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Collision of strong gravitational and electromagnetic waves in the expanding universe

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An exact analytical model of the process of collision and nonlinear interaction of gravitational and/or electromagnetic soliton waves and strong nonsoliton electromagnetic traveling waves of arbitrary profile propagating in the expanding universe (the symmetric Kasner spacetime) is presented. In contrast to intuitive expectations that rather strong traveling waves can destroy the soliton, it occurs that the soliton survives during its interaction with electromagnetic waves of arbitrary amplitude and profile, but its parameters begin to evolve under the influence of this interaction. If a traveling electromagnetic wave possesses a finite duration, the soliton parameters after interaction take constant values again, but these values in general are different from those before the interaction. Based on exact solutions of the Einstein-Maxwell equations, our model demonstrates a series of nonlinear phenomena, such as (a) creation of gravitational waves in the collision of two electromagnetic waves, (b) creation of electromagnetic soliton waves in the collision of a gravitational soliton with traveling electromagnetic waves, (c) scattering of a part of a soliton wave in the direction of propagation of a traveling electromagnetic wave, and (d) quasiperiodic oscillating character of fields in the wave interaction region and multiple mutual transformations of gravitational and electromagnetic waves in this region. The figures illustrate these features of nonlinear wave interactions in general relativity.

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I. INTRODUCTION AND RESULTS

Since the beautiful discovery just a hundred years ago of Einstein's general relativity theory, one of the most interesting problems has been understanding the different aspects of behavior of strong gravitational fields and their interactions with various matter fields. Most of the information known today was obtained using nonlinear geometric analysis of spacetime structures, approximation methods, or numerical simulations [1]. In most cases, however, this information possesses a character that is too global and very restricted, lacking the most interesting details. On the other hand, the most detailed information concerning the properties of strong gravitational fields can be obtained from the exact solutions, especially if these include some constant or functional parameters providing flexibility in these models.

During the centennial history of general relativity, many exact solutions of the Einstein equations were found, but there were no solutions describing the nonlinear interactions, mutual scattering, and mutual transformations of arbitrary strong gravitational and electromagnetic waves propagating and colliding in curved spacetimes. All these phenomena are described by a large class of exact solutions of Einstein-Maxwell equations presented in this paper. Typical pictures of such interactions are shown in Fig. 1 and Fig. 2.

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First of all, we have to clarify that, although in our class of exact solutions the function $\phi(t-x)$ can be chosen arbitrarily, Fig. 1 and Fig. 2 show the dynamics of waves for the simplest choice of this function for which $\phi(u)$ is a linear function in some interval $u_1 < u < u_2$ and possesses

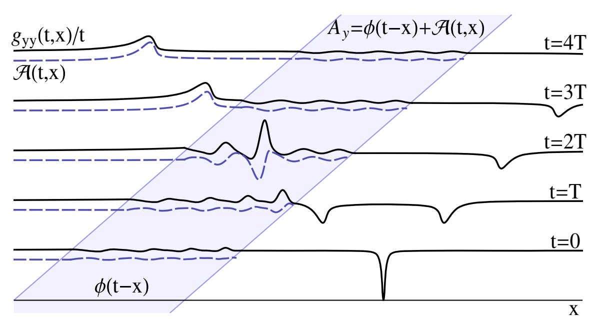

FIG. 1. Nonlinear interaction of gravitational soliton waves (solid lines to the right from dark stripe) with traveling pure electromagnetic waves of arbitrary profile $\phi(t-x)$ (dark stripe) in the symmetric Kasner spacetime background. This interaction creates gravitational (solid lines) and electromagnetic (dashed lines) waves in the wave interaction region (within the stripe). Oscillations of fields in this region, such that each local maximum of amplitude of one field is located near a local minimum of the other, mean the presence of multiple mutual transformations of gravitational and electromagnetic waves. After interaction (i.e. to the left of the stripe), the gravitational part of the soliton was changed and its electromagnetic counterpart was created. A part of the soliton gravitational wave is scattered in the positive x direction (along the stripe).

