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Uniqueness of photon sphere for Einstein-Maxwell-dilaton black holes with arbitrary coupling constant

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The uniqueness of a static asymptotically flat photon sphere for a static black hole solution in the Einstein-Maxwell-dilaton theory with an arbitrary coupling constant is proposed. Using the conformal positive energy theorem, we show that the dilaton photon sphere subject to the nonextremality condition constitutes a cylinder over a topological sphere.

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I. INTRODUCTION

Gravity theories like general relativity and its modifications predict the existence of a spacetime region in which photon orbits are closed. The aforementioned regions play an essential role in gravitational lensing, one of the main tools in astrophysical observations [1–6].

The photon sphere can be considered a timelike hypersurface on which the bending angle is unboundedly large. Compact objects like black holes, neutron stars, wormholes, and others ought to be, in principle, surrounded by a photon sphere. On the other hand, as was revealed in Refs. [7–9], the photon spheres are connected with quasinormal modes for the compact objects in question. Their presence is the main factor in their stability.

Moreover, it was found that photon spheres possess some very intriguing features, such as the lapse function constancy. They are totally umbilical hypersurfaces with constant mean curvature and constant surface gravity [10–12]. The properties in question very much resemble the characteristic features of black hole event horizons. In the case of black hole physics, the no-hair theorem's mathematical formulation, the uniqueness theorem resolves the problem of classification of domains of outer communication of suitably regular black hole spacetimes.

The first attempts to classify nonsingular static black hole solutions in Einstein gravity were undertaken in [13], and some others mathematical refinements were presented in Refs. [14–20]. The complete classification of static vacuum and electrovacuum black hole solutions was finished in [21,22], where the condition of nondegeneracy of the event horizon was removed and it was proved that all degenerate components of the black hole event horizon have charges of the same signs. As far as stationary axisymmetric black holes are concerned, the problem turned out to be far more complicated [23] and the complete uniqueness proof was achieved by Mazur [24] and Bunting [25] (for a review of the uniqueness of black hole solutions, see [26] and references therein). Contemporary unification schemes such as M/string theories triggered the efforts to classify higher-dimensional charged black holes with both nondegenerate and degenerate components of the event horizon, which was proposed in Refs. [27–29]. On the other hand, some progress concerning the nontrivial case of the n-dimensional rotating black object (black holes, black rings, or black lenses) uniqueness theorem was presented in [30], while the behavior of matter fields in the spacetime of higher-dimensional black holes was examined in [31]. The desire to construct a consistent quantum gravity theory also raised interest in mathematical aspects of black holes in the low-energy limit of the string theories and supergravity [32,33]. Various modifications of Einstein gravity such as the Gauss-Bonnet extension were examined from the point of view of the black hole uniqueness theorem. The strictly stationary static vacuum spacetimes was discussed in [34], while it turned out that up to the small curvature limit, the static uncharged or electrically charged Gauss-Bonnet black hole is diffeomorphic to the Schwarzschild-Tangherlini or Reissner-Nordström black hole solution, respectively [35]. For black holes appearing in Chern-Simons modified gravity, it was proved that a static asymptotically flat black hole solution was unique in Schwarzschild spacetime [36], while the electrically charged black hole in the theory in question was diffeomorphic to the Reissner-Nordström black hole [37].

The analogy of the photon sphere and the black hole event horizon arises as a tantalizing question if the presence of a photon sphere uniquely characterizes the spacetimes with asymptotical charges. This problem was tackled for the first time in [12], where it was shown that an asymptotically flat vacuum Einstein equation with mass and photon sphere is isometric to the Schwarzschild solution characterizing the same mass. Recently, modified version of arguments presented by Bunting and Massod [38] were applied in the proof of the uniqueness of the Einstein vacuum photon sphere and the electrovacuum one [39]. On the other hand, the uniqueness of the static

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Einstein scalar and Einstein-Maxwell spacetimes with a photon sphere were given in Refs. [40,41]. In Ref. [42], the general classification of photon spheres (covering black hole and non-black-hole spacetimes) in Einstein-Maxwell-dilaton gravity with arbitrary coupling constant was proposed, subject to the auxiliary condition that the lapse function regularly foliates the spacetime outside the photon sphere. In our paper, we relax the additional condition and consider the case of the uniqueness theorem for the dilaton black hole photon sphere in the dilaton gravity with arbitrary coupling constant. Namely, we shall pay attention to the system described by the standard action

$$I = \int d^4x\sqrt{-\hat{g}}[{}^{(4)}R - e^{-2\alpha\phi}F_{\mu\nu}F^{\mu\nu} - 2\nabla_\mu\phi\nabla^\mu\phi], \quad (1)$$

