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New constraint on effective field theories of the QCD flux tube

M. Baker

Department of Physics, University of Washington P.O. Box 351650, Seattle, Washington 98195, USA

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Effective magnetic $SU(N)$ gauge theory with classical Z_N flux tubes of intrinsic width $\frac{1}{M}$ is an effective field theory of the long-distance quark-antiquark interaction in $SU(N)$ Yang-Mills theory. Long-wavelength fluctuations of the Z_N vortices of this theory lead to an effective string theory. In this paper, we clarify the connection between effective field theory and effective string theory, and we propose a new constraint on these vortices. We first examine the impact of string fluctuations on the classical dual superconductor description of confinement. At interquark distances $R \sim \frac{1}{M}$, the classical action for a straight flux tube determines the heavy quark potentials. At distances $R \gg \frac{1}{M}$, fluctuations of the flux tube axis $\tilde{x}$ give rise to an effective string theory with an action $S_{\rm eff}(\tilde{x})$, the classical action for a curved flux tube, evaluated in the limit $\frac{1}{M} \to 0$. This action is equal to the Nambu-Goto action. These conclusions are independent of the details of the Z_N flux tube. Further, we assume the QCD flux tube satisfies the additional constraint, $\int_0^\infty rdr \frac{T_{\theta\theta}(r)}{r^2} = 0$, where $\frac{T_{\theta\theta}(r)}{r^2}$ is the value of the $\theta\theta$ component of the stress tensor at a distance r from the axis of an infinite flux tube. Under this constraint, the string tension σ equals the force on a quark in the chromoelectric field $\vec{E}$ of an infinite straight flux tube, and the Nambu-Goto action can be represented in terms of the chromodynamic fields of effective magnetic $SU(N)$ gauge theory, yielding a field theory interpretation of effective string theory.

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I. INTRODUCTION

A. Background

1. Dual superconductor mechanism of confinement

In the dual superconductor mechanism for confinement [1–3], a dual Meissner effect confines color electric flux to a narrow flux tube connecting a quark-antiquark pair and, as a consequence, the energy of the pair increases linearly with their separation, confining the quarks in hadrons.

The Abelian Higgs model is an example of a relativistic field theory having confining vortex solutions [4]. The $U(1)$ gauge symmetry is completely broken by scalar Higgs fields ϕ, which vanish on the axis of the flux tube and increase to their nonvanishing vacuum value ϕ_0 at large distances from the vortex. Interpreting the $U(1)$ symmetry as a magnetic gauge symmetry coupling "dual" potentials to magnetically charged Higgs fields with magnetic coupling constant g, the flux tube then carries electric flux $\frac{2\pi}{g}$ confining a "quark" and an "antiquark" attached to its ends [1].

2. Effective field theory of dual superconductivity

Spontaneously broken magnetic $SU(N)$ gauge theory, describing non-Abelian "dual" potentials C_μ coupled to magnetically charged adjoint representation scalar Higgs fields ϕ_i, provides a non-Abelian example of an effective field theory of the long-distance quark-antiquark interaction in $SU(N)$ Yang-Mills theory [5,6]. "Dual" potentials or "electric vector potentials" C_μ were first defined kinematically by Mandelstam [7] in terms of 't Hooft loops [8], operators which create vortices of magnetic flux. The spatial components of the field tensor $G_{\mu\nu}$, constructed from the potentials C_μ, determine the color electric field $\vec{E}$ and the spacetime components, the color magnetic field $\vec{B}$. The fields $\vec{E}$ and $\vec{B}$ evaluated at the position of the quarks can be identified with the corresponding chromodynamic fields of the underlying $SU(N)$ Yang-Mills theory [9].

This effective field theory possesses (i) the $SU(N)$ symmetry of Yang-Mills theory and (ii) the same low energy spectrum; i.e., it contains no massless particles and has Z_N electric flux tube solutions. The gauge coupling constant is denoted g_m, and the magnitude of the vacuum value of the Higgs field is denoted ϕ_0. The mass $M \sim g_m\phi_0$ of the vector particle arising from the non-Abelian Higgs mechanism determines the flux tube intrinsic width $\frac{1}{M}$. The energy per unit length of the classical flux tube, the string tension $\sigma \sim \# \frac{M^2}{g_m^2}$.

3. Effective string theory from effective field theory

When the distance R between the quark and antiquark is much larger than $\frac{1}{M}$, long-wavelength fluctuations of the Z_N vortices become important and lead to an effective string theory of these fluctuations [10]. The action $S_{\rm eff}(\tilde{x})$ of this effective string theory equals $S^{\rm class}(\tilde{x})$, the classical action for a curved vortex sheet $\tilde{x}$, evaluated in the limit $\frac{1}{M} \to 0$.

This action equals the Nambu-Goto action with the classical string tension. $S_{\rm eff}(\tilde{x})$ is then equal to the Nambu-Goto action.

B. Effective string theory

The long-distance $q\bar{q}$ interaction is usually described by effective string theory [11–13] with an action $S_{\rm eff}(\tilde{x})$ in which the string tension σ is an independent parameter. The heavy quark potential $V(R)$ is an expansion in powers of $\frac{1}{\sigma R^2}$. The leading terms in this expansion are the linear potential and the universal Lüscher term [11]:

$$V(R) = \sigma R - \frac{\pi}{12R} + \cdots. \tag{1}$$

Effective string theory has since been developed extensively. It has been shown [14,15] that consistency with Poincaré symmetry requires that the expansion of the ground state heavy quark potential in powers of $\frac{1}{\sigma R^2}$ coincides to order $\frac{1}{R^5}$ with the potential generated by the Nambu-Goto action. (Boundary terms in $S_{\rm eff}$ give corrections of order $\frac{1}{R^4}$.)

Since the Nambu-Goto action is the action of the effective string theory obtained from effective field theory, this result implies that effective field theory accounts for the contributions of string fluctuations to the ground state heavy quark potential to order $\frac{1}{R^5}$. Higher-order terms in this long-distance expansion are not taken into account by effective field theory.

C. The goal of this paper

The purpose of this paper is twofold: (i) to clarify the connection between effective field theory and effective string theory, and (ii) to propose a new constraint on the structure of the QCD flux tube.

D. Impact of string fluctuations on the flux tube picture

We first examine the impact of string fluctuations on the classical description of confinement. At distances $R \sim \frac{1}{M}$, the classical action for a straight flux tube determines the heavy quark potential $V(R)$. Calculations [16,17] of heavy quark potentials in the model introduced in [5] were consistent with early lattice simulations [18] with $M \sim 2\sqrt{\sigma}$ [19].[1]

At distances $R \gg \frac{1}{M}$, where corrections due to string fluctuations become important, effective string theory determines the heavy quark potential. In an intermediate range of distances between approximately $\frac{1}{M}$ and $\frac{2}{\sqrt{\sigma}}$ both

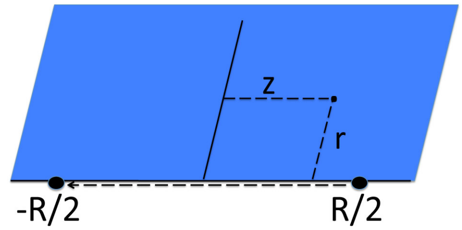

FIG. 1. Half plane passing through the axis of the flux tube. Equation (6) is the condition that the torque per unit length acting across any such (r, z) plane must vanish as $R \to \infty$.

the flux tube intrinsic width and the effect of string fluctuations must be taken into account. Both effects were considered in the recent analysis [22] of lattice simulations of field distributions surrounding a quark-antiquark pair for a range of values of their separation.

E. A constraint on the confining flux tubes

The motivation for our constraint is based on the following expression for the string tension σ, derived in Sec. IV B and valid for any form of the Higgs potential $V(\phi_i)$ for which the $\frac{SU(N)}{Z_N}$ symmetry of the effective field theory is completely broken:

$$\sigma = 2{\rm tr}\left[-\frac{2\pi}{g_m} Y \vec{E}(r=0)\right] \cdot \hat{e}_z - 2\pi\tau, \tag{2}$$

where

$$2{\rm tr}\left[-\frac{2\pi}{g_m} Y \vec{E}(r=0)\right] \equiv \vec{F} \tag{3}$$

is the chromodynamic force on a quark in the color field $\vec{E}(r=0)$ on the axis of an infinite Z_N flux tube. (Both the quark color charge $-\frac{2\pi}{g_m}Y$ and the color field $\vec{E}(r=0)$ have N components and the trace in (3) is a sum of the products of these components.) τ is the torque per unit length on any r, z half plane ($\theta =$ constant, $r > 0$) passing through the axis of the flux tube (Fig. 1) and is given by

$$\tau \equiv \int_0^\infty r dr \frac{T_{\theta\theta}(r)}{r^2}, \tag{4}$$

where $\frac{T_{\theta\theta}(r)}{r^2}$ is the value of the $\theta\theta$ component of the stress tensor at a distance r from the flux tube axis (the z axis). $T_{\theta\theta}(r)$ defines an azimuthal pressure $p(r)$,

$$p(r) \equiv \frac{T_{\theta\theta}(r)}{r^2}, \tag{5}$$

and τ is the radial moment of this pressure distribution.

Equation (2) is the work-energy relation for a flux tube. The work per unit length needed to move a quark along the

[1] Since $SU(3)$ lattice simulations [20] of pure gauge theory yield a deconfinement temperature $T_C \approx 0.65\sqrt{\sigma} \sim \frac{M}{3}$ there is an interval of temperatures where we expect that effective magnetic gauge theory is also applicable in the deconfined phase [21].

flux tube axis is $\vec{F}\cdot\hat{e}_z$. The work per unit length required to remove the field energy in a sector $\Delta\theta$ of the flux tube while maintaining the quark-antiquark separation is $-\Delta\theta\tau$, so that $-2\pi\tau$ is required to remove all the field energy. The flux tube energy per unit length σ is then the sum (2) of these two contributions to the work per unit length.

The torque per unit length τ is a new long-distance parameter of effective field theory relating the string tension to the color field on the flux tube axis via (2). We assume that the value $\tau = 0$ characterizes the structure of the QCD flux tube, distinguishing it from the flux tubes arising from other field theories: i.e.,

$$\tau \equiv \int_0^\infty r dr \frac{T_{\theta\theta}(r)}{r^2} = 0. \tag{6}$$

If the constraint (6) is met, then by (2) the string tension σ is equal to the force on a quark in the chromoelectric field $\vec{E}(r=0)$ on the axis of an infinite flux tube:

$$\sigma = 2\mathrm{tr}\left[-\frac{2\pi}{g_m} Y\vec{E}(r=0)\right]\cdot\hat{e}_z = \vec{F}\cdot\hat{e}_z. \tag{7}$$

Our conjecture is that the equivalent conditions (6) and (7) characterize the QCD flux tube.

