Formulas for flavor neutrino masses and their application to texture two zeros

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We demonstrate the usefulness of flavor neutrino masses expressed in terms of M_{ee}, $M_{e\mu}$ and $M_{e\tau}$. The analytical expressions for the flavor neutrino masses, mass eigenstates and physical CP-violating Majorana phases for texture two zeros are obtained exactly.

I. INTRODUCTION

The atmospheric neutrino oscillations have been experimentally confirmed by the Super-Kamiokande Collaboration [1]. A similar oscillation phenomenon, the solar neutrino oscillation, has been long suggested [2] and has finally been confirmed by various collaborations [3].

Theoretically, the neutrino oscillations are realized if neutrinos have different masses and can be explained by mixings of three flavor neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$. These mixings are well described by a unitary matrix U [4] involving three mixing angles θ_{12}, θ_{23}, θ_{13}, which converts three neutrino mass eigenstates (ν_1, ν_2, ν_3) with masses (m_1, m_2, m_3) into $(\nu_e, \nu_\mu, \nu_\tau)$. Furthermore, leptonic CP violation is induced if U contains phases which are given by one CP-violating Dirac phase δ and two CP-violating Majorana phases α_2, α_3 [5].

There have been various discussions of the flavor neutrino mass matrix to ensure the appearance of the observed neutrino mixings and masses [6]. The flavor neutrino mass matrix M is parametrized by

$$M = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix}, \qquad (1)$$

where $\mathrm{diag}(m_1, m_2, m_3) = U^T M U$. The elements of the mass matrix, M_{ij} $(i, j = e, \mu, \tau)$, are functions of the mixing angles, mass eigenstates and CP phases:

$$M_{ij} = f(\theta_{12}, \theta_{23}, \theta_{13}, m_1, m_2, m_3, \delta, \alpha_2, \alpha_3). \qquad (2)$$

Our recent discussions [7] have found that $M_{\mu\mu}$, $M_{\tau\tau}$, $M_{\mu\tau}$ as well as m_1, m_2, m_3 are expressed in terms of M_{ee}, $M_{e\mu}$, $M_{e\tau}$ as follows:

$$M_{\mu\mu,\tau\tau,\mu\tau} = f(M_{ee}, M_{e\mu}, M_{e\tau}, \theta_{12}, \theta_{23}, \theta_{13}, \delta) \qquad (3)$$

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and

$$m_{1,2,3} = f(M_{ee}, M_{e\mu}, M_{e\tau}, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_2, \alpha_3). \qquad (4)$$

Various specific textures of flavor neutrino mass matrices for the maximal CP violation and the maximal atmospheric neutrino mixing have been obtained by using the relation of Eq. (3) [7].

In this article, we would like to remind the reader about the useful formulas for the flavor neutrino masses expressed in terms of M_{ee}, $M_{e\mu}$, $M_{e\tau}$. In Sec. II, we show the useful formulas. In Sec. III, to demonstrate the usefulness of flavor neutrino masses expressed in terms of M_{ee}, $M_{e\mu}$ and $M_{e\tau}$, we use the formulas to find exact analytical expressions for flavor neutrino masses, mass eigenstates and CP-violating Majorana phases in the texture two zeros. To confirm results of our discussions more concretely, we study the following three subjects based on numerical and semi-analytical calculations: (1) the consistency between our results and conclusions obtained from other papers, (2) the upper limit of the effective Majorana neutrino mass for the neutrino less double beta decay for maximal Dirac CP violation and (3) the dependence of the CP-violating Majorana phases on the CP-violating Dirac phase. Section IV is devoted to a summary.

II. FORMULAS

A. $M_{\mu\mu,\tau\tau,\mu\tau}$

We summarize the useful formulas found in Ref. [7]. We adopt the standard parametrization of the unitary matrix $U = U_0 K$ [8] with $U_0 = U_0^{\mathrm{PDG}}$ where

$$\begin{aligned} U_0^{\mathrm{PDG}} = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \\ &\times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \qquad (5)$$

and

$$K = \text{diag}(e^{i\phi_1/2}, e^{i\phi_2/2}, e^{i\phi_3/2}), \qquad (6)$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ (as well as $t_{ij} = \tan\theta_{ij}$) and θ_{ij} represents the ν_i-ν_j mixing angle $(i, j = 1,2,3)$. The phases $\phi_{1,2,3}$ stand for the Majorana phases. The physical Majorana CP violation is specified by two combinations made of ϕ_1, ϕ_2, ϕ_3 such as $\alpha_i = \phi_i - \phi_1$ in place of ϕ_i in K [5]:

$$K = \text{diag}(1, e^{i\alpha_2/2}, e^{i\alpha_3/2}). \qquad (7)$$

Defining

$$M_+ = s_{23}M_{e\mu} + c_{23}M_{e\tau},$$
$$M_- = c_{23}M_{e\mu} - s_{23}M_{e\tau}, \qquad (8)$$