where $\hat{g}_{ij}$ stands for the four-dimensional metric tensor, $F_{\mu\nu}$ is the strength tensor of the $U(1)$-gauge Maxwell field, ϕ is the dilaton field, and α is the coupling constant. Because our considerations will be bounded with the static spacetime, let us suppose that there exists a smooth Riemannian manifold and a smooth lapse function $N \to M^3 \to R^+$, such that $M^4 = R \times M^3$. The line element of the above manifold is provided by the following:

$$ds^2 = \hat{g}_{\alpha\beta}dx^\alpha dx^\beta = -N^2dt^2 + g_{ij}dx^i dx^j. \quad (2)$$

Moreover, we introduce an asymptotically timelike Killing vector field $k_\alpha = (\frac{\partial}{\partial t})_\alpha$, such that the $U(1)$-gauge Maxwell and the dilaton field are invariant under the action generated by this Killing vector field, i.e., $\mathcal{L}_kF_{\alpha\beta} = 0$, $\mathcal{L}_k\phi = 0$. The above conditions define the $U(1)$ gauge field and dilaton field staticity. The above notions of staticities, i.e., metric staticity and field staticity, are consistent. It follows from the fact that one has the Ricci-static spacetime which means that the Ricci one-form is proportional to the Kiling vector field k_α, which directly follows from the equations of motion and the field staticity. On the other hand, it can be proved that the static spacetime is Ricci-static [26]. We shall assume, further, that the three-dimensional submanifold (M^3, g_{ij}) is simply connected. This fact enables one to define the electric field having potential ψ in the standard form $E_\beta = -F_{\beta\gamma}k^\gamma = \nabla_\beta\psi$. For the reader's convenience, we also quote the equations of motion for the system in question,

$${}^{(g)}\nabla_i{}^{(g)}\nabla^iN = \frac{e^{-2\alpha\phi}}{N}{}^{(g)}\nabla_i\psi^{(g)}\nabla^i\psi, \quad (3)$$

$$N^{(g)}\nabla_i{}^{(g)}\nabla^i\psi = {}^{(g)}\nabla_i\psi^{(g)}\nabla^iN + 2\alpha N^{(g)}\nabla_k\psi^{(g)}\nabla^k\phi, \quad (4)$$

$$N^{(g)}\nabla_iN^{(g)}\nabla^i\phi + N^{2(g)}\nabla_m{}^{(g)}\nabla^m\phi - \alpha e^{-2\alpha\phi(g)}\nabla_c\psi^{(g)}\nabla^c\psi = 0, \quad (5)$$

$$\begin{aligned}{}^{(g)}R_{ij} - \frac{1}{N}{}^{(g)}\nabla_i{}^{(g)}\nabla_jN &= \frac{e^{-2\alpha\phi}}{N^2}(g_{ij}{}^{(g)}\nabla_m\psi^{(g)}\nabla^m\psi - 2^{(g)}\nabla_i\psi^{(g)}\nabla_j\psi)\\ &\quad + 2^{(g)}\nabla_i\phi^{(g)}\nabla_j\phi, \end{aligned} \quad (6)$$

where the covariant derivative with respect to the metric tensor g_{ij} is denoted by ${}^{(g)}\nabla$, while ${}^{(g)}R_{ij}$ is the Ricci tensor defined in M^3 space.

Our paper is organized as follows. In Sec. II, after describing the basic features of the *photon sphere* in Einstein-Maxwell-dilaton (EMD) gravity with arbitrary coupling constant, we conduct the uniqueness proof using the conformal positive energy theorem. Section III concludes our investigations.

II. UNIQUENESS

Before we proceed to the main subject of our work, let us recall some basic facts which will be useful in our construction of the proof. First of all, let us assume that the spacetime under consideration will be asymptotically flat, which means that the spacetime contains a data set $(\Sigma_{\text{end}}, g_{ij}, K_{ij})$ with gauge fields such that Σ_{end} is diffeomorphic to R^3 minus a ball and the following asymptotic conditions are provided:

$$\begin{aligned}&|g_{ij} - \delta_{ij}| + r|\partial_a g_{ij}| + \cdots + r^k|\partial_{a_1\ldots a_k}g_{ij}|\\ &\quad + r|K_{ij}| + \cdots + r^k|\partial_{a_1\ldots a_k}K_{ij}| \leq \mathcal{O}\left(\frac{1}{r}\right),\end{aligned} \quad (7)$$

$$|F_{\alpha\beta}| + r|\partial_aF_{\alpha\beta}| + \cdots + r^k|\partial_{a_1\ldots a_k}F_{\alpha\beta}| \leq \mathcal{O}\left(\frac{1}{r^2}\right), \quad (8)$$

$$\phi + r\partial_a\phi + r^k\partial_{a_1\ldots a_k}\phi \leq \mathcal{O}\left(\frac{1}{r^2}\right). \quad (9)$$