Condition (6) means that the work per unit length required to remove the field energy available after a quark-antiquark pair have been separated by a distance R approaches 0 in the limit $R \gg \frac{1}{M}$. Then the long-distance heavy quark potential σR becomes equal to the work $\vec{F}\cdot R\hat{e}_z$ needed to separate the quark-antiquark pair a distance R in the field $\vec{E}(r=0)$ on the axis of an infinite flux tube, which is condition (7).

F. Outline of this paper

In Sec. II, we provide the background and notation used in the paper, and we discuss Z_N flux tubes and their coupling to a quark-antiquark pair. We review the transition from effective field theory to effective string theory [10] in Sec. III and discuss the interplay between the width due to string fluctuations and the intrinsic width of the flux tube.

In Sec. IV, we derive a generalization of (2) to curved vortex sheets $\tilde{x}$ to obtain an expression for $S^{\mathrm{class}}(\tilde{x})$, the classical action for the vortex sheet $\tilde{x}$ determining the action of the effective string theory. We use this expression in Sec. V, where we impose our constraint (6) on flux tubes. Making use of Poincaré invariance, we then obtain a representation of the Nambu-Goto action as an integral over the chromodynamic force on the vortex sheet. This representation is the generalization of (7) to curved vortex sheets, and gives a field theory interpretation of effective string theory.

In Sec. VI, we examine this picture in a particular $SU(3)$ example [5] where explicit classical Z_3 flux tube solutions have been found. The constraint $\tau = 0$ fixes the value of a parameter κ in the Higgs potential of the non-Abelian theory. This parameter plays the role of the Landau-Ginzburg parameter of the Abelian Higgs model. In the Summary we discuss the possibility of testing the conjecture (6) using lattice simulations.

II. EFFECTIVE MAGNETIC $SU(N)$ GAUGE THEORY

We consider effective field theories coupling magnetic $SU(N)$ gauge potentials C_μ to adjoint representation scalar fields ϕ_i. The gauge coupling constant is g_m. The magnetic gauge potentials C_μ and Higgs fields ϕ_i are elements of the Lie Algebra of $SU(N)$. We use a timelike metric: $C_\mu = (C_0, -\vec{C}) = \sum_a C_\mu^a T_a$, $\phi_i = \sum_a \phi_i^a T_a$, where the $T_a = \lambda_a/2$ are the fundamental representation generators normalized so that

$$2\mathrm{tr} T_a T_b = \delta_{ab}. \tag{8}$$

The effective Lagrangian is

$$\mathcal{L}_{\mathrm{eff}}(C_\mu, \phi_i) = 2\mathrm{tr}\left(-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{2}(\mathcal{D}_\mu\phi_i)^2\right) - V(\phi_i), \tag{9}$$

with

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - i g_m [C_\mu, C_\nu], \tag{10}$$

and

$$\mathcal{D}_\mu \phi_i = \partial_\mu \phi_i - i g_m [C_\mu, \phi_i]. \tag{11}$$

The components of the field tensor $G^{\mu\nu}$ define color electric and magnetic fields $\vec{E}$ and $\vec{B}$:

$$E^k = \frac{1}{2}\epsilon_{klm} G^{lm}, \qquad B^k = G^{k0}. \tag{12}$$

$V(\phi_i)$ is an $SU(N)$ invariant Higgs potential which has an absolute minimum at a nonvanishing value ϕ_{i0} of the Higgs fields such that in the confining vacuum,

$$C_\mu = 0, \qquad \phi_i = \phi_{i0}, \tag{13}$$

the $\frac{SU(N)}{Z_N}$ symmetry is completely broken and all particles become massive. The number of Higgs fields and the form of the Higgs potential are otherwise unspecified. In Sec. VI, we will write down a specific $SU(N)$ Higgs potential for which explicit Z_3 flux tube solutions were found.[2]

[2]For a general discussion of magnetic vortices in non-Abelian gauge theory, see [23].

A. Z_N electric flux tubes

Effective magnetic gauge theory has electric Z_N flux tube solutions for which, at large distances r from the flux tube axis, C_μ and ϕ_i approach a gauge transformation $\Omega(\theta)$ of the vacuum fields [23]:

$$C_\mu \to \frac{i}{g_m}\Omega^{-1}(\theta)\partial_\mu\Omega(\theta), \qquad \phi_i(x) \to \Omega^{-1}(\theta)\phi_{i0}\Omega(\theta). \tag{14}$$

In order that the Higgs field be single valued on any path encircling the z axis, the matrix $\Omega^{-1}(\theta=2\pi)\Omega(\theta=0)$ must commute with all the ϕ_{i0} and, since the gauge symmetry is completely broken, must be an element of Z_N: $\Omega(\theta=2\pi)=\exp(2\pi ik/N)\Omega(\theta=0)$, $k=0,1,2,\ldots,N-1$.

We can choose a gauge where Ω is Abelian. For a Z_N flux tube with $k=1$, we take

$$\Omega(\theta)=\exp(i\theta Y), \tag{15}$$

where Y is a diagonal matrix. Its first $N-1$ elements $= 1/N$ and its Nth element $= -(N-1)/N$. (There are N physically equivalent coupling choices for Y related to each other by a gauge transformation [23]). With the choice (15) for $\Omega(\theta)$,

$$C_\mu \to \frac{-\partial_\mu\theta}{g_m}Y, \qquad \text{as } r\to\infty, \tag{16}$$

so that

$$\vec{C} \to \frac{1}{g_m r}\hat{e}_\theta Y, \qquad \text{as } r\to\infty. \tag{17}$$

Integrating $\vec{C}$ around a path at large r surrounding the z axis gives

$$\exp\left(ig_m\oint\vec{C}\cdot d\vec{l}\right)=\exp(2\pi iY)=\exp\left(\frac{2\pi i}{N}\right), \tag{18}$$

reflecting the one unit of Z_N electric flux passing through the xy plane.

We assume that there is a classical solution where the gauge potential $\vec{C}$ is everywhere proportional to the matrix Y:

$$\vec{C}=C(r)\hat{e}_\theta Y. \tag{19}$$

The flux tube electric field (12) also lies along the Y direction in color space:

$$\vec{E}(r)=-\vec{\nabla}\times\vec{C}=-\frac{1}{r}\frac{d(rC(r))}{dr}Y\hat{e}_z, \tag{20}$$

The Higgs fields ϕ_i have the form

$$\phi_i=\Omega^{-1}(\theta)\phi_i(r)\Omega(\theta), \quad \text{where } \phi_i(r)\to\phi_{i0}, \quad r\to\infty. \tag{21}$$

In order that the flux tube have finite energy, the Higgs fields ϕ_i, for which $[C_\mu,\phi_i]\neq 0$, must vanish on the flux tube axis $r=0$.

The vector mass M generated by the Higgs condensate, which determines the intrinsic width $\frac{1}{M}$ of the flux tube, is obtained by replacing ϕ_i by ϕ_{i0} and C_μ by Y in (9), is

$$M^2=g_m^2\sum_i\frac{2\mathrm{tr}[iY,\phi_{i0}]^2}{2\mathrm{tr}Y^2}. \tag{22}$$

B. Coupling of Z_N flux tubes to quarks

Classical Z_N vortices of magnetic $\frac{SU(N)}{Z_N}$ gauge theory carrying one unit of Z_N flux couple to a quark-antiquark pair in the fundamental representation of $SU(N)$ via a Dirac string $G^s_{\mu\nu}$, carrying color charge $\frac{2\pi}{g_m}Y$, which is nonvanishing on some line connecting the pair.

Long-wavelength fluctuations of the axis of the flux tube sweep out a spacetime surface $\tilde{x}^\mu(\sigma,\tau)$ bounded by the loop Γ formed by the world lines of the quark and antiquark at the ends of the vortex. We assume that the classical solution C_μ having a vortex on the sheet $\tilde{x}^\mu(\sigma,\tau)$ is also proportional to the matrix Y:

$$C_\mu=C_\mu(x,\tilde{x})Y. \tag{23}$$

(For $SU(3)$, we have obtained an explicit solution (96), (97) where C_μ has the form (23) with $Y=\frac{\lambda_8}{\sqrt{3}}$.)

The Higgs fields ϕ_i, for which $[Y,\phi_i]\neq 0$, contribute to the magnetic current density, the source of the potential C_μ, and must vanish on $\tilde{x}^\mu(\sigma,\tau)$. We choose a gauge where the surface swept out by the Dirac string coincides with the vortex sheet $\tilde{x}^\mu(\sigma,\tau)$. The corresponding Dirac polarization tensor $G^s_{\mu\nu}=G^s_{\mu\nu}(x,\tilde{x})$ is [24]

$$G^s_{\mu\nu}(x,\tilde{x})=-\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\int d\tau\int d\sigma\sqrt{-g}t^{\alpha\beta}\delta(x-\tilde{x}(\sigma,\tau))\frac{2\pi}{g_m}Y, \tag{24}$$

where g is the determinant of the induced metric g_{ab},

$$g_{ab}=\frac{\partial\tilde{x}^\mu}{\partial\xi^a}\frac{\partial\tilde{x}_\mu}{\partial\xi^b}, \qquad \xi^1=\tau, \qquad \xi^2=\sigma, \tag{25}$$

$\tilde{x}^\mu(\xi)\equiv\tilde{x}^\mu(\sigma,\tau)$ is a parametrization of the vortex sheet, and

$$t^{\alpha\beta}=\frac{1}{\sqrt{-g}}\left(\frac{\partial\tilde{x}^\alpha}{\partial\tau}\frac{\partial\tilde{x}^\beta}{\partial\sigma}-\frac{\partial\tilde{x}^\alpha}{\partial\sigma}\frac{\partial\tilde{x}^\beta}{\partial\tau}\right) \tag{26}$$

is the tensor specifying the orientation of the surface $\tilde{x}^\mu(\sigma,\tau)$ in four-dimensional spacetime. It is invariant under a reparametrization of the surface $\tilde{x}^\mu$ and normalized so that $t^{\alpha\beta}t_{\alpha\beta} = -2$.