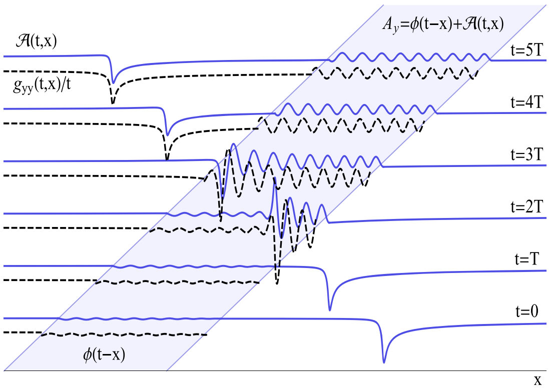

FIG. 2. Nonlinear interaction of electromagnetic soliton waves (solid lines to the right from the dark stripe) with traveling electromagnetic nonsoliton waves of an arbitrary profile $\phi(t-x)$ (within the stripe) in the symmetric Kasner spacetime background. This interaction creates gravitational (dashed lines) and electromagnetic (solid lines) waves in the wave interaction region (within the stripe). Oscillations of fields in this region, such that each local maximum of amplitude of one field is located near a local minimum of amplitude of the other, mean the presence of effect of multiple mutual transformations of gravitational and electromagnetic waves. After interaction (the region to the left of the stripe), the electromagnetic part of the soliton is changed and its gravitational counterpart is created. A part of a soliton wave was scattered to the positive x direction (along the stripe).

constant values outside it. For other choices of $\phi(u)$ in this interval, the dynamics of the fields is qualitatively similar to that shown in Fig. 1 and Fig. 2.

We have to notice also that "pure electromagnetic waves" here and below mean that, in these waves, the transverse components of the metric (g_{yy}, g_{yz}, g_{zz}) are the same as for the background; i.e., the gravitational wave is absent. However, these electromagnetic waves produce some running longitudinal deformations of spacetime (of metric components on a (t,x) plane).

Figures 1 and 2 show only the dynamics of one of the transverse components of the metric $g_{yy}(t,x)$ and the component $A_y(t,x)$ of the electromagnetic potential. The dynamics of other transverse metric components g_{yz}, g_{zz} and the electromagnetic potential $A_z(t,x)$ are qualitatively similar to those shown above. The cases shown on the pictures are the subcases of our general class of solutions in which the soliton may possess initially both gravitational and electromagnetic counterparts and the nonsoliton electromagnetic wave may have any profile determined by an arbitrary function $\phi(t-x)$.

A. On previous studies of the interaction of gravitational and electromagnetic fields

In the framework of perturbation theory, the first evidence of the possibility of the transformation of photons to gravitons in a strong magnetic field was found by Gertsenshtein [2]. Later, more detailed information of this kind was obtained from the studies of a coupled system of small perturbations of gravitational and electromagnetic fields on charged black hole backgrounds. However, a discussion of all the literature on this topic and the corresponding specific results are not in the scope of the present paper. Here we mention only the paper [3], where in WKB approximation it was shown that the interaction of gravitational and electromagnetic waves propagating in the external fields of charged black holes possesses the character of quasiperiodic mutual transformation, and the length of the spatial modulation of each of the waves caused by these transformations was calculated. General perturbative approaches to the studies of the interaction of these waves in arbitrary external gravitational and electromagnetic fields in WKB approximation were suggested in coordinate form in [4] and in Newman-Penrose formalism in [5].

The nonlinear interactions of gravitational and electromagnetic fields in different situations are described by a large variety of the exact solutions of Einstein-Maxwell equations known today. Most of the solutions derived before the end of the 1970s had been collected in [6]. More detailed discussions of the physical and geometrical properties of these as well as of many later published solutions were presented in [7]. Among the earliest examples are the exact solutions constructed in [8,9] for closed homogeneous anisotropic cosmological models which contain electromagnetic and scalar fields.

Many solutions for colliding plane waves in general relativity, which include the solutions for gravitational and electromagnetic waves, can be found in [10]. However, a great majority of these solutions describe plane-fronted waves with parallel rays colliding on the Minkowski background, where the mutual focusing of colliding waves and the inevitable emergence of curvature singularities in the wave interaction region does not allow one to have a complete picture of wave dynamics in the whole spacetime until the null infinity. In contrast to these, in our solutions the background spacetime expansion prevents the creation of such singularities and provides a complete regular picture of interacting wave dynamics.

An interesting class of solutions for gravitational and electromagnetic (and scalar) waves in closed cosmological models was constructed and discussed in [11,12] as a subclass of a more general class of solutions called electromagnetic Einstein-Rosen waves that were studied previously in [13–15]. In this class of solutions of Einstein-Maxwell equations, the field variables responsible for the transverse part of the metric and for the nonvanishing components of the electromagnetic potential are considered as functionally dependent. The dynamical equations for these fields reduces to only one linear equation identical to the equation for pure vacuum Einstein-Rosen waves. In [11,12], these solutions were used successfully for the construction of standing waves in closed cosmological models [16].

