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Rotating compact bodies with a disk surface layer

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The Senovilla family for a subclass of Petrov type-D stationary axisymmetric differentially rotating perfect fluids is considered. A scheme is presented to construct from a solution an interior of a rotating compact body satisfying dominant energy conditions and with a boundary of vanishing pressure. The equatorial disk of the body is a surface layer due to a jump in the second fundamental form. However, unlike previous results, the body is free from curvature singularities.

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I. INTRODUCTION

Many years ago, Senovilla [1] introduced a simple formulation for the problem of a Petrov type-D stationary axisymmetric differentially rotating perfect fluid, with the 4-velocity lying in the 2-planes spanned by the two principal null directions of the Weyl tensor and with the vanishing magnetic part of the Weyl tensor with respect to the fluid 4-velocity. A detailed qualitative study of this family [2] has shown that, by clever interpretation of the coordinates, solutions could represent interiors of compact rotating bodies with various configurations. However, all configurations have two curvature singularities located at the axis of symmetry. This paper presents a scheme for constructing, from members of the family, compact rotating bodies that are free from those curvature singularities. The scheme avoids the singularities by restricting the solution to two regular bounded regions of the spacetime. The two, apparently separate, regions are glued together across a hypersurface Σ that is to be identified as the equatorial disk of the body. The first fundamental form is continuous at Σ. However, the second fundamental form has a jump there, and hence Σ is a surface layer. By applying an appropriate transformation to cylindrical-like coordinates, the solution could be interpreted as an interior of a differentially rotating spheroid with positive pressure throughout and with a boundary of vanishing pressure. In Sec. II, features of Senovilla's family are outlined. In Sec. III, the scheme for constructing compact bodies is presented. In Sec. IV, properties of the surface layer are derived. Section V gives explicit forms and properties of the subfamily of rigidly rotating solutions. In Sec. VI, the scheme is applied to construct rigidly rotating compact bodies. The paper ends with some concluding remarks.

II. SENOVILLA'S FAMILY OF DIFFERENTIALLY ROTATING SOLUTIONS

In Senovilla's formulation, the metric is given by [1]

$$\mathrm{d}s^2 = \frac{1}{N^2}\left(\mathrm{d}y^2 + \frac{\mathrm{d}x^2}{mh+s^2} + h\mathrm{d}\phi^2 - 2s\mathrm{d}t\mathrm{d}\phi - m\mathrm{d}t^2\right), \tag{1}$$

where h, m, and s are functions of the radial coordinate x and N is a function of the axial coordinate y. There exist two commuting Killing vectors: ∂_t and ∂_ϕ. For $h, m > 0$, and hence $mh + s^2 > 0$, ∂_t is timelike, ∂_ϕ is spacelike, and the signature is $+2$.

Without loss of generality, the axis of symmetry may be set at $x = 0$, where regularity requires [2]

$$h(0) = 0, \qquad s(0) = 0, \qquad m(0) > 0. \tag{2}$$

The singularity of g_{xx} at the axis is just a coordinate singularity that can be removed by the gauge $x \to x^2$.

The requirement of positive h in a region about the axis of symmetry implies

$$h'(0) \geq 0, \tag{3}$$

where a prime denotes differentiation with respect to x.

The condition of elementary flatness at the axis reduces to

$$\lim_{x\to 0} mh'^2 = 4. \tag{4}$$

For a perfect fluid satisfying dominant energy conditions, the field equations imply

$$N = \cosh ay, \qquad a = \text{const} > 0, \tag{5}$$

where a constant of integration has been set to 1. The solution is symmetric with respect to the plane $y = 0$.

Some of the field equations reduce to a pair of nonlinear differential equations:

$$m''h'' + s''^2 = 0, \tag{6}$$

$$(mh + s^2)'' + 4a^2 = m'h' + s'^2. \tag{7}$$

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This system has two equations in three functions, and hence an additional assumption is needed to obtain a solution, as emphasized in Ref. [1].

It has been shown in Ref. [2] that the formula for the 4-velocity and the field equations imply

$$h'' < 0, \qquad m'' > 0. \tag{8}$$

The remaining field equations (with $8\pi G = c = 1$) determine the angular velocity ω, the pressure p, and the density μ by the formulas

$$\omega = -m''/s'', \tag{9}$$

$$p = \frac{1}{4}[(m'h' + s'^2 + 4a^2)\cosh^2 ay - 12a^2], \tag{10}$$

$$\mu = p + 6a^2. \tag{11}$$

In general, ω is a function of x (differential rotation). Equation (11) is the fluid's equation of state, which satisfies the energy condition $\mu > p$.