and

$$M_{\mu\tau}^{(0)} = -\left(\frac{1}{c_{13}\tan 2\theta_{12}} + \frac{t_{13}e^{-i\delta}}{\tan 2\theta_{23}}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{\tan 2\theta_{13}} + \frac{1}{2}t_{13}e^{i\delta}\right)M_+$$
$$- \frac{1 - e^{-2i\delta}}{2}M_{ee}, \qquad (9)$$

we can derive

$$M_{\mu\mu} = \left(\frac{1}{c_{13}\tan 2\theta_{12}} - \frac{t_{13}e^{-i\delta}}{\sin 2\theta_{23}}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{\tan 2\theta_{13}} - \frac{1}{2}t_{13}e^{i\delta}\right)M_+$$
$$+ \frac{1 + e^{-2i\delta}}{2}M_{ee} - M_{\mu\tau}^{(0)}\cos 2\theta_{23},$$

$$M_{\tau\tau} = \left(\frac{1}{c_{13}\tan 2\theta_{12}} + \frac{t_{13}e^{-i\delta}}{\sin 2\theta_{23}}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{\tan 2\theta_{13}} - \frac{1}{2}t_{13}e^{i\delta}\right)M_+$$
$$+ \frac{1 + e^{-2i\delta}}{2}M_{ee} + M_{\mu\tau}^{(0)}\cos 2\theta_{23},$$

$$M_{\mu\tau} = M_{\mu\tau}^{(0)}\sin 2\theta_{23}, \qquad (10)$$

and

$$m_1 e^{-i\phi_1} = -\frac{t_{12}}{c_{13}}M_- - t_{13}e^{i\delta}M_+ + M_{ee},$$
$$m_2 e^{-i\phi_2} = \frac{1}{c_{13}t_{12}}M_- - t_{13}e^{i\delta}M_+ + M_{ee},$$
$$m_3 e^{-i\phi_3} = \frac{1}{t_{13}}e^{-i\delta}M_+ + e^{-2i\delta}M_{ee}. \qquad (11)$$

Equations (10) and (11) are the useful formulas which we introduce in the Introduction.

B. Different parametrizations of U

There are alternative parametrizations of the unitary matrix U. There are nine parametrizations of U_0 in the general case [9]. Because the reactor mixing angle is small $(\sin^2\theta_{13} \ll 1)$, the following four patterns [as well as the standard pattern U_0^{PDG} in Eq. (5)] are of great interest:

$$U_0^{\text{I}} = R_{23}(\theta_{23})R_{12}(\theta_{12})R_{13}(\theta_{13})$$
$$\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \qquad (12)$$

$$U_0^{\text{II}} = R_{23}(\theta_{23})R_{12}(\theta_{12})R_{23}(\theta_{13})$$
$$\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{13} & s_{13}e^{-i\delta} \\ 0 & -s_{13}e^{i\delta} & c_{13} \end{pmatrix}, \qquad (13)$$

$$U_0^{\text{III}} = R_{13}(\theta_{13})R_{23}(\theta_{23})R_{12}(\theta_{12})$$
$$\equiv \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$
$$\times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (14)$$

and

$$U_0^{\text{IV}} = R_{12}(\theta_{13})R_{23}(\theta_{23})R_{12}(\theta_{12})$$
$$\equiv \begin{pmatrix} c_{13} & s_{13}e^{-i\delta} & 0 \\ -s_{13}e^{i\delta} & c_{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$
$$\times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (15)$$

Since experimental analyses of neutrino oscillations are based on $U_0^{\text{PDG}}K$, numerical values of the mixing angles other than $U_0^{\text{PDG}}K$ need to be modified as

$$\sin^2\theta_{12}^{\text{PDG}} = \frac{|(U_0^{\text{I}})_{12}|^2}{|(U_0^{\text{I}})_{11}|^2 + |(U_0^{\text{I}})_{12}|^2},$$

$$\sin^2\theta_{23}^{\text{PDG}} = \frac{|(U_0^{\text{I}})_{23}|^2}{|(U_0^{\text{I}})_{23}|^2 + |(U_0^{\text{I}})_{33}|^2},$$

$$\sin^2\theta_{13}^{\text{PDG}} = |(U_0^{\text{I}})_{13}|^2, \tag{16}$$

where $\theta_{12,23,13}^{\text{PDG}}$ are $\theta_{12,23,13}$ in U_0^{PDG}, which are directly compared with experimental data.

It is possible to express $M_{\mu\mu,\mu\tau,\tau\tau}$ in terms of $M_{ee,e\mu,e\tau}$ in these different U's. As an example, in the case of U_0^{I}, we find that

$$M_{\mu\mu} = \left(\frac{1}{\tan 2\theta_{12}} - \frac{1}{2}t_{12}e^{-2i\delta}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{c_{12}\tan 2\theta_{13}} - \frac{t_{12}}{\sin 2\theta_{23}}\right)M_+$$
$$+ \frac{1+e^{-2i\delta}}{2}M_{ee} - M_{\mu\tau}^{(0)}\cos 2\theta_{23},$$

$$M_{\tau\tau} = \left(\frac{1}{\tan 2\theta_{12}} - \frac{1}{2}t_{12}e^{-2i\delta}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{c_{12}\tan 2\theta_{13}} + \frac{t_{12}}{\sin 2\theta_{23}}\right)M_+$$
$$+ \frac{1+e^{-2i\delta}}{2}M_{ee} + M_{\mu\tau}^{(0)}\cos 2\theta_{23},$$