The class of electromagnetic Einstein-Rosen waves, unlike the class of solutions constructed in the present paper, cannot provide us with a complete picture of the processes of collision of the gravitational and electromagnetic waves. From the physical point of view, the solutions of this class do not carry some important imprints of the character of the nonlinear interaction of gravitational and electromagnetic waves in general. The linearity of the dynamical equation for electromagnetic Einstein-Rosen waves means, in particular, that in the case of the collision of incident waves from this class, the wave which left the interaction region does not carry any information about other waves with which it has interacted. The interaction of waves in this case emerges only in their influence on the longitudinal part of the metric, but direct interactions of transverse modes and their mutual scattering are absent. Besides that, in the class of electromagnetic Einstein-Rosen waves, we cannot consider such physically important situations as the collision of two pure electromagnetic waves (i.e., of electromagnetic waves which are not accompanied by some transverse gravitational waves) or collision of pure gravitational and pure electromagnetic waves because in the electromagnetic Einstein-Rosen solutions, the gravitational and electromagnetic parts of waves are in a fixed functional dependence on each other. However, all these difficulties are absent in the class of solutions constructed in the present paper.

The decades after the end of the 1970s were marked by the discoveries of powerful methods for solving Einstein's field equations, such as soliton generating transformations and various integral equation methods (see [17,18] and the references there). However, the power of these methods (based on the integrability of Einstein's field equations for spacetimes with certain symmetries) has not been used fully for the construction of nontrivial physically interesting solutions for interacting waves. A rich new class of such solutions presented in this paper demonstrates the yet hidden power of these methods and allows us to elucidate a number of new interesting features of nonlinear dynamics of interacting waves.

B. The main results

In this paper, we present a class of exact solutions of Einstein-Maxwell equations which describe the collision and nonlinear interaction of one-soliton gravitational and electromagnetic waves and nonsoliton pure electromagnetic traveling waves of arbitrary amplitudes and profiles propagating in the symmetric Kasner background. These solutions include a number of soliton parameters and an arbitrary real function of a null coordinate $u = t - x$,

$$\ell + is, c, d \quad \text{and} \quad \phi(u),$$

which govern the initial location of a soliton along the x axis (ℓ), its shape (s), and the amplitudes of its gravitational (c) and electromagnetic (d) components, while the arbitrarily chosen function $\phi(u)$ determines the amplitude and profile of electromagnetic nonsoliton waves. In the physically most interesting cases, $\phi(u)$ takes constant values outside some interval $u_1 < u < u_2$; i.e., the nonsoliton wave is of finite duration. In the simplest case, $\phi(u)$ is a linear function of u for $u_1 < u < u_2$. Just this choice of $\phi(u)$ was used for nonsoliton waves in Fig. 1 and Fig. 2.

(i) These solutions demonstrate a qualitatively new picture of nonlinear interactions of soliton and nonsoliton waves. In contrast to intuitive expectations that rather strong traveling nonsoliton waves can destroy the soliton, it occurs that in the collision with nonsoliton waves of arbitrary amplitudes and profiles, the soliton survives, but the values of its parameters c and d after the collision differ from the values of these parameters before the collision (compare the soliton wave profiles to the left and to the right of the stripe in Fig. 1 or in Fig. 2).

(ii) Inside the wave interaction region $u_1 < u < u_2$, the analytical form of the solution remains the same, but the soliton parameters c and d, which were constants for $u < u_1$ begin to evolve: $c \to \hat{c}(u)$, $d \to \hat{d}(u)$ until the soliton leaves the wave interaction region, i.e. until $u > u_2$, and so that

$$\begin{aligned}\hat{c} &= c\cosh S - id\sinh S,\\ \hat{d} &= ic\sinh S + d\cosh S,\end{aligned} \quad S(u) = \sqrt{2}\int_{u1}^{u} \frac{\phi'(u)du}{\sqrt{w_o - u}}, \tag{1}$$

where $w_o = \ell + is$. For $u < u_1$ where $\phi(u) = 0$, we have $\hat{c} = c$ and $\hat{d} = d$. For $u > u_2$ where $\phi(u) = \text{const}$, these parameters become constants $\hat{c}(u) = \hat{c}(u_2)$, $\hat{d}(u) = \hat{d}(u_2)$, and the wave becomes a pure soliton again. However, in general, $\hat{c}(u_2) \neq c$ and $\hat{d}(u_2) \neq d$.

(iii) In the case $d = 0$, the initial soliton ($u < u_1$) is pure vacuum. Then for $u > u_2$, we have $\hat{c} = c\cosh S(u_2)$ and $\hat{d} = ic\sinh S(u_2)$. The condition $\hat{d} \neq 0$ for $u > u_2$ means that some electromagnetic part of the soliton wave was created in this interaction (see Fig. 1).