Using Eq. (7) in Eq. (10), the pressure takes the form

$$p = \frac{1}{4}[(mh + s^2)'' + 8a^2]\cosh^2 ay - 3a^2. \tag{12}$$

The pressure is monotonically increasing with $|y|$, going to ∞ as $|y| \to \infty$. On the other hand, it is monotonically increasing (decreasing) with x when $(mh + s^2)'''$ is positive (negative).

The equipressure surfaces with $p = p_0 = \text{const}$ are given by

$$\cosh^2 ay = \frac{4(p_0 + 3a^2)}{(mh + s^2)'' + 8a^2}. \tag{13}$$

In general, the pressure takes positive, zero, and negative values.

The limit surface, where the pressure vanishes, is given by

$$B_\pm\colon\ y = \pm f(x) \coloneqq \pm\frac{1}{a}\cosh^{-1}\sqrt{\frac{12a^2}{(mh + s^2)'' + 8a^2}}. \tag{14}$$

Then, $B = B_+ \cup B_-$ exists on some interval $x \geq 0$ in either of two qualitatively different cases:

(i) Case 1:

$$(mh + s^2)''|_{x=0} < 4a^2 \quad \text{and} \quad (mh + s^2)'''|_{x=0} > 0, \tag{15}$$

in which case $f(x)$ is decreasing with x, vanishing when $(mh + s^2)'' = 4a^2$.

(ii) Case 2:

$$(mh + s^2)''|_{x=0} \leq 4a^2 \quad \text{and} \quad (mh + s^2)'''|_{x=0} < 0, \tag{16}$$

in which case $f(x)$ is increasing with x.

In either case, $B_\pm$ intersects the axis of symmetry at $y = \pm y_0 \coloneqq \pm f(0)$. However, in case 2, when $(mh + s^2)''|_{x=0} = 4a^2$, the limit surface passes through the origin $x = y = 0$.

III. CONSTRUCTION OF DIFFERENTIALLY ROTATING COMPACT BODIES

By using the gauge $\sin(aY) = \tanh(ay)$, with Y interpreted as an angle in $[-\pi/2a, \pi/2a]$, it has been shown [2] that solutions of Senovilla's family could represent interiors of compact rotating bodies with various configurations. However, each configuration has two curvature singularities at the two points $Y = \pm\pi/2a$ located, respectively, at the north and south poles of the body.

Now, we suggest a scheme for constructing a rotating compact body not containing the above two singular points. The coordinates x and y will be interpreted as markers related to the coordinate radial distances from, respectively, the axis of symmetry and the *center* of the body. First, we restrict x to an interval,

$$0 \leq x \leq x_\mathrm{a}, \tag{17}$$

where the signature of the metric is maintained and a limit surface exists. Second, we restrict y to extend from the limit surface B_+ up and from B_- down, where the pressure is positive. By this step, we avoid regions of negative pressure, which appear as holes in some configurations in Ref. [2]. Third, we specify a finite positive pressure p_c at the center $(0, y_\mathrm{c})$ where, using Eq. (13),

$$y_\mathrm{c} = \frac{1}{a}\cosh^{-1} 2\sqrt{\frac{p_\mathrm{c} + 3a^2}{(mh + s^2)'' + 8a^2}} > y_0. \tag{18}$$

Then, we restrict the solution to a region $I = I_+ \cup I_-$ consisting of the two disjoint regions (with $0 \leq \phi < 2\pi$)

$$I_+\colon\ 0 \leq x \leq x_\mathrm{a}, \qquad f(x) \leq y \leq u(x), \tag{19}$$

$$I_-\colon\ 0 \leq x \leq x_\mathrm{a}, \qquad -u(x) < y \leq -f(x). \tag{20}$$

The function $u(x)$ is a smooth monotonic function on its domain $[0, x_\mathrm{a}]$, the graph of which passes through the two points $(0, y_\mathrm{c})$ and $(x_\mathrm{a}, f(x_\mathrm{a}))$, and hence lies above that of $f(x)$. Thus, u is required to satisfy the conditions