The action $S[C_\mu, \phi_i]$ describing field configurations having a vortex on the sheet $\tilde{x}^\mu(\sigma,\tau)$ coupling the dual potential (23) to $G^s_{\mu\nu}$ is

$$S[C_\mu, \phi_i] = \int dx \mathcal{L}_{\text{eff}}(C_\mu, \phi_i, G^s_{\mu\nu}(x,\tilde{x})), \tag{27}$$

where the Lagrangian $\mathcal{L}_{\text{eff}}(C_\mu, \phi_i, G^s_{\mu\nu}(x,\tilde{x}))$ is obtained by replacing, in the Lagrangian (9), the dual field strength tensor $G_{\mu\nu}$ by

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + G^s_{\mu\nu}(x,\tilde{x}). \tag{28}$$

(The nonlinear term $-ig_m[C_\mu, C_\nu]$ in (10) does not contribute to the field tensor (28) in the gauge where the classical solution (23) is Abelian.)

After having partially fixed the gauge by the choice (23), the action (27) has a residual invariance under the Abelian gauge transformation $\Omega = \exp[i\Lambda(x)Y]$:

$$C_\mu \to C_\mu - \frac{1}{g_m}\partial_\mu\Lambda(x)Y, \qquad \phi_i \to \Omega^{-1}\phi_i\Omega. \tag{29}$$

III. EFFECTIVE FIELD THEORY

The 't Hooft loop acting in effective magnetic gauge theory creates a vortex of electric flux, and its expectation value determines the Wilson loop $W(\Gamma)$ of Yang-Mills theory, calculated in magnetic gauge theory. $W(\Gamma)$ is the partition function of the effective field theory in the presence of a Dirac string; i.e., $W(\Gamma)$ is a path integral over all field configurations C_μ, ϕ_i having a vortex on any surface $\tilde{x}^\mu(\sigma,\tau)$ bounded by the loop Γ formed from the world lines of a quark-antiquark pair [9],

$$W(\Gamma) = \int \mathcal{D}C_\mu \mathcal{D}\phi_i \exp(i[S(C_\mu,\phi_i) + S_{gf}]), \tag{30}$$

where $S(C_\mu,\phi_i)$ is the action (27), while S_{gf} is a gauge fixing term. The path integral (30) is cut off at a scale Λ, which must be less than the mass of the lightest glue ball, the lightest particle which has been integrated out in obtaining $\mathcal{L}_{\text{eff}}$. Λ must also be somewhat greater than the vector mass M in order to resolve distances of the order of the flux tube radius.

Identification of the partition function (30) and the Wilson loop $W(\Gamma)$ implies that the expectation value of the field tensor $G_{\mu\nu}$ at the position of the quarks can be identified with the corresponding expectation values of the color fields of Yang Mills theory [9]. (For a static quark-antiquark pair separated by a distance R, the loop Γ is a rectangle in the zt plane and $W(\Gamma)$, evaluated in the limit as the elapsed time $T \to \infty$, determines the static heavy quark potential $V(R)$.)

We now briefly summarize the results of [10], where the field theory path integral (30) was transformed into a partition function of an effective string theory of vortices.

A. From effective field theory to effective string theory

To transform $W(\Gamma)$ into a path integral over vortex sheets $\tilde{x}^\mu(\xi)$, we carry out the functional integration in two stages:

(1) We first fix the location $\tilde{x}^\mu(\xi)$ of a particular vortex.

We integrate over field configurations in (30) having a vortex on this particular surface. The integration over these configurations is proportional to $e^{iS_{\text{eff}}(\tilde{x})}$, defining the action of the effective string theory $S_{\text{eff}}(\tilde{x})$, and the constraint on this integration introduces a Fadeev-Popov determinant into the functional integral (30).

The one-loop calculation of (30) in an expansion around the classical solution includes a contribution from field modes generated by moving the position of the vortex. This contribution is cancelled by the Fadeev-Popov determinant, so that only massive modes contribute to the one-loop integration. Since (30) is cut off at a scale Λ which is only slightly larger than the mass M of the vector particle, the lightest particle in the effective field theory, the one-loop corrections to $W(\Gamma)$ are negligible at the distance scales $\sim\frac{1}{M}$ described by effective field theory. $S_{\text{eff}}(\tilde{x})$ can then be approximated by $S^{\text{class}}(\tilde{x})$, the value of the action at the classical configuration $(C^{\text{class}}_\mu(x,\tilde{x}), \phi^{\text{class}}_i(x,\tilde{x}))$ minimizing the action (27) for a fixed position $\tilde{x}^\mu(\xi)$ of the vortex:

$$S_{\text{eff}}(\tilde{x}) \approx S(\tilde{x}, C^{\text{class}}_\mu(x,\tilde{x}), \phi^{\text{class}}_i(x,\tilde{x})) \equiv S^{\text{class}}(\tilde{x}). \tag{31}$$

(2) We then integrate over all surfaces $\tilde{x}^\mu(\xi)$.

We choose a particular parametrization of $\tilde{x}^\mu$ in terms of the amplitudes $f^1(\xi)$ and $f^2(\xi)$ of the two transverse fluctuations of the vortex sheet,

$$\tilde{x}^\mu = x^\mu(\xi, f^1(\xi), f^2(\xi)), \tag{32}$$

This gives $W(\Gamma)$ the form of a path integral of an effective string theory of vortices:

$$W(\Gamma) = \int Df^1 Df^2 \Delta \exp(iS_{\text{eff}}(\tilde{x}), \tag{33}$$

where

$$\Delta \equiv \mathcal{D}et\left[\frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{-g}}\frac{\partial x^\mu}{\partial f^1}\frac{\partial x^\nu}{\partial f^2}\frac{\partial \tilde{x}^\alpha}{\partial \xi^1}\frac{\partial \tilde{x}^\beta}{\partial \xi^2}\right] \quad (34)$$

is the determinant produced by gauge fixing the reparametrization symmetry. The path integral representation (33) for $W(\Gamma)$ is invariant under reparametrizations of the vortex sheet $\tilde{x}^\mu(\xi)$, and is restricted to wavelengths longer than $\frac{1}{\Lambda}$.

The action of the effective string theory $S_{\text{eff}}(\tilde{x})$ is the action (31) of the effective magnetic gauge theory evaluated at a classical solution for a curved vortex sheet $\tilde{x}$. Since the contribution of string fluctuations to the heavy quark interaction determined by the path integral (33) is applicable only for quark-antiquark separations $R \gg \frac{1}{M}$, in this integral the action $S^{\text{class}}(\tilde{x})$ must be evaluated in the limit $\frac{1}{M} \to 0$, i.e., $S_{\text{eff}}(\tilde{x}) = S^{\text{class}}(\tilde{x})|_{\frac{1}{M}=0}$. In this limit, $S^{\text{class}}(\tilde{x})$ depends only upon a single-dimensional parameter, the classical string tension σ, and by Poincaré symmetry it must equal the Nambu-Goto action $S_{\text{NG}}(\tilde{x}^\mu)$:

$$S^{\text{class}}(\tilde{x})|_{\frac{1}{M}=0} = S_{\text{NG}}(\tilde{x}^\mu) \equiv -\sigma \int d^2\xi \sqrt{-g(\tilde{x}^\mu(\xi))}. \quad (35)$$

The action of the effective string theory obtained from effective field theory is then the Nambu-Goto action. Since deviations from the Nambu-Goto action give contributions to the ground state heavy quark potential that fall off faster than $\frac{1}{R^5}$ [14,15], effective field theory describes the expansion of ground state heavy quark potential to order $\frac{1}{R^5}$. Higher-order terms in this long-distance expansion are not taken into account by effective field theory and are not considered in this paper.

With the use of analytic regularization to renormalize $S_{\text{eff}}(\tilde{x})$ no additional dimensional parameters appear in the resulting static potential $V(R)$, and the string tension σ retains its classical value as the energy per unit length of the flux tube [25].

For a loop Γ describing the motion of a quark-antiquark pair separated by a fixed distance and rotating with constant angular velocity, $W(\Gamma)$ determines the leading semiclassical correction to the classical formula for meson Regge trajectories [26].

B. Width from string fluctuations

For distances much larger than $\frac{1}{\sqrt{\sigma}}$, string fluctuations determine the flux tube width and lead to a logarithmic increase of the mean square width $w^2(R/2)$ of the flux tube at its midpoint [27];

$$w^2(R/2) = \frac{d-2}{2\pi\sigma}\log\frac{R}{r_0}. \quad (36)$$

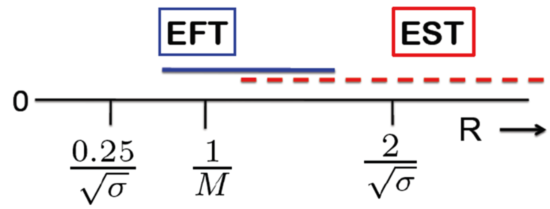

FIG. 2. Schematic showing approximate domains of applicability of effective field theory (EFT) (solid blue line) and effective string theory (EST) (red dashed line).

($\frac{1}{r_0}$ can be interpreted as the cutoff Λ of the effective field theory. Fluctuations of wavelengths less than $\frac{1}{\Lambda}$ produce a divergent contribution to $w^2(R/2)$.)

This prediction has been tested by very accurate lattice simulations [28] of the mean square flux tube width in $d = 2+1$ $SU(2)$ Yang-Mills theory extending to distances $R \approx \frac{36}{\sqrt{\sigma}}$. These simulations gave excellent agreement with the prediction (36) for distances $R > \frac{1.5}{\sqrt{\sigma}}$ with the choice $r_0 = \frac{0.364}{\sqrt{\sigma}}$ corresponding to a value of $\Lambda \sim 2.75\sqrt{\sigma} \approx 1.4M$. However for distances $R < \frac{1.5}{\sqrt{\sigma}}$ the lattice simulations of $w^2(R/2)$ lie above the leading-order prediction (36) of effective string theory. This excess may be interpreted as a manifestation of the flux tube intrinsic width at $q\bar{q}$ separations $R < \frac{1.5}{\sqrt{\sigma}}$.

C. The intrinsic width of the flux tube

The intrinsic width produces an uncertainty of order $\frac{1}{M}$ in the position of the vortex, so that for quark-antiquark separations $R \sim \frac{1}{M}$ string fluctuations do not contribute to the path integral (30). The Wilson loop (30) can then be replaced by its minimum value, fixed by the value of the classical action for a flat vortex sheet connecting a static quark-antiquark pair. $W(\Gamma)$ then yields $V^{\text{class}}(R)$, the heavy quark potential in the classical approximation.

