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Dynamical structure of pure Lovelock gravity

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We study the dynamical structure of pure Lovelock gravity in spacetime dimensions higher than four using the Hamiltonian formalism. The action consists of a cosmological constant and a single higher-order polynomial in the Riemann tensor. Similarly to the Einstein-Hilbert action, it possesses a unique constant curvature vacuum and charged black hole solutions. We analyze physical degrees of freedom and local symmetries in this theory. In contrast to the Einstein-Hilbert case, the number of degrees of freedom depends on the background and can vary from zero to the maximal value carried by the Lovelock theory.

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I. INTRODUCTION

Lovelock-Lanczos gravity [1,2] is a natural generalization of general relativity to higher dimensions. It provides the most general gravity action, yielding the second-order field equations in the metric $g_{\mu\nu}(x)$. In a $(d+1)$-dimensional spacetime, the action is given by

$$I[g] = \int d^{d+1}x \sum_{k=0}^{[d/2]} \alpha_k \mathcal{L}_k. \qquad (1.1)$$

Each term in the sum is characterized by the coupling constant α_k multiplied by the dimensionally continued Euler density $\mathcal{L}_k$ of order k in the curvature,

$$\mathcal{L}_k = \frac{1}{2^k}\sqrt{-g}\delta^{\mu_1\cdots\mu_{2k}}_{\nu_1\cdots\nu_{2k}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2k-1}\nu_{2k}}_{\mu_{2k-1}\mu_{2k}}. \qquad (1.2)$$

Here $R^{\alpha}{}_{\beta\mu\nu}$ is the Riemann curvature tensor and $\delta^{\mu_1\cdots\mu_{2k}}_{\nu_1\cdots\nu_{2k}}$ is the totally antisymmetric generalized Kronecker delta of order k defined as the determinant of the $k\times k$ matrix $[\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_k}_{\nu_k}]$. This kind of action, polynomial in curvature, is of significant interest in theoretical physics because it describes a wide variety of models. It has been shown in Refs. [3,4] that, for arbitrary constants α_k, a degeneracy may appear in the space of solutions because the metric is not fully fixed by the field equations. For instance, if the action has nonunique degenerate vacua, then the temporal component g_{tt} of any static spherically symmetric ansatz remains arbitrary [5]. This problem can be avoided by a special choice of the coefficients α_k. The most simple example is given by the Einstein-Hilbert (EH) term alone, which has the unique Minkowski vacuum. Presence of positive or negative cosmological constant terms gives the theory a unique de Sitter (dS) or anti-de Sitter (AdS) vacuum, respectively. Adding the Gauss-Bonnet term with $\alpha_2 \neq 0$ produces two different (A)dS vacua, which can become degenerate for the critical value of the parameter where the two vacua coincide [6].

Another way to fix the coefficients α_k is to have a unique vacuum in the theory that is degenerated, which leads to Chern-Simons gravity in odd dimensions and Born-Infeld gravity in even dimensions [7]. In those theories all couplings are expressed only in terms of the gravitational interaction and the cosmological constant. Also, choosing the coefficients up to a certain order $k = 1, \ldots, [d/2] \equiv N$ leads to a family of nonequivalent theories whose black hole solutions were studied in [8] and also in [9] for the maximal case with $k = N$.

Recently, another possibility has been suggested, where instead of the full Lovelock series, only two terms in the sum are considered in the action: the cosmological constant and a polynomial in the curvature of order p. These pure Lovelock (PL) gravities [10] remarkably admit nondegenerate vacua in even dimensions, while in odd dimensions they have a unique nondegenerate dS and AdS vacuum. Their black hole solutions are asymptotically indistinguishable from the ones appearing in general relativity [5]. This is the case even though the action and equations of motion are free of the linear Einstein-Hilbert term. This similar asymptotic behavior of the two theories seems to also extend to the level of the dynamics and a number of physical degrees of freedom in the bulk.

The properties of PL gravity have been discussed in the literature. The stability of PL black holes has been analyzed in [11]. Application of gauge-gravity duality to phase transitions in quantum field theories dual to pure Gauss-Bonnet AdS gravity was studied in Ref. [12]. It can be shown that in any dimension $d+1$ there is a special power p such that the black hole entropy behaves as in any

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particular lower dimension. In the case of the maximum power, $p = N$, such as the five-dimensional pure Gauss-Bonnet action, the black holes exhibit a peculiar thermodynamical behavior [5,13], where temperature and entropy bear the same relation to horizon radius as in the case for 3D and 4D, respectively. Thermodynamical parameters are thus universal in terms of horizon radius for all odd $D = 2N + 1$ and even $D = 2N + 2$ dimensions.

Dynamical aspects of PL theory were analyzed in Ref. [14] in terms of analogs of the Riemann and Weyl tensors for Nth-order PL gravity. It turns out that it is possible to define an Nth-order Riemann curvature with the property that the trace of its Bianchi derivative yields the same divergence-free (analogue of Einstein tensor) second-rank tensor as the one obtained by the corresponding Lovelock polynomial action. Thus, one can obtain the gravitational equations for PL gravity [15,16] in the same way as one does for the Einstein equations from the Bianchi identity. However, there is one crucial difference, which is that the second Bianchi identity (i.e., vanishing of the Bianchi derivative) is only satisfied by the Riemann tensor and not by its Nth-order analogue. The former therefore has a direct link to the metric, while for the latter this relation is more involved. What yields the divergence-free tensor is the vanishing of the trace of the Bianchi derivative, and not necessarily the derivative itself. From this perspective, PL gravity could be seen as kinematic, which means that the Nth-order Riemann tensor is entirely given in terms of the corresponding Ricci tensor in all critical odd $D = 2N + 1$ dimensions, and it becomes dynamic in the even $D = 2N + 2$ dimensions. This might uncover a universal feature of gravitational dynamics in all critical odd and even dimensions, making it drastically different in critical odd dimensions. More precisely, the PL vacuum is flat with respect to the Nth-order Riemann tensor but not relative to the Riemann tensor. This suggests that there are no dynamical degrees of freedom in the critical odd dimensions relative to the former, but this may not be the case for the latter.

On the other hand, it has been argued in Ref. [17] that the metric Lovelock theory should have the same number of degrees of freedom as the higher-dimensional Einstein-Hilbert gravity, namely, $D(D-3)/2$. This is different than expected from our previous discussion, which suggested fewer physical fields. However, the number of degrees of freedom can change with the backgrounds. For example, Lovelock-Chern-Simons gravity has different numbers of degrees of freedom in different sectors of the phase space [18,19]. Because of the nonlinearity of the theory, the symplectic matrix might have different ranks depending on the background [20], causing more symmetries and less degrees of freedom in some of them, which was explicitly demonstrated in Chern-Simons supergravity [21]. The constraints can also become functionally dependent in certain symmetric backgrounds [22].

Therefore, we provide a detailed analysis of the dynamical structure of PL theory by explicitly performing Hamiltonian analysis and exploring to what extent it is similar to general relativity and whether it exhibits any additional universal features.

II. PURE LOVELOCK GRAVITY

We focus on pure Lovelock gravity of order p in $(d+1)$ dimensions, whose action consists of the unique Lovelock term $\mathcal{L}_p$ and the cosmological constant $\mathcal{L}_0$,

$$I[g] = -\kappa \int d^{d+1}x\sqrt{-g}\left(\frac{1}{2^p}\delta^{\mu_1\cdots\mu_{2p}}_{\nu_1\cdots\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} - 2\Lambda\right), \tag{2.1}$$

where $\alpha_p = -\kappa$ and $\alpha_0 = 2\kappa\Lambda$. The gravitational constant κ has dimension of $(\text{length})^{d+1-2p}$ and the cosmological constant has dimension of $(\text{length})^{-2p}$, and not $(\text{length})^{-2}$ as in general relativity. Varying the action with respect to the metric $g_{\mu\nu}(x)$, one obtains equations of motion in the form

$$^{(p)}G^\mu_\nu + \Lambda\delta^\mu_\nu = 0, \tag{2.2}$$

where $\Lambda = 0$ or $\Lambda = \frac{(\pm 1)^p d!}{2(d-2p)!\ell^{2p}}$, and the generalized Einstein tensor is symmetric to pth order in the curvature,

$$^{(p)}G^\mu_\nu = -\frac{1}{2^{p+1}}\delta^{\mu\nu_1\cdots\nu_{2p}}_{\nu\mu_1\cdots\mu_{2p}}R^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}}. \tag{2.3}$$

The form of Λ given above is such that $\Lambda = 0$ has a Minkowski metric as a particular solution, whereas $\Lambda \neq 0$ has dS (sign $+$) and AdS (sign $-$) space of the radius ℓ as solutions of the PL field equations.

Because of the presence of local symmetries in the theory, not all components of the metric are physical. In order to determine dynamically propagating fields in the bulk, we turn to the Hamiltonian formalism, which provides a systematic method to separate physical variables from the nonphysical ones. However, applying the canonical analysis to the PL action in the metric formalism is technically involved, even though it only depends on velocities. A reason for this is that it is higher order in curvature.

On the other hand, if we write the action (2.1) in the Palatini formalism $\tilde{I}[g,\Gamma]$, where the metric $g_{\mu\nu}$ and affine connection $\Gamma^\lambda_{\mu\nu}$ are treated as independent fundamental fields, then the theory naturally includes torsional degrees of freedom. Then, the vanishing torsion would just correspond to a particular solution of the field equations, whereas in general relativity, it is the only solution. A wider space of solutions can be avoided by introducing a Lagrange multiplier that forces the torsion to vanish, in such a way that the field equations become the ones of PL gravity in Riemann space.

In the next section, we reformulate the PL gravity in first-order formalism, linear in velocities, which makes it much simpler to apply the Hamiltonian analysis.

A. First-order formalism

The fundamental fields in the first-order formalism, vielbein $e^a_\mu(x)$ and spin connection $\omega^{ab}_\mu(x)$, are related to the fields in the tensorial formalism through the relations $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ and $\Gamma^\lambda_{\mu\nu} = \omega^{ab}_\nu e^\lambda_a e_{a\mu} + e^\lambda_a \partial_\nu e^a_\mu$, where $a, b = 0, 1, \ldots, d$ are the Lorentz indices. Note that the change of variables $(g, \Gamma) \to (e, \omega)$ is not unique but is determined up to Lorentz rotations. With the new fields, we obtain the Riemann curvature tensor $R^{ab}_{\mu\nu}$ and the torsion tensor $T^a_{\mu\nu}$ as

$$\begin{aligned} R^{ab}_{\mu\nu} &= \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu b}\omega^{bc}_\nu - \omega^a_{\nu b}\omega^{bc}_\mu, \\ T^a_{\mu\nu} &= D_\mu e^a_\nu - D_\nu e^a_\mu, \end{aligned} \tag{2.4}$$

where $D = D(\omega)$ is a covariant derivative with respect to the spin connection acting on the Lorentz indices only, e.g., $D_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu$.