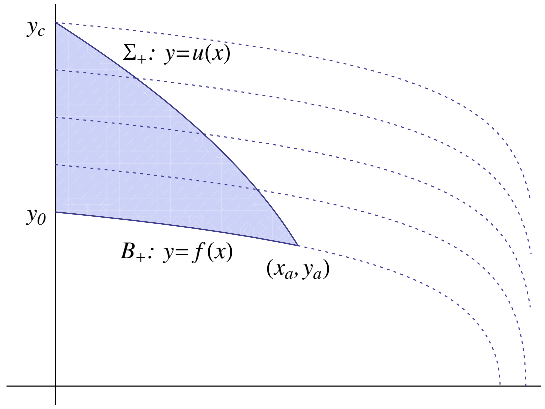

FIG. 1. The region I_+ in case 1.

$$\begin{aligned} u(0) &= y_c, \\ u(x_a) &= y_a \coloneqq f(x_a), \\ u(x) &> f(x) \quad \forall\ x \in (0, x_a). \end{aligned} \tag{21}$$

The compact body will be constructed, around the axis, by joining the two regions $I_\pm$ at the surface $\Sigma_\pm : y = \pm u(x)$, which will be identified as the *equatorial disk*. The body will be bounded by the limit surfaces $B_\pm : y = \pm f(x)$, which will be connected at the *equator* $x = x_a, y = \pm y_a$. Smoothness of the boundary requires that $B_\pm$ and $\Sigma_\pm$ be orthogonal at the equator. Thus, besides Eq. (21), u is required to satisfy the additional condition

$$u'(x_a) = -\frac{1}{f'(x_a)(mh + s^2)(x_a)}. \tag{22}$$

Otherwise, u is arbitrary.

Figures 1 and 2 show a section of the region I_+ at $\phi = \text{const}$, for a solution of cases 1 and 2 respectively. The dotted curves are the traces of some equipressure surfaces. The region I_- is the reflection of I_+ in the hyperplane $y = 0$.

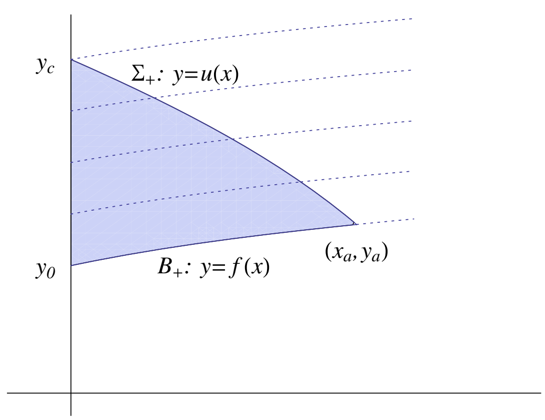

FIG. 2. The region I_+ in case 2.

Finally, the solution in I can be cast in cylindrical-like coordinates (r, z, ϕ) by applying an appropriate coordinate transformation that maps the surface $y = u(x)$ onto the equatorial disk $z = 0$, for example,

$$r^2 = x, \qquad z = \pm\sqrt{u(x) - y}, \qquad (x, y, \phi) \in I_\pm, \tag{23}$$

whereby I_+ and I_- are mapped, respectively, onto the upper $(z \geq 0)$ and lower $(z < 0)$ halves of the (connected) spheroidal region (with $0 \leq \phi < 2\pi$)

$$I' : \ y_0 - y_c \leq f(r^2) - u(r^2) + z^2 \leq 0. \tag{24}$$

Now, we examine whether the rotating body is oblate. The proper equatorial radius is

$$R_e = \int_0^{x_a} N^{-1} \left[u'(x) + \frac{1}{\sqrt{mh + s^2}} \right] dx, \tag{25}$$

and the proper polar radius is

$$R_p = \int_{y_0}^{y_c} N^{-1} dy. \tag{26}$$

Unless $u(x) = y_c = y_a = \text{const}$, $u(x)$ is invertible, and the equatorial disk may be given by $x = U(y)$, in which case

$$R_e = \text{sgn}(y_c - y_a) \int_{y_a}^{y_c} N^{-1} \left[1 + \frac{(dU/dy)^2}{[mh + s^2]_{x=U(y)}} \right]^{1/2} dy.$$

If the solution has a limit surface of case 1, then

$$y_c > y_0 = f(0) > y_a,$$

which implies $R_e > R_p$, and hence the body is oblate.