Recent very accurate lattice simulations [22,29] of field and energy distributions in $SU(3)$ flux tubes find values of the intrinsic width characterizing these distributions that corresponds to a mass M of approximately 900 MeV. Since $M \sim 2\sqrt{\sigma}$, there is an interval of intermediate distances $R \sim \frac{1}{M}$ lying in the range where the predictions of effective field theory at the classical level are not washed out by string fluctuations. (The lattice simulations of heavy quark potentials [18], were carried out at these distances.) In this interval, denoted [EFT] in Fig. 2, the classical flux tube picture should be manifest, while effective string theory should be used in the distance range $R > \frac{2}{\sqrt{\sigma}}$ (denoted [EST] in the figure).

Effective string theory must be used to fit more recent simulations [30] of heavy quark potentials for values of R extending to 1.2 fm $> \frac{2}{\sqrt{\sigma}}$. In the intermediate range of distances depicted in Fig. 2 both the flux tube intrinsic width and the effect of string fluctuations must be taken into account.

IV. THE CLASSICAL ACTION FOR $SU(N)$ VORTICES

We now obtain a representation for the classical action of curved vortex sheets and a corresponding representation for flat sheets. We will use these representations, together with Poincaré invariance, to obtain information about the classical action of a general vortex sheet from the action of a flat sheet.

Equation (31) gives the action $S_{\mathrm{eff}}(\tilde{x}^\mu)$ of the effective string theory as the action (27) of the effective field theory, evaluated at a classical solution having a vortex at $\tilde{x}^\mu$. To find the nonperturbative contribution to this action we separate C_μ into a perturbative contribution C_μ^D and a nonperturbative contribution c_μ:

$$C_\mu = C_\mu^D + c_\mu = (C_\mu^D(x,\tilde{x}) + c_\mu(x,\tilde{x}))Y. \tag{37}$$

The perturbative vector potential C_μ^D gives the Maxwell field $G^{\mu\nu}_{\mathrm{MAX}}$ of the external $q\bar{q}$ pair generated by the coupling of the dual potentials to $G^s_{\mu\nu}$ [24]:

$$G^{\mu\nu}_{\mathrm{MAX}} = \partial^\mu C^{D\nu} - \partial^\nu C^{D\mu} + G^{s\mu\nu}. \tag{38}$$

The corresponding dual field tensor $G^{\mu\nu}$ assumes the form

$$G^{\mu\nu} = G^{\mu\nu}_{\mathrm{MAX}} + G^{\mu\nu}_{\mathrm{class}}, \tag{39}$$

where

$$G^{\mu\nu}_{\mathrm{class}}(x,\tilde{x}) = \partial^\mu c^\nu - \partial^\nu c^\mu = (\partial^\mu c^\nu(x,\tilde{x}) - \partial^\nu c^\mu(x,\tilde{x}))Y, \tag{40}$$

is the nonperturbative field tensor satisfying the classical equation of motion:

$$\partial_\nu G^{\nu\mu}_{\mathrm{class}} = ig_m[\phi_i, \mathcal{D}^\mu \phi_i] \equiv j^\mu, \tag{41}$$

defining the magnetic current density j^μ. For consistency, the non-Abelian Higgs fields ϕ_i must have a color structure such that j^μ is also proportional to the matrix Y.

The action (27), evaluated at the classical solution, separates into the sum:

$$S[C_\mu, \phi_i, \tilde{x}^\mu] = S^{\mathrm{MAX}}(\Gamma) + S^{\mathrm{class}}(\tilde{x}^\mu), \tag{42}$$

where $S^{\mathrm{MAX}}(\Gamma)$ is (up to a color factor) the Maxwell action, and $S^{\mathrm{class}}(\tilde{x}^\mu)$ is the nonperturbative contribution to the action:

$$S^{\mathrm{class}}(\tilde{x}) = \int dx \left\{ 2\mathrm{tr}\left[-\frac{1}{4} G^{\mu\nu}_{\mathrm{class}} G_{\mu\nu\mathrm{class}} + \frac{1}{2}(\mathcal{D}_\mu \phi_i)^2 \right] - V(\phi_i) \right\}. \tag{43}$$

(The classical action is related to the Hamiltonian: $-\partial S/\partial t = H$ [31].)

Using the equation of motion (41) in (43) to write $2S^{\mathrm{class}}(\tilde{x})$, and subtracting $S^{\mathrm{class}}(\tilde{x})$ in the original form, gives

$$S^{\mathrm{class}}(\tilde{x}) = \int dx \left\{ 2\mathrm{tr}\left[-\frac{1}{2} G^s_{\mu\nu} G^{\mu\nu}_{\mathrm{class}} + C_\mu j^\mu + \frac{1}{4} G^{\mu\nu}_{\mathrm{class}} G_{\mu\nu\mathrm{class}} + \frac{1}{2}(\mathcal{D}_\mu \phi_i)^2 \right] - V(\phi_i) \right\}. \tag{44}$$

(There is also a term on the right-hand side of (44) proportional to $G^{\mathrm{MAX}}_{\mu\nu} G^{\mu\nu}_{\mathrm{class}}$, which vanishes after integration by parts and use of Maxwell's equations.) Then, use of the identity

$$\mathrm{tr}\left(C^\mu j_\mu + \frac{1}{2}(\mathcal{D}_\mu \phi_i)^2 \right) \equiv \mathrm{tr}\left(\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{[ig_m C_\mu, \phi_i]^2}{2} \right) \tag{45}$$

to rewrite (44) gives the following representation of the classical action:

$$-S^{\mathrm{class}}(\tilde{x}) = \int dx 2\mathrm{tr}\left(\frac{1}{2} G^s_{\mu\nu} G^{\mu\nu}_{\mathrm{class}} \right) + S_g(\tilde{x}) - S_\phi(\tilde{x}), \tag{46}$$

where

$$S_g(\tilde{x}) = \int dx 2\mathrm{tr}\left[-\frac{1}{4} G^{\mu\nu}_{\mathrm{class}} G_{\mu\nu\mathrm{class}} - \frac{g_m^2 [C_\mu, \phi_i]^2}{2} \right], \tag{47}$$

and

$$S_\phi(\tilde{x}) = \int dx \left[2\mathrm{tr}\frac{(\partial_\mu \phi_i)^2}{2} - V(\phi_i) \right]. \tag{48}$$

The first term in (46),

$$\int dx 2\mathrm{tr}\left[\frac{1}{2}G^s_{\mu\nu}G^{\mu\nu}_{\rm class}\right] = -\frac{1}{4}\int d\tau \int d\sigma\sqrt{-g}\epsilon_{\mu\nu\lambda\alpha}2\mathrm{tr}\left(\frac{2\pi}{g_m}YG^{\mu\nu}_{\rm class}(x,\tilde{x}|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)})\right)t^{\lambda\alpha}(\sigma,\tau) \equiv \int d\tau W(\tau), \tag{49}$$

the integrated work required to separate the quark-antiquark pair along the vortex sheet $\tilde{x}^\mu(\sigma,\tau)$ in the fully developed field $G^{\mu\nu}_{\rm class}(x,\tilde{x})$, and the second term,

$$S_g(\tilde{x}) - S_\phi(\tilde{x}) = \int dx\left(2\mathrm{tr}\left[-\frac{1}{4}G^{\mu\nu}_{\rm class}G_{\mu\nu\,\rm class} - \frac{g_m^2[C_\mu,\phi_i]^2}{2}\right] - \left[2\mathrm{tr}\frac{(\partial_\mu\phi_i)^2}{2} - V(\phi_i)\right]\right), \tag{50}$$

is the net additional integrated field energy available from the process of creating the vortex sheet, i.e. it is the difference between $-S^{\rm class}(\tilde{x})$, the integrated work needed to separate the quarks in the developing color fields, and the corresponding integrated work $\int d\tau W(\tau)$ in the fully developed chromodynamic field $G^{\mu\nu}_{\rm class}$.

With a parametrization where $\frac{\partial\tilde{x}^0}{\partial\sigma}|_\tau = 0$, (49) takes the form:

$$\int dx 2\mathrm{tr}\left[\frac{1}{2}G^s_{\mu\nu}G^{\mu\nu}_{\rm class}\right] = \int d\tau\int d\sigma\frac{\partial\vec{\tilde{x}}}{\partial\sigma}\cdot\vec{F}_{\rm class}(\sigma,\tau,\tilde{x})\left(\frac{\partial\tilde{x}^0}{\partial\tau}\right), \tag{51}$$

where

$$\vec{F}_{\rm class}(\sigma,\tau,\tilde{x}) \equiv 2\mathrm{tr}\left[-\frac{2\pi}{g_m}Y(\vec{E}_{\rm class}(x,\tilde{x}) + \vec{v}\times\vec{B}_{\rm class}(x,\tilde{x}))\right]|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)}, \tag{52}$$

$$E^k_{\rm class}(x,\tilde{x}) = \frac{1}{2}\epsilon_{klm}(G^{lm}_{\rm class}(x,\tilde{x})), \qquad B^k_{\rm class}(x,\tilde{x}) = G^{k0}_{\rm class}(x,\tilde{x}) \tag{53}$$

are the classical chromoelectric and chromomagnetic fields, and

$$\vec{v}(\sigma,\tau) = \frac{\partial\vec{\tilde{x}}(\sigma,\tau)}{\partial\tau}\bigg/\frac{\partial\tilde{x}^0}{\partial\tau} \tag{54}$$

is the velocity of the sheet.

A. The heavy quark potential in the classical approximation

The classical action $S^{\rm class}(\tilde{x})$, evaluated for a flat vortex sheet connecting a static quark at $\vec{x}_1 = \frac{R}{2}\hat{e}_z$ and an antiquark at $\vec{x}_2 = -\frac{R}{2}\hat{e}_z$, determines $V^{\rm class}(R)$, the approximation to the heavy quark potential, where string fluctuations are neglected. For this sheet the components (24) of $G^s_{\mu\nu}(x,\tilde{x})$ are given by

$$G^s_{k0} = 0, \qquad G^s_{lm} = \frac{1}{2}\epsilon_{lmn}E^{sm},$$
$$\vec{E}^s = -\frac{2\pi}{g_m}\delta(x)\delta(y)[\theta(z+R/2) - \theta(z-R/2)]\hat{e}_zY. \tag{55}$$

The vector potential (23) becomes

$$\vec{C} = C(r,z)\hat{e}_\theta Y, \qquad C_0 = 0. \tag{56}$$

The spatial components of the tensor $G_{\mu\nu}$ (28) yield the static chromoelectric field $\vec{E}$:

$$\vec{E} = -\vec{\nabla}\times\vec{C} + \vec{E}^s. \tag{57}$$

In cylindrical coordinates, $g_{00} = 1$, $g_{zz} = g_{rr} = -1$, $g_{\theta\theta} = -r^2$, $g \equiv \det g_{\mu\nu} = -r^2$, and the components of $\vec{E}$ are

$$E_z \equiv -\frac{G_{r\theta}}{r} = \frac{1}{r}\frac{\partial C_\theta}{\partial r} + E^s_z, \qquad E_r \equiv -\frac{G_{\theta z}}{r} = -\frac{1}{r}\frac{\partial C_\theta}{\partial z}, \tag{58}$$

with

$$C_\theta = -rC(r,z)Y. \tag{59}$$

The Higgs fields $\phi_i = \phi_i(r,z)$ are independent of θ and t.