Naively, the first-order PL action can be cast in the form

$$\tilde{I}[e, \omega] = \int d^{d+1}x (\alpha_0 \mathcal{L}_0 + \alpha_p \mathcal{L}_p), \tag{2.5}$$

where we rescaled $\alpha_k \to -\frac{\alpha_k}{(d+1-2k)!}$ and $\mathcal{L}_k \to -(d+1-2k)!\mathcal{L}_k$, and the Euler densities now become polynomials in R and e,

$$\begin{aligned} \mathcal{L}_0 &= \epsilon_{a_1 \cdots a_{d+1}} \epsilon^{\mu_1 \cdots \mu_{d+1}} e^{a_1}_{\mu_1} \cdots e^{a_{d+1}}_{\mu_{d+1}} \sim e^{d+1}, \\ \mathcal{L}_p &= \frac{1}{2^p} \epsilon_{a_1 \cdots a_{d+1}} \epsilon^{\mu_1 \cdots \mu_{d+1}} R^{a_1 a_2}_{\mu_1 \mu_2} \cdots R^{a_{2p-1} a_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots \\ &\quad \times e^{a_{d+1}}_{\mu_{d+1}} \sim R^p e^{d+1-2p}. \end{aligned} \tag{2.6}$$

Notation for the Levi-Civita symbol $\epsilon^{\mu_1 \cdots \mu_{d+1}}$ is given in the Appendix. The coupling constants become

$$\alpha_0 = \frac{2\Lambda\kappa}{(d+1)!}, \qquad \alpha_p = -\frac{\kappa}{(d+1-2p)!}. \tag{2.7}$$

However, the field equations obtained from the action (2.5) after varying it in e^a_μ and ω^{ab}_μ are, respectively,

$$\begin{aligned} 0 = \epsilon_{a a_1 \cdots a_d} \epsilon^{\mu\mu_1 \cdots \mu_d} \bigg(&\frac{1}{2^p} R^{a_1 a_2}_{\mu_1 \mu_2} \cdots R^{a_{2p-1} a_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots \\ &\times e^{a_d}_{\mu_d} + \frac{\alpha_0 (d+1)}{\alpha_p (d+1-2p)} e^{a_1}_{\mu_1} \cdots e^{a_d}_{\mu_d} \bigg), \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 = \epsilon_{a b a_2 \cdots a_d} \epsilon^{\mu\mu_1 \cdots \mu_d} \bigg(&\frac{1}{2^p} R^{a_2 a_3}_{\mu_2 \mu_3} \cdots \\ &\times R^{a_{2p-2} a_{2p-1}}_{\mu_{2p-2} \mu_{2p-1}} T^{a_{2p}}_{\mu_{2p} \mu_1} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots e^{a_d}_{\mu_d} \bigg). \end{aligned} \tag{2.9}$$

These equations are not equivalent to the PL field equations (2.2) because the Riemann spaces for which $T^a_{\mu\nu} = 0$ are not the only solutions of Eq. (2.9) when $d + 1 > 4$ and $p > 1$. Thus, treating $(e^a_\mu, \omega^{ab}_\mu)$ as independent fields changes the dynamics of the system. In order to use first-order formalism and, at the same time, obtain field equations of pure Lovelock gravity, where $T^a_{\mu\nu} = 0$ is the unique solution, we introduce a Lagrange multiplier $\lambda^{\mu\nu}_a$ that forces the torsion tensor to vanish through a constraint. The new action reads

$$I[e, \omega, \lambda] = \int d^{d+1}x \left(\alpha_0 \mathcal{L}_0 + \alpha_p \mathcal{L}_p + \frac{1}{2} T^a_{\mu\nu} \lambda^{\mu\nu}_a \right). \tag{2.10}$$

The field $\lambda^{\mu\nu}_a(x)$ is antisymmetric in the indices $[\mu\nu]$.

Although the proposed action is explicitly torsionless, it does not imply that the equations of motion give the dynamics equivalent to the PL one. An example of the system where the addition of the constraint $T^a \lambda_a$ modifies the dynamics of the theory is topologically massive gravity, where it introduces a term involving the Cotton tensor [23–25]. There, the term with the multiplier has nontrivial implications on the derivation of conserved charges [26]. Therefore, the influence of a multiplier has to be well understood on the level of the field equations.

The action (2.10) reaches an extremum on the equations of motion,

$$\begin{aligned} \delta e^a_\mu \colon \ 0 = \epsilon_{a a_1 \cdots a_d} \epsilon^{\mu\mu_1 \cdots \mu_d} \bigg[&\frac{\alpha_p}{2^p} (d+1-2p) R^{a_1 a_2}_{\mu_1 \mu_2} \cdots \\ &\times R^{a_{2p-1} a_{2p}}_{\mu_{2p-1} \mu_{2p}} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots e^{a_d}_{\mu_d} \\ &+ \alpha_0 (d+1) e^{a_1}_{\mu_1} \cdots e^{a_d}_{\mu_d} \bigg] + D_\nu \lambda^{\mu\nu}_a, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \delta \omega^{ab}_\mu \colon \ 0 = \frac{1}{2^p} &\epsilon_{a b a_2 \cdots a_d} \epsilon^{\mu\mu_1 \cdots \mu_d} R^{a_2 a_3}_{\mu_2 \mu_3} \cdots \\ &\times R^{a_{2p-2} a_{2p-1}}_{\mu_{2p-2} \mu_{2p-1}} T^{a_{2p}}_{\mu_{2p} \mu_1} e^{a_{2p+1}}_{\mu_{2p+1}} \cdots e^{a_d}_{\mu_d} \\ &+ \frac{1}{2} (e_{b\nu} \lambda^{\mu\nu}_a - e_{a\nu} \lambda^{\mu\nu}_b), \end{aligned} \tag{2.12}$$

$$\delta \lambda^{\mu\nu}_a \colon \ 0 = T^a_{\mu\nu}. \tag{2.13}$$

In addition, the curvature and torsion tensors satisfy the first and second Bianchi identities,

$$D_\mu T^a_{\rho\sigma} + D_\rho T^a_{\sigma\mu} + D_\sigma T^a_{\mu\rho} = R^{ab}_{\mu\rho} e_{b\sigma} + R^{ab}{}_{\rho\sigma} e_{b\mu} + R^{ab}{}_{\sigma\mu} e_{b\rho},$$

$$D_\mu R^{ab}{}_{\rho\sigma} + D_\rho R^{ab}{}_{\sigma\mu} + D_\sigma R^{ab}{}_{\mu\rho} = 0. \tag{2.14}$$

When the torsion tensor vanishes, the field equation (2.12) becomes

$$0 = e_{b\nu}\lambda^{\mu\nu}_a - e_{a\nu}\lambda^{\mu\nu}_b, \tag{2.15}$$

where $d(d+1)^2/2$ components of $\lambda^{\mu\nu}_a$ can be solved as

$$\lambda^{\mu\nu}_a = 0. \tag{2.16}$$

This result is obtained by rewriting (2.15) with the Lorentz indices as $\lambda_{a,bc} - \lambda_{c,ba} = 0$, and combining it with two other expressions obtained by performing the permutation of indices, which directly leads to $\lambda_{a,bc} = 0$ and therefore (2.16). Using (2.16), the last equation (2.11) is indeed equivalent to the Lovelock field equations in Riemann space. The Bianchi identities (2.14) in that case read

$$R^a{}_{(\sigma\mu\rho)} = 0, \qquad D_{(\mu} R^{ab}_{\rho\sigma)} = 0. \tag{2.17}$$

III. ACTION IN THE TIMELIKE FOLIATION

The Hamiltonian formalism is not explicitly covariant because it presents all the quantities in the timelike foliation $x^\mu = (t, x^i)$, where $x^0 = t \in \mathbb{R}$ is the temporal coordinate and x^i $(i = 1, \ldots, d)$ are local coordinates at the spatial section Σ.

In the tangent space, we decompose the indices as $a = (0, \bar{a})$. The vielbein e^a_μ is invertible on $\mathbb{R} \times \Sigma$, and its inverse is e^μ_a. We require that $e^t_0 \neq 0$ and that the d-dimensional vielbein $e^{\bar{a}}_i$ is also invertible with the inverse

$${}^{(d)}e^i_{\bar{a}} = e^i_{\bar{a}} - \frac{e^i_0 e^t_{\bar{a}}}{e^t_0}. \tag{3.1}$$

In order to introduce canonical variables in the action (2.10), we have to define the action in configurational space, that is, in terms of the fields e^a_μ, ω^{ab}_μ and its velocities $\dot{e}^a_\mu, \dot{\omega}^{ab}_\mu$. To this end, we have the splitting of the fields in the timelike foliation

$$e^a_\mu \to (e^a_t, e^a_i), \qquad \omega^{ab}_\mu \to (\omega^{ab}_t, \omega^{ab}_i),$$

and similarly for the multiplier $\lambda^{\mu\nu}_a \to (\lambda^{ti}_a \equiv \lambda^i_a, \lambda^{ij}_a)$. It is worthwhile noticing that ω^{ab}_t transforms as a tensor of rank 2 under local Lorentz transformations on Σ and ω^{ab}_i as the Lorentz gauge connection.