On the other hand, if the solution has a limit surface of case 2, then

$$y_0 < y_a \quad \text{and} \quad y_0 < y_c, \quad \text{but} \quad y_c <=> y_a.$$

Hence, the body may be oblate or prolate, depending on the specific solution.

IV. SURFACE LAYER AT THE EQUATORIAL DISK

With the two hypersurfaces $\Sigma_\pm : y = \pm u(x)$ now identified as the hypersurface gluing the two regions $I_\pm$, we have to examine junction conditions. The metric tensor is obviously continuous across $\Sigma_\pm$. On the other hand, the extrinsic curvature tensors on $\Sigma_\pm$, imbedded in the spacetime of Eq. (1), are given by

$$K^\pm_{ij} = -\nabla_b n^\pm_a \frac{\partial x^a_\pm}{\partial \xi^i} \frac{\partial x^b_\pm}{\partial \xi^j}, \tag{27}$$

where $x^a_\pm = (x, \pm u(x), \phi, t), \xi^i = (x, \phi, t)$, and $n^\pm_a$ are the unit normals pointing, respectively, into I_+ and out of I_-, given by

$$n^\pm_a = \frac{1}{\cosh au\sqrt{1+u'^2(mh+s^2)}}(\pm u', -1, 0, 0).$$

The nonzero components of the extrinsic curvature tensors reduce to the formulas

$$\begin{aligned} K^\pm_{xx} &= \mp\frac{2a\sqrt{1+u'^2(mh+s^2)}\tanh au}{2(mh+s^2)\cosh au} \\ &\mp\frac{2u''(mh+s^2)+u'(mh+s^2)'}{2(mh+s^2)\cosh au\sqrt{1+u'^2(mh+s^2)}} \\ K^\pm_{\phi\phi} &= \mp\frac{2ah\tanh au + h'u'(mh+s^2)}{2\cosh au\sqrt{1+u'^2(mh+s^2)}} \\ K^\pm_{\phi t} &= \pm\frac{2as\tanh au + s'u'(mh+s^2)}{2\cosh au\sqrt{1+u'^2(mh+s^2)}} = K^\pm_{t\phi} \\ K^\pm_{tt} &= \pm\frac{2am\tanh au + m'u'(mh+s^2)}{2\cosh au\sqrt{1+u'^2(mh+s^2)}}. \end{aligned} \tag{28}$$

As expected by symmetry, $K^\pm_{ij}$ are opposite in sign so that $\mathbf{K}$ is continuous if and only if it vanishes. Even with $u(x)$ arbitrary, the condition $\mathbf{K} = \mathbf{0}$ leads to an overdetermined problem. Hence, the extrinsic curvature tensor is, in general, discontinuous, and $\Sigma_\pm$ is a surface layer [3]. The jump

$$[K_{ij}] = K^+_{ij} - K^-_{ij} = 2K^+_{ij}$$

determines the surface stress-energy tensor through the Lanczos equation

$$-S_{ij} = [K_{ij}] - g^3_{ij}\mathrm{Tr}[\mathbf{K}], \tag{29}$$

where g^3_{ij} is the 3-metric induced on Σ.

Then, the equatorial disk has surface energy density and radial and rotational pressures, given, respectively, by

$$\begin{aligned} \sigma &= \frac{[\psi u'^2]'\cosh au}{u'[1+\psi u'^2]^{3/2}} + \frac{4a\sinh au}{\sqrt{1+\psi u'^2}} \\ &+ \frac{u'[\psi(\psi''+8a^2)]'\cosh au}{(\psi''+8a^2)\sqrt{1+\psi u'^2}}, \end{aligned} \tag{30}$$

$$p_r = -\frac{\psi' u'\cosh au + 4a\sinh au}{\sqrt{1+\psi u'^2}}, \tag{31}$$

$$\begin{aligned} p_\phi &= -\frac{[\psi u'^2]'\cosh au}{u'[1+\psi u'^2]^{3/2}} - \frac{4a\sinh au}{\sqrt{1+\psi u'^2}} \\ &- \frac{u'(\psi' h - \psi h')\cosh au}{h\sqrt{1+\psi u'^2}}, \end{aligned} \tag{32}$$

where $\psi := mh + s^2$.