$$M_{\mu\tau} = M_{\mu\tau}^{(0)}\sin 2\theta_{23}, \tag{17}$$

where

$$M_{\mu\tau}^{(0)} = -\left(\frac{1}{\tan 2\theta_{12}} + \frac{1}{2}t_{12}e^{-2i\delta}\right)M_-$$
$$+ \left(\frac{e^{-i\delta}}{c_{12}\tan 2\theta_{13}} - \frac{t_{12}}{\tan 2\theta_{23}}\right)M_+$$
$$- \frac{1-e^{-2i\delta}}{2}M_{ee}. \tag{18}$$

Roughly speaking, except for the sign of δ, Eq. (17) is obtained from Eq. (10) by the replacement of $M_{ij} \to e^{i\delta}M_{ij}$ $(i,j = \mu, \tau)$, $M_+ \leftrightarrow M_-$ and $\theta_{12} \leftrightarrow \theta_{13}$. The replacement of $\theta_{12} \leftrightarrow \theta_{13}$ can be understood as the effect of the interchanged rotations between $R_{13}(\theta_{13})$ and $R_{12}(\theta_{12})$ to get U_0^{I} from U_0^{PDG}. We have also numerically evaluated $\sin^2\theta_{12,23,13}$ and δ and have observed that $\sin^2\theta_{12,23}$ and δ indicate no significant deviation due to the smallness of $\sin^2\theta_{13}$. On the other hand, $|(U_0^{\text{I}})_{13}| = c_{12}s_{13}$ requires larger values of s_{13} to yield the same magnitude of

$\sin^2\theta_{13}^{\text{PDG}}$ owing to the suppression by c_{12}^2. The neutrino masses turn out to be

$$m_1 e^{-i\phi_1} = -t_{12}M_- + \left[\frac{1}{2}\left(\frac{1}{2} - e^{2i\delta}\right)t_{12}t_{23} - \frac{t_{13}}{c_{12}}e^{i\delta}\right]M_+$$
$$+ M_{ee},$$

$$m_2 e^{-i\phi_2} = -t_{12}M_- + \frac{t_{23}}{2c_{12}}M_+ + M_{ee},$$

$$m_3 e^{-i\phi_3} = -t_{12}e^{-2i\delta}M_-$$
$$+ \left[\frac{1}{2}\left(\frac{1}{2}e^{-2i\delta} - 1\right)t_{12}t_{23} + \frac{1}{t_{13}c_{12}}e^{-i\delta}\right]M_+$$
$$+ e^{-2i\delta}M_{ee}. \tag{19}$$

There is no simple relationship between Eqs. (11) and (19) realized by replacement of the masses and the mixing angles.

It will be interesting to discuss the phase structure of M determined by these different U's because there appears to be different dependence of δ from the one derived by U_0^{PDG}. Since predictions of $m_{1,2,3}$ in various U's are expected to be quite different from those by U_0^{PDG} as has been learned from the comparison between Eqs. (11) and (19), the Majorana phases may give different behaviors from those predicted by U_0^{PDG}. However, discussing experimentally allowed regions for CP-violating phases including δ as well as $\theta_{12,23,13}$ in $U_0^{\text{I,II,III,IV}}$ is beyond the scope of the present discussions.

Moreover, there is another parametrization induced by three Dirac CP-violating phases γ_{23}, γ_{13}, γ_{12} without explicitly referring to Majorana phases [10]:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23}^* & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & \tilde{s}_{13} \\ 0 & 1 & 0 \\ -\tilde{s}_{13}^* & 0 & c_{13} \end{pmatrix}$$
$$\times \begin{pmatrix} c_{12} & \tilde{s}_{12} & 0 \\ -\tilde{s}_{12}^* & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

where $\tilde{s}_{ij} = s_{ij}e^{-i\gamma_{ij}}$. All three Dirac CP-violating phases $\gamma_{23,13,12}$ are physical. These Dirac CP-violating phases are related to the phases in the PDG version as $\gamma_{13} = \delta - \alpha_3/2$, $\gamma_{12} = \alpha_2/2$ and $\gamma_{23} = (\alpha_3 - \alpha_2)/2$ [11]. The advantages of choosing $\gamma_{23,13,12}$ instead of δ, α_2, α_3 have been discussed in Refs. [10,11]. For an example, this alternative parametrization provides that only the two Majorana phases appear in the effective Majorana neutrino mass for the neutrinoless double beta decay. Since constraints on δ and α_2, α_3 to be derived in Sec. III can be readily converted to those on γ's, we use the PDG parametrization to compare our results with conclusions obtained from other papers.