Since $L = \int d^dx \mathcal{L}$, the Lagrangian scalar density of (2.10) can be written in a compact way,

$$\mathcal{L} = \frac{1}{2}\dot{\omega}^{ab}_i \mathcal{L}^i_{ab} + \dot{e}^a_i \lambda^i_a + \frac{1}{2}\omega^{ab}_t \mathcal{S}_{ab} + e^a_t \mathcal{S}_a + \frac{1}{2} T^a_{ij}\lambda^{ij}_a. \tag{3.2}$$

We neglect all boundary terms. In the action above, we introduce the quantities which do not depend on velocities and timelike components,

$$\mathcal{L}^i_{ab} = \frac{p\alpha_p}{2^{p-2}} \epsilon_{aba_2\cdots a_d} \epsilon^{ii_2\cdots i_d} R^{a_2a_3}_{i_2i_3} \cdots R^{a_{2p-2}a_{2p-1}}_{i_{2p-2}i_{2p-1}} e^{a_{2p}}_{i_{2p}} \cdots e^{a_d}_{i_d}, \tag{3.3}$$

$$\mathcal{S}_a = \mathcal{H}_a + D_i\lambda^i_a, \tag{3.4}$$

$$\mathcal{S}_{ab} = \mathcal{H}_{ab} + e_{bi}\lambda^i_a - e_{ai}\lambda^i_b, \tag{3.5}$$

where

$$\mathcal{H}_a = \epsilon_{aa_1\cdots a_d}\epsilon^{i_1\cdots i_d}\Big[(d+1)\alpha_0 e^{a_1}_{i_1}\cdots e^{a_d}_{i_d} + \frac{\alpha_p}{2^p}(d+1-2p) R^{a_1a_2}_{i_1i_2}\cdots R^{a_{2p-1}a_{2p}}_{i_{2p-1}i_{2p}} e^{a_{2p+1}}_{i_{2p+1}}\cdots e^{a_d}_{i_d}\Big],$$

$$\mathcal{H}_{ab} = \frac{p\alpha_p}{2^{p-1}}(d+1-2p)\epsilon_{aba_2\cdots a_d}\epsilon^{i_1\cdots i_d} R^{a_2a_3}_{i_2i_3}\cdots \times R^{a_{2p-2}a_{2p-1}}_{i_{2p-2}i_{2p-1}} T^{a_{2p}}_{i_1i_{2p}} e^{a_{2p+1}}_{i_{2p+1}}\cdots e^{a_d}_{i_d}. \tag{3.6}$$

The Lagrangian (3.2) is similar to the one in Chern-Simons theory, whose Hamiltonian analysis was studied in Ref. [19].

IV. HAMILTONIAN ANALYSIS IN FIVE DIMENSIONS

Let us start with the simplest case of a five-dimensional pure Gauss-Bonnet action $(d = 4,\ p = 2)$,

$$I = \int d^5x \Big[\epsilon_{\mathrm{abcde}}\epsilon^{\mu\nu\rho\sigma\gamma}\Big(\alpha_0 e^a_\mu e^b_\nu e^c_\rho e^d_\sigma e^e_\gamma + \frac{\alpha_2}{4} R^{ab}_{\mu\nu} R^{cd}_{\rho\sigma} e^e_\gamma\Big) + \frac{1}{2} T^a_{\mu\nu}\lambda^{\mu\nu}_a\Big]. \tag{4.1}$$

The Lagrangian has the form (3.2) with particular tensors

$$\begin{aligned}
\mathcal{L}^i_{ab} &= 2\alpha_2 \epsilon^{ijkl}\epsilon_{\mathrm{abcde}} R^{cd}_{jk} e^e_l,\\
\mathcal{S}_{ab} &= \mathcal{H}_{ab} + e_{bi}\lambda^i_a - e_{ai}\lambda^i_b,\\
\mathcal{S}_a &= \mathcal{H}_a + D_i\lambda^i_a,\\
\mathcal{H}_{ab} &= \alpha_2 \epsilon_{bacde}\epsilon^{ijkl} R^{cd}_{ij} T^e_{kl},\\
\mathcal{H}_a &= \epsilon_{abcde}\epsilon^{ijkl}\Big(5\alpha_0 e^b_i e^c_j e^d_k e^e_l + \frac{\alpha_2}{4} R^{bc}_{ij} R^{de}_{kl}\Big),
\end{aligned} \tag{4.2}$$

and the multipliers are conveniently written as

$$\lambda_a^i = \frac{1}{3!}\epsilon^{ijkl}\lambda_{a,jkl}, \qquad \lambda_a^{ij} = \frac{1}{2!}\epsilon^{ijkl}\lambda_{a,tkl}. \tag{4.3}$$

If we denote the generalized coordinates by $q^M(x)$ and the corresponding conjugated momenta by $\pi_M(x)$,

$$\begin{aligned} q^M &= \{e_t^a, e_i^a, \omega_t^{ab}, \omega_i^{ab}, \lambda_a^i, \lambda_a^{ij}\}, \\ \pi_M &= \{\pi_a^t, \pi_a^i, \pi_{ab}^t, \pi_{ab}^i, p_i^a, p_{ij}^a\}, \end{aligned} \tag{4.4}$$

we can use the definition $\pi_M = \frac{\partial \mathcal{L}}{\partial \dot{q}^M}$ to find $\pi_{ab}^i = \mathcal{L}_{ab}^i$ and $\pi_a^i = \lambda_a^i$, while all other momenta are zero. Thus, the Hessian matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^M \partial \dot{q}^N}$ is not invertible, and we cannot express all velocities in terms of the momenta. In turn, we get the constraints, which are called

$$\text{primary constraints: } \Phi_M = \{\phi_a^t, \phi_a^i, \phi_{ab}^t, \phi_{ab}^i, p_i^a, p_{ij}^a\}. \tag{4.5}$$

They are defined on the phase space as

$$\begin{aligned} \phi_a^t &= \pi_a^t \approx 0, & \phi_a^i &= \pi_a^i - \lambda_a^i \approx 0, \\ \phi_{ab}^t &= \pi_{ab}^t \approx 0, & \phi_{ab}^i &= \pi_{ab}^i - \mathcal{L}_{ab}^i \approx 0, \\ p_i^a &\approx 0, & p_{ij}^a &\approx 0. \end{aligned} \tag{4.6}$$

The surface $\Phi_M \approx 0$ in the phase space is called the primary constraint surface, Γ_P. The weak equality $f(q,\pi) \approx 0$ on Γ_P implies that a phase-space function f vanishes on Γ_P, but its derivatives (variations) are nonvanishing. This is different than the strong equality, $f(q,\pi) = 0$, where both f and its variations vanish on Γ_P. This distinction is relevant for the definition of the Poisson brackets, since $f \approx 0$ does not imply $\{f, \cdots\} \approx 0$.

To simplify our notation, we write the arguments of the phase-space functions symbolically, assuming that all quantities are defined at the same instant, $x^0 = x'^0 = t$,

$$\begin{aligned} &A = A(x), \qquad B' = B(x'), \qquad \partial_i = \frac{\partial}{\partial x^i}, \qquad \partial_i' = \frac{\partial}{\partial x'^i}, \\ &\delta = \delta(\vec{x} - \vec{x}'), \qquad \delta_{cd}^{ab} = \delta_c^a \delta_d^b - \delta_d^a \delta_c^b. \end{aligned} \tag{4.7}$$

The fundamental Poisson brackets (PBs) that are different than zero are

$$\begin{aligned} \{e_\mu^a, \pi_b'^\nu\} &= \delta_b^a \delta_\mu^\nu \delta, \\ \{\omega_\mu^{ab}, \pi_{cd}'^\nu\} &= \delta_{cd}^{ab} \delta_\mu^\nu \delta, \\ \{\lambda_a^i, p_j'^b\} &= \delta_b^a \delta_j^i \delta, \\ \{\lambda_a^{ij}, p_{kl}'^b\} &= \delta_b^a \delta_{kl}^{ij} \delta. \end{aligned} \tag{4.8}$$

The symplectic matrix Ω_{MN} of the primary constraints reads

$$\{\Phi_M, \Phi_N'\} = \Omega_{MN}\delta, \tag{4.9}$$

and it is antisymmetric, $\Omega_{MN} = -\Omega_{NM}$. The only (independent) submatrices of the symplectic matrix that are different than zero are

$$\begin{aligned} \{\phi_{ab}^i, \phi_{cd}'^j\} &= \Omega_{abcd}^{ij}\delta = -8\alpha_2 \epsilon^{ijkl}\epsilon_{abcde} T_{kl}^e \delta, \\ \{\phi_{ab}^i, \phi_c'^j\} &= \Omega_{abc}^{ij}\delta = -2\alpha_2 \epsilon^{ijkl}\epsilon_{abcde} R_{kl}^{de} \delta, \\ \{\phi_a^i, p_j'^b\} &= -\delta_a^b \delta_j^i \delta. \end{aligned} \tag{4.10}$$

The canonical Hamiltonian, $\mathcal{H}_C = \pi_M \dot{q}^M - \mathcal{L}$, defined on Γ_P is

$$\mathcal{H}_C(p,q) = -\frac{1}{2}\omega_t^{ab}\mathcal{S}_{ab} - e_t^a \mathcal{S}_a - \frac{1}{2}T_{ij}^a \lambda_a^{ij}, \tag{4.11}$$

and the total Hamiltonian, defined on the full phase space Γ, is obtained by introducing the indefinite multipliers $u^M(x)$,

$$\mathcal{H}_T(p,q,u) = \mathcal{H}_C(p,q) + u^M \Phi_M(p,q), \tag{4.12}$$

where $u^M = \{u_t^a, u_i^a, u_t^{ab}, u_i^{ab}, v_a^i, v_a^{ij}\}$. Evolution of any quantity $A(q(x), \pi(x)) = A(x)$ in the phase space is given by

$$\begin{aligned} \dot{A} &= \int d\vec{x}' (\{A, \mathcal{H}_C'\} + u'^M \{A, \Phi_M'\}) \\ &\approx \int d\vec{x}' \{A, \mathcal{H}_T'\}. \end{aligned} \tag{4.13}$$

This allows us to identify some field velocities with the Hamiltonian multipliers,

$$\dot{\omega}_i^{ab} = u_i^{ab}, \quad \dot{e}_i^a = u_i^a, \quad \dot{\lambda}_a^{ij} = v_a^{ij}, \quad \dot{\lambda}_a^i = v_a^i. \tag{4.14}$$

Consistency of the theory requires that the primary constraints remain on the constraint surface during their evolution, that is,

$$\dot{\Phi}_M = \int d\vec{x}' \{\Phi_M, \mathcal{H}_C'\} + \Omega_{MN} u^N \approx 0. \tag{4.15}$$

These consistency conditions will either solve some multipliers or lead to the secondary constraints, or they will be identically satisfied.