The matter of the surface layer is then, in general, an anisotropic fluid. This is similar to the Bonnor solution [4] for a rotating dust cloud with a nondust disk surface layer.

Now, to keep zero normal pressure at the boundary of the body, the disk radial pressure, p_r in Eq. (31), should vanish at the equator. Hence, we impose the condition

$$u'(x_\mathrm{a}) = -\frac{4a\tanh ay_\mathrm{a}}{(mh+s^2)'(x_\mathrm{a})}, \tag{33}$$

which, using the last condition in Eq. (21), leads to

$$4a\tanh ay_\mathrm{a} f'(x_\mathrm{a})(mh+s^2)(x_\mathrm{a}) = (mh+s^2)'(x_\mathrm{a}). \tag{34}$$

This equation relates the solution parameters. It is a restriction on the solutions that could be candidates for our scheme.

To apply the scheme to a specific solution, we consider the particular case of rigid rotation.

V. SUBFAMILY OF RIGIDLY ROTATING SOLUTIONS

For the particular case of rigid rotation, $\omega = \text{const}$, by choosing t such that the 4-velocity is proportional to ∂_t, Senovilla [5] found the general solution with no higher symmetry,

$$h = C\ln\left(\frac{x}{B}\right) - 4a^2x, \qquad m = x, \qquad s = s_0 = \text{const},$$
$$C \geq 0, \qquad B > 0. \tag{35}$$

For our purposes, we now obtain the solution in an equivalent form. Equations (6) and (9) give $s'' = \omega h''$ and $m'' = -\omega^2 h''$, which, using Eq. (2), respectively, integrate to

$$s = \omega(h + \gamma x), \qquad m = 1 + \beta x - \omega^2 h, \tag{36}$$

where β and γ are constants and the constant term in m has been set to 1 by a scaling of t.

Substitution from Eq. (36) reduces Eq. (7) to the linear differential equation in h,

$$(1 + bx)h'' + bh' + c = 0, \tag{37}$$

where

$$b := 2\omega^2\gamma + \beta, \qquad c := 4a^2 + \omega^2\gamma^2 > 4a^2. \tag{38}$$

When $b = 0$, Eq. (37) reduces to $h'' = \text{const}$, which leads to the solution given in Ref. [2] with a multitransitive isometry G_4 acting on timelike hypersurfaces. For $b \neq 0$, the general solution of Eq. (37) satisfying the condition (2) is

$$h = -\left(\frac{c}{b}\right)x + \ln(1 + bx), \tag{39}$$

where, taking account of Eq. (8), the second constant of integration has been set to 1 by a scaling of ϕ.

The condition (4) implies

$$c = b(b - 2), \tag{40}$$

and hence, for $c > 4a^2$, either $b > 1 + \sqrt{1 + 4a^2}$ or $b < 1 - \sqrt{1 + 4a^2}$.

Using Eqs. (36), (38), (39), and (40), the general solution of the system (6) and (7), for rigid rotation with no higher symmetry, simplifies to the form

$$\begin{aligned} h &= (2 - b)x + \ln(1 + bx), \\ m &= 1 + [b + \omega^2(b - 2 - 2\gamma)]x - \omega^2 \ln(1 + bx), \\ s &= \omega[(2 - b + \gamma)x + \ln(1 + bx)], \end{aligned} \tag{41}$$

which is valid for the interval

$$\begin{cases} 0 \le x < \infty, & b > 1 + \sqrt{1 + 4a^2}, \\ 0 \le x < -1/b, & b < 1 - \sqrt{1 + 4a^2}. \end{cases} \tag{42}$$

The static limit, when $\omega = 0$, was given by Barnes [6].

The function m is even in ω, and s is odd, as expected. Also, m is positive. However, h is positive and increasing with x only on the interval

$$x < x_{\mathrm{h}} := \frac{2}{b(b - 2)}, \tag{43}$$

which is a subinterval of Eq. (42).

Substituting

$$(mh + s^2)'' = \frac{b^2}{1 + bx} - 8a^2 \tag{44}$$

into Eq. (12) gives the pressure

$$p = \frac{1}{4}\left(\frac{b^2 \cosh^2 ay}{1 + bx} - 12a^2\right), \tag{45}$$

which is increasing with $|y|$ and increasing (decreasing) with x when b is negative (positive).

