What is Δm^2_{ee}?

Stephen Parke*

Theoretical Physics Department, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510, USA

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The current short baseline reactor experiments, Daya Bay and RENO (Double Chooz) have measured (or are capable of measuring) an effective Δm^2 associated with the atmospheric oscillation scale of 0.5 km/MeV in electron antineutrino disappearance. In this paper, I compare and contrast the different definitions of such an effective Δm^2 and argue that the simple, L/E independent definition given by $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, i.e. "the ν_e weighted average of Δm^2_{31} and Δm^2_{32}," is superior to all other definitions and is useful for both short baseline experiments mentioned above and for the future medium baseline experiments JUNO and RENO-50.

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I. INTRODUCTION

The short baseline reactor experiments, Daya Bay [1], RENO [2], and Double Chooz [3], have been very successful in determining the electron neutrino flavor content of the neutrino mass eigenstate with the smallest amount of ν_e, the state usually labeled ν_3. The parameter which controls the size of this flavor content is the mixing angle θ_{13}, in the standard PDG convention,[1] and the current measurements indicate that $\sin^2 2\theta_{13} \approx 0.09$ with good precision (~5%).

The mass of the ν_3 eigenstate has a mass squared splitting from the other two mass eigenstates, ν_1 and ν_2, of approximately $\pm 2.4\times 10^{-3}$ eV^2 given by $\Delta m^2_{31} \equiv m^2_3 - m^2_1$ and $\Delta m^2_{32} \equiv m^2_3 - m^2_2$; the sign determines the atmospheric mass ordering. The mass squared difference between ν_2 and ν_1, $\Delta m^2_{21} \equiv m^2_2 - m^2_1 \approx +7.5\times 10^{-5}$ eV^2, is about 30 times smaller than both Δm^2_{31} and Δm^2_{32}, and hence $\Delta m^2_{31} \approx \Delta m^2_{32}$. However, the difference between Δm^2_{31} and Δm^2_{32} is ~3%.

Recently, two of these reactor experiments, Daya Bay (see [4–6]) and RENO [7], have extended their analysis of their data, from just fitting $\sin^2 2\theta_{13}$ to a two-parameter fit of both $\sin^2 2\theta_{13}$ and an effective Δm^2. The measurement uncertainty on this effective Δm^2 is approaching the difference between Δm^2_{31} and Δm^2_{32}. So it is now a pertinent question: "What is the physical meaning of this effective Δm^2?" Clearly, the effective Δm^2 measured by these experiments is some combination of Δm^2_{31} and Δm^2_{32}. Answering the question "What is the combination of Δm^2_{31} and Δm^2_{32} that is measured in such a short baseline reactor experiment?" is the primary purpose of this paper,

*parke@fnal.gov

[1]A more informative notation for mixing angles $(\theta_{12}, \theta_{13}, \theta_{23})$ is $(\theta_{e2}, \theta_{e3}, \theta_{\mu 3})$, respectively, such that $U_{e2} = \cos\theta_{e3}\sin\theta_{e2}$, $U_{e3} = \sin\theta_{e3}e^{-i\delta}$ and $U_{\mu 3} = \cos\theta_{e3}\sin\theta_{\mu 3}$.

The outline of this paper is as follows: in Sec. II, I review the $\bar{\nu}_e$ survival probability as calculated in terms of an effective Δm^2 which naturally arises in this calculation; then this definition is applied to the short baseline reactor experiments, L/E < 1 km/GeV. In Sec. III, I compare and contrast other possible definitions of an effective Δm^2, including two new ones as well as the two definitions invented by the Daya Bay Collaboration. The new effective Δm^2's are essentially equal to the effective Δm^2 of Sec. II whereas the two invented by Daya Bay are L/E dependent and their original definition is discontinuous. This is followed by a conclusion and Appendixes A and B.

II. $\bar{\nu}_e$ SURVIVAL PROBABILITY IN VACUUM

The exact $\bar{\nu}_e$ survival probability in vacuum (see Fig. 1) is given by[2]

$$\begin{aligned} P_x(\bar{\nu}_e \to \bar{\nu}_e) &= 1 - 4|U_{e2}|^2|U_{e1}|^2\sin^2\Delta_{21} \\ &\quad - 4|U_{e3}|^2|U_{e1}|^2\sin^2\Delta_{31} \\ &\quad - 4|U_{e3}|^2|U_{e2}|^2\sin^2\Delta_{32} \\ &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\ &\quad - \sin^2 2\theta_{13}(\cos^2\theta_{12}\sin^2\Delta_{31} \\ &\quad + \sin^2\theta_{12}\sin^2\Delta_{32}), \end{aligned} \tag{1}$$

using $\Delta_{ij} \equiv \Delta m^2_{ij}\mathrm{L}/4\mathrm{E}$.