When the symplectic matrix has zero modes and $\{\Phi_M, \mathcal{H}_C\} \neq 0$, the consistency conditions lead to the secondary constraints,

$$\dot{\phi}_a^t = \mathcal{S}_a \approx 0, \tag{4.16}$$

$$\dot{\phi}_{ab}^t = \mathcal{S}_{ab} \approx 0, \tag{4.17}$$

$$\dot{p}_{ij}^a = T_{ij}^a \approx 0. \tag{4.18}$$

Other consistency conditions solve Hamiltonian multipliers, such as $\dot{p}_i^a \approx 0$, which gives

$$u_i^a = D_i e_t^a - \omega_t^{ab} e_{bi}. \tag{4.19}$$

On the other hand, from $\dot{\phi}_a^i \approx 0$ we solve the multiplier,

$$v_a^i = -\epsilon_{abcde}\epsilon^{ijkl}[20\alpha_0 e_t^b e_j^c e_k^d e_l^e + \alpha_2 R_{kl}^{de}(u_j^{bc} - D_j\omega_t^{bc})] + \omega_{ta}^b \lambda_b^i + D_j \lambda_a^{ij}. \tag{4.20}$$

Using the Bianchi identities, $D_j\epsilon_{abcde} = 0$ and the property that any totally antisymmetric tensor of rank 6 defined in five dimensions must vanish, that is,

$$-\epsilon_{bcdef}e_{aj} + \epsilon_{cdefa}e_{bj} - \epsilon_{defab}e_{cj} + \epsilon_{efabc}e_{dj} - \epsilon_{fabcd}e_{ej} + \epsilon_{abcde}e_{fj} = 0,$$

the last consistency condition for ϕ_{ab}^i becomes

$$0 \approx \dot{\phi}_{ab}^i \approx e_{at}\lambda_b^i - e_{bt}\lambda_a^i - \lambda_a^{ij}e_{bj} + \lambda_b^{ij}e_{aj}. \tag{4.21}$$

One can show, in a similar way as for Eq. (2.16), that the constraints (4.17) and (4.21) are now equivalent to zero multipliers $\lambda_a^i \approx 0$ and $\lambda_a^{ij} \approx 0$.

So far, we have found the following,

$$\text{secondary constraints:}\ \mathcal{S}_a \approx 0, \qquad T_{ij}^a \approx 0, \qquad \lambda_a^i \approx 0, \qquad \lambda_a^{ij} \approx 0, \tag{4.22}$$

and we determined the multipliers u_i^a and v_a^i. The functions $\{u_t^a, u_t^{ab}, u_i^{ab}, v_a^{ij}\}$ remain arbitrary. A submanifold $\Gamma_S \subset \Gamma$ defines the secondary constraint surface, where all constraints discovered so far vanish.

To ensure that the secondary constraints λ evolve on the constraint surface Γ_S, we require that $\dot{\lambda}_a^i = v_a^i$ and $\dot{\lambda}_a^{ij} = v_a^{ij}$ vanish. This leads to $v_a^i = 0$, which by Eq. (4.20) can be equivalently expressed as

$$\chi_a^i = -\epsilon_{abcde}\epsilon^{ijkl}[20\alpha_0 e_t^b e_j^c e_k^d e_l^e + \alpha_2 R_{kl}^{de}(u_j^{bc} - D_j\omega_t^{bc})] \approx 0 \quad \text{and} \quad v_a^{ij} = 0. \tag{4.23}$$

Before we continue, we notice that the pairs of conjugated variables (λ, p), which are constraints, have PBs whose right-hand side (symplectic form) is invertible on Γ_S. Thus, they are second-class constraints that do not generate any symmetry but represent redundant, nonphysical quantities. They can be eliminated by defining the reduced phase space Γ^* with the Poisson brackets replaced by the Dirac brackets,

$$\{A, B'\}^* = \{A, B'\} + \int dy \Big[\{A, \lambda_a^i(y)\}\{p_i^a(y), B'\} - \{A, p_i^a(y)\}\{\lambda_a^i(y), B'\} + \frac{1}{2}\{A, \lambda_a^{ij}(y)\}\{p_{ij}^a(y), B'\} - \frac{1}{2}\{A, p_{ij}^a(y)\}\{\lambda_a^{ij}(y), B'\}\Big]. \tag{4.24}$$

It is straightforward to check (and it is a general property of the Dirac brackets) that the use of $\{,\}^*$ turns the weak equality into the strong equality on Γ^*,

$$\lambda_a^i = 0, \qquad p_i^a = 0, \quad \text{on } \Gamma^*, \qquad \lambda_a^{ij} = 0, \qquad p_{ij}^a = 0, \quad \text{on } \Gamma^*, \tag{4.25}$$

because $\{\lambda_a^i, p_j^{\prime b}\}^* = 0$ and $\{\lambda_a^{ij}, p_{kl}^{\prime b}\}^* = 0$ on Γ^*. The remaining generalized coordinates of the space Γ^* are $(e_\mu^a, \omega_\mu^{ab}, \pi_a^\mu, \pi_{ab}^\mu)$, and their Dirac brackets remain unmodified (they are equal to the Poisson brackets). From now on, we drop the star from the Dirac brackets.

Let us analyze the consistency condition of $\mathcal{S}_a$. Using $\dot{R}_{ij}^{bc} = D_i u_j^{bc} - D_j u_i^{bc}$ and $\dot{e}_i^a = u_i^a$, we get

$$\dot{\mathcal{S}}_a = \epsilon_{abcde}\epsilon^{ijkl}[20\alpha_0 D_i e_t^b e_j^c e_k^d e_l^e + \alpha_2 D_i U_j^{bc} R_{kl}^{de} + \omega_{tf}{}^b(20\alpha_0 e_i^f e_j^c e_k^d e_l^e + \alpha_2 R_{ij}^{fc} R_{kl}^{de})], \tag{4.26}$$

where we denoted

$$U_i^{ab} = u_i^{ab} - D_i\omega_t^{ab} \tag{4.27}$$

and used $[D_i, D_j]\omega_t^{bc} = R_{ij}^{bf}\omega_{tf}{}^c - R_{ij}^{cf}\omega_{tf}{}^b$. It can be recognized from the Lagrangian formalism that $U_i^{ab} \equiv R_{ti}^{ab}$ because the Hamiltonian prescription treats all time derivatives as new functions. Next, we use a combinatorial identity, valid for any completely antisymmetric tensor Σ^{cdef},

$$0 = D_t\epsilon_{acdef}\Sigma^{cdef} = (\epsilon_{bcdef}\omega_{ta}{}^a + 2\epsilon_{abdef}\omega_{tc}{}^b + 2\epsilon_{acdbf}\omega_{te}{}^b)\Sigma^{cdef}.$$

For a particular choice of $\Sigma^{fcde} = 20\alpha_0 e_i^f e_j^c e_k^d e_l^e + \alpha_2 R_{ij}^{fc} R_{kl}^{de}$, we obtain that $\mathcal{S}_a$ does not leave the surface Γ_S during its evolution,

$$\dot{\mathcal{S}}_a = -D_i\chi_a^i - \omega_{ta}{}^b\mathcal{S}_b \approx 0. \tag{4.28}$$

Furthermore, we also have to require the same for the torsion tensor,

$$\dot{T}_{ij}^a = D_i u_j^a - D_j u_i^a + u_i^{ab}e_{bj} - u_j^{ab}e_{bi} \approx 0. \tag{4.29}$$

With the help of Eq. (4.19), we rewrite the last equation as

$$0 \approx \dot{T}^a_{ij} \approx R^{ab}_{ij} e_{bt} + U^a{}_{ji} - U^a{}_{ij}. \tag{4.30}$$

Here, the vielbein projects the Lorentz indices to the spacetime ones, $U^a{}_{ji} = U^{ab}_i e_{bj}$. The above equation gives 30 algebraic equations in 40 unknown functions $U^a{}_{ij}$, which can be decomposed into $16 + 24$ components $(U^0_{ij}, U^{\bar{a}}_{ij})$. The final solution is

$$U^a{}_{[ij]} = \frac{1}{2} R^{ab}_{ij} e_{bt} = \frac{1}{2} R^a_{tij} \Rightarrow U^\mu{}_{[ij]} = \left(0, \frac{1}{2} R^k{}_{tij}\right). \tag{4.31}$$

In that way, the $6 + 4$ coefficients $(U^0_{[ij]}, U^{\bar{a}}_{[ij]})$ become completely determined by the consistency of $\dot{T}^0_{ij}$ and $\dot{T}_{[ki]j}$. Since $U^a{}_{ij} = R^a{}_{itj}$, the above relation just represents the first Bianchi identity for the components (tij) rederived in the Hamiltonian way.

The 20 components of $\dot{T}^a_{ij}$ that do not solve the corresponding multipliers are exactly the ones that are symmetric in the first two indices,

$$\begin{aligned}\dot{T}_{(ki)j} &\approx \frac{1}{2}(e_{ak} \dot{T}^a_{ij} + e_{ai} \dot{T}^a_{kj}) \\ &\approx \frac{1}{2} R_{ktij} + U_{k[ji]} + \frac{1}{2} R_{itkj} + U_{i[jk]} = 0,\end{aligned} \tag{4.32}$$

which vanish due to the known $U_{i[jk]}$. Thus, these components do not lead to new conditions. We conclude that 30 equations $\dot{T}^a_{ij} = 0$ solve only 10 antisymmetric components u^{ab}_i and the remaining 20 equations do not give anything new—they are automatically satisfied.

Thanks to the relation (4.31) and because the curvature R^{ab}_{ij} satisfies the first Bianchi identity, we can collect all first Bianchi identities in a covariant way, $\mathcal{B}_{\mu\nu\alpha\beta} = R_{\mu(\nu\alpha\beta)} = 0$, where the components of the tensor $\mathcal{B}$ are

$$\begin{aligned} 0 &= \mathcal{B}^a_{ijk} \equiv R^a_{(ijk)}, \\ 0 &= \mathcal{B}^a_{tij} \equiv e_{bt} R^{ab}_{ij} - 2U^a_{[ij]}. \end{aligned} \tag{4.33}$$

This has an important consequence on the number of *linearly independent* multipliers U^{ab}_i. Namely, we can prove that

$$R_{\mu\nu\alpha\beta} - R_{\alpha\beta\mu\nu} = \frac{1}{2}(\mathcal{B}_{\mu\nu\alpha\beta} + \mathcal{B}_{\beta\mu\nu\alpha} - \mathcal{B}_{\alpha\beta\mu\nu} - \mathcal{B}_{\nu\alpha\beta\mu}) = 0, \tag{4.34}$$

so the Riemann curvature is symmetric, $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$, or

$$R_{titj} = R_{tjti}, \qquad R_{tijk} = R_{jkti}, \qquad R_{ijkl} = R_{klij}. \tag{4.35}$$

The last relation in (4.35) does not give any further information because it is just the Bianchi identity on Σ. The first one, instead, shows that not all coefficients $U_{\mu ij} = e_{a\mu} U^a_{ij}$ are independent because U_{tij} are symmetric, $U_{tij} = U_{tji} = e_{at} e_{bi} R^{ab}_{tj}$. The second condition in (4.35) is equivalent to

$$U_{jki} = e_{at} e_{bi} R^{ab}_{jk}, \tag{4.36}$$

in a way that is consistent with (4.31). The only remaining unknown multipliers are 10 symmetric components $U_{t(ij)}$, leading to the final expression for U^{ab}_i as

$$\begin{aligned} U^{ab}_i &= U_{\mu\nu i} e^{a\mu} e^{b\nu} \\ &= U_{t(ij)}(e^{ta} e^{jb} - e^{tb} e^{ja}) + e_{ct} e_{di} R^{cd}_{jk} e^{ja} e^{kb}, \\ U_{tij} &= U_{tji}. \end{aligned} \tag{4.37}$$

From the point of view of the irreducible components of U^{ab}_i, we can see the 10 components of $U_{t(ij)}$ as the only unsolved part in the table below.