The decomposition (37) of C_μ takes the form:

$$C_0 = 0, \qquad \vec{C} = \vec{C}^D + \vec{c} \equiv (C^D(r,z) + c(r,z))\hat{e}_\theta Y, \tag{60}$$

where

$$C^D(r,z) = \frac{1}{4\pi r}\left[\frac{z-R/2}{\sqrt{r^2+(z-R/2)^2}} - \frac{z+R/2}{\sqrt{r^2+(z+R/2)^2}}\right]\frac{2\pi}{g_m} \tag{61}$$

is the perturbative potential of the quark sources generated by the Dirac string (55), and $c(r,z)$ is the nonperturbative potential generated by the induced currents (41).

The color electric field (57) becomes the sum of a Coulomb field $\vec{E}_C$ and a nonperturbative contribution $\vec{E}_{\rm class}$:

$$\vec{E} = \vec{E}_C(\vec{x},R) + \vec{E}_{\rm class}(\vec{x},R), \tag{62}$$

where

$$\vec{E}_C = \frac{1}{4\pi}\left(\frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3} - \frac{\vec{x}-\vec{x}_2}{|\vec{x}-\vec{x}_2|^3}\right)\frac{2\pi}{g_m}Y,$$
$$\vec{E}_{\rm class}(\tilde{x},R) = -\vec{\nabla}\times\vec{c}. \tag{63}$$

At large distances $\vec{E}_{\rm class}$ screens the Coulomb field while the Higgs fields approach their vacuum values ϕ_{i0}, so that the boundary conditions are

$$\vec{c}\to -\vec{C}^D, \qquad \phi_i \to \phi_{i0}, \qquad r\to\infty \quad \text{or} \quad z\to\infty. \tag{64}$$

(42), evaluated for static quarks yields the heavy quark potential as the sum of a Coulomb potential $V^C(R)$ and a nonperturbative potential $V^{\rm class}(R)$, where

$$-S^{\rm MAX} = TV^C(R), \qquad V^C(R) = -2{\rm tr}\left(\frac{2\pi Y}{g_m}\right)^2\left(\frac{1}{4\pi R}\right); \tag{65}$$

$$-S^{\rm class}(\tilde{x}) = TV^{\rm class}(R), \qquad V^{\rm class}(R) = \int d\vec{x}T_{00}(\vec{x},R), \tag{66}$$

and $T_{00}(\vec{x},R)$ is the nonperturbative contribution to the energy density:

$$T_{00}(\vec{x},R) = 2{\rm tr}\left[\frac{\vec{E}_{\rm class}(\vec{x},R)^2}{2} + \frac{g_m^2\vec{C}(\vec{x})^2[iY,\phi_i(\vec{x})]^2}{2}\right] + 2{\rm tr}\left[\frac{(\vec{\nabla}\phi_i(\vec{x}))^2}{2}\right] + V(\phi_i). \tag{67}$$

(49), evaluated for static quarks, becomes

$$\int dx\left[2{\rm tr}\frac{1}{2}G^s_{\mu\nu}G^{\mu\nu}_{\rm class}\right] \equiv TW(R), \tag{68}$$

where

$$W(R) = \int_{-R/2}^{R/2} dz 2{\rm tr}\left[-\frac{2\pi}{g_m}Y\hat{e}_z\cdot\vec{E}_{\rm class}(r=0,z,R)\right], \tag{69}$$

the work required to separate a quark-antiquark pair a distance R in the field $\vec{E}_{\rm class}(\vec{x},R)$.

(50), evaluated for static quarks, becomes the relation

$$(S_g(\tilde{x}) - S_\phi(\tilde{x})) = -T\int d\vec{x}\frac{T_{\theta\theta}(\vec{x},R)}{r^2} = -2\pi T\int dzrdr\frac{T_{\theta\theta}(r,z,R)}{r^2}, \tag{70}$$

where $\frac{T_{\theta\theta}(\vec{x},R)}{r^2}$ is the $\theta\theta$ component of the stress tensor for finite values of R:

$$\frac{T_{\theta\theta}(\vec{x},R)}{r^2} = 2{\rm tr}\left[\frac{\vec{E}_{\rm class}(\vec{x},R)^2}{2} + \frac{g_m^2\vec{C}(\vec{x})^2[iY,\phi_i(\vec{x})]^2}{2}\right] - \left(2{\rm tr}\left[\frac{(\vec{\nabla}\phi_i(\vec{x}))^2}{2}\right] + V(\phi_i)\right). \tag{71}$$

(71) expresses $T_{\theta\theta}$ as the difference between a repulsive gauge contribution and the attractive Higgs contribution produced by the circulating magnetic currents generated by the Higgs condensate.

Using (65), (68) and (70) the decomposition (46) of $S^{\rm class}(\tilde{x})$ becomes a corresponding decomposition of the heavy quark potential:

$$V^{\rm class}(R) = W(R) - \int d\vec{x}\frac{T_{\theta\theta}(\vec{x},R)}{r^2}. \tag{72}$$

1. *Physical interpretation of the representation (72) of* $V^{\rm class}(\boldsymbol{R})$

The quantity $\int dzrdr\frac{T_{\theta\theta}(r,z,R)}{r^2}$ is the total torque $\mathcal{T}(R)$ acting across any (r,z) plane bounded by the axis of the flux tube. (See Fig. 1.) Then $\mathcal{T}(R)\Delta\theta$ is the work required to remove the field energy in a sector of the flux tube of angular width $\Delta\theta$ between two (r,z) planes while maintaining the quark-antiquark separation R. Since the torque is independent of θ, the work required to remove all the field energy while maintaining the quark-antiquark separation R is just $2\pi\mathcal{T}(R) = \int d\vec{x}\frac{T_{\theta\theta}(\vec{x},R)}{r^2}$. If $\mathcal{T}(R) > 0$ (net repulsion) it takes work to remove the field energy.

The heavy quark potential $V^{\rm class}(R)$ is the energy available for doing work when a separated quark-antiquark pair come together. Equation (72) expresses $V^{\rm class}(R)$ as the difference between $W(R)$, the work necessary to separate the pair in the fixed field $\vec{E}_{\rm class}(\vec{x},R)$, and $2\pi\mathcal{T}(R)$, the work necessary to remove the field energy created by their separation.

B. Limit $R \to \infty$ $(R \gg \frac{1}{M})$

As $R \to \infty$,

$$C^D(r,z) \to -\frac{1}{g_m r}, \qquad c(r,z) \to C(r), \qquad \phi_i(r,z) \to \phi_i(r), \tag{73}$$

$$\vec{E}_{\rm class}(\vec{x},R) \to \vec{E}(r) = -\frac{1}{r}\frac{d(rC(r))}{dr} Y\hat{e}_z, \tag{74}$$

$$W(R) \to 2{\rm tr}\left[-\frac{2\pi}{g_m} Y\hat{e}_z \cdot \vec{E}(r=0)\right]R, \quad T_{00}(\vec{x},R) \to T_{00}(r), \quad T_{\theta\theta}(\vec{x},R) \to T_{\theta\theta}(r), \tag{75}$$

$$V^{\rm class}(R) = \int d\vec{x} T_{00}(\vec{x},R) \quad \to \textstyle\int_0^\infty 2\pi r dr T_{00}(r) R = \sigma R, \tag{76}$$

$$2\pi \mathcal{T}(R) = \int d\vec{x} \frac{T_{\theta\theta}(\vec{x},R)}{r^2} \to \int_0^\infty 2\pi r dr \frac{T_{\theta\theta}(r)}{r^2} R = 2\pi\tau R; \tag{77}$$

where τ is the torque per unit length (4), and

$$T_{00}(r) = 2{\rm tr}\left[\frac{1}{2}\vec{E}^2(r) + g_m^2\left(C(r) - \frac{1}{g_m r}\right)^2 [iY,\phi_i]^2\right] + 2{\rm tr}\left[\frac{1}{2}\left(\frac{d\phi_i(r)}{dr}\right)^2\right] + V(\phi_i),$$
$$\frac{T_{\theta\theta}(r)}{r^2} = 2{\rm tr}\left[\frac{1}{2}\vec{E}^2(r) + g_m^2\left(C(r) - \frac{1}{g_m r}\right)^2 [iY,\phi_i]^2\right] - 2{\rm tr}\left[\frac{1}{2}\left(\frac{d\phi_i(r)}{dr}\right)^2\right] - V(\phi_i). \tag{78}$$

Taking the large R limit of (72), using (75), (76) and (77) yields Eq. (2), as stated in the Introduction. Equation (2) links the string tension σ to the field $\vec{E}(r=0)$ on the axis of an infinite flux tube via the parameter τ, and has the physical interpretation discussed in the Introduction and in the previous section.

Using the fact that σ is the long-distance force on a quark,

$$\sigma = 2{\rm tr}\left[-\frac{2\pi}{g_m} Y\vec{E}_{\rm class}\left(r=0, z=\pm\frac{R}{2}; R\right)\right] \cdot \hat{e}_z, \qquad R \gg \frac{1}{M}, \tag{79}$$

we can write (2) in an alternate form:

$$2{\rm tr}\left[-\frac{2\pi}{g_m} Y\vec{E}_{\rm class}\left(r=0, z=\pm\frac{R}{2}; R\right)\right] = 2{\rm tr}\left[-\frac{2\pi}{g_m} Y\vec{E}(r=0)\right] - 2\pi\tau\hat{e}_z, \qquad R \gg \frac{1}{M}. \tag{80}$$

The field on the axis of an infinite flux tube $\vec{E}(r=0)$ is equal to the field of a quark and antiquark, $\vec{E}_{\rm class}(r=0,z;R)$, evaluated in the central region $|z| \ll \frac{R}{2}$, far from the positions of the quarks. Consequently, (80) has the equivalent form:

$$2\pi\tau\hat{e}_z = 2{\rm tr}\left[-\frac{2\pi}{g_m} Y\vec{E}_{\rm class}\left(r=0, z=\pm\frac{R}{2}; R\right)\right] - 2{\rm tr}\left[-\frac{2\pi}{g_m} Y\vec{E}_{\rm class}(r=0,z;R)\right], \quad |z| \ll \frac{R}{2}, \qquad R \gg \frac{1}{M}. \tag{81}$$

The torque per unit length τ thus determines the difference between the value of the field $\vec{E}_{\rm class}$ at the positions of the quarks and its value midway between them. ((81) is an equivalent characterization of the parameter τ.)