It was shown in [8] that to an excellent accuracy

$$\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32} \approx \sin^2\Delta_{ee}$$

[2]The standard PDG conventions with the kinematical phase are given by $\Delta_{ij} \equiv \Delta m^2_{ij}\mathrm{L}/4\mathrm{E}$ or $1.267\Delta m^2_{ij}\mathrm{L/E}$ depending on whether one is using natural or (eV^2, km, MeV) units. Also, matter effects shift the Δm^2 by $(1 + \mathcal{O}(E/10\ \mathrm{GeV}))$, where $E < 10$ MeV, so are negligible for typical reactor neutrino experiments.

where

$$\Delta m_{ee}^2 \equiv \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2$$

for L/E < 0.8 km/MeV. A variant of this derivation is given in Appendix A.

However, in this article we will use an exact formulation given in [9], which follows Helmholtz [10] in combining the two oscillation frequencies, proportional to Δ_{31} and Δ_{32}, into one frequency plus a phase. The exact survival probability is given by (see Appendix B)

$$P_x(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \frac{1}{2}\sin^2 2\theta_{13}\left(1 - \sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}\cos\Omega\right) \quad (2)$$

with $\Omega = (\Delta_{31} + \Delta_{32}) + \arctan(\cos 2\theta_{12}\tan\Delta_{21})$.

Ω consists of two parts: one that is even under the interchange of Δ_{31} and Δ_{32} and is linear in L/E, $(\Delta_{31} + \Delta_{32})$, and the other which is odd under this interchange and contains both linear and higher (odd) powers in L/E, $\arctan(\cos 2\theta_{12}\tan\Delta_{21})$; remember that $\Delta_{21} = \Delta_{31} - \Delta_{32}$.

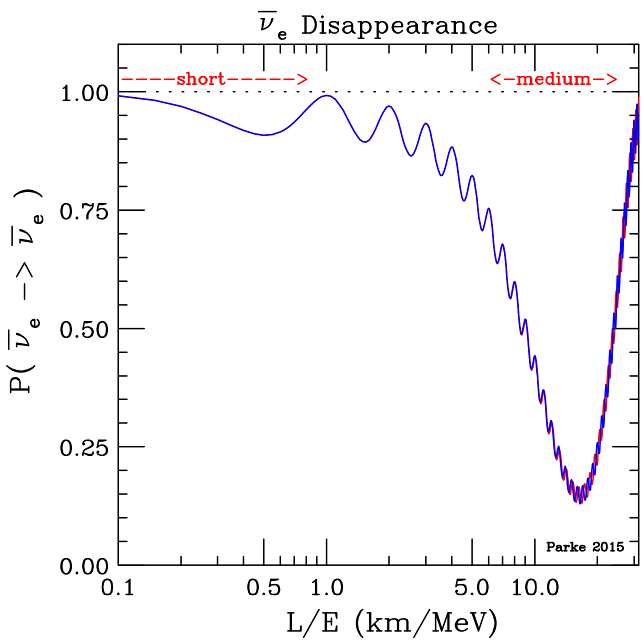

FIG. 1. The vacuum survival probability for $\bar{\nu}_e$ as a function of L/E. Blue is for the normal mass ordering (NO) and red is the inverted mass ordering (IO) with Δm_{31}^2 and Δm_{32}^2 chosen in such a fashion that the two survival probabilities are identical at small L/E, i.e. $\Delta m_{31}^2(IO) = -\Delta m_{31}^2(NO) + 2\sin^2\theta_{12}\Delta m_{21}^2$. Near the solar oscillation minimum, L/E ~ 15 km/MeV, the phase of the θ_{13} oscillations advances (retards) for the normal (inverted) mass ordering and the two oscillation probabilities are distinguishable, in principle. Also near the solar minimum, the amplitude of the θ_{13} oscillations is significantly reduced compared to smaller values of L/E. The short baseline experiments, Daya Bay, RENO and Double Chooz, probe L/E < 0.8 km/MeV and the medium baseline, JUNO and RENO-50, probe 6 < L/E < 25 km/MeV, as indicated.