Multiplier $U_{\mu ij}$:	$U_{t[ij]}$	$U_{tk}{}^k$	${}^S U_{tij}$	${}^A U_{kij}$	${}^S U_{kij}$	${}^T U_{kij}$
40 components:	6	1	9	4	4	16
Solved by:	$\dot{T}^0_{ij}$	arbitrary	arbitrary	$\dot{T}_{[ijk]}$	Bianchi	Bianchi

As it is well known, the irreducible components of the rank-2 tensor U_{tij} are as follows: its antisymmetric part $U_{t[ij]}$, the trace $U_{tk}{}^k$ and the symmetric traceless component ${}^S U_{tij} = U_{t(ij)} - \frac{1}{4} g_{ij} U_{tk}{}^k$. On the other hand, the irreducible components of the rank-3 tensor $U_{i(jk)}$ are as follows: its vectorial component (trace) $U_i \equiv U_{jk}{}^j$, also written as ${}^S U_{ijk} = g_{ij} U_k + g_{ik} U_j$, the axial-vector component ${}^A U_{ijk} = U_{[ijk]}$ and the tensorial one ${}^T U = U - {}^A U - {}^S U$.

The last equation to analyze is $\chi^i_a \approx 0$. It can be combined together with $\mathcal{H}_a \approx 0$ into

$$\begin{aligned} \chi^\lambda_a &= (\mathcal{H}_a, \chi^i_a) \\ &= \epsilon_{abcde} \epsilon^{\lambda\mu\nu\alpha\beta} \left(5\alpha_0 e^b_\mu e^c_\nu e^d_\alpha e^e_\beta + \frac{\alpha_2}{4} R^{bc}_{\mu\nu} R^{de}_{\alpha\beta}\right) \approx 0, \end{aligned} \tag{4.38}$$

in which we recognize the generalized Einstein equations with cosmological constant (2.2). In contrast to the Einstein-Hilbert case [27,28], the multipliers in Eq. (4.23) cannot be fully solved because they are nonlinear in the fields, causing ambiguities. In fact, if we write it as

$$2\alpha_2\epsilon_{abcde}\epsilon^{ijkl}R_{kl}^{de}U_j^{bc} = -40\alpha_0\epsilon_{abcde}\epsilon^{ijkl}e_t^b e_j^c e_k^d e_l^e, \tag{4.39}$$

then the rank of the matrix $\Omega_{abc}^{ij} = -2\alpha_2\epsilon_{abcde}\epsilon^{ijkl}R_{kl}^{de}$ explicitly depends on the background considered. More concretely, replacing the solution for the multiplier (4.37) in (4.39), we obtain a set of algebraic equations

$$M_a^{i(jm)}U_{tjm} = A_a^i, \quad \text{or} \quad MU = A, \tag{4.40}$$

where the matrix of the system is obtained by symmetrization of

$$M_\mu^{ijm} = -\Omega_{abc}^{ij}e_\mu^a e^{tb}e^{mc} = \frac{2\alpha_2}{|e|}g_{\mu n}\epsilon^{mnpq}\epsilon^{ijkl}R_{pqkl}. \tag{4.41}$$

The nonhomogeneous part of the system is

$$A_\mu^i = |e|(120\alpha_0\delta_\mu^i - \alpha_2\epsilon_{\mu mn\nu\lambda}\epsilon^{ijkl}R^{\nu\lambda}{}_{kl}R_{tj}{}^{mn}). \tag{4.42}$$

In the context of Eq. (4.40), M is the 20×10 matrix that acts on the 10-component column U. When the rank of M is maximal, that is, 10, then all components of U can be determined. This is the case for AdS space, as we show below. A situation is completely different in flat space, where the equation becomes homogeneous ($A=0$) and the rank of $M=0$ is zero. In that case, all 10 components of the vector $U_{t(jm)}$ remain arbitrary. In the EH case this matrix always has maximal rank because it does not depend on the curvature. In higher-dimensional PL gravity, M is again polynomial in the curvature and may have different ranks. It is a generic feature of Lovelock gravity, and it has already been noted for the Lovelock-Chern-Simons case [19].

We are interested in the $\Lambda\neq 0$ backgrounds, where black hole solutions exist. We restrict our theory to the part of the phase space where the rectangular 20×10 matrix $M_a^{i(jm)}(x)$ has maximal rank, 10, for all x. In that case, only the *left* inverse of M exists, which is the 10×20 matrix $\Delta_{(kl)i}^a(x)$ of the rank 10 defined by

$$\frac{1}{2}\Delta_{(kl)i}^a M_a^{i(jm)} = \delta_k^j\delta_l^m + \delta_l^j\delta_k^m. \tag{4.43}$$

The matrix Δ depends on e_μ^a and ω_μ^{ab}. Then Eq. (4.40) can be solved and the multipliers are

$$U_{tij} = \Delta_{(ij)k}^a A_a^k. \tag{4.44}$$

To show that the chosen subspace contains a nonempty set of solutions, we consider the AdS background

$$\bar{R}_{\mu\nu\alpha\beta} = -\frac{1}{\ell^2}(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \bar{g}_{\nu\alpha}\bar{g}_{\mu\beta}), \tag{4.45}$$

with $\alpha_0 = \frac{1}{5\ell^4}$ and $\alpha_2 = -1$. Then

$$\bar{M}_\mu^{i(jm)} = \frac{4}{\ell^2|\bar{e}|}{}^{(4)}\bar{g}\bar{g}_{\mu n}(\bar{g}^{im}\bar{g}^{jn} + \bar{g}^{ij}\bar{g}^{nm} - 2\bar{g}^{in}\bar{g}^{jm}), \tag{4.46}$$

where ${}^{(4)}\bar{g} = \det[\bar{g}_{kl}]$ is the determinant of the spatial background induced metric ${}^{(4)}\bar{g}_{ij} = \bar{g}_{ij}$ and its inverse is ${}^{(4)}g^{ij} = g^{ij} - \frac{g^{ti}g^{tj}}{g^{tt}}$.

Now we linearize Eq. (4.40) around this background, i.e., $(\bar{M}+\delta M)(\bar{U}+V) = \bar{A}+\delta A$, where $\delta U_{t(jm)} = V_{jm}$. We multiply the zero order, $\bar{M}\bar{U} = \bar{A}$, by $\bar{e}_k^a$ and obtain

$$\bar{U}_{tij} = \frac{5\alpha_0\ell^2}{\alpha_2}\frac{\bar{g}\bar{g}_{ij}}{{}^{(4)}\bar{g}} = -\frac{1}{\ell^2}\bar{g}_{tt}\bar{g}_{ij}, \tag{4.47}$$

where we replaced the values of the constants α_k. We also used the identity $|\bar{e}|^2 = -\bar{g} = -\bar{g}_{tt}{}^{(4)}\bar{g}$.

Projecting $\bar{M}\bar{U} = \bar{A}$ by $\bar{e}_t^a$, we find that (4.47) is satisfied. One can obtain the same result from the definition $\bar{U}_{tij} = \bar{R}_{titj}$ coming from Eqs. (4.27) and (4.45).

The linear order equation, $\bar{M}V + \delta M\bar{U} = \delta A$, projected by $\bar{e}_\mu^a$ reads

$$\bar{M}_\mu^{i(jm)}V_{tjm} = C_\mu^i, \tag{4.48}$$

where we defined $C_\mu^i = -\delta M_\mu^{i(jm)}\bar{U}_{tjm} + e_\mu^a\delta A_a^i$. After replacing the matrix M [see Eq. (4.46)], we find

$$V_t{}^i{}_\mu - \delta_\mu^i V_t{}^j{}_j = \frac{|e|\ell^2}{2{}^{(4)}\bar{g}}C_\mu^i. \tag{4.49}$$

In that way, all 10 symmetric multipliers $U = \bar{U}+V$ are uniquely solved in the AdS background with

$$V_t{}^i{}_j = \frac{|e|\ell^2}{2{}^{(4)}\bar{g}}\left(C_j^i - \frac{1}{3}\delta_j^i C_k^k\right). \tag{4.50}$$

It is straightforward to check that the remaining equations, $\bar{M}_t^{i(jm)}V_{tjm} = C_t^i$, are automatically satisfied. Therefore, we explicitly find the matrix $\bar{\Delta}_{(kl)i}^a$ in the AdS background.

It is easy to prove in a similar way that the static black holes also belong to the chosen region of the phase space where the left inverse $\bar{\Delta}_{(kl)i}^a$ exists. Namely, as in AdS space, the black hole curvature $R^{\mu\nu}_{\alpha\beta}$ has each component proportional to $\delta^{\mu\nu}_{\alpha\beta}$, with different factors. An explicit check confirms that M has maximal rank for static pure Gauss-Bonnet black holes.

With all constraints identified and the Hamiltonian multipliers solved, we can obtain the information about the degrees of freedom and local symmetries in the theory in a particular class of backgrounds, where M has maximal rank during the whole evolution of the fields.