We recall that an embedded timelike hypersurface will be called a photon surface if any null geodesics initially tangent to it, remain tangent as long as it exists. On the other hand, by the "photon sphere," we mean a photon surface for which the lapse function N is constant and the auxiliary conditions for the fields emerging in the theories in question are satisfied. It turns out that an arbitrary spherically symmetric static spacetime admits a photon sphere subject to the condition [10]

$$g_{tt}\partial_r g_{\theta\theta} = g_{\theta\theta}\partial_r g_{tt}. \quad (10)$$

Our main aim is to study various features of a photon sphere in the spacetime of the electrically charged dilaton black hole, where the line element is given by

$$ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} dt^2 + \frac{dr^2}{(1-\frac{r_+}{r})(1-\frac{r_-}{r})^{\frac{1-\alpha^2}{1+\alpha^2}}} + r\left(r-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}(d\theta^2+\sin^2\theta d\varphi^2). \quad (11)$$

The dilaton black hole event horizon is located at r_+, while in the case of r_-, we have another singularity, but it can be ignored it because of the fact that $r_- < r_+$. On the other hand, the dilaton field is given by the relation $e^{2\phi} = e^{-2\phi_0}(1-\frac{r_-}{r})^{\frac{2\alpha}{1+\alpha^2}}$, where ϕ_0 is the dilaton field value as $r \to \infty$. The mass M and the charge Q are related by the relations $M = \frac{r_+}{2} + \frac{1-\alpha^2}{1+\alpha^2}\frac{r_-}{2}$ and $Q^2 = \frac{r_+ r_-}{2} e^{2\phi_0}$. For such a black hole, the photon sphere lies outside the black hole event horizon r_+ and forms the timelike hypersurface at $r = r_{phs}$ [10]:

$$r_{phs} = \frac{1}{4}\left[\left(\frac{3-\alpha^2}{1+\alpha^2}\right)r_- + 3r_+\sqrt{\left(\left(\frac{3-\alpha^2}{1+\alpha^2}\right)r_- + 3r_+\right)^2 - \frac{32 r_+ r_-}{1+\alpha^2}}\right]. \quad (12)$$

It can be seen that for $r_+ > A r_-$, where $A = \frac{2(7+3\alpha^2)(1+\alpha^2)}{(3-\alpha^2)^2+9(1+\alpha^2)}$, we obtain a single timelike photon sphere.

In such a spacetime, we define a dilaton-electric static system as a time slice of the static spacetime $(R\times M^3, -N^2dt^2 + g_{ij}dx^i dx^j)$. Then one defines the photon surface, the main ingredient of our considerations. Namely, let $(M^3, g_{ij}, N, \psi, \phi)$ be a dilaton-electric system bounded with a static spacetime defined above, with the line element given by Eq. (2). Keeping in mind the aforementioned definition of a photon sphere, it will be subject to a timelike embedded hypersurface $(P^3, h_{ij}) \hookrightarrow (R\times M^3, -N^2dt^2 + g_{ij}dx^i dx^j)$ if the embedding is umbilic and the lapse function, electric one-form, and dilaton form $d\phi$ are normal to P^3. The photon sphere emerges as the inner boundary of the spacetime in question [38]. Namely,

$$(P^3, h_{ij}) = (R\times\Sigma^2, -N^2dt^2 + \sigma_{ij}dx^i dx^j) = \cup_{i=1}^{I}(R\times\Sigma_i^2, -N_i^2dt^2 + \sigma_{ij}^{(i)}dx^i dx^j), \quad (13)$$

where each P_i^3 is a connected component of P^3.

In order to conduct the uniqueness proof of the photon sphere in EMD gravity, we shall follow the reasoning presented in [38]. In the first step, we define the dilaton electrostatic system $(M^3, g_{ij}, N, \psi, \phi)$ which will be asymptotic to the dilaton black hole solution and will have a Killing horizon boundary. It can be done by gluing pieces of the (spatial) dilaton black hole manifold of the dilaton field of adequate mass, charge, and value. In order to create a new horizon boundary corresponding to each Σ_i^2, we attach (glue) at each photon sphere base Σ_i^2 a neck piece of the dilaton black hole manifold of $\mu_i > 0, Q_i, \phi$ (the cylindrical piece) between the photon sphere in question and its event horizon. Away from the gluing surface, the manifold will have non-negative scalar curvature, it will be smooth, and the metric lapse function, electric potential, and dilaton field will also be smooth away from the glued surfaces. In the next step, we double the glued manifolds under consideration and assert that the emergent system will be smooth across the boundary. In the last step, we perform the adequate conformal transformations in order to apply the conformal positive energy theorem, which will complete the proof.