V. A NEW CONSTRAINT ON THE QCD FLUX TUBE

We now assume that the value $\tau = 0$ characterizes the QCD flux tube and examine the consequences of this constraint.

If $\tau = 0$ the string tension equals the color charge $\frac{2\pi}{g_m} Y$ of the quark multiplied by the field $\vec{E}(r=0)$ on the axis of an infinite Z_N flux tube [Eq. (7)]; i.e., the force on a quark in the field of the "string" connecting the pair.

Further, (81) becomes the equality

$$\vec{E}_{\rm class}\left(r=0, z=\pm\frac{R}{2}; R\right) = \vec{E}_{\rm class}(r=0,z;R), \qquad |z| \ll \frac{R}{2}, \qquad R \gg \frac{1}{M}, \tag{82}$$

so that the field at the positions of the quarks equals the field in the middle of the flux tube.

A nonvanishing value of τ necessitates a variation of $\vec{E}_{\rm class}(r=0,z;R)$ along the line connecting the pair. The condition $\tau = 0$ allows this field to remain constant for all z including points close to the positions of the quarks. (Expressed in this way one might speculate that the condition $\tau = 0$ imposed on the effective field theory reflects a flicker of the short distance asymptotic freedom of the fundamental theory visible in the effective field theory.)

A. The action of the effective string theory

Poincaré invariance implies that the action of the effective string theory obtained from effective field theory $S_{\rm eff}(\tilde{x}) = S^{\rm class}(\tilde{x})|_{\frac{1}{M}=0} = S_{\rm NG}(\tilde{x})$ for any value of τ [Eq. (35)]. We will now show that under the condition $\tau = 0$, $S^{\rm class}(\tilde{x})|_{\frac{1}{M}=0}$ has a representation in terms of the chromodynamic fields of magnetic $SU(N)$ gauge theory. This will give a field theory interpretation of effective string theory.

For long straight strings (70) and (77) show that the term linear in R in $S_g(\tilde{x}) - S_\phi(\tilde{x})$ is proportional to τ. Hence, for curved strings, by Poincaré symmetry the term having the Nambu-Goto form in $S_g(\tilde{x}) - S_\phi(\tilde{x})$ is also proportional to τ [32]. Thus if $\tau = 0$, $S_g(\tilde{x}) - S_\phi(\tilde{x})$ does not contain a term proportional to the Nambu-Goto action, and can be neglected on the right-hand side of Eq. (46) for $S^{\rm class}(\tilde{x})$; its contribution to $S_{\rm eff}(\tilde{x})$ generates terms in the ground state heavy quark potential that fall off faster than $\frac{1}{R^5}$ [14,15]. Then (46) takes the form

$$S^{\rm class}(\tilde{x})|_{\frac{1}{M}=0} = -\int dx 2{\rm tr}\left[\frac{1}{2}G^s_{\mu\nu}G^{\mu\nu}_{\rm class}\right] = \frac{1}{4}\int d\tau \int d\sigma \sqrt{-g}\epsilon_{\mu\nu\lambda\alpha}2{\rm tr} \times \left(\frac{2\pi}{g_m}YG^{\mu\nu}_{\rm class}(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)}\right)t^{\lambda\alpha}(\sigma,\tau), \quad (83)$$

an integral of the field tensor $G^{\mu\nu}_{\rm class}(x,\tilde{x})$ evaluated on the vortex sheet $x^\mu = \tilde{x}^\mu(\sigma,\tau)$. Equation (83) gives the Nambu-Goto action a representation solely in terms of the chromodynamic fields of the four-dimensional effective field theory.

Writing (83) in a parametrization where $\frac{\partial \tilde{x}^0}{\partial \sigma}|_{\tau=0}$, using (51) and (52) gives

$$S_{\rm NG}(\tilde{x}) = -\int d\tau \int d\sigma \frac{\partial \vec{\tilde{x}}}{\partial \sigma}\cdot \vec{F}_{\rm class}(\sigma,\tau,\tilde{x})\left(\frac{\partial \tilde{x}^0}{\partial \tau}\right), \quad (84)$$

where $\vec{F}_{\rm class}(\sigma,\tau,\tilde{x})$ is the chromodynamic force (52) acting along the string. (84) is the representation of the Nambu-Goto action in terms of fields and is the generalization of the relation (7) to curved vortex sheets.

B. The relation between fields and surfaces

For a curved vortex sheet Lorentz invariance and reparametrization invariance imply that $\epsilon_{\mu\nu\lambda\alpha}G^{\lambda\alpha}_{\rm class}(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)}$ must be proportional to the tensor (26) describing the orientation of the world sheet $\tilde{x}^\mu(\sigma,\tau)$:

$$2{\rm tr}\left(\frac{2\pi}{g_m}Y\frac{1}{2}\epsilon_{\mu\nu\lambda\alpha}G^{\lambda\alpha}_{\rm class}(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)}\right) = \sigma t_{\mu\nu}(\sigma,\tau). \quad (85)$$

Consistency of (85) evaluated for a long straight vortex with (7) fixes the string tension σ as the coefficient of $t_{\mu\nu}$. Taking into account nonleading terms in $\frac{1}{M}$ would introduce higher-dimensional tensors and new parameters on the right-hand side of (85).[3]

Therefore, to leading order in $\frac{1}{M}$, the values of the chromodynamic fields $G^{\mu\nu}_{\rm class}$ on the vortex sheet are determined in terms of the string tension σ and the geometry of the vortex sheet. Using (85) in (83) and the normalization $t^{\mu\nu}t_{\mu\nu} = -2$ of the surface tensor yields the Nambu-Goto action directly.

Expressing (85) in terms of the color electric and magnetic components (53) of $G^{\mu\nu}_{\rm class}$ gives the values of the fields $\vec{E}_{\rm class}$ and $\vec{B}_{\rm class}$ on the vortex sheet:

$$2{\rm tr}\left[-\frac{2\pi}{g_m}YE^k_{\rm class}\right]\Bigg|_{x^\mu=\tilde{x}^\mu} = \sigma t^{0k}(\sigma,\tau),$$
$$2{\rm tr}\left[\frac{2\pi}{g_m}YB^k_{\rm class}\right]\Bigg|_{x^\mu=\tilde{x}^\mu} = \frac{\sigma}{2}\epsilon_{klm}t^{lm}(\sigma,\tau). \quad (86)$$

We choose $\tau = t$, $\sigma = z$, and a parametrization $\tilde{x}^\mu(z,t)$ of the vortex sheet in terms of the two transverse fluctuations $\vec{x}^i_\perp(z,t)$, $i = 1, 2$:

$$\tilde{x}^\mu(z,t) = x^\mu(t,z,\vec{x}^1_\perp(z,t),\ \vec{x}^2_\perp(z,t)). \quad (87)$$

The color fields evaluated on the vortex sheet are corresponding functions of z and t:

$$\vec{E}^i_{\rm class}(x,\tilde{x})|_{x^\mu\equiv\tilde{x}^\mu(z,t)} \equiv \vec{E}^i_{\rm class}(z,t),$$
$$\vec{B}^i_{\rm class}(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(z,t)} \equiv \vec{B}^i_{\rm class}(z,t), \quad (88)$$

Equation (86), with the use of (26) and (87), determines the fields $\vec{E}_{\rm class}(z,t)$ and $\vec{B}_{\rm class}(z,t)$ in terms of the transverse fluctuations $\vec{x}^i_\perp(z,t)$ for $-R/2 \le z \le R/2$:

$$2{\rm tr}\left[-\frac{2\pi Y}{g_m}\vec{E}^i_{\rm class}(z,t)\right] = \frac{\sigma}{\sqrt{-g}}\frac{\partial \vec{x}^i_\perp}{\partial z},$$
$$2{\rm tr}\left[\frac{2\pi Y}{g_m}\vec{B}^i_{\rm class}(z,t)\right] = \frac{\sigma}{\sqrt{-g}}\left(\hat{e}_z \times \frac{\partial \vec{x}_\perp}{\partial t}\right)^i,$$
$$2{\rm tr}\left[-\frac{2\pi Y}{g_m}\vec{B}^z_{\rm class}(z,t)\right] = \frac{\sigma}{2\sqrt{-g}}\left(\frac{\partial \vec{x}_\perp}{\partial t}\times\frac{\partial \vec{x}_\perp}{\partial z}\right)\cdot\hat{e}_z,$$
$$2{\rm tr}\left[-\frac{2\pi Y}{g_m}\vec{E}^z_{\rm class}(z,t)\right] = \frac{\sigma}{\sqrt{-g}}. \quad (89)$$

The Wilson loop $W(\Gamma)$ written in the parametrization (87) is

[3]Relations between fields and surfaces, postulated on the basis of symmetry, with account taken of nonleading terms and limited to the positions of the quarks, have been used to calculate heavy quark potentials [33].

$$W(\Gamma) = \int \mathcal{D}\vec{x}^1_\perp \mathcal{D}\vec{x}^2_\perp \exp\left[-i\sigma \int dt \int_{-R/2}^{R/2} dz \sqrt{-g(\tilde{x}^\mu(z,t))}\right]. \quad (90)$$

If $\tau = 0$ the Nambu-Goto action has the representation (84), so that the Wilson loop (90) can be used along with the relations (89) to calculate correlation functions of the fields $\vec{E}_{\rm class}(z,t)$ and $\vec{B}_{\rm class}(z,t)$ and physical quantities dependent on them.

We now describe the picture that results from the condition $\tau = 0$ in a particular model.

VI. $SU(N)$ VORTICES IN A PARTICULAR MODEL

The effective Lagrangian in the model [5] has the form (9) with three scalar Higgs fields and a Higgs potential $V(\phi_i)$ generated from one-loop contributions to the scalar 2-point and 4-point functions in effective $SU(N)$ magnetic gauge theory:

$$V(\phi_i) = \mu^2 N \sum_i 2{\rm tr}[\phi_i^2] + \frac{4N\lambda}{3}\left({\rm tr}\left(\sum_{ij}\phi_i^2\phi_j^2\right) + \frac{1}{N}\left({\rm tr}\left(\sum_i \phi_i^2\right)\right)^2 + \frac{2}{N}\sum_{ij}({\rm tr}\phi_i\phi_j)^2\right), \quad (91)$$

where the parameter μ^2 has dimensions of mass squared and λ is dimensionless.