V. DEGREES OF FREEDOM AND SYMMETRIES

The next step in the Hamiltonian analysis is to separate first- and second-class constraints. The first-class constraints generate local symmetries, and the second-class constraints eliminate nonphysical fields not related to the symmetries. If there are N_1 first-class and N_2 second-class constraints in the phase space with N generalized coordinates, then a physical number of degrees of freedom is given by the Dirac formula

$$N^* = N - N_1 - \frac{1}{2}N_2. \tag{5.1}$$

Thus, determination of a class of constraints is of essential importance for identification of the physical fields living on the reduced phase space Γ^*. Furthermore, first-class constraints are related to the existence of indefinite multipliers in a theory, and their numbers should match since each first-class constraint appearing in the Hamiltonian is multiplied by an arbitrary function. Let us recall from the previous section that the solved multipliers are $\{u^a_i, U^{ab}_i, v^i_a = 0, v^{ij}_a = 0\}$, and the unsolved ones u^{ab}_t and u^a_t are related to the local symmetries, Lorentz transformations and diffeomorphisms. In addition, we do not know the explicit form of all multipliers U^{ab}_i because $U_{t(ij)}$ depends on the background. It is then expected that we will not be able to obtain a closed, background-independent form of all generators.

To find first-class constraints, it is helpful to write the total Hamiltonian density $\mathcal{H}_T$ with solved multipliers because it is known that this is a first-class quantity (it commutes with all constraints); therefore, only first-class constraints will naturally appear as combinations of other constraints. Thus, replacing the solutions (4.19), (4.23) and (4.27) in $\mathcal{H}_T$, we obtain the Hamiltonian density

$$\begin{aligned}\mathcal{H} = &-\frac{1}{2}\omega^{ab}_t J_{ab} - e^a_t J_a + u^a_t \pi^t_a + \frac{1}{2}u^{ab}_t \pi^t_{ab}\\ &+\frac{1}{2}U^{ab}_i \phi^i_{ab} + \partial_i \mathcal{D}^i,\end{aligned} \tag{5.2}$$

where the constraints $(\mathcal{H}_a, \mathcal{H}_{ab})$ are replaced by the new ones (J_a, J_{ab}),

$$\begin{aligned}J_{ab} &= \mathcal{H}_{ab} - e_{ai}\pi^i_b + e_{bi}\pi^i_a + D_i\phi^i_{ab}\\ &= -e_{ai}\pi^i_b + e_{bi}\pi^i_a + D_i\pi^i_{ab},\\ J_a &= \mathcal{H}_a + D_i\pi^i_a.\end{aligned} \tag{5.3}$$

The total divergence $\mathcal{D}^i = e^a_t\pi^i_a + \frac{1}{2}\omega^{ab}_t\phi^i_{ab}$ can be neglected, as it contributes only to a boundary term in the total Hamiltonian.

The functions (J_a, J_{ab}) are not guaranteed yet to be first class because we still have to replace U^{ab}_i. But to evaluate $U \cdot \phi$, we have to choose a particular background for $U_{t(ij)}$, so we will not write it explicitly, as we prefer to keep the background-independent expressions. A more detailed analysis shows that after using Eqs. (4.37) and (4.44), the multipliers can be written as

$$\begin{aligned}\frac{1}{2}U^{ab}_i\phi^i_{ab} &= \Delta^c_{(ij)k}A^k_c g^{tl} e^a_l e^{jb}\phi^i_{ab}\\ &\quad - e^a_t\left(\frac{1}{2}R_{acjk}e^{kb}e^c_i e^{jd}\phi^i_{bd} - \Delta^c_{(ij)k}A^k_c g^{tt}e^{jb}\phi^i_{ab}\right)\\ &= -\frac{1}{2}\omega^{ab}_t\Delta J_{ab} - e^a_t\Delta J_a,\end{aligned} \tag{5.4}$$

so, in general, this expression can affect the generators J_a and J_{ab} because $\Delta^c_{(ij)k}$ is a function of e^a_t and ω^{ab}_t. The first-class generators that appear in the Hamiltonian (5.2) are $\mathcal{J}_{ab} = J_{ab} + \Delta J_{ab}$ and $\mathcal{J}_a = J_a + \Delta J_a$. Note that these corrections contain the nonlinear R^2 terms and the background-dependent $\Delta(e, \omega)$. This is similar to what happens in the $R + T^2 + R^2$ theory [29]. Because of the complexity of the problem, in the next step we will not account for the $U \cdot \phi$ term.

The temporal components of the fields, ω^{ab}_t and e^a_t, are Lagrangian multipliers because they are not dynamical, and in the Hamiltonian notation, they are arbitrary functions multiplying the constraints. Therefore, the Hamiltonian (5.2) can be seen as the extended Hamiltonian, which contains constraints of all generations, both primary and secondary. Furthermore, since only first-class constraints are associated with indefinite multipliers, we can identify them as

$$\text{first-class constraints:}\ \mathcal{J}_a, \mathcal{J}_{ab}, \pi^t_a, \pi^t_{ab},$$

and there are $N_1 = (5 + 10) \times 2 = 30$ of them. With respect to the second-class constraints, from (4.22) we know that $T^a_{ij} \approx 0$ is satisfied, but some components of the torsion tensor are first class and some are second-class constraints. They cannot be separated explicitly. For example, 10 functions $\mathcal{H}_{ab}$ are linear combinations of T^a_{ij}. This means, in order to define $\mathcal{J}_{ab}$ in terms of $\mathcal{H}_{ab}$, we had to change the basis of the constraints. In doing so, it is important that the regularity conditions are satisfied, ensuring that all constraints are linearly independent on the phase space because they have the maximal rank of the Jacobian with respect to the phase-space variables. In our case, we replaced the initial set of 30 constraints T^a_{ij} by a new one $(\mathcal{H}_{ab}, \mathcal{T}_z)$. Then, the regularity conditions require that $\mathfrak{R}\mathrm{ank}[\frac{\partial(\mathcal{H}_{ab},\mathcal{T}_z)}{\partial(q^M,p_N)}] = 30$, what means that there must be 20 second-class constraints $\mathcal{T}_z$. We denote them by $\mathcal{T}_z = \{\tilde{T}^a_{ij}\}$ regardless of their tensorial properties, to remember that they are redundant torsional components which do not generate any local symmetry. Thus, we represented T^a_{ij} by an equivalent set of the $10 + 20$ constraints $(\mathcal{H}_{ab}, \tilde{T}^a_{ij})$. Then we can identify the remaining set of constraints as

$$\text{second-class constraints:}\ \tilde{T}^a_{ij}, \phi^i_a, \phi^i_{ab},$$

and there are $N_2 = 20 + 20 + 40 = 80$ of them. Then, counting the degrees of freedom is straightforward: for $N = 25 + 50 = 75$ dynamical fields $(e^a_\mu, \omega^{ab}_\mu)$, the Dirac formula (5.1) gives the number of degrees of freedom

$$N^* = 5. \tag{5.5}$$

This is the same number as in the five-dimensional Einstein-Hilbert theory, and the maximal number that a PL gravity can contain. One of these degrees of freedom is the radial one. This can be proved by performing the Hamiltonian analysis of the action in minisuperspace approximation, which involves only the relevant degrees of freedom, similar to Ref. [21]. Fundamental fields in this approximation are the most general ones among $g_{\mu\nu}$ and $T_{\mu\nu\lambda}$ that have the same isometries. The identified radial degree of freedom corresponds to the metric component $g_{tt} = -1/g_{rr}$.

If the rank of M in (4.40) is smaller than maximal, then some functions $U_{t(ij)}$ remain arbitrary, reflecting the fact that there are more local symmetries in the theory and less degrees of freedom. In the extreme case, when the rank of M is zero, all $U_{t(ij)}$ are indefinite, so there are 10 additional local symmetries because second-class constraints are converted into the first class; thus, $N_1 \to N_1 + 10 = 40$ and $N_2 \to N_2 - 10 = 70$. This implies that in the flat background the theory has $N^* - 10 + \frac{1}{2}10 = 0$ degrees of freedom. Indeed, in Ref. [30], the PL gravity with $\Lambda = 0$ was studied in five dimensions, and it was pointed out that it is kinematic. However, it was also shown that the LL term alone did not admit a linear approximation, and it was suggested that a linear Einstein-Hilbert term would make the theory more physical. In our case, the cosmological term has this purpose. In general, the number of degrees of freedom in five-dimensional PL gravity varies in the range

$$0 \leq N^* \leq 5. \tag{5.6}$$

Let us analyze the local symmetries and their generators. The first-class constraint $G \approx 0$ acts on the fundamental field q through the smeared generator $G[\lambda] = \int d^4x \lambda G$, and the field transforms as $\delta q = \{q, G[\lambda]\}$. In our case, the generator for all first-class constraints is

$$\begin{aligned} G[\Lambda,\dot{\Lambda},\epsilon,\dot{\epsilon}] &= \int d^4x \left[\frac{1}{2}\Lambda^{ab}(J_{ab} - e_{at}\pi^t_b + e_{bt}\pi^t_a) \right. \\ &\quad \left. -\frac{1}{2}D_0\Lambda^{ab}\pi^t_{ab} + \epsilon^a J_a - D_0\epsilon^a\pi^t_a\right] \\ &= \int d^4x \left(\frac{1}{2}\Lambda^{ab}\tilde{J}_{ab} - \frac{1}{2}\dot{\Lambda}^{ab}\pi^t_{ab} + \epsilon^a\tilde{J}_a - \dot{\epsilon}^a\pi^t_a\right), \end{aligned} \tag{5.7}$$

where we redefined $J_{ab} \to J_{ab} - e_{at}\pi^t_b + e_{bt}\pi^t_a$ and $J_a \to J_a + \omega_{ta}{}^b\pi^t_b$ in order to covariantize the $e \cdot \pi$ term. It is equivalent to the redefinition of the multipliers, so that

$$\begin{aligned} \tilde{J}_{ab} &= J_{ab} - e_{at}\pi^t_b + e_{bt}\pi^t_a + \omega^c_{ta}\pi^t_{cb} - \omega^c_{tb}\pi^t_{ac}, \\ \tilde{J}_a &= J_a + \omega_{ta}{}^b\pi^t_b. \end{aligned} \tag{5.8}$$

The parameters $\dot{\Lambda}^{ab}$ and $\dot{\epsilon}^a$ are required by Castellani's construction of the generators [31] (for an alternative method, see [32,33]) to replace the independent parameters by the first-class constraints π^t_{ab} and π^t_a. A reason for this is that in the Hamiltonian formalism, all PB are taken at the same time, and the time derivatives of parameters are treated as the new, independent functions; for example, $D_0\Lambda^{ab}$ is linearly independent of Λ^{ab}. In addition, the Castellani method gives a procedure to determine these parameters in a way that recovers covariance of the Lagrangian theory. Direct calculation shows that, up to the background-dependent term $U \cdot \phi$, the given generators indeed satisfy Castellani's conditions.