III. TEXTURE TWO ZEROS

A. Candidates of texture two zeros

Texture two zeros for the flavor neutrino mass matrix M_{ij} ($i, j = e, \mu, \tau$) have been extensively studied in the literature (see, for example, Refs. [12–15]). There are 15 possible combinations of two vanishing independent elements in the flavor neutrino mass matrix. If we require a nonvanishing Majorana effective mass for the neutrinoless double beta decay and maximal or approximately maximal CP violation, the interesting textures are the following four [13,14] (or less [15]):

$$B_1 : \begin{pmatrix} M_{ee} & M_{e\mu} & 0 \\ M_{e\mu} & 0 & M_{\mu\tau} \\ 0 & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix},$$

$$B_2 : \begin{pmatrix} M_{ee} & 0 & M_{e\tau} \\ 0 & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & 0 \end{pmatrix},$$

$$B_3 : \begin{pmatrix} M_{ee} & 0 & M_{e\tau} \\ 0 & 0 & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix},$$

$$B_4 : \begin{pmatrix} M_{ee} & M_{e\mu} & 0 \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ 0 & M_{\mu\tau} & 0 \end{pmatrix}. \tag{21}$$

The texture zeros in the Majorana neutrino mass matrix can be realized by imposing appropriate symmetries on the Lagrangian of neutrino mass models such as an Abelian flavor symmetry (e.g., the cyclic group Z_n) [12,13,16] as well as a non-Abelian symmetry [15]. The stability of texture zeros with respect to higher order corrections is protected by these symmetries [12,13].

For example, B_1, B_2, B_3 and B_4 textures are obtained by imposing the discrete cyclic group Z_3 [13] in the framework of type-I [17] plus type-II [18] seesaw mechanism. There are other symmetry realizations of B_1, B_2, B_3 and B_4 textures. The B_1 and B_2 textures have been realized by using A_4 or its Z_3 subgroup [19]. The B_3 and B_4 textures have been obtained by softly breaking the $L_\mu - L_\tau$ symmetry [20].

B. Analytical solutions

The mass matrices based on the texture two zeros are good examples to demonstrate the usefulness of the formulas in Eqs. (10) and (11). For B_1 texture, because of $M_{e\tau} = M_{\mu\mu} = 0$, it is evident that

$$M_a = f_a(\theta_{12}, \theta_{23}, \theta_{13}, \delta) M_{ee}, \tag{22}$$

$$m_j e^{-i\phi_j} = f_j(\theta_{12}, \theta_{23}, \theta_{13}, \delta) M_{ee}, \tag{23}$$

where $a = e\mu, \mu\tau, \tau\tau$ and $j = 1, 2, 3$. The nonvanishing flavor neutrino masses $M_{e\mu}, M_{\mu\tau}, M_{\tau\tau}$ and mass eigenstates m_1, m_2, m_3 are the functions of only M_{ee} (with mixing angles and phases fixed). More concretely, with the following definitions in mind,

$$A_1 = c_{23}^2 + s_{23}^2 e^{-2i\delta},$$

$$A_2 = s_{23}^2 + c_{23}^2 e^{-2i\delta},$$

$$B_1 = \frac{2c_{23}^2}{c_{13} \tan 2\theta_{12}} - t_{13} \sin 2\theta_{23} e^{-i\delta},$$

$$B_2 = \frac{2s_{23}^2}{c_{13} \tan 2\theta_{12}} + t_{13} \sin 2\theta_{23} e^{-i\delta},$$

$$B_3 = \frac{\sin 2\theta_{23}}{c_{13} \tan 2\theta_{12}} + t_{13} \cos 2\theta_{23} e^{-i\delta},$$

$$C_1 = 2\left(\frac{s_{23}^2 e^{-i\delta}}{\tan 2\theta_{13}} - \frac{t_{13} c_{23}^2 e^{i\delta}}{2} \right),$$

$$C_2 = 2\left(\frac{c_{23}^2 e^{-i\delta}}{\tan 2\theta_{13}} - \frac{t_{13} s_{23}^2 e^{i\delta}}{2} \right),$$

$$C_3 = \sin 2\theta_{23} \left(\frac{e^{-i\delta}}{\tan 2\theta_{13}} + \frac{t_{13} e^{i\delta}}{2} \right), \tag{24}$$

we obtain

$$M_{e\mu} = -\frac{A_1}{c_{23} B_1 + s_{23} C_1} M_{ee},$$

$$M_{\mu\tau} = \left(-A_1 \frac{-c_{23} B_3 + s_{23} C_3}{c_{23} B_1 + s_{23} C_1} - \frac{1 - e^{-2i\delta}}{2} \sin 2\theta_{23} \right) M_{ee},$$

$$M_{\tau\tau} = \left(-A_1 \frac{c_{23} B_2 + s_{23} C_2}{c_{23} B_1 + s_{23} C_1} + A_2 \right) M_{ee}, \tag{25}$$

and

$$m_1 e^{-i\phi_1} = \left(A_1 \frac{\frac{t_{12} c_{23}}{c_{13}} + t_{13} s_{23} e^{i\delta}}{c_{23} B_1 + s_{23} C_1} + 1 \right) M_{ee},$$

$$m_2 e^{-i\phi_2} = \left(A_1 \frac{-\frac{c_{23}}{c_{13} t_{12}} + t_{13} s_{23} e^{i\delta}}{c_{23} B_1 + s_{23} C_1} + 1 \right) M_{ee},$$

$$m_3 e^{-i\phi_3} = \left(A_1 \frac{-\frac{s_{23}}{t_{13}} e^{-i\delta}}{c_{23} B_1 + s_{23} C_1} + e^{-2i\delta} \right) M_{ee}, \tag{26}$$

for B_1 texture.