A. Asymptotically flat manifold with minimal boundary and non-negative scalar curvature

We commence with the definition of the Komar-type charge in the form provided by

$$Q_i = -\frac{1}{4\pi}\int_{\Sigma_i^2} dA \frac{e^{-\alpha\phi} n^a E_a^{(i)}}{N} = -\frac{e^{-\alpha\phi} r_i^2 n^a E_a^{(i)}}{N_i}, \quad (14)$$

where n^a is a unit normal to P^3. The above relation corresponds to the dilaton-elecric charge, while the definition of the dilaton charge yields

$$q_i = -\frac{1}{4\pi}\int_{\Sigma_i^2} dA n^j \nabla_j \phi^{(i)} = -r_i^2 n^j \nabla_j \phi^{(i)}. \quad (15)$$

On the other hand, using Eq. (36) from Ref. [42], the above definitions enable us to find that

$$\frac{4}{3} = H_i^2 r_i^2 + \frac{4}{3}\frac{e^{-2\alpha\phi}}{N_i^2} n^j E_k^{(i)} n^j E_k^{(i)} - \frac{4}{3} r_i^2 n^{m(g)} \nabla_m \phi^{(i)} n^{a(g)} \nabla_a \phi^{(i)}, \quad (16)$$

where H_i stands for the mean curvature of each ith component of P^3. The relation (16) can be rewritten in the form as follows:

$$\frac{4}{3} = H_i^2 r_i^2 + \frac{4}{3}\left(\frac{Q_i}{r_i}\right)^2 - \frac{4}{3}\left(\frac{q_i}{r_i}\right)^2. \quad (17)$$

Then, following Ref. [38], we define the mass μ_i on each Σ_i^2 and intervals I_i:

$$\mu_i = \frac{r_i}{3}, \qquad I_i = [s_i = 2\mu_i, r_i = r_{phs}(Q_i, q_i, \mu_i)] \subset R. \quad (18)$$

Next, we glue in to each boundary component of Σ_i^2 a cylinder $I_i \times \Sigma_i^2$. The photon sphere component in question, i.e., $\Sigma_i^2 \subset M^3$, is related to the level $\{r_i\} \times \Sigma_i^2$

of the above-constructed cylinder. In what follows, we shall call this surface still Σ_i^2. By virtue of the aforementioned procedure, we obtain the manifold $\tilde{M}^3$ which has the inner boundary

$$B = \cup_{i=1}^{I} \{s_i\} \times \Sigma_i^2.$$

In the next step, we shall build an electrodilaton system smooth away from the gluing surface Σ_i^2. It should be also geodesically complete up to the corresponding boundary B. Just on the cylinder $I_i \times \Sigma_i^2$, one defines the line element provided by

$$\begin{aligned} ds^2|_{I_i\times\Sigma_i^2} = \gamma_{ij}dx^i dx^j &= \frac{dr^2}{f_i^2(r)} + \frac{g_i(r)}{r_i^2}\sigma_i \\ &= \frac{dr^2}{f_i^2(r)} + g_i(r)d\Omega^2, \end{aligned} \tag{19}$$

where we have denoted

$$\begin{aligned} f_i(r) &= \left(1 - \frac{r_{+(i)}(Q_i,\mu_i)}{r}\right)\left(1 - \frac{r_{-(i)}(Q_i,\mu_i)}{r}\right)^{\frac{1-\alpha^2}{2(1+\alpha^2)}}, \\ g_i(r) &= r\left(r - \frac{r_{-(i)}(Q_i,\mu_i)}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \end{aligned} \tag{20}$$

and $\sigma_i = r_i^2 d\Omega$.

To conclude, it was glued in the portion of the spatial dilaton black hole system possessing mass $\mu_i > 0$ and charge Q_i subject to the nonextremality condition. It was done from the gluing surface to the photon sphere in question.