In the confining vacuum the Higgs condensate ϕ_{i0} has the color structure:

$$\phi_{10} = \phi_0 J_x, \qquad \phi_{20} = \phi_0 J_y, \qquad \phi_{30} = \phi_0 J_z, \quad (92)$$

where J_x, J_y, and J_z are the three generators of the N-dimensional irreducible representation of the three-dimensional rotation group corresponding to angular momentum $J = \frac{N-1}{2}$. Since any matrix which commutes with all three generators J_i must be a multiple of the unit matrix, there is no $SU(N)$ transformation which leaves all three ϕ_i invariant and the dual $\frac{SU(N)}{Z_N}$ gauge symmetry is completely broken.

The Higgs potential has an absolute minimum at $\phi_i = \phi_{i0}$ with $\phi_0^2 = -\frac{9\mu^2}{8(N^2-1)\lambda}$. The difference ϵ_V between the energy density of the symmetry breaking vacuum $\phi_i = \phi_{i0}$ and the perturbative vacuum $\phi_i = 0$ is the minimum value $V(\phi_{i0})$ of the Higgs potential:

$$\epsilon_V = V(\phi_{i0}) = \frac{-\lambda}{9}((N(N^2-1)\phi_0^2)^2. \quad (93)$$

A. The classical action for $SU(3)$ vortices

For $SU(3)$,

$$J_x = \lambda_7, \quad J_y = -\lambda_5, \quad J_z = \lambda_2, \quad Y = \frac{\lambda_8}{\sqrt{3}}, \quad (94)$$

and the vector mass (22) has the value

$$M = \sqrt{6} g_m \phi_0. \quad (95)$$

We make the following ansatz for the classical solution:

$$\phi_1 = \phi_1(x,\tilde{x})\frac{(\lambda_7 - i\lambda_6)}{2} + \phi_1^*(x,\tilde{x})\frac{(\lambda_7 + i\lambda_6)}{2},$$
$$\phi_2 = \phi_2(x,\tilde{x})\frac{(-\lambda_5 - i\lambda_4)}{2} + \phi_2^*(x,\tilde{x})\frac{(-\lambda_5 + i\lambda_4)}{2},$$
$$\phi_3 = \phi_3(x,\tilde{x})\lambda_2, \quad (96)$$

$$C_\mu = C_\mu(x)Y = (C_\mu^D(x,\tilde{x}) + c_\mu(x,\tilde{x}))Y,$$
$$G_{\mu\nu{\rm class}} = (\partial_\mu c_\nu(x) - \partial_\nu c_\mu(x))Y \equiv G_{\mu\nu{\rm class}}(x,\tilde{x})Y. \quad (97)$$

There are two other solutions, physically equivalent to (96), related by gauge transformations taking $Y \to -\frac{Y+\lambda_3}{2}$ or $Y \to \frac{\lambda_3 - Y}{2}$, corresponding to the other two quark colors [23].

The commutation relations

$$[Y, \lambda_7 - i\lambda_6] = \lambda_7 - i\lambda_6,$$
$$[Y, -\lambda_5 - i\lambda_4] = -(-\lambda_5 - i\lambda_4), \qquad [Y, \lambda_2] = 0 \quad (98)$$

yield

$$\mathcal{D}_\mu\phi_1 = (\partial_\mu - ig_m C_\mu(x))\phi_1(x)\frac{(\lambda_7 - i\lambda_6)}{2} + (\partial_\mu + ig_m C_\mu(x))\phi_1^*(x)\frac{(\lambda_7 + i\lambda_6)}{2},$$
$$\mathcal{D}_\mu\phi_2 = (\partial_\mu + ig_m C_\mu(x))\phi_2(x)\frac{(-\lambda_5 - i\lambda_4)}{2} + (\partial_\mu - ig_m C_\mu(x))\phi_2^*(x)\frac{(-\lambda_5 + i\lambda_4)}{2}, \quad (99)$$

so that the Higgs fields ϕ_1 and ϕ_2 carry Y charge ± 1 and that ϕ_3 carries Y charge 0.

The consistency requirement that the magnetic current density (41) be proportional to Y forces

$$\phi_1(x,\tilde{x}) = \phi_2^*(x,\tilde{x}) \equiv \phi(x,\tilde{x}), \quad (100)$$

and yields

$$j^\mu = 6g_m\left(\frac{\phi^*(x)D^\mu\phi(x) - \phi(x)(D^\mu\phi(x))^*}{2i}\right)Y, \quad (101)$$

where

$$D_\mu\phi(x) \equiv (\partial_\mu - ig_m C_\mu(x))\phi(x). \quad (102)$$

Using the color ansatz (23) and (96) in (43) and (91), making use of (100) and subtracting off the vacuum energy density ϵ_V gives $S^{\rm class}(\tilde{x})$ the form:

$$S^{\mathrm{class}}(\tilde{x}) = \int dx \bigg[\frac{4}{3}\left(-\frac{1}{4}G_{\mu\nu\mathrm{class}}(x)G^{\mu\nu}_{\mathrm{class}}(x)\right) + 4(D_\mu\phi(x))(D^\mu\phi(x))^* + 2\partial_\mu\phi_3(x)\partial^\mu\phi_3(x) - V(\phi,\phi_3)\bigg], \quad (103)$$

where

$$V(\phi,\phi_3) = \frac{22\lambda}{3}(2(|\phi|^2-\phi_0^2)^2 + (\phi_3^2-\phi_0^2)^2) + \frac{14\lambda}{3}(2|\phi|^2+\phi_3^2-3\phi_0^2)^2. \quad (104)$$

The corresponding field equations are

$$\partial^\mu G^{\mathrm{class}}_{\mu\nu}(x,\tilde{x}) = \partial^\mu\partial_\mu c_\nu - \partial_\nu\partial^\mu c_\mu = 6g_m\left(\frac{\phi^*\partial_\nu\phi - \phi\partial_\nu\phi^*}{2i} - g_m C_\nu\phi^*\phi\right), \quad (105)$$

and

$$-D_\mu D^{\mu *}\phi(x) = \frac{1}{4}\frac{\delta V}{\delta\phi^*(x)},$$
$$-\partial_\mu\partial^\mu\phi_3(x) = \frac{1}{2}\frac{\delta V}{\delta\phi_3(x)}. \quad (106)$$

At large distances the Higgs fields are a gauge transformation of the vacuum solution (92). With an appropriate gauge transformation (29) the field $\phi(x,\tilde{x})$ can be made real. The boundary conditions at large distances are then

$$\phi(x,\tilde{x}) \to \phi_0, \qquad \phi_3(x,\tilde{x}) \to \phi_0,$$
$$c_\mu(x,\tilde{x}) \to -C^D_\mu(x,\tilde{x}). \quad (107)$$

On the vortex sheet $\tilde{x}^\mu(\sigma,\tau)$ where $C^D_\mu(x,\tilde{x})$ is singular the boundary conditions are:

$$\phi(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)} = 0,$$
$$\phi_3(x,\tilde{x})|_{x^\mu=\tilde{x}^\mu(\sigma,\tau)} = \mathrm{finite}. \quad (108)$$

Equations (105) and (106) were solved for the flat vortex sheet (55), and the resulting static heavy quark potential $V^{\mathrm{class}}(R)$ determined in [6].

B. Static flux tube solutions

For the infinite Z_3 flux tube the vector potential $\vec{C}$ has the form (19) with $Y=\frac{\lambda_8}{\sqrt{3}}$, and the Higgs fields (21) are obtained by making the gauge transformation $\Omega(\theta) = \exp[i\theta Y]$ to the color ansatz (96) with $\phi_1(x,\tilde{x}) = \phi_2^*(x,\tilde{x}) = \phi(r)$, $\phi_3(x,\tilde{x}) = \phi_3(r)$;

$$\exp[-i\theta Y]\phi_1\exp[i\theta Y] = \phi(r)\exp[-i\theta]\frac{\lambda_7 - i\lambda_6}{2} + \phi(r)\exp[i\theta]\frac{\lambda_7 + i\lambda_6}{2},$$
$$\exp[-i\theta Y]\phi_2\exp[i\theta Y] = \phi(r)\exp[i\theta]\frac{-\lambda_5 - i\lambda_4}{2} + \phi(r)\exp[-i\theta]\frac{-\lambda_5 + i\lambda_4}{2},$$
$$\exp[-i\theta Y]\phi_3\exp[i\theta Y] = \phi_3(r)\lambda_2. \quad (109)$$

Then the Higgs fields in the infinite flux tube (109) have the color structure (96) with

$$\phi_1(x,\tilde{x}) = \phi_2^*(x,\tilde{x}) = \phi(x,\tilde{x}) = \phi(r)\exp(-i\theta),$$
$$\phi_3(x,\tilde{x}) = \phi_3(r). \quad (110)$$

(109) gives the specific form of (21) for the $SU(3)$ flux tube in the gauge where $C_\mu = C_\mu(x)Y$, with $Y=\frac{\lambda_8}{\sqrt{3}}$. Replacing Y by $-\frac{Y+\lambda_3}{2}$ or by $\frac{\lambda_3 - Y}{2}$ on the right-hand side of (23) (corresponding to the other two quark colors) yields three physically equivalent vortices, each carrying one unit of Z_3 flux, related by $SU(3)$ gauge transformations.

We rescale the flux tube fields, choosing the flux tube radius $\frac{1}{M}$ as the scale of length, making the replacement

$$r \to r/M, \qquad C(r) \to \frac{MC(r)}{g_m}, \qquad \phi(r) \to \phi_0\phi,$$
$$\phi_3(r) \to \phi_0\phi_3, \qquad \phi_0^2 = \frac{M^2}{6g_m^2},$$

and define a rescaled Higgs potential $W(\phi,\phi_3)$:

$$W(\phi,\phi_3) \equiv \frac{1}{96}\frac{V(\phi_0\phi,\phi_0\phi_3)}{M^4 g_m^2} = \frac{\kappa^2}{200}(11[2(\phi^2-1)^2 + (\phi_3^2-1)^2] + 7[2(\phi^2-1)+(\phi_3^2-1)]^2) = \kappa^2\left(\frac{(\phi^2-1)^2}{4} + 9\frac{(\phi_3^2-1)^2}{100} - 7\frac{(\phi_3^2-1)(1-\phi^2)}{50}\right), \quad (111)$$

with

$$\kappa^2 \equiv \frac{25}{9}\frac{\lambda}{g_m^2}. \quad (112)$$

Note that with $\phi_3(r)$ replaced by 1 in (111), $W(\phi,\phi_3)$ becomes $\frac{\kappa^2}{4}(\phi^2-1)^2$, the Higgs potential of the Abelian Higgs model with Landau-Ginzburg parameter κ.