Similarly, the nonvanishing flavor neutrino masses as well as mass eigenstates for B_2, B_3 and B_4 textures are expressed in terms of M_{ee} as

$$M_{e\tau} = \frac{A_2}{s_{23}B_2 - c_{23}C_2}M_{ee},$$

$$M_{\mu\mu} = \left(A_2\frac{-s_{23}B_1 + c_{23}C_1}{s_{23}B_2 - c_{23}C_2} + A_1\right)M_{ee}, M_{\mu\tau}$$

$$= \left(A_2\frac{s_{23}B_3 + c_{23}C_3}{s_{23}B_2 - c_{23}C_2} - \frac{1 - e^{-2i\delta}}{2}\sin 2\theta_{23}\right)M_{ee},$$

$$(27)$$

and

$$m_1 e^{-i\phi_1} = \left(A_2\frac{\frac{t_{12}s_{23}}{c_{13}} - t_{13}c_{23}e^{i\delta}}{s_{23}B_2 - c_{23}C_2} + 1\right)M_{ee},$$

$$m_2 e^{-i\phi_2} = \left(A_2\frac{-\frac{s_{23}}{c_{13}t_{12}} - t_{13}c_{23}e^{i\delta}}{s_{23}B_2 - c_{23}C_2} + 1\right)M_{ee},$$

$$m_3 e^{-i\phi_3} = \left(A_2\frac{\frac{c_{23}}{t_{13}}e^{-i\delta}}{s_{23}B_2 - c_{23}C_2} + e^{-2i\delta}\right)M_{ee},$$

$$(28)$$

for B_2 texture,

$$M_{e\tau} = \frac{A_1}{s_{23}B_1 - c_{23}C_1}M_{ee},$$

$$M_{\mu\tau} = \left(A_1\frac{s_{23}B_3 + c_{23}C_3}{s_{23}B_1 - c_{23}C_1} - \frac{1 - e^{-2i\delta}}{2}\sin 2\theta_{23}\right)M_{ee},$$

$$M_{\tau\tau} = 2\left(A_1\frac{(s_{23}B_3 + c_{23}C_3)\cos 2\theta_{23} - s_{23}t_{13}e^{-i\delta}}{\sin 2\theta_{23}(s_{23}B_1 - c_{23}C_1)}\right.$$

$$\left. - \frac{1 - e^{-2i\delta}}{2}\cos 2\theta_{23}\right)M_{ee},$$

$$(29)$$

and

$$m_1 e^{-i\phi_1} = \left(A_1\frac{\frac{t_{12}s_{23}}{c_{13}} - t_{13}c_{23}e^{i\delta}}{s_{23}B_1 - c_{23}C_1} + 1\right)M_{ee},$$

$$m_2 e^{-i\phi_2} = \left(A_1\frac{-\frac{s_{23}}{c_{13}t_{12}} - t_{13}c_{23}e^{i\delta}}{s_{23}B_1 - c_{23}C_1} + 1\right)M_{ee},$$

$$m_3 e^{-i\phi_3} = \left(A_1\frac{\frac{c_{23}}{t_{13}}e^{-i\delta}}{s_{23}B_1 - c_{23}C_1} + e^{-2i\delta}\right)M_{ee},$$

$$(30)$$

for B_3 texture and

$$M_{e\mu} = -\frac{A_2}{c_{23}B_2 + s_{23}C_2}M_{ee},$$

$$M_{\mu\mu} = \left(-A_2\frac{c_{23}B_1 + s_{23}C_1}{c_{23}B_2 + s_{23}C_2} + A_1\right)M_{ee},$$

$$M_{\mu\tau} = \left(-A_2\frac{-c_{23}B_3 + s_{23}C_3}{c_{23}B_2 + s_{23}C_2} - \frac{1 - e^{-2i\delta}}{2}\sin 2\theta_{23}\right)M_{ee},$$

$$(31)$$

and

$$m_1 e^{-i\phi_1} = \left(A_2\frac{\frac{t_{12}c_{23}}{c_{13}} + t_{13}s_{23}e^{i\delta}}{c_{23}B_2 + s_{23}C_2} + 1\right)M_{ee},$$

$$m_2 e^{-i\phi_2} = \left(-A_2\frac{\frac{c_{23}}{c_{13}t_{12}} - t_{13}s_{23}e^{i\delta}}{c_{23}B_2 + s_{23}C_2} + 1\right)M_{ee},$$

$$m_3 e^{-i\phi_3} = \left(-A_2\frac{\frac{s_{23}}{t_{13}}e^{-i\delta}}{c_{23}B_2 + s_{23}C_2} + e^{-2i\delta}\right)M_{ee}, \quad (32)$$

for B_4 texture.