(iv) In the case $c = 0$, the initial soliton wave($u < u_1$) is purely electromagnetic. For $u > u_2$, we have $\hat{c} = -id\sinh S(u_2)$ and $\hat{d} = d\cosh S(u_2)$. The condition $\hat{c} \neq 0$ for $u > u_2$ means that a collision of the soliton and nonsoliton electromagnetic waves created the gravitational part of the soliton wave (see Fig. 2).

(v) In the wave interaction region $u_1 < u < u_2$, the fields possess an oscillating character such that each local maximum of the amplitude of the incoming wave is located near a local minimum of the amplitude of the created wave (see Fig. 1 and Fig. 2). These oscillations point to the existence of the phenomenon of multiple mutual transformations of gravitational and electromagnetic waves during their nonlinear interactions.

(vi) The nonlinear interaction of a soliton with a nonsoliton electromagnetic wave gives rise also to the effect of the nonlinear scattering of waves so that a part of the initial soliton is pushed to propagate as gravitational and electromagnetic waves in the direction of propagation of the nonsoliton wave.

In the remaining part of the paper, we describe the symmetric Kasner background, the electrovacuum one-soliton waves, and nonsoliton traveling electromagnetic waves in this background, as well as the new class of solutions for nonlinear interaction of these waves.

II. NEW INTERACTING WAVE SOLUTIONS

A. The symmetric Kasner universe

For the background where the waves will propagate and interact with its curvature and with each other, we choose a vacuum Kasner universe with two equal exponents $(p_1, p_2, p_3) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ which we take in the form

$$ds^2 = -\frac{1}{\sqrt{t}} du dv + t(dy^2 + dz^2), \qquad (2)$$

where the null coordinates are $u = t - x$ and $v = t + x$. For waves propagating in the x direction, the expansion with time of this background in the transverse directions prevents a creation of caustics and curvature singularities and this allows to see a complete regular dynamics of waves, their mutual scattering and transformations.

B. Traveling nonsoliton electromagnetic waves in the symmetric Kasner universe

The class of solutions of Einstein-Maxwell equations which describe plane-fronted traveling nonsoliton pure electromagnetic waves with arbitrary amplitudes, profiles, and polarizations propagating in the symmetric Kasner universe along its axis of symmetry was constructed in [19]. In what follows, we consider only the case of such waves with linear polarizations. In this case,

$$\begin{cases} ds^2 = -\frac{1}{\sqrt{t}} e^{4\mathcal{B}(u)} du dv + t(dy^2 + dz^2), \\ A_i = \{0, 0, \phi(u), 0\}, \quad \mathcal{B}(u) = \int \phi'^2(u) du \end{cases}, \qquad (3)$$

where $A_y = \phi(u)$ is an arbitrary real function of $u = t - x$.

C. Soliton waves in the symmetric Kasner universe

Another type of wave that may exist in the expanding Kasner universe (besides the traveling waves considered above) are the soliton gravitational and electromagnetic waves determined by soliton solutions of the Einstein-Maxwell equations which were found in [20]. Omitting the details (and referring the readers to [20,21]), we present here the electrovacuum one-soliton solution on the symmetric Kasner background (2):

$$\begin{cases} ds^2 = -f du dv + t\left[h_{yy} dy^2 + 2h_{yz} dy dz + h_{zz} dz^2\right], \\ A_i = \{0, 0, A_y, A_z\}, \end{cases} \qquad (4)$$

where the metric functions possess the expressions

$$h_{yy} = \frac{(\mathcal{G} - 4isc)(\bar{\mathcal{G}} + 4is\bar{c})}{\mathcal{D}\bar{\mathcal{D}}}, \qquad h_{yz} = -\frac{4s(c\bar{\mathcal{G}} + \bar{c}\mathcal{G})}{\mathcal{D}\bar{\mathcal{D}}},$$
$$h_{zz} = \frac{(\mathcal{G} + 4isc)(\bar{\mathcal{G}} - 4is\bar{c})}{\mathcal{D}\bar{\mathcal{D}}}, \qquad f = \frac{1}{\sqrt{t}} \cdot \frac{\mathcal{D}\bar{\mathcal{D}}}{4r_+ r_-} \qquad (5)$$

and the electromagnetic potential components are the real parts of the corresponding complex potentials:

$$\{A_y, A_z\} = \Re\{\Phi, \tilde{\Phi}\},$$
$$\Phi = -2sd\frac{\mathcal{F}}{\mathcal{D}},$$
$$\tilde{\Phi} = 2isd\frac{\tilde{\mathcal{F}}}{\mathcal{D}}. \qquad (6)$$