The next problem will be to show that $\gamma_{ij}|_{I_i\times\Sigma_i^2}$ is smooth away from the gluing surface, and it is a function of the $C^{1,1}$ class across Σ_i^2. Let us introduce the function [38]

$$\xi : \tilde{M}^3 \to R : p \to \begin{cases} N(p) & \text{if } p \in M^3, \\ \frac{3m_i}{r_i} f_i(r(p)) & \text{if } p \in I_i \times \Sigma_i^2. \end{cases} \tag{21}$$

One will apply it as a smooth collar function across the gluing surfaces. By the construction, the function ξ is smooth away from Σ_i^2, for all $i \in \{1,\ldots,I\}$. The choice of the conformal factor $\frac{3m_i}{r_i}$ as well as the relation-binding ith mean curvature with N_i, i.e., $N_i H_i = 2n^{(a)(g)}\nabla_a N_i$ (for the derivation of this equation see Refs. [38,41]) imply that ξ has the same constant value at each of the sides of Σ_i^2. Hence, it is well defined across Σ_i^2.

The unit normal to Σ_i^2 towards the dilaton black hole side has the form

$$n_r = \xi_i(r_i)\partial_r. \tag{22}$$

The definitions of m_i, μ_i and charges ensure that the normal derivative of ξ is the same positive constant on both sides of Σ_i^2. It means that this fact allows one to implement the function ξ as a smooth coordinate function in the neighborhood of each analyzed $\Sigma_i^2 \subset \tilde{M}^3$.

In order to show that ξ is a $C^{1,1}$ class function, we shall take into account local coordinates on Σ_i^2 as well as a flow to a neighborhood of $\Sigma_i^2 \subset \tilde{M}^3$ along the level set flow defined by ξ. Then it is enough to show that, for all $A, K = 1, 2$, the components of the metric tensor $\tilde{q}_{AB}, \tilde{q}_{A\xi}, \tilde{q}_{\xi\xi}$ are $C^{1,1}$ class functions, with respect to the local coordinates (x_A, ξ) across the aforementioned ξ function level set of Σ_i^2.

Because of the fact that ∂_ξ is given by

$$\partial_\xi = \frac{1}{n^{a(\tilde{q})}\nabla_a \xi} n^{j(\tilde{q})}\nabla_j, \tag{23}$$

the continuity of $\tilde{q}_{ij}$ in the (x_A, ξ) coordinate system and smoothness in the tangential directions along Σ_i^2 is seen. Then the metric tensor components imply

$$\tilde{q}_{AB} = r_i^2 \Omega_{AB}, \quad \tilde{q}_{A\xi} = 0, \quad \tilde{q}_{\xi\xi} = \frac{1}{(n^{a(\tilde{q})}\nabla_a \xi)^2}, \tag{24}$$

on Σ_i^2 (from both sides).

Further, we calculate the derivative of $\tilde{q}_{AB}$. It yields

$$\partial_\xi(\tilde{q}_{AB}) = \frac{2}{n^{a(\tilde{q})}\nabla_a \xi}\tilde{h}_{AB}. \tag{25}$$

In the proceeding sections, we show the umbilicity of every component of any dilaton photon sphere, as well as the fact that the mean curvature of every photon sphere is determined by its radius and charges (up to the signs). Having in mind the exact form of the metric tensor $\tilde{q}_{ij}$, one can conclude that $\tilde{h} = \pm 1/2 H_i \sigma_i = \pm 1/2 H_i r_i^2 \Omega$ hold on both sides of the dilaton photon sphere. As far as the sign is concerned, from the side of M^3, $H_i > 0$ (H_i is calculated with respect to n^a being directed towards the asymptotic end). On the dilaton black hole side, the mean curvature of the dilaton photon surface is directed towards infinity and, thus, into M^3. It is also positive. Therefore, in both considered cases, $\tilde{h}_{AB}$ and thus $\partial_\xi(\tilde{q}_{AB})$ coincide from the two sides of Σ_i^2 [38,39].

The relation $\partial_\xi(\tilde{q}_{A\xi}) = 0$ holds on both sides of Σ_i^2 (by the construction $\tilde{q}_{A\xi} = 0$). It remains to show that $\partial_\xi(\tilde{q}_{\xi\xi})$ coincides on both sides of the hypersurface in question. In order to do so, let us calculate

$$\partial_\xi(\tilde{q}_{\xi\xi}) = -\frac{2}{(n^{a(\tilde{q})}\nabla_a \xi)^5}(n^{j(\tilde{q})}\nabla_j)(n^{b(\tilde{q})}\nabla_b \xi) \tag{26}$$

from both sides of Σ_i^2.