The physical CP-violating Majorana phases α_2, α_3 depend on the CP-violating Dirac phase δ in all cases of B_1, B_2, B_3 and B_4. Since the mass eigenstate m_j is obtained as Eq. (23), the Majorana phase ϕ_j depends on not only δ but also $\arg(M_{ee})$ as follows:

$$\phi_j = -\arg(m_j e^{-i\phi_j}) = -\arg(f_j(\delta)) - \arg(M_{ee}). \quad (33)$$

On the contrary, the physical CP-violating Majorana phase α_j is specified by two combinations made of ϕ_1, ϕ_2, ϕ_3 such as

$$\alpha_j = \phi_j - \phi_1 = \arg\left(\frac{f_1(\delta)}{f_j(\delta)}\right), \quad (34)$$

which depends on the CP-violating Dirac phase δ. For example, the functions f_1, f_2 and f_3 are obtained from Eq. (26) as

$$f_1 = A_1\frac{\frac{t_{12}c_{23}}{c_{13}} + t_{13}s_{23}e^{i\delta}}{c_{23}B_1 + s_{23}C_1} + 1,$$

$$f_2 = A_1\frac{-\frac{c_{23}}{c_{13}t_{12}} + t_{13}s_{23}e^{i\delta}}{c_{23}B_1 + s_{23}C_1} + 1,$$

$$f_3 = A_1\frac{-\frac{s_{23}}{t_{13}}e^{-i\delta}}{c_{23}B_1 + s_{23}C_1} + e^{-2i\delta}, \quad (35)$$

for B_1 texture. The function of f_j for other textures is immediately obtained from Eqs. (28), (30) and (32).

The analytical expressions for the flavor neutrino masses, mass eigenstates and the physical CP-violating Majorana phases are obtained exactly. We observed that the flavor neutrino masses and mass eigenstates for B_1, B_2, B_3 and B_4 depend on M_{ee} (with mixing angles and phases fixed). Also, the physical CP-violating Majorana phases depend on the CP-violating Dirac phase δ. This clear understanding is obtained from the observation based on the formulas in Eqs. (10) and (11).

C. Numerical calculations

Although we have reached our main goal of this paper to see the usefulness of the formulas, some numerical and semi-analytical calculations may be required to confirm results of our discussions.

First, we show the consistency between our results and conclusions obtained from other papers. The results from the precise parameter search for M_{ij} [eV] are reported by Dev et al. [13]. For B_1 texture, they obtained

$$M_{e\mu} = 0.00018 + 0.00258i,$$
$$M_{\mu\tau} = -0.11001 - 0.00287i,$$
$$M_{\tau\tau} = -0.01105 - 0.00024i, \qquad (36)$$

for $M_{ee} = 0.10479 + 0.00127i$ with $s_{12}^2 = 0.32$, $s_{23}^2 = 0.476$, $s_{13}^2 = 0.0246$ and $\delta = -90.65°$. The predicted values from our exact expression in Eq. (25) are

$$M_{e\mu} = 0.000185 + 0.00258i,$$
$$M_{\mu\tau} = -0.110 - 0.00287i,$$
$$M_{\tau\tau} = -0.0110 - 0.000237i, \qquad (37)$$

for the same values of M_{ee}, s_{12}^2, s_{23}^2, s_{13}^2 and δ. The mass eigenstates (m_1, m_2, m_3) are estimated by Meloni et al. [14] and, for B_1 texture, they reported

$$(m_1, m_2, m_3) = (6.27, 6.33, 7.97) \times 10^{-2} \text{ eV}, \qquad (38)$$

for $M_{ee} = 6.32 \times 10^{-2}$ eV with $s_{12}^2 = 0.306$, $s_{23}^2 = 0.446$, $s_{13}^2 = 0.0229$, $\delta = -91.7°$, $\alpha_2 = 5.61°$ and $\alpha_3 = 183.8°$. We have obtained the same masses from our exact expression in Eq. (26). Similar consistency for the remaining textures, B_2, B_3 and B_4, is also guaranteed.

Second, we estimate the upper limit of $|M_{ee}|$ by the expression in Eq. (23) with observed data of $\sum m_\nu = \sum_j |m_j e^{-i\phi_j}|$. To calculate $f_j(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$, we use the following values of three mixing angles [21]:

$$\sin^2\theta_{12} = 0.304,$$
$$\sin^2\theta_{23} = 0.452 \quad \text{or} \quad 0.579,$$
$$\sin^2\theta_{13} = 0.0219. \qquad (39)$$

The octant of θ_{23} (i.e., lower octant $\theta_{23} < 45°$ or upper octant $\theta_{23} > 45°$) is still an unresolved problem. According to Dev et al. [13] and Meloni et al. [14], we employ the values of θ_{23} given for both octants which are shown in Table I. For the CP-violating Dirac phase, the T2K Collaboration has reported the result of their new analysis [22] and has shown that the likelihood maximum is reached at $\delta = 270°$ and $\sin^2\theta_{23} = 0.528$.