In these expressions, the following notations are used:

$$\mathcal{D} = \bar{K}_+ K_- - c\bar{c} K_+ \bar{K}_- + d\bar{d} \bar{K}_+ \bar{K}_-,$$
$$\mathcal{G} = K_+ K_- - c\bar{c} \bar{K}_+ \bar{K}_- + d\bar{d} K_+ \bar{K}_-,$$
$$\mathcal{F} = \bar{K}_+ - \bar{c}\bar{K}_-, \quad \tilde{\mathcal{F}} = \bar{K}_+ + \bar{c}\bar{K}_-. \qquad (7)$$

In these expressions, $K_\pm$ denote two complex functions,

$$K_+ = \sqrt{R_+} + \frac{is}{\sqrt{R_+}}, \qquad K_- = \sqrt{R_-} + \frac{is}{\sqrt{R_-}}, \qquad (8)$$

and the real functions $R_\pm$ and $r_\pm$ are defined as

$$R_+ = \ell - u + r_+, \qquad r_+ = \sqrt{(\ell - u)^2 + s^2},$$
$$R_- = \ell + v + r_-, \qquad r_- = \sqrt{(\ell + v)^2 + s^2}. \qquad (9)$$

This solution depends on free constant parameters $\{\ell, s, c, d\}$ where the first two parameters are real and the other two are complex. The parameter ℓ determines a shift of the whole configuration along the x axis, s determines the shape of the soliton, c is responsible for gravitational parts of the soliton, and d for electromagnetic parts of the soliton. In particular, in the case $d = 0$, we have the pure vacuum one-soliton configuration which decays with time into two incident gravitational waves propagating in opposite x directions (Fig. 1, the region to the right of the stripe) and the case $c = 0$ corresponds to pure electromagnetic one-soliton wave which propagates along the x axis (Fig. 2, the region to the right of the stripe).

D. Interaction of gravitational and electromagnetic soliton waves with electromagnetic nonsoliton traveling waves

For construction of the solution, which describes the interaction of soliton waves (4)–(9) with traveling waves (3), we apply the electrovacuum soliton generating transformations [20,21] choosing the spacetime (3) as the background for solitons. It is remarkable that the corresponding solution occurs to have the same form (4)–(9), but in all these expressions, the soliton parameters c and d should be substituted by the evolving parameters $\hat{c}(u)$ and $\hat{d}(u)$ and, after this substitution, the following corrections should be made for complex electromagnetic potentials in (6) and the conformal factor f in (5):

$$\begin{matrix} c \to \hat{c}(u), & \Phi \to \Phi + \phi(u), \\ d \to \hat{d}(u), & \tilde{\Phi} \to \tilde{\Phi} + i\phi(u), \end{matrix} \quad f \to e^{4\mathcal{B}(u)} f$$

where $\hat{c}(u)$, $\hat{d}(u)$, and $\mathcal{B}(u)$ were defined in (1) and (3).

III. CONCLUDING REMARKS

In this paper, we present a new large class of exact solutions of the Einstein-Maxwell equations which describe the nonlinear interaction of gravitational and electromagnetic soliton waves with nonsoliton electromagnetic waves of arbitrary amplitudes and profiles propagating in (and interacting with) the expanding spatially homogeneous universe represented by axisymmetric vacuum Kasner spacetime. It is worth emphasizing that the main purpose of this paper was not a construction of some viable cosmological model but the investigation of the process of collision and nonlinear interaction of arbitrarily strong gravitational and electromagnetic waves, their mutual scattering, and mutual transformations.

From a mathematical point of view, the choice of the expanding spacetime as the background for waves (instead of, e.g., the Minkowski spacetime) was very important because the expansion of this background prevents the emergence of caustics and curvature singularities caused by mutual focusing of colliding waves. This leads to a regular global picture of wave dynamics. Besides that, the choice of the axisymmetric Kasner background provides us with a rich class of solutions for traveling electromagnetic nonsoliton waves which exist on this background. The solutions of this class are very simple and depend explicitly on an arbitrary function of a null coordinate which determines the arbitrarily chosen amplitudes and profiles of these waves. (Similar classes of explicitly calculated solutions for waves on the other backgrounds are not known.)

From a physical point of view, however, the process of the collision of such waves is a local process during which the global structure of the universe is not so important. Therefore, one may expect that the main features of this process will be similar for the collision of high-energy waves of finite durations on the other backgrounds.

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