The gauge transformations generated by $G[\Lambda, \dot{\Lambda}, \epsilon, \dot{\epsilon}]$ have the form

$$\begin{aligned} \delta e^a_\mu &= \Lambda^{ab}e_{b\mu} - D_\mu\epsilon^a, \\ \delta\omega^{ab}_\mu &= -D_\mu\Lambda^{ab}. \end{aligned} \tag{5.9}$$

The $\Lambda^{ab}(x)$ is recognized as a Lorentz gauge parameter. The local transformations with the parameter $\epsilon^a(x)$ are related to the diffeomorphisms on shell, and their explicit form cannot be written because it depends on the background.

The nonvanishing brackets between the constraints $\{\tilde{J}_{ab}, \tilde{J}_a, \pi^t_a, \pi^t_{ab}, T^a_{ij}, \phi^i_a, \phi^i_{ab}\}$ contain the Lorentz algebra

$$\{\tilde{J}_{ab}, \tilde{J}'_{cd}\} = (\eta_{ad}\tilde{J}_{bc} + \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac})\delta, \tag{5.10}$$

where the brackets with $\tilde{J}_{ab}$ vanish weakly with all other constraints, so they are explicitly first class,

$$\begin{aligned}
\{\tilde{J}_{ab},\pi'^t_c\} &= (\eta_{bc}\pi^t_a-\eta_{ac}\pi^t_b)\delta,\\
\{\tilde{J}_{ab},\pi'^t_{cd}\} &= (\eta_{ad}\pi^t_{bc}+\eta_{bc}\pi^t_{ad}-\eta_{ac}\pi^t_{bd}-\eta_{bd}\pi^t_{ac})\delta,\\
\{\tilde{J}_{ab},\tilde{J}'_c\} &= (\eta_{bc}\tilde{J}_a-\eta_{ac}\tilde{J}_b)\delta,\\
\{\tilde{J}_{ab},\phi'^i_c\} &= (\eta_{bc}\phi^i_a-\eta_{ac}\phi^i_b)\delta,\\
\{\tilde{J}_{ab},\phi'^i_{cd}\} &= (\eta_{ad}\phi^i_{bc}+\eta_{bc}\phi^i_{ad}-\eta_{ac}\phi^i_{bd}-\eta_{bd}\phi^i_{ac})\delta,\\
\{\tilde{J}_{ab},T'^c_{ij}\} &= (\delta^c_bT_{aij}-\delta^c_aT_{bij})\delta. \qquad (5.11)
\end{aligned}$$

For completeness, we also list the other nonvanishing brackets among the constraints,

$$\{\tilde{J}_a,\tilde{J}'_b\} = -\frac{15\alpha_0}{4\alpha_2}\Omega^{ij}_{abij}\delta,$$

where $\Omega^{ij}_{abij}=\Omega^{ij}_{abcd}e^c_ie^d_j$ and, by introducing $K^i_{abc}=4\alpha_2\epsilon^{ijkl}\epsilon_{abcde}\omega^d_{j\,f}R^{fe}_{kl}+\eta_{ab}\pi^i_c-\eta_{ac}\pi^i_b$,

$$\begin{aligned}
\{\tilde{J}_a,\pi'^t_{bc}\} &= (\eta_{ab}\pi^t_c-\eta_{ac}\pi^t_b)\delta,\\
\{\tilde{J}_a,\phi'^i_b\} &= -120\alpha_0|e|(e^t_ae^i_b-e^t_be^i_a)\delta,\\
\{\tilde{J}_a,\phi'^i_{bc}\} &= \Omega^{ij}_{abc}\partial_j\delta+K^i_{abc}\delta,\\
\{\tilde{J}_a,T'^b_{ij}\} &= R^b{}_{aij}\delta,\\
\{T^a_{ij},\phi'^l_c\} &= -\delta^a_c\delta^{lk}_{ij}\partial_k\delta+(\omega^a_{i\ c}\delta^l_j-\omega^a_{j\ c}\delta^l_i)\delta,\\
\{\phi^j_{ab},T'^c_{kl}\} &= (-e_{dl}\delta^{cd}_{ab}\delta^j_k+e_{dk}\delta^{cd}_{ab}\delta^j_l)\delta. \qquad (5.12)
\end{aligned}$$

As already mentioned, the symplectic form in PL gravity is nonlinear in the curvature so its rank depends on the particular background. This implies that the second-class constraints cannot, in general, be separated from the first-class constraints. The constraints whose brackets do not vanish explicitly on the constraint surface are the ones given by Eq. (5.12).

VI. HAMILTONIAN ANALYSIS OF PL GRAVITY IN $(d+1)$ DIMENSIONS

In this section we only give the main results of the Hamiltonian analysis in $d+1$ dimensions and point out the differences with respect to the five-dimensional case. The generalized coordinates q^M, momenta π_M and primary constraints have the form (4.4)–(4.6), where now the indices run in the wider range $i=1,\ldots,d$ and $a=0,\ldots,d$. The symplectic matrix has the components

$$\begin{aligned}
\Omega^{ij}_{abcd} &= 4(p-1)\beta_p\epsilon^{ijki_4\cdots i_d}\epsilon_{abcda_4\cdots a_d}R^{a_4a_5}_{i_4i_5}\cdots\\
&\quad\times R^{a_{2p-2}a_{2p-1}}_{i_{2p-2}i_{2p-1}}T^{a_{2p}}_{ki_{2p}}e^{a_{2p+1}}_{i_{2p+1}}\cdots e^{a_d}_{i_d},\\
\Omega^{ij}_{abc} &= \beta_p\epsilon^{iji_3\cdots i_d}\epsilon_{abca_3\cdots a_d}R^{a_3a_4}_{i_3i_4}\cdots R^{a_{2p-1}a_{2p}}_{i_{2p-1}i_{2p}}e^{a_{2p+1}}_{i_{2p+1}}\cdots e^{a_d}_{i_d}, \qquad (6.1)
\end{aligned}$$

where $\beta_p=-2^{2-p}(d+1-2p)p\alpha_p$ is a real constant. The matrix Ω^{ij}_{abcd} is identically zero only in the Einstein-Hilbert gravity ($p=1$). In general ($p>1$), the matrix Ω^{ij}_{abcd} only weakly vanishes for the PL gravity (2.10). The phase-space functions $\mathcal{L}^i_{ab}$, $\mathcal{S}_a$and $\mathcal{S}_{ab}$ in higher dimensions become

$$\begin{aligned}
\mathcal{L}^i_{ab} &= -\frac{1}{d+1-2p}\Omega^{ij}_{abc}e^c_j,\\
\mathcal{S}_a &= \mathcal{H}_a+D_i\lambda^i_a,\\
\mathcal{S}_{ab} &= \mathcal{H}_{ab}+e_{bi}\lambda^i_a-e_{ai}\lambda^i_b, \qquad (6.2)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_{ab} &= -\frac{1}{2}\Omega^{ij}_{abc}T^c_{ij},\\
\mathcal{H}_a &= (d+1)\alpha_0\epsilon_{aa_1\cdots a_d}\epsilon^{i_1\cdots i_d}e^{a_1}_{i_1}\cdots e^{a_d}_{i_d}-\frac{1}{4p}\Omega^{ij}_{abc}R^{bc}_{ij}. \qquad (6.3)
\end{aligned}$$

Using the Hamiltonian (4.11) and (4.12), we find the following secondary constraints from the condition of vanishing $\dot{p}^a_{ij}$, $\dot{\phi}^t_a$ and $\dot{\phi}^t_{ab}$,

$$T^a_{ij}\approx 0, \qquad \mathcal{S}_a\approx 0, \qquad (6.4)$$

$$\mathcal{S}_{ab}\approx e_{bi}\lambda^i_a-e_{ai}\lambda^i_b\approx 0. \qquad (6.5)$$

The requirement of vanishing $\dot{\phi}^i_a$ and $\dot{p}^a_i$ solves the multipliers v^i_a and u^a_i,

$$v^i_a = e^b_t\Sigma^i_{ab}+\frac{1}{2}\Omega^{ij}_{cda}U^{cd}_j+\omega^b_{t\ a}\lambda^i_b+D_j\lambda^{ij}_a\approx 0, \qquad (6.6)$$

$$u^a_i = D_ie^a_t-\omega^{ab}_te_{bi}, \qquad (6.7)$$

where it was convenient to define $U^{ab}_i=u^{ab}_i-D_i\omega^{ab}_t$ and

$$\begin{aligned}
\Sigma^i_{ab} &= \epsilon_{aba_2\cdots a_d}\epsilon^{ii_2\cdots i_d}\Big[-d(d+1)\alpha_0e^{a_2}_{i_2}\cdots e^{a_{2p+1}}_{i_{2p+1}}\\
&\quad+\frac{d-2p}{4p}\beta_pR^{a_2a_3}_{i_2i_3}R^{a_4a_5}_{i_4i_5}\cdots R^{a_{2p}a_{2p+1}}_{i_{2p}i_{2p+1}}\Big]e^{a_{2p+2}}_{i_{2p+2}}\cdots e^{a_d}_{i_d}. \qquad (6.8)
\end{aligned}$$

In odd-dimensional spaces with $d=2p$, the last line in Σ^i_{ab} vanishes. This is the case of five-dimensional pure Gauss-Bonnet gravity analyzed in the previous sections.