The key point is the separation of the kinematic phase, Ω, into an effective 2Δ (linear in L/E) and a phase, ϕ. For short baseline experiments, it is natural to expand Ω in a power series in L/E and identify the coefficient of the linear term in L/2E as the effective Δm^2 and include all the higher order terms in the phase.[3] Then,

$$\Omega = 2\Delta_{ee} + \phi \quad (3)$$

$$\text{where } \Delta m_{ee}^2 \equiv \left.\frac{\partial\Omega}{\partial(\mathrm{L}/2\mathrm{E})}\right|_{\frac{\mathrm{L}}{\mathrm{E}}\to 0} = \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2 \quad (4)$$

$$\text{and} \quad \phi \equiv \Omega - 2\Delta_{ee} = \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}. \quad (5)$$

With this separation, $2\Delta_{ee}$ varies at the atmospheric scale, 0.5 km/MeV, whereas ϕ varies at the solar oscillation scale, 15 km/MeV, and

$$\phi = 0, \quad \frac{\partial\phi}{\partial(\mathrm{L}/2\mathrm{E})} = 0 \quad \text{and} \quad \frac{\partial^2\phi}{\partial(\mathrm{L}/2\mathrm{E})^2} = 0 \quad \text{at} \quad \frac{\mathrm{L}}{\mathrm{E}} = 0;$$

therefore, in a power series in L/E, ϕ starts at $(\Delta m_{21}^2 \mathrm{L/E})^3$ [see Eq. (11)].

Since Ω only appears as $\cos\Omega$, it is useful to redefine $\Omega = 2|\Delta_{ee}| \pm \phi$, so that the sign associated with the mass ordering appears only in front of ϕ. If and only if this sign is determined can the mass ordering be determined in ν_e disappearance experiments.

There are three things worth noting about writing the exact ν_e survival probability as in Eq. (2), with Ω given by Eq. (3):

[3] Appendix A of [9] contains a discussion of an effective Δm^2 as a function of L/E for arbitrary L/E. At L/E = 0 this definition is identical to Δm_{ee}^2.

(i) The effective atmospheric Δm^2 associated with θ_{13} oscillation is a simple combination of the fundamental parameters [see Eq. (4) above or in Ref. [8] as they are identical],

$$\begin{aligned}\Delta m^2_{ee} &= \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}\\ &= \Delta m^2_{31} - \sin^2\theta_{12}\Delta m^2_{21}\\ &= \Delta m^2_{32} + \cos^2\theta_{12}\Delta m^2_{21}\\ &= m_3^2 - (\cos^2\theta_{12}m_1^2 + \sin^2\theta_{12}m_2^2).\end{aligned}$$

Thus Δm^2_{ee} is simply the "ν_e average of Δm^2_{31} and Δm^2_{32}," since the ν_e ratio of ν_1 to ν_2 is $\cos^2\theta_{12}$ to $\sin^2\theta_{12}$, and determines the L/E scale associated with the θ_{13} oscillations.

(ii) The modulation of the amplitude associated with the θ_{13} oscillation is manifest in the square root multiplying the $\cos\Omega$ oscillating term, where

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} = \begin{cases} 1 & \text{at } \Delta_{21} = n\pi \\ \cos 2\theta_{12} \approx 0.4 & \text{at } \Delta_{21} = (2n+1)\pi/2 \end{cases} \quad (6)$$

for $n = 0, 1, 2, \ldots$. Thus, at solar oscillation minima, when $\Delta_{21} = 0, \pi, 2\pi, \ldots$, the oscillation amplitude is just $\sin^2 2\theta_{13}$, whereas at solar oscillation maxima, when $\Delta_{21} = \pi/2, 3\pi/2, \ldots$, the oscillation amplitude is $\cos 2\theta_{12}\sin^2 2\theta_{13}$ i.e. reduced by approximately 60%.

(iii) The phase, ϕ, causes an advancement (retardation) of the θ_{13} oscillation for the normal (inverted) mass ordering of the neutrino mass eigenstates. ϕ is a "rounded" staircase function,[4] which is zero and has zero first and second derivatives at $\mathrm{L/E} = 0$ ($\Delta_{21} = 0$), but then between $\mathrm{L/E} \sim 10-20$ km/MeV ($\Delta_{21} \sim \frac{\pi}{3} - \frac{2\pi}{3}$) rapidly jumps by $2\pi\sin^2\theta_{12}$, and this pattern is repeated for every increase of $\mathrm{L/E} \sim 30$ km/MeV (Δ_{21} by π), i.e.

$$\phi(\Delta_{21} \pm \pi) = \phi(\Delta_{21}) \pm 2\pi\sin^2\theta_{12}; \quad (7)$$

see Fig. 2. Also shown on the same plot is $2|\Delta_{ee}|$ divided by 80. This number 80 was chosen so that $2|\Delta_{ee}|$ fits on the same plot and to demonstrate that $2|\Delta_{ee}| \geq 80\phi$ so that the shift in phase caused by ϕ is never bigger than a 1.25% effect. Also for $\mathrm{L/E} < 5$ km/MeV, the shift in phase is much smaller than this; see next section.