Figure 1 shows the $\sum m_\nu - |M_{ee}|$ planes for $\delta = 270°$. The upper panel shows the relation for the NO, $m_1 < m_2 < m_3$, while the lower panel shows the relation for the IO, $m_3 < m_1 < m_2$. The upper (lower) horizontal dashed and dotted lines in the figures show the observed upper (lower) limit of the sum of the light neutrino masses $\sum m_\nu$ from the recent results of the Planck [23] (neutrino oscillation [21]) experiments. The upper limit is

TABLE I.   Octant of θ_{23} for B_1, B_2, B_3 and B_4 textures (ℓ: lower octant $\theta_{23} < 45°$; u: upper octant $\theta_{23} > 45°$) [13,14].

Ordering	B_1	B_2	B_3	B_4
NO	ℓ	u	ℓ	u
IO	u	ℓ	u	ℓ

$\sum m_\nu \leq 0.17$ eV. The lower limits are $\sum m_\nu \gtrsim m_1 + \Delta m_{21} + \Delta m_{31} \cong 0.05$ eV for NO (with $m_1 = 0$) and $\sum m_\nu \gtrsim 2|\Delta m_{32}| + m_3 \cong 0.1$ eV for IO (with $m_3 = 0$) where we have used the values of the squared mass differences, $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, as [21]

$$\Delta m_{21}^2 = 7.5 \times 10^{-5} \text{ eV}^2 \quad \text{for NO and IO},$$
$$\Delta m_{31}^2 = 2.46 \times 10^{-3} \text{ eV}^2 \quad \text{for NO},$$
$$\Delta m_{32}^2 = -2.45 \times 10^{-3} \text{ eV}^2 \quad \text{for IO}. \qquad (40)$$

The lines for B_1 and B_3 as well as B_2 and B_4 almost overlap each other because the neutrino mass spectrum is quasidegenerate as $m_1 \simeq m_2 \simeq m_3 t_{23}^2$ for B_1, B_3 or $m_1 \simeq m_2 \simeq m_3/t_{23}^2$ for B_2, B_4 [12,14]. The upper limit of $|M_{ee}|$ is
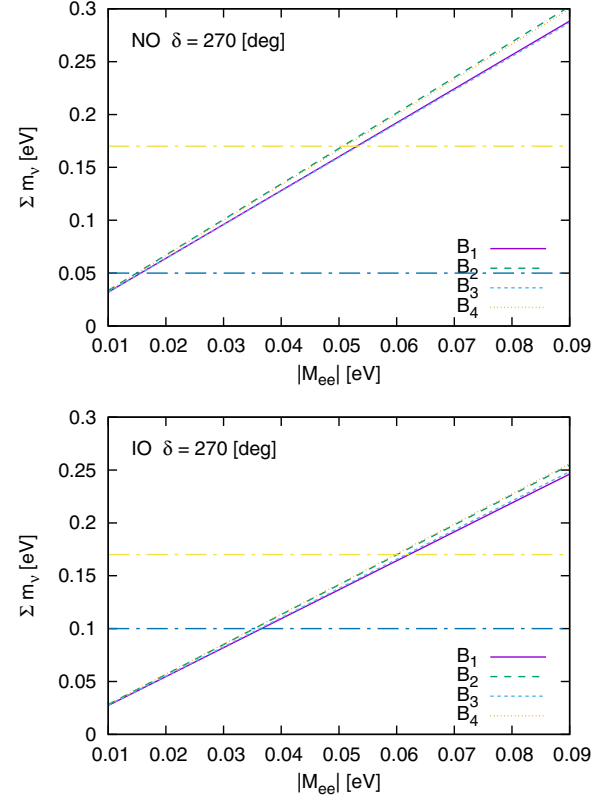