The equipressure surfaces with $p = p_0 = \mathrm{const}$ are given by

$$\cosh^2 ay = \frac{4}{b^2}(p_0 + 3a^2)(1 + bx). \tag{46}$$

The limit surface is given by

$$B\colon\ x = F(y) := \frac{1}{b}\left(\frac{b^2 \cosh^2 ay}{12a^2} - 1\right)$$

or, equivalently,

$$B\colon\ y = \pm f(x) := \pm \frac{1}{a} \cosh^{-1}\left(\frac{2a}{|b|}\sqrt{3(1 + bx)}\right). \tag{47}$$

Then, B exists on some interval $x \ge 0$ in either of two qualitatively different cases:

$$\text{case 1:}\ -2\sqrt{3}a < b < 0, \tag{48}$$

in which case $f(x)$ is decreasing with x and vanishing when $x = x_1 := (b^2 - 12a^2)/(12a^2 b)$, or

$$\text{case 2:}\ 2 < b \le 2\sqrt{3}a, \tag{49}$$

in which case $f(x)$ is increasing with x.

In either case, the limit surface B intersects the axis of symmetry $x = 0$ at $y = \pm y_0 := \pm(1/a)\cosh^{-1}(2\sqrt{3}a/|b|)$. However, in case 2, when $b = 2\sqrt{3}a$, the limit surface passes through the origin $x = y = 0$.

The solution (5,41) has four parameters, a, b, ω, and γ, which, using Eqs. (40) and (38), should satisfy the conditions

$$4a^2 < b(b - 2), \tag{50}$$

$$b(b - 2) = 4a^2 + \omega^2\gamma^2. \tag{51}$$

VI. CONSTRUCTION OF RIGIDLY ROTATING COMPACT BODIES

Now, we apply the scheme of Sec. III to the rigidly rotating solution (5,41). In the process, we need to adjust the parameters at hand so that all conditions are satisfied. By choosing a solution of case 2, the solution parameters should satisfy the conditions (49–51). On the other hand, the scheme introduces a function $u(x)$ and two parameters $x_{\mathrm{a}}, y_{\mathrm{c}}$ which should satisfy the conditions (21,22,34,43). The condition (43) leads to

$$x_{\mathrm{a}} = \frac{2\epsilon}{b(b - 2)}, \qquad 0 < \epsilon < 1. \tag{52}$$

Using Eq. (41), the condition (34) reduces to

$$(8a^2 + 2b - 2b^2)x_{\mathrm{a}} + b(1 + bx_{\mathrm{a}})\ln(1 + bx_{\mathrm{a}}) = 2, \tag{53}$$

which depends on a, b, ϵ only. To adjust the parameters, we fix ϵ and solve (53) for a in terms of b. Then, we pick a pair $\{a, b\}$ that satisfies the conditions (49,50). That pair determines x_{a} and $\omega\gamma$. Finally, the remaining condition (22) determines y_{c}.

As a specific example, with $y_a = f(x_a)$, let us take

$$u(x) \coloneqq \sqrt{y_c{}^2 - x/\alpha}, \qquad \alpha \coloneqq \frac{x_a}{y_c{}^2 - y_a{}^2} > 0, \tag{54}$$

which satisfies the conditions (21). Then, the condition (22) reduces to

$$(2-b)x_a - 4a^2 x_a{}^2 + b(1+bx_a)\ln(1+bx_a) = \frac{bx_a y_a \sinh y_a \cosh y_a}{3a(y_c^2 - y_a^2)}, \tag{55}$$

which determines y_c. Then, using Eq. (45), $p_c = p(0, y_c)$.