[4]In the limit $\sin^2\theta_{12} \to \frac{1}{2}$, one recovers the well-known result that this rounded staircase function becomes a true staircase or step function.

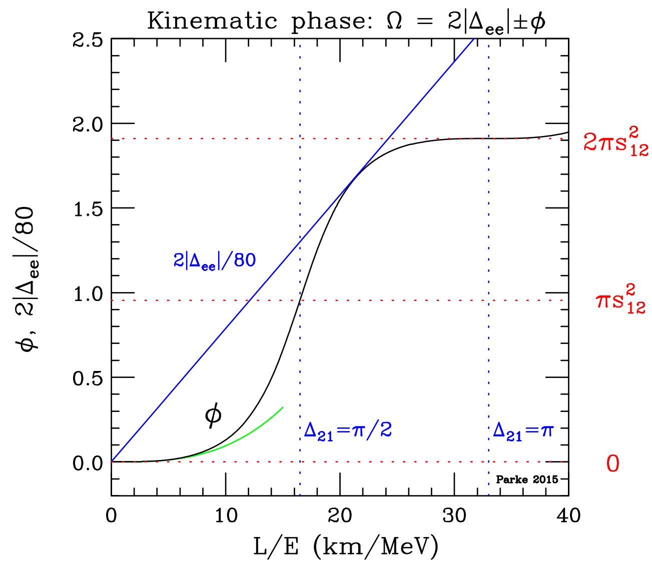

FIG. 2. The L/E dependence of the two components that make up the kinematic phase $\Omega = 2|\Delta_{ee}| \pm \phi$ associated with the θ_{13} oscillation [Eq. (20)]. ϕ is the black staircase function which increases by $2\pi\sin^2\theta_{12}$ for every increase in Δ_{21} by π; see Eq. (7). The blue straight line is $2|\Delta_{ee}|/80$, which is always greater than or equal to ϕ. The green curve is the Δ^3_{21} approximation to ϕ given in Eq. (11), which is an excellent approximation for $\mathrm{L/E} < 8$ km/MeV.

A. Short baseline experiments ($0 < \mathrm{L/E} < 1$ km/MeV)

For reactor experiments with baselines less than 2 km, the exact expression Eq. (2) contains elements which require measurement uncertainties on the oscillation probability to better than one part in 10^4. This is way beyond the capability of the current or envisaged experiments. This occurs because for experiments at these baselines some elements of Eq. (2) dependent on higher powers of Δ_{21}. Note the following conditions on the kinematic phases are satisfied,

$$0 < |\Delta_{31}| \approx |\Delta_{32}| < \pi \Rightarrow 0 < \Delta_{21} < 0.1. \quad (8)$$

These elements are

(i) The modulation of the θ_{13} oscillation amplitude which when expanded in powers of Δ_{21} is given by

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} = 1 - 2\sin^2\theta_{12}\cos^2\theta_{12}\Delta^2_{21} + \mathcal{O}(\Delta^4_{21}) \quad (9)$$

$$= 1 + \mathcal{O}(< 10^{-3}). \quad (10)$$

Remember, this amplitude modulation factor is multiplied by $\frac{1}{2}\sin^2 2\theta_{13} \sim 0.05$, reducing the effect of the amplitude modulation to less than one part in 10^4.

(ii) The advancement or retardation of the kinematic phase, Ω, caused by ϕ whose sign depends on the mass ordering. For small values of Δ_{21} the advancing/retarding phase can be written as

$$\phi = \frac{1}{3}\cos 2\theta_{12}\sin^2 2\theta_{12}\Delta_{21}^3 + \mathcal{O}(\Delta_{21}^5). \quad (11)$$

Then using this approximation in the kinematic phase Ω, we have

$$\begin{aligned}&\cos(2|\Delta_{ee}| \pm \phi)\\ &= \cos(2|\Delta_{ee}|)\cos\phi \mp \sin(2|\Delta_{ee}|)\sin\phi\\ &= \cos(2|\Delta_{ee}|) \mp \frac{1}{3}\cos 2\theta_{12}\sin^2 2\theta_{12}\Delta_{21}^3\sin(2|\Delta_{ee}|)\\ &\quad + \mathcal{O}(\Delta_{21}^5)\\ &= \cos(2|\Delta_{ee}|) + \mathcal{O}(< 10^{-4}). \quad (12)\end{aligned}$$

Again remember that we have a further reduction by $\frac{1}{2}\sin^2 2\theta_{13} \sim 0.05$, making the phase advancement or retardation significantly smaller than even the amplitude modulation for these experiments.