Let us recall that for the isometric embedding with a unit normal n_i and the second fundamental form K_{ab}, for every smooth function θ, we have

$$(D, h_{ij}, A^n) \hookrightarrow (\nabla, g_{ij}, B^{n+1}),$$
$$\nabla_a \nabla^a \theta = D_m D^m \theta + (n^a \nabla_a)(n^j \nabla_j)\theta + K_a{}^a n^i \nabla_i \theta. \tag{27}$$

Using the identity (27), we obtain

$$(n^{k(\tilde{q})}\nabla_k)(n^{a(\tilde{q})}\nabla_a \xi) = {}^{(\tilde{q})}\nabla_a {}^{(\tilde{q})}\nabla^a N - H_i n^{c(\tilde{q})}\nabla_c N. \tag{28}$$

By virtue of the fact that ξ is constant on Σ_i^2 and having in mind equation of motion (3), we receive

$$(n^{k(\tilde{q})}\nabla_k)(n^{a(\tilde{q})}\nabla_a \xi) = \frac{e^{-2\alpha\phi}}{N} {}^{(\tilde{q})}\nabla_m \psi {}^{(\tilde{q})}\nabla^m \psi - H_i n^{c(\tilde{q})}\nabla_c N, \tag{29}$$

on both sides of Σ_i^2. Moreover, we recall that $n^{a(\tilde{q})}\nabla\psi, N, H_i, n^{a(\tilde{q})}\nabla_a \xi$ are continuous across Σ_i^2, which in turn implies that $(n^{a(\tilde{q})}\nabla_a)(n^{a(\tilde{q})}\nabla_a \xi)$ is continuous across Σ_i^2. Consequently, we concludes that $\tilde{q}$ is a $C^{1,1}$ class across Σ_i^2 and, for arbitrary $i \in \{1, \ldots, I\}$, the set $(\tilde{M}^3, \tilde{q}_{AB}, N, \psi, \phi)$ belongs to the same class.

B. Conformal transformations leading to non-negativity of scalar curvature and vanishing of the ADM mass

In this section, we shall consider basic conformal transformations which lead to the conformal positive theorem being the key ingredient in the proof of the uniqueness of the black hole dilaton *photon sphere* [32]. For the brevity of the notation, in this section, we write Σ instead of Σ_i^2.

To proceed further, let us introduce the definitions of the crucial quantities in the proof of the uniqueness. Namely, they can be written as follows:

$$\Phi_1 = \frac{1}{2}\left[e^{\alpha\phi}N + \frac{1}{e^{\alpha\phi}N} - (1+\alpha^2)\frac{\psi^2}{e^{\phi}N}\right], \tag{30}$$

$$\Phi_0 = \sqrt{1+\alpha^2}\frac{\psi}{e^{\alpha\phi}N}, \tag{31}$$

$$\Phi_{-1} = \frac{1}{2}\left[e^{\alpha\phi}N - \frac{1}{e^{\alpha\phi}N} - (1+\alpha^2)\frac{\psi^2}{e^{\phi}N}\right], \tag{32}$$

and

$$\Psi_1 = \frac{1}{2}\left[e^{-\frac{\phi}{\alpha}}N + \frac{e^{\frac{\phi}{\alpha}}}{N}\right], \tag{33}$$

$$\Psi_{-1} = \frac{1}{2}\left[e^{-\frac{\phi}{\alpha}}N - \frac{e^{\frac{\phi}{\alpha}}}{N}\right]. \tag{34}$$

It worth pointing out that defining the metric tensor $\eta_{AB} = \mathrm{diag}(1, -1, -1)$, we get that $\Phi_A \Phi^A = \Psi_A \Psi^A = -1$, where $A = -1, 0, 1$. Having in mind the conformal transformation provided by

$$\tilde{g}_{ij} = N^2 g_{ij}, \tag{35}$$

one can introduce the symmetric tensors written in terms of Φ_A in the following form:

$$\tilde{G}_{ij} = \tilde{\nabla}_i \Phi_{-1} \tilde{\nabla}_j \Phi_{-1} - \tilde{\nabla}_i \Phi_0 \tilde{\nabla}_j \Phi_0 - \tilde{\nabla}_i \Phi_1 \tilde{\nabla}_j \Phi_1, \tag{36}$$

and similarly for the potential Ψ_A,

$$\tilde{H}_{ij} = \tilde{\nabla}_i \Psi_{-1} \tilde{\nabla}_j \Psi_{-1} - \tilde{\nabla}_i \Psi_1 \tilde{\nabla}_j \Psi_1, \tag{37}$$

where by $\tilde{\nabla}_i$ we have denoted the covariant derivative with respect to the metric $\tilde{g}_{ij}$. Consequently, according to the relations (36) and (37), the field equations may be cast in the forms

$$\tilde{\nabla}^2 \Phi_A = \tilde{G}_i{}^i \Phi_A, \qquad \tilde{\nabla}^2 \Psi_A = \tilde{H}_i{}^i \Psi_A. \tag{38}$$