We are now ready to apply the scheme. First, we restrict x to the interval $[0, x_a]$. Then, we restrict the solution to a region $I = I_+ \cup I_-$ consisting of the two disjoint regions (with $0 \le \phi < 2\pi$)

$$I_+\colon\ 0 \le x \le x_a, \qquad f(x) \le y \le u(x), \tag{56}$$

$$I_-\colon\ 0 \le x \le x_a, \qquad -u(x) < y \le -f(x). \tag{57}$$

Finally, we apply the coordinate transformation

$$x = \alpha r^2, \qquad y = \pm\sqrt{y_c{}^2 - r^2 - z^2}, \qquad (x, y, \phi) \in I_\pm, \tag{58}$$

whereby the surface $y = u(x)$ is mapped onto the equatorial disk $z = 0$ and I_+ and I_- are mapped, respectively, onto the upper $(z \ge 0)$ and lower $(z < 0)$ halves of the (connected) spheroidal region (with $0 \le \phi < 2\pi$)

$$I'\colon\ 12a^2 \le -12a^2 b\alpha r^2 + b^2\cosh^2 a\sqrt{y_c{}^2 - r^2 - z^2} \le b^2\cosh^2 a y_c. \tag{59}$$

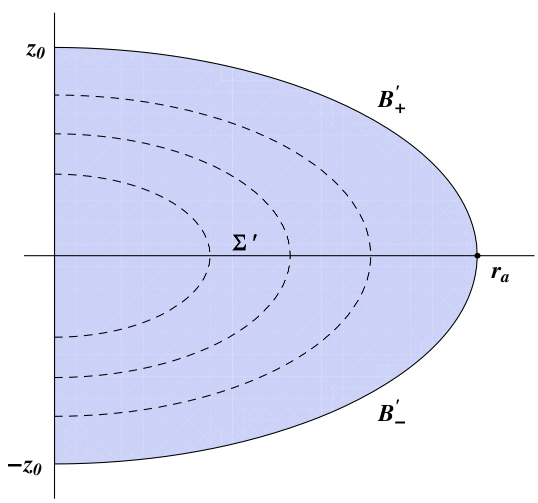

FIG. 3. The image region I'.

Figure 3 shows a section of the image region I' at $\phi = \text{const}$. A prime denotes the image of a surface. The dotted curves are the traces of images of some equipressure surfaces.

Thus, the solution could be interpreted as a compact rigidly rotating body of perfect fluid satisfying dominant energy conditions. The pressure and density are decreasing outward. The boundary of the body is smooth. The body has a surface layer at the equatorial disk. The surface energy density and radial and rotational pressures are given by Eqs. (30)–(32). The radial pressure vanishes at the equator. The constrained parameters ϵ and b determine the other parameters except individual ω and γ. However, the product $\omega\gamma$ is determined so that the angular velocity ω can arbitrarily be specified, with γ determined accordingly.

VII. CONCLUSION

The need for restricting the region of validity of an interior solution may be demonstrated by referring to the well-known interior Schwarzschild solution. Initially, the pressure may be derived in the form

$$p = \mu_0\left(\frac{2\alpha}{3\alpha - \sqrt{1-y}} - 1\right), \qquad 0 \le y \le 1, \qquad 0 \le \alpha \le 1,$$

so that it takes positive, zero, and negative values; it has a singularity at $y = 1 - 9\alpha^2$; and a limit surface exists at $y = 1 - \alpha^2$. With the mass function given, the interior is restricted between the limit surface and the center where the mass vanishes.

Despite the existence of a surface layer, the rotating body is free from curvature singularities. In a sense, it is more physically acceptable than all configurations given in Ref. [2]. Many interior solutions with boundary surface layers have been constructed, for example, as models of rotating neutron stars with surface layer crust [7,8]. However, a few interior solutions with disk surface layers have been obtained, for example, rotating dust clouds [4,9] and a rigidly rotating charged dust cylinder [10].

Besides the rigidly rotating solution, only three explicit solutions within Senovilla's family have been obtained [1,11,12] with various features. However, as shown in Ref. [12], the solution of Ref. [1] has $p < 0$, the one of Ref. [11] has $p > \mu$, and that of Ref. [12] has no limit surface. Only those solutions satisfying dominant energy conditions and having a limit surface may be candidates for the application of the scheme presented above.

It should be stressed that the function $u(x)$ is largely arbitrary. However, no $u(x)$ can be found to make the extrinsic tensor continuous across Σ. Nevertheless, $u(x)$ may be utilized to derive new solutions satisfying some desirable weaker conditions. More precisely, we now have four functions, m, h, s, and u, governed by only two equations (6), (7). Then, we may impose two additional equations, such as the vanishing of the surface pressures (31,32).

ACKNOWLEDGMENTS

This work gives a new interpretation to an old solution. The new interpretation of coordinates has led to an essentially different physical interpretation of the solution.

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