FIG. 1. $\sum m_\nu - |M_{ee}|$ planes for $\delta = 270°$. The upper panel shows the relation for the normal ordering (NO) while the lower panel shows the relation for inverted ordering (IO). The upper (lower) horizontal dashed and dotted lines in the figures show the observed upper (lower) limit of the sum of the light neutrino masses from the recent results of the Planck (neutrino oscillations) experiments.
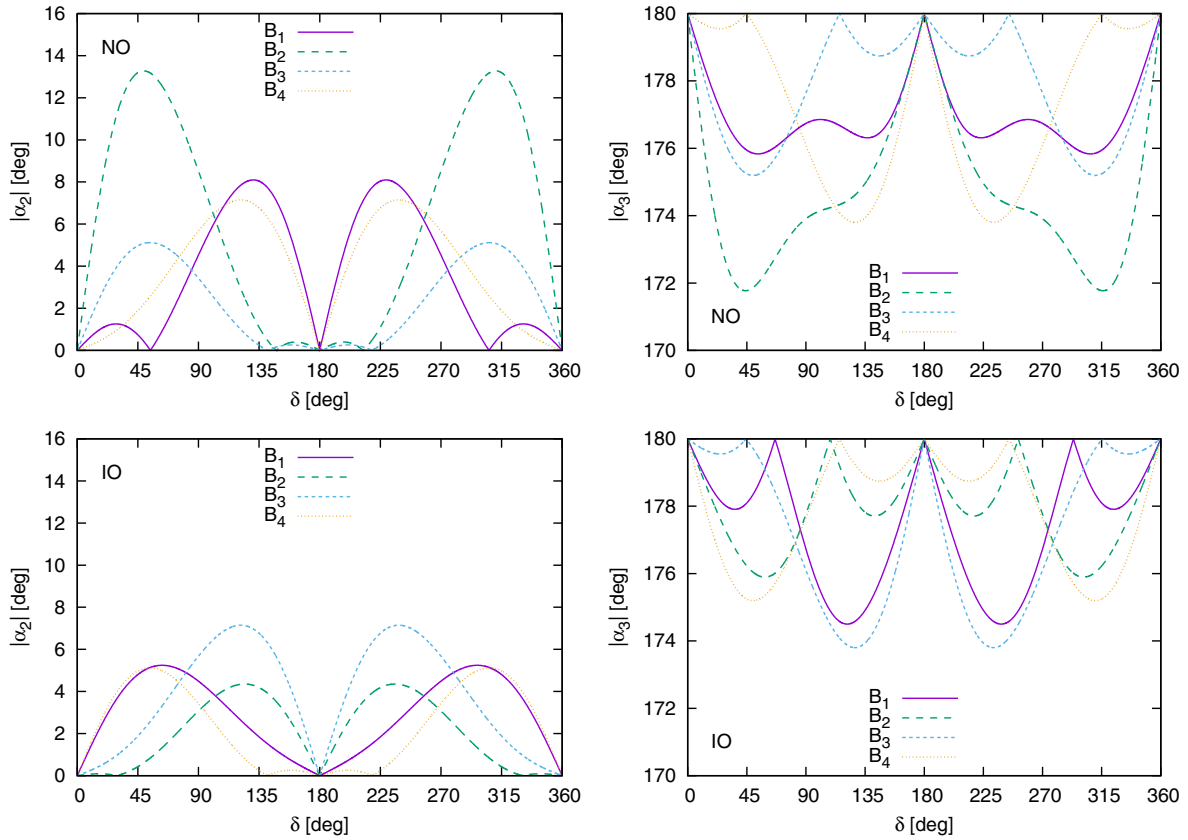
FIG. 2.   The dependence of the Majorana CP-violating phases α_2 and α_3 on the CP-violating Dirac phase δ. The upper two panels show the magnitude of α_2 (left) and α_3 (right) for the NO while the lower two panels show the magnitude of α_2 (left) and α_3 (right) for the IO.

around 0.05 eV (NO) or 0.06 eV (IO) for maximal CP violation $\delta = 270°$. These upper limits are consistent with the result in Ref. [14].

The theoretically expected half-life of the neutrinoless double beta decay is proportional to the effective Majorana neutrino mass $m_{\beta\beta}$ where $|m_{\beta\beta}| = |M_{ee}| = |\sum_{i=1}^{3} U_{ei}^2 m_i|$. The estimated magnitude of the effective Majorana mass from the experiments is $|M_{ee}| = 0.20$–2.5 eV [24]. In future experiments, a desired sensitivity of $|M_{ee}| \simeq$ a few 10^{-2} eV will be reached [25].

Finally, we study the dependence of the CP-violating Majorana phases α_2 and α_3 on the CP-violating Dirac phase δ. In Fig. 2, the upper two panels show the magnitude of α_2 (left) and α_3 (right) for the NO, while the lower two panels show the magnitude of α_2 (left) and α_3 (right) for the IO. The behavior of the curves in Fig. 2 can be understand by the semi-analytical expressions. For example, we obtain

$$\frac{f_1}{f_2} \simeq -2.29 + \frac{1}{0.287 - 0.0421e^{i\delta}} + \frac{1}{-0.611 + 2.20e^{i\delta}},$$

$$\frac{f_1}{f_3} \simeq -0.797 - 0.247e^{i\delta}$$

$$+ \frac{1}{-0.669 - 5.43e^{i\delta} + (0.512 - 1.91e^{i\delta})^{-1}}, \quad (41)$$

with the mixing angles in Eq. (39) for B_1 texture in the case of NO (we use a lower octant of θ_{23}), and $f_1/f_2 = 2.42$ and $f_1/f_3 = -1.19$ for $\delta = 0°$ yield $\alpha_2 = 0°$ and $\alpha_3 = 180°$, respectively.

IV. SUMMARY

We have demonstrated the usefulness of flavor neutrino masses expressed in terms of M_{ee}, $M_{e\mu}$ and $M_{e\tau}$. The analytical expressions for the flavour neutrino masses and mass eigenstates for texture two zeros are obtained exactly. These flavor neutrino masses and mass eigenstates depend on M_{ee} (with mixing angles and phases fixed). The analytical expressions for the physical CP-violating Majorana phases are also obtained exactly. These phases depend on the CP-violating Dirac phase.

To confirm results of our discussions more concretely, we have discussed the following three subjects based on numerical and semi-analytical calculations: (1) the consistency between our results and conclusions obtained from other papers, (2) the upper limit of the effective Majorana neutrino mass for the neutrinoless double beta decay for maximal Dirac CP violation and (3) the dependence of the CP-violating Majorana phases on the CP-violating Dirac phase.

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