The rescaled expressions for $T_{00}(r)$ and $\frac{T_{\theta\theta}(r)}{r^2}$ (78) are:

$$T_{00}(r) = \frac{4}{3}\frac{M^4}{g_m^2}\left(\frac{1}{2}\left(\frac{1}{r}\left(\frac{d(rC)}{dr}\right)^2 + \frac{1}{2}\left(C - \frac{1}{r}\right)^2\phi^2 + \frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + \frac{1}{4}\left(\frac{d\phi_3}{dr}\right)^2 + W(\phi,\phi_3),\right.\right.$$

$$\frac{T_{\theta\theta}(r)}{r^2} = \frac{4}{3}\frac{M^4}{g_m^2}\left(\frac{1}{2}\left(\frac{1}{r}\left(\frac{d(rC)}{dr}\right)^2 + \frac{1}{2}\left(C - \frac{1}{r}\right)^2\phi^2 - \frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 - \frac{1}{4}\left(\frac{d\phi_3}{dr}\right)^2 - W(\phi,\phi_3)\right)\right). \quad (113)$$

The rescaled static field equations obtained from $T_{00}(r)$ are

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rC)}{dr}\right) = \left(C - \frac{1}{r}\right)\phi^2, \quad (114)$$

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \phi\left(C - \frac{1}{r}\right)^2 + \kappa^2\phi\left[(\phi^2 - 1) + \frac{7}{25}(\phi_3^2 - 1)\right] = 0, \quad (115)$$

and

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_3}{dr}\right) + \frac{2\kappa^2}{25}\phi_3[7(\phi^2 - 1) + 9(\phi_3^2 - 1)] = 0. \quad (116)$$

with boundary conditions

$$C(r) \to \frac{1}{r}, \quad \phi(r) \to 1, \quad \phi_3(r) \to 1 \quad \text{as } r \to \infty,$$
$$C \to 0, \quad \phi(r) \to 0, \quad \phi_3(r) \to \text{finite as } r \to 0. \quad (117)$$

The numerical solution of (114), (115) and (116) shows that $\phi(r) < 1$ and $\phi_3(r) > 1$ everywhere; hence the term coupling ϕ and ϕ_3 in $W(\phi, \phi_3)$ is attractive. This additional attraction reduces the energy of the Z_3 vortex below that of the Abelian configuration with $\phi_3(r) = 1$, viewed as an unstable configuration of the non-Abelian vortex.

Evaluation of (113) at the classical solution yields expressions for the string tension σ and the torque per unit length τ as the sum and difference, respectively, of a gauge contribution $\sigma_g(\kappa)$ and a Higgs contribution $\sigma_h(\kappa)$:

$$\sigma = \int_0^\infty 2\pi r T_{00}(r) dr = \frac{4}{3}\frac{M^2}{g_m^2}(\sigma_g(\kappa) + \sigma_h(\kappa)) \equiv \frac{4}{3}\frac{M^2}{g_m^2}\sigma(\kappa), \quad (118)$$

$$2\pi\tau = \int_0^\infty 2\pi r dr \frac{T_{\theta\theta}(r)}{r^2} = \frac{4}{3}\frac{M^2}{g_m^2}(\sigma_g(\kappa) - \sigma_h(\kappa)), \quad (119)$$

where

$$\sigma_g(\kappa) = \int_0^\infty 2\pi r dr \left(\frac{1}{2}\left(\frac{1}{r}\left(\frac{d(rC)}{dr}\right)^2 + \frac{1}{2}\left(C - \frac{1}{r}\right)^2\phi^2\right)\right), \quad (120)$$

and

$$\sigma_h(\kappa) = \int_0^\infty 2\pi r dr \left(\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + \frac{1}{4}\left(\frac{d\phi_3}{dr}\right)^2 + W(\phi,\phi_3)\right). \quad (121)$$

The condition $\tau = 0$ becomes $\sigma_g(\kappa) = \sigma_h(\kappa)$.

C. Results for $T_{\theta\theta}$ in Z_3 flux tubes

Figure 3 shows $\frac{T_{\theta\theta}(r)}{r} \equiv rp(r)$ evaluated at the classical solution for three values of κ^2. The condition (6), $\sigma_g(\kappa) = \sigma_h(\kappa)$, yields $\kappa^2 \approx 0.6$, and $\sigma(\kappa^2 = 0.6) \approx 3.1$. (The value of $\kappa^2 = 0.6$ lies close to the value $\kappa^2 = \frac{5}{9}$ used in [19] in comparing calculations of heavy quark potentials in this model with lattice simulations [18].) For $\kappa^2 \approx 0.6$, the stress tensor component $T_{\theta\theta}(r) = 0$ at $r \equiv r^* = \frac{1.7}{M}$. There is repulsion for $r < r^*$, where $T_{\theta\theta}(r) > 0$, and attraction for $r > r^*$, where $T_{\theta\theta}(r) < 0$. It is then natural to identify r^* as a boundary separating the repulsive interior of the flux tube from its attractive exterior.

(6) is also satisfied by the flux tubes of the Abelian-Higgs model with $\kappa^2 = \frac{1}{2}$. These are BPS states [34] describing an Abelian magnetic superconductor on the border between type I and type II. In this situation $T_{\theta\theta}(r) = 0$ for all r [35], so that the profile of $T_{\theta\theta}(r)$ does not reveal a boundary. The difference between the non-Abelian and Abelian vortices is caused by the additional attractive interaction among the octet of scalar particles which breaks the supersymmetry [36] giving rise to the BPS vortex of the Abelian Higgs model. For $\kappa^2 \approx 0.6$, where $\tau = 0$, this additional interaction is approximately compensated for by the additional gauge repulsion associated with the fact that $\kappa^2 > \frac{1}{2}$.

VII. SUMMARY

A. Relation between effective field theory and effective string theory

We have started with magnetic $SU(N)$ gauge theory as an effective field theory of the long-distance heavy quark interaction in Yang-Mills theory. At interquark distances $R \sim \frac{1}{M}$ the classical action for a straight flux tube describes the heavy quark potential.

When the distance R between the quark and the antiquark is much larger than the intrinsic width $\frac{1}{M}$ of the classical Z_N flux tube, long-distance fluctuations of the axis must be taken into account and give rise to an effective string theory. To leading order in $\frac{1}{M}$ the action of the effective string theory $S_{\rm eff}(\tilde{x})$ is the classical action for a curved vortex sheet $\tilde{x}$, evaluated in the limit $\frac{1}{M} \to 0$. This action is equal to the Nambu-Goto action with a string tension given by the energy per length of an infinite straight flux tube.

B. The constraint $\tau = 0$ and its consequences

We have introduced a new long-distance parameter, the torque per unit length τ [Eq. (4)], linking the string tension to the chromoelectric field on the axis of an infinite straight flux tube [Eq. (2)]. For large R, the parameter τ determines the difference between the chromoelectric field of a quark-antiquark pair at the positions $|z| = \pm \frac{R}{2}$ of the quarks and its value at points $|z| \ll \frac{R}{2}$ in the middle of the flux tube [Eq. (81)].

In this paper we have assumed the value $\tau = 0$ characterizes the QCD flux tube. Under this constraint the chromodynamic fields $\vec{E}$ and $\vec{B}$ on a curved vortex sheet $\tilde{x}$ are determined, to leading order in $\frac{1}{M}$, in terms of the string tension [Eq. (89)], and the Nambu-Goto action is expressed in terms of these fields on $\tilde{x}$ [Eq (84)].

Imposition of the condition $\tau = 0$ on the flux tubes in a particular $SU(3)$ model [5] gives a physical picture of these flux tubes in which the behavior of the moment of the pressure [Eq. (5)] defines a boundary separating a repulsive interior from an attractive exterior (Fig. 3).

C. Testing the constraint

Testing our conjecture is a problem. (The fit of early lattice simulations of heavy quark potentials and flux tube energy distributions to classical calculations of these quantities discussed in Sec. VI kept κ fixed and, therefore, provides only a crude test.)

Recent lattice simulations [22,29] of field and energy distributions can be used to test the consistency of the condition $\tau = 0$, taking into account string fluctuations in the interpretation of the lattice data. The results of these simulations can be compared with the relations (89) expressing the fields on the vortex sheet in terms of the string tension, generalizing (7) to curved sheets. Comparison with lattice data, of the predicted ratio of the field at the center of a flux tube to the string tension, provides the most direct test of the constraint (6). Further lattice data and analysis is necessary to put strong limits on τ. Testing our conjecture is a problem that remains to be solved.

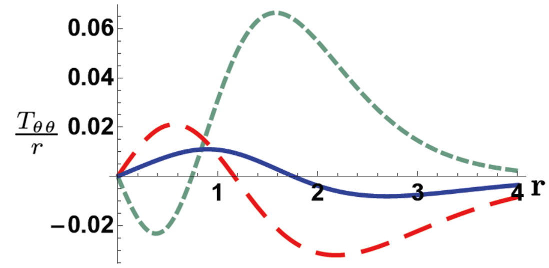

FIG. 3. The torque per unit area, $T_{\theta\theta}(r)/r$. Red, long dashed, $\kappa^2 = 0.5$; blue, thick, $\kappa^2 = 0.59$; green, short dashed, $\kappa^2 = 0.8$.

D. Discussion

According to the correspondence (89), the world sheet variables $\tilde{x}^\mu$ of effective string theory are associated with chromodynamic fields of effective magnetic gauge theory on this sheet. This association, combined with the correspondence of these fields with the underlying fields of Yang Mills theory [9], provides a relation between effective string theory and long-distance Yang Mills theory.

The location of the string can thus be thought of as the axis of a classical flux tube, and the fields associated with the string regarded as the classical chromodynamic fields on that axis. We have shown that this is possible if the flux tube structure is constrained by the condition $\tau = 0$. Our conjecture is that the QCD flux tube has the requisite structure.

We have obtained the constraint $\tau = 0$ on the structure of the QCD flux tube by requiring that no field energy is created by the separation of a quark-antiquark pair. In this situation the heavy quark potential, the energy available for doing work when the pair is released, is equal to the energy available when the pair is released in the fixed field created by their separation. The string tension is then equal to the charge on the quark multiplied by the field on the axis of an infinite flux tube; i.e., the field of the "string" connecting the quark and antiquark.

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