at $T = 0$. This tensor can be written as

$$T^\mu_\nu = \frac{c^2}{\mu_0} \frac{\partial \Phi}{\partial \mu_0} v^\mu v_\nu + \frac{\omega^\mu \omega_\nu}{\omega} \frac{\partial \Phi}{\partial \omega} - \left(\Phi - \omega \frac{\partial \Phi}{\partial \omega} \right) \delta^\mu_\nu, \quad \text{(F1)}$$

where μ_0 is the invariant chemical potential; Φ is the invariant pressure; v^μ is the superfluid velocity normalized by the condition $v_\mu v^\mu = \mu_0^2/c^2$ (see Eq. (54) of Ref. [18]); and δ^μ_ν is the Kronecker symbol. Finally, $\omega = \sqrt{-\omega_\mu \omega^\mu}$, where the four-vector ω^μ is the generalization of curl $\boldsymbol{V}_{\rm s}$ to the relativistic case; it is given by the formula (74) of Ref. [18]. Below we assume that the metric is flat and equals $g_{\mu\nu} = (c^2, -1, -1, -1)$ (see a formula after Eq. (27) in Ref. [18]).

Consider a tensor $T^{\mu\nu}$ in the coordinate frame in which $v^\mu = (\mu_0/c^2, 0, 0, 0)$. In this frame the time component of the four-vector ω^0 vanishes, $\omega^0 = 0$ (see Eq. (74) of Ref. [18]), hence the energy density ε, given by the component T^0_0 of the tensor T^μ_ν, equals

$$\begin{aligned} \varepsilon &= T^0_0 \\ &= \frac{c^2}{\mu_0} \frac{\partial \Phi}{\partial \mu_0} \frac{\mu_0^2}{c^2} - \left(\Phi - \omega \frac{\partial \Phi}{\partial \omega} \right) \\ &= -\Phi + \mu_0 \frac{\partial \Phi}{\partial \mu_0} + \omega \frac{\partial \Phi}{\partial \omega}. \end{aligned} \quad \text{(F2)}$$

This expression can be rewritten if one introduces the mass density $\rho = m_0 n$, where m_0 is the mass of a free particle and n is the number density. As follows from the formula (76) of Ref. [18] for j^μ (j^μ is the density of the mass 4-flux), in the chosen coordinate frame $\rho = j^0 = \partial \Phi / \partial \mu_0$, thus equation (F2) can be represented as

$$\varepsilon = -\Phi + \mu_0 \rho + \omega \frac{\partial \Phi}{\partial \omega}. \quad \text{(F3)}$$

This formula seems to be incorrect (contradicts other equations of Ref. [18]). The simplest way to demonstrate this is to look at the nonrelativistic limit of the vortex hydrodynamics of Ref. [18]. In this limit Eq. (F3) should reduce to the corresponding nonrelativistic expression for the energy density if we set to zero the superfluid velocity $\boldsymbol{V}_{\rm s}$ in the latter expression (in other words, we consider a point in space in which $\boldsymbol{V}_{\rm s} = 0$ at some particular moment of time).

The nonrelativistic expression for the energy density was obtained in Ref. [1] and is presented in the monograph by Khalatnikov [3] on p. 101,

$$E_0 = -P + TS + \breve{\mu}_{\rm Kh} \rho + (\boldsymbol{V}_{\rm n} - \boldsymbol{V}_{\rm s}) \boldsymbol{j}_0, \quad \text{(F4)}$$

where E_0 is the nonrelativistic energy density measured in the frame in which $\boldsymbol{V}_{\rm s} = 0$ (not to be confused with E_0 from Appendix A); P is the pressure defined as in Ref. [3]; $\breve{\mu}_{\rm Kh}$ is the nonrelativistic chemical potential, which equals $\breve{\mu}_{\rm Kh} = \partial E_0 / \partial \rho$ (in the monograph by Khalatnikov [3] this potential is denoted by μ); $j_0 = \rho_{\rm n}(\boldsymbol{V}_{\rm n} - \boldsymbol{V}_{\rm s})$; $\rho_{\rm n}$ is the normal density. In our case we have $T = 0$, thus $\boldsymbol{j}_0 = 0$ and (F4) can be rewritten as

$$E_0 = -P + \breve{\mu}_{\rm Kh} \rho. \quad \text{(F5)}$$

The nonrelativistic energy E_0 and the pressure P are related to their relativistic counterparts by the formulas

$$\begin{aligned} \varepsilon &= E_0 + \rho c^2, \\ \Phi &= P, \end{aligned} \quad \text{(F6)}$$

where the last equality is needed to reproduce the correct nonrelativistic stress tensor Π^{ik} from Eq. (F1). Moreover, μ_0 can be presented as $\mu_0 = c^2 + \delta\mu_0$, where $\delta\mu_0$ is a small correction. As a result, one obtains that the formula (F3) transforms to the form

[21] See Appendix B of that reference.

$$E_0 = -P + \delta\mu_0\rho + \omega\frac{\partial\Phi}{\partial\omega}. \tag{F7}$$

Comparing (F5) and (F7) one sees that

$$\delta\mu_0 = \frac{1}{\rho}\left(\breve{\mu}_{\rm Kh}\rho - \omega\frac{\partial\Phi}{\partial\omega}\right), \tag{F8}$$

i.e., $\delta\mu_0 \neq \breve{\mu}_{\rm Kh}$. In other words, the "invariant" chemical potential μ_0 is not simply given by the partial derivative $\partial\varepsilon/\partial\rho$, where ε is the energy density measured in the frame in which $v^\mu = (\mu_0/c^2, 0, 0, 0)$. This is a strange result (in which frame is then μ_0 specified as a derivative of the energy density with respect to the mass density?) that contradicts, in particular, the nonrelativistic superfluid equation (80) presented in Ref. [18]. In order to make the Eq. (80) of Ref. [18] compatible with the corresponding equation of nonrelativistic HVBK-hydrodynamics (written for a point in space in which $\boldsymbol{V}_{\rm s} = 0$ at some particular moment of time; see the equation 16.40 of the monograph [3] by Khalatnikov with $\rho_s = \rho$ and $\beta' = \beta = \gamma = 0$), it is necessary to have $\nabla v_0 = \nabla\breve{\mu}_{\rm Kh}$, i.e., $\delta\mu_0 = \breve{\mu}_{\rm Kh}$ (since in the chosen reference frame $v_0 = \mu_0 = c^2 + \delta\mu_0$), in contradiction with (F8).

We come to conclusion that the vortex hydrodynamics of Ref. [18] (and hence Ref. [19]) is internally inconsistent: Eq. (F3) is not correct (the last term in its right-hand side is superfluous), which means that the energy-momentum tensor (F1) should be modified.

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