It can be verified by the direct calculations that the Ricci curvature tensor with respect to the conformally rescaled metric $\tilde{g}_{ij}$ is given by the relation

$$\tilde{R}_{ij} = \frac{2}{1+\alpha^2}(\tilde{G}_{ij} + \alpha^2 \tilde{H}_{ij}). \tag{39}$$

As far as the conformal positive energy theorem is concerned, one assumes that we have two asymptotically flat Riemannian three-dimensional manifolds $(\Sigma^\Phi, {}^{(\Phi)}g_{ij})$ and $(\Sigma^\Psi, {}^{(\Psi)}g_{ij})$. Moreover, we establish the conformal transformation of the form ${}^{(\Psi)}g_{ij} = \Omega^{2(\Phi)}g_{ij}$, connecting the adequate metric tensors of the manifolds in question. It implies that the corresponding masses obey the relation of the form ${}^{\Phi}m + \beta {}^{\Psi}m \geq 0$ if ${}^{(\Phi)}R + \beta\Omega^{2(\Psi)}R \geq 0$, for some positive constant β. The aforementioned inequalities are satisfied if the three-dimensional Riemannian manifolds are flat [43].

To proceed further, due to the requirement of the conformal positive energy theorem, we introduce conformal transformations fulfilling the following:

$${}^{(\Phi)}g_{ij}^{\pm} = {}^{(\Phi)}\omega_{\pm}^2 \tilde{g}_{ij}, \qquad {}^{(\Psi)}g_{ij}^{\pm} = {}^{(\Psi)}\omega_{\pm}^2 \tilde{g}_{ij}. \tag{40}$$

Their conformal factors are subject to the relations of the forms:

$${}^{(\Phi)}\omega_{\pm} = \frac{\Phi_1 \pm 1}{2}, \qquad {}^{(\Psi)}\omega_{\pm} = \frac{\Psi_1 \pm 1}{2}. \tag{41}$$

Next, we implement the standard procedure of pasting $(\Sigma_{\pm}^\Phi, {}^{(\Phi)}g_{ij}^{\pm})$ and $(\Sigma_{\pm}^\Psi, {}^{(\Psi)}g_{ij}^{\pm})$ across their shared minimal boundary [38]. We have four manifolds: $(\Sigma_+^\Phi, {}^{(\Phi)}g_{ij}^+)$, $(\Sigma_-^\Phi, {}^{(\Phi)}g_{ij}^-)$, $(\Sigma_+^\Psi, {}^{(\Psi)}g_{ij}^+)$, $(\Sigma_-^\Psi, {}^{(\Psi)}g_{ij}^+)$. Pasting them across shared minimal boundaries $\mathcal{B}^\Psi$ and $\mathcal{B}^\Phi$, one can construct complete regular hypersurfaces $\Sigma^\Phi = \Sigma_+^\Phi \cup \Sigma_-^\Phi$ and $\Sigma^\Psi = \Sigma_+^\Psi \cup \Sigma_-^\Psi$. Having two regular hypersurfaces, one has to check that each total gravitational mass on Σ^Φ and on Σ^Ψ vanishes.

In order to find this result, we shall use the conformal positive theorem [43]. On this account, it is customary to define another conformal transformation described by the relation

$$\hat{g}_{ij}^\pm = [({}^{(\Phi)}\omega_\pm)^2 ({}^{(\Psi)}\omega_\pm)^{2\alpha^2}]^{\frac{1}{2}} \tilde{g}_{ij}. \tag{42}$$

It follows that the Ricci curvature tensor on the space under consideration can be written in the form as

$$\begin{aligned}(1+\alpha^2)\hat{R}_\pm = {}& [{}^{(\Phi)}\omega_\pm^2 \, {}^{(\Psi)}\omega_\pm^{2\alpha^2}]^{-\frac{1}{2}} ({}^{(\Phi)}\omega_\pm^2 \, {}^{(\Phi)}R_\pm + {}^{(\Psi)}\omega_\pm^2 \, {}^{(\Psi)}R_\pm) \\ &+ \frac{2\alpha^2}{1+\alpha^2} (\hat{\nabla}_i \ln {}^{(\Phi)}\omega_\pm - \hat{\nabla}_i \ln {}^{(\Psi)}\omega_\pm) \\ &\times (\hat{\nabla}^i \ln {}^{(\Phi)}\omega_\pm - \hat{\nabla}^i \ln {}^{(\Psi)}\omega_\pm). \end{aligned} \tag{43}$$