We ask that the constraint $\phi^i_{ab}\approx 0$ vanishes during its time evolution in $(d+1)$-dimensional spacetime, leading to

$$\dot{\phi}^i_{ab} \approx -\frac{1}{2}\omega^{cd}_t(\Omega^{ij}_{cda}e_{bj} - \Omega^{ij}_{cdb}e_{aj} + \Omega^{ij}_{abc}e_{dj} - \Omega^{ij}_{ab,d}e_{cj}) + e_{ta}\lambda^i_b - e_{tb}\lambda^i_a - \lambda^{ij}_a e_{bj} + \lambda^{ij}_b e_{aj}. \quad (6.9)$$

However, the first line identically vanishes due to the combinatorial identity

$$\epsilon_{ba_1\cdots a_d}e_{aj} - \epsilon_{aa_1\cdots a_d}e_{bj} + \epsilon_{aba_2\cdots a_d}e_{a_1j} - \cdots + (-1)^{d+1}\epsilon_{abca_1\cdots a_{d-1}}e_{a_dj} = 0, \quad (6.10)$$

and the second line of (6.9) with Eq. (6.5) can be equivalently written as $\lambda^i_a \approx 0$ and $\lambda^{ij}_a \approx 0$, so that we find

$$\text{secondary constraints}: \mathcal{S}_a \approx 0, \quad T^a_{ij} \approx 0, \quad \lambda^i_a \approx 0, \quad \lambda^{ij}_a \approx 0. \quad (6.11)$$

Next, we require that the secondary constraints also evolve on the constraint surface. Thus, the requirement of vanishing $\dot{\lambda}^{ij}_a$ and $\dot{\lambda}^i_a$ solves the multipliers $v^{ij}_a = 0$ and $v^i_a = 0$, but because the form of v^i_a is already known from Eq. (6.6), we obtain the algebraic equation for the multipliers U^{cd}_j,

$$0 \approx \chi^i_a = e^b_t\Sigma^i_{ab} + \frac{1}{2}\Omega^{ij}_{cda}U^{cd}_j. \quad (6.12)$$

By replacing $U^{ab}_i = R^{ab}_{ti}$, we can prove that the above expression combined with $\mathcal{H}_a$ is equivalent to the Lagrangian equations,

$$0 \approx \chi^i_a = (\mathcal{H}_a, \chi^i_a) = (d+1)\alpha_0\epsilon_{aa_1\cdots a_d}\epsilon^{\lambda\mu_1\mu_2\cdots\mu_d}e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}\cdots e^{a_d}_{\mu_d} + \frac{\alpha_p}{2^p}(d+1-2p)\epsilon^{\lambda\mu_1\cdots\mu_d}\epsilon_{aa_1\cdots a_d}R^{a_1a_2}_{\mu_1\mu_2}\cdots \times R^{a_{2p-1}a_{2p}}_{\mu_{2p-1}\mu_{2p}}e^{a_{2p+1}}_{\mu_{2p+1}}\cdots e^{a_d}_{\mu_d}. \quad (6.13)$$

Further calculation can be simplified by observing that, as in five dimensions, the pairs of conjugated constraints, (λ^i_a, p^b_j) and $(\lambda^{ij}_a, p^b_{kl})$, are second class. This means they can be eliminated from the phase space by defining the reduced phase space Γ^*, where the Poisson brackets are replaced by the Dirac brackets (4.24). The coordinates of the space Γ^* are $(e^a_\mu, \omega^{ab}_\mu, \pi^\mu_a, \pi^\mu_{ab})$, and their Dirac brackets are equal to the Poisson brackets. From now on, we drop the star in the Dirac brackets and continue working on Γ^*.

The evolution of $\mathcal{S}_a$ can be obtained, after the long, but straightforward calculation, with the help of the identity $0 = D\epsilon_{aba_1\cdots a_d}$, which implies

$$\omega_{ta}{}^bR^{a_1a_2}_{i_1i_2}\Omega^{i_1i_2}_{ba_1a_2} = -\omega_{tf}{}^{a_1}[2p\Omega^{i_1i_2}_{aa_1a_2}R^{fa_2}_{i_1i_2} + (d-2p)\Omega^{i_2i_3k}_{aa_1a_2a_3}R^{a_2a_3}_{i_2i_3}e^f_k]. \quad (6.14)$$

Then, we find that $\mathcal{S}_a$ never leaves the constraint surface,

$$\dot{\mathcal{S}}_a \approx -D_i\chi^i_a - \omega_{ta}{}^b\mathcal{H}_b \approx 0. \quad (6.15)$$

Finally, the consistency condition of T^a_{ij} gives $\frac{(d+1)d(d-1)}{2}$ algebraic equations for $\frac{d^2(d+1)}{2}$ unknown functions $U^a{}_{ij} = U^{ab}_j e_{bi}$,

$$0 \approx \dot{T}^a_{ij} \approx R^{ab}_{ij}e_{bt} + U^a{}_{ji} - U^a{}_{ij}. \quad (6.16)$$

This form is the same as in five dimensions, so we skip the detailed analysis (4.31)–(4.36) and conclude that the antisymmetric parts $U^a{}_{[ij]}$ of the multipliers are solved, $U_{t(ij)}$ remain unknown and the others are not independent due to the Bianchi identity. The final expression for U^{ab}_i is given by Eq. (4.37). The result for the coefficients U can be summarized in the following table.

Multiplier $U_{\mu ij}$:	$U_{t[ij]}$	$U_{t(ij)}$	${}^AU_{kij}$	${}^SU_{kij}$	${}^TU_{kij}$
$\frac{d^2(d+1)}{2}$ components:	$\frac{d(d-1)}{2}$	$\frac{d(d+1)}{2}$	d	d	$\frac{d(d^2-d-4)}{2}$
Solved by:	$\dot{T}^0_{ij}$	arbitrary	$\dot{T}_{[ijk]}$	Bianchi	Bianchi

Solutions of $U_{t(ij)}$ depend on the equation $\chi^i_a = 0$ given by Eq. (6.12), which after using (4.37) becomes

$$M^{i(jm)}_a U_{tjm} = A^i_a. \quad (6.17)$$

The tensor M has the form (4.41). This is the $d(d+1) \times \frac{d(d+1)}{2}$ matrix of order $p-1$ in the Riemann tensor. The nonhomogenous part of the equation, on the other hand, is

$$A^i_a = \Sigma^i_{ab}e^b_t + \frac{1}{2}\Omega^{ij}_{abc}e^{kb}e^{lc}R_{tjkl}. \quad (6.18)$$

Higher-order dependence of M in the curvature means that its rank can change throughout the phase space. When $\Lambda \neq 0$, there is always the region of Γ^* where the rank of M is maximal, that is, $\frac{d(d+1)}{2}$ which enables us to solve all $\frac{d(d+1)}{2}$ coefficients $U_{t(ij)}$. This completes the constraint analysis, which has the same structure as in five dimensions. Arbitrary multipliers are associated with the first-class constraints, and the rest are second-class constraints.

Therefore, we have $N = \frac{(d+1)^2(d+2)}{2}$ fundamental fields $(e^a_\mu, \omega^{ab}_\mu)$ in the PL gravity on the reduced space,

$N_1 = (d+1)(d+2)$ first-class constraints $(J_a, J_{ab}, \pi^t_a, \pi^t_{ab})$ and $N_2 = d^2(d+1)$ second-class constraints $(\tilde{T}^a_{ij}, \phi^i_a, \phi^i_{ab})$. Therefore, the number of physical fields in the bulk in this particular background is

$$N^* = \frac{(d+1)(d-2)}{2}. \qquad (6.19)$$

In other backgrounds we can have less degrees of freedom, so that the number of degrees of freedom in a higher-dimensional PL gravity is $0 \leq N^* \leq \frac{(d+1)(d-2)}{2}$. The first-class constraints and gauge generators have the same form as before; only the matrix M and the tensor A^a_μ that appear in (4.40) are of order $p-1$ in the curvature, and we will not write them explicitly—it is straightforward to repeat the previous calculation here.

VII. DISCUSSION

We performed a Hamiltonian analysis of PL gravity in any dimension $D \geq 5$. This Lovelock gravity is not a mere correction of the Einstein-Hilbert theory because it does not even contain the linear term in the scalar curvature. Instead, its kinetic term is described by a pth-order polynomial in the Riemann tensor such that the equations of motion remain of second order in the metric. When the cosmological constant is included, PL gravity has the unique dS and/or AdS vacuum.

The first-order formalism was used to deal with nonlinearities involved in the theory. We ensured that spacetime is Riemannian by introducing the constraint that forced the torsion to vanish.

The detailed analysis revealed that the number of symmetries and degrees of freedom in this theory depends on the background. In the generic case, which includes (A) dS space and spherically symmetric, static black holes, the theory contains $D(D-3)/2$ degrees of freedom, which is the same as in general relativity. But in contrast to relativity, a change of the background can increase the amount of local symmetries in the theory and convert previously physical fields into nonphysical ones, even leading to a topological theory (with no degrees of freedom in the bulk). This is typical for Lovelock theories. In the PL case, this change of degrees of freedom is kept under control through the matrix M, whose rank can be between 0 and $D(D-1)/2$, which yields between 0 and $D(D-3)/2$ degrees.

A constraint analysis probes a number of physical components of the metric field $g_{\mu\nu}$, which is directly related to the Riemann tensor. Its relation to the PL Riemann tensor is indirect and not anchored to any metric or connection in a straightforward way. It turns out that the maximum possible number of physical fields does not depend on a particular Lovelock theory, as was pointed out earlier in Ref. [17]. This reflects the fact that so long as the equations of motion are second order, the metric degrees of freedom would be the same for Einstein as well as Lovelock theories.

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APPENDIX: CONVENTIONS

We use the signature of the Minkowski metric $\eta_{ab} = \mathtt{diag}(-+++\cdots+)$.

The Levi-Civita symbols in $d+1$ and d dimensions are defined by

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d+1}} = \epsilon^{\mu_1 \cdots \mu_{d+1}} d^{d+1}x, \qquad \epsilon^{t i_1 i_2 \cdots i_d} = \epsilon^{i_1 i_2 \cdots i_d}. \qquad (A1)$$

The generalized Kronecker delta of rank s is constructed as the determinant

$$\delta^{\nu_1 \cdots \nu_s}_{\mu_1 \cdots \mu_s} = \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{\nu_s}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_s}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_s} & \delta^{\nu_2}_{\mu_s} & \cdots & \delta^{\nu_s}_{\mu_s} \end{vmatrix}. \qquad (A2)$$

If the range of indices is D, a contraction of $k \leq s$ indices in the Kronecker delta of rank s produces a delta of rank $s-k$,

$$\delta^{\nu_1 \cdots \nu_k \cdots \nu_s}_{\mu_1 \cdots \mu_k \cdots \mu_s} \delta^{\mu_1}_{\nu_1} \cdots \delta^{\mu_k}_{\nu_k} = \frac{(D-s+k)!}{(D-s)!} \delta^{\nu_{k+1} \cdots \nu_s}_{\mu_{k+1} \cdots \mu_s}. \qquad (A3)$$

Other identities involving the Levi-Civita symbol and the generalized Kronecker delta are

$$\begin{aligned} \epsilon_{\nu_1 \ldots \nu_{d+1}} \epsilon^{\mu_1 \ldots \mu_{d+1}} &= -\delta^{\mu_1 \ldots \mu_{d+1}}_{\nu_1 \ldots \nu_{d+1}}, \\ \epsilon_{a_1 \cdots a_{d+1}} e^{a_1}_{\mu_1} \cdots e^{a_{d+1}}_{\mu_{d+1}} &= |e| \epsilon_{\mu_1 \ldots \mu_{d+1}}, \end{aligned} \qquad (A4)$$

where $|e| = \det[e^a_\alpha]$.

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