Further, direct calculations reveal that Eq. (43) can be cast as follows:

$$\begin{aligned} &{}^{(\Phi)}\omega_\pm^2 \, {}^{(\Phi)}R_\pm + \alpha^2 \, {}^{(\Psi)}\omega_\pm^2 \, {}^{(\Psi)}R_\pm \\ &= 2\left| \frac{\Phi_0 \tilde{\nabla}_i \Phi_{-1} - \Phi_{-1} \tilde{\nabla}_i \Phi_0}{\Phi_1 \pm 1} \right|^2 \\ &\quad + 2\left| \frac{\Psi_0 \tilde{\nabla}_i \Psi_{-1} - \Psi_{-1} \tilde{\nabla}_i \Psi_0}{\Psi_1 \pm 1} \right|^2, \end{aligned} \tag{44}$$

with the terms on the right-hand side of the relation non-negative.

On the other hand, the conformal positive energy theorem enables us to claim that ${}^{(\Phi)}\omega = \text{const}{}^{(\Psi)}\omega$, as well as $\Phi_0 = \text{const}\,\Phi_{-1}$ and $\Psi_0 = \text{const}\,\Psi_{-1}$. Moreover, each of the manifolds $(\Sigma^\Phi, {}^\Phi g_{ij})$, $(\Sigma^\Psi, {}^\Psi g_{ij})$, and $(\hat{\Sigma}, \hat{g}_{ij})$ are flat. Just the manifold (Σ, g_{ij}) is conformally flat. The metric tensor $\hat{g}_{ij}$ can be written in a conformally flat form. Namely, let us define

$$\hat{g}_{ij} = \mathcal{U}^{4\,(\Phi)} g_{ij}, \tag{45}$$

where one sets $\mathcal{U} = ({}^\Phi\omega_\pm V)^{-1/2}$. The fact that the Ricci scalar in the $\hat{g}_{ij}$ metric is equal to zero implies that the equations of motion of the system in question reduce to the Laplace equation on the three-dimensional Euclidean manifold,

$$\nabla_i \nabla^i \mathcal{U} = 0, \tag{46}$$

where ∇ is the connection on a flat manifold. Next, it yields that the expression for the flat base space is valid; i.e., one obtains the following:

$${}^{(\Phi)}g_{ij} dx^i dx^j = \tilde{\rho}^2 d\mathcal{U}^2 + \tilde{h}_{AB} dx^A dx^B. \tag{47}$$

The photon sphere will be located at some constant value of $\mathcal{U}$. The radius of the photon sphere can be given at the fixed value of the ρ coordinate [12]. All these allow that, on the hypersurface Σ, the metric tensor can be given in the form of

$$\hat{g}_{ij} dx^i dx^j = \rho^2 dV^2 + h_{AB} dx^A dx^B,$$

and a connected component of the photon surface can be identified at a fixed value of the ρ coordinate.

In order to proceed further, let us assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ consist of two solutions of the boundary value problem of the system in question. Using the Green identity and integrating over the volume element, we arrive at the relation

$$\begin{aligned} &\left(\int_{r\to\infty} - \int_{\mathcal{H}} \right) (\mathcal{U}_1 - \mathcal{U}_2) \frac{\partial}{\partial r} (\mathcal{U}_1 - \mathcal{U}_2) dS \\ &= \int_\Omega \left| \nabla(\mathcal{U}_1 - \mathcal{U}_2) \right|^2 d\Omega. \end{aligned} \tag{48}$$

In view of the last equation, the surface integrals disappear due to the imposed boundary conditions. On the other hand, by virtue of the above relation, one finds that the volume integral must be identically equal to zero. Taking all the above into account, we can assert that the following theorem holds:

Theorem: Let us consider the set $(M^3, g_{ij}, N, \psi, \phi)$ being the system asymptotic to the dilaton black hole spacetime and possessing the photon sphere $(P^3, h_{ij}) \hookrightarrow (R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$, which can be regarded as the inner boundary of $R \times M^3$. Suppose further that M and Q are the ADM mass and the total charge of $(R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$. Then $(R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$ is isometric to the region of $r \geq r_{phs}$, exterior to the photon sphere in the electrically charged dilaton black hole spacetime. The photon sphere in question is connected, and it constitutes a cylinder over a topological sphere.

III. CONCLUSIONS

In our paper, we have elaborated the uniqueness of a static asymptotically flat black hole photon sphere in the Einstein-Maxwell-dilaton theory of gravity with arbitrary coupling constant α. Using the conformal positive energy theorem, we show that the region exterior to the photon sphere of the adequate radius in the electrically charged dilaton black hole spacetime is connected, and it authorizes a cylinder over a topological sphere. The proof is valid for the nonextremal dilaton black hole photon sphere.

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