Using this information in the ν_e survival probability, we can replace Eq. (2) by

$$\begin{aligned}P_{\text{short}}(\bar{\nu}_e \to \bar{\nu}_e) = 1 &- \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21}\\ &- \sin^2 2\theta_{13}\sin^2|\Delta_{ee}|, \quad (13)\end{aligned}$$

which is accurate to better than one part in 10^{-4}. In Fig. 3 the fractional difference between Eqs. (2) and (13) is shown for an experiment with a baseline of 1.6 km. Since the measurement uncertainty on the ν_e survival probability is much greater ($>0.01\%$) than the difference between the exact [Eq. (2)] and the approximate [Eq. (13)] survival probabilities, use of either will result in the same measured values of the parameters $\sin^2 2\theta_{13}$ and $|\Delta m_{ee}^2|$; i.e. the measurement uncertainties will dominate.

If new, extremely precise, short baseline experiments ever need a more accurate survival probability, one could easily add the first correction of the amplitude modulation, giving

$$\begin{aligned}P_{\text{xshort}}(\bar{\nu}_e \to \bar{\nu}_e) = 1 &- \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21}\\ &- \sin^2 2\theta_{13}[\sin^2|\Delta_{ee}|\\ &+ \sin^2\theta_{12}\cos^2\theta_{12}\Delta_{21}^2\cos(2|\Delta_{ee}|)] \quad (14)\end{aligned}$$

and this would improve the accuracy of the approximation to better than one part in 10^5.

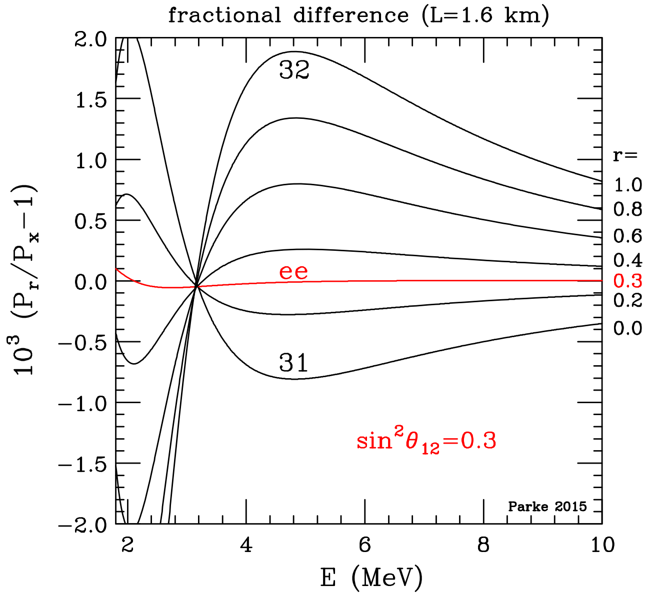

FIG. 3. The fractional difference between the exact survival probability, Eq. (1), and a sequence of approximate survival probabilities, where $\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}$ is replaced with $\sin^2(\Delta m_{rr}^2 L/4E)$ with $\Delta m_{rr}^2 \equiv (1-r)\Delta m_{31}^2 + r\Delta m_{32}^2$. Clearly, $r = \sin^2\theta_{12}$ minimizes the absolute value of the fractional difference between the exact and approximate survival probabilities. Thus, the approximation of replacing $\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}$ with $\sin^2\Delta_{ee}$ gives an approximate survival probability that is better than one part in 10^4 over the L/E range of the Daya Bay, RENO and Double Chooz experiments.

An alternative way to derive these approximate survival probabilities, Eqs. (13) and (14), is given in Appendix A.

III. OTHER POSSIBLE DEFINITIONS OF AN EFFECTIVE Δm^2

A. A new definition of the effective Δm^2

Another possible way to define an effective Δm^2 (here I will use the symbol Δm_{XX}^2) is as follows:

$$\Delta m_{XX}^2 \equiv \sqrt{\cos^2\theta_{12}(\Delta m_{31}^2)^2 + \sin^2\theta_{12}(\Delta m_{32}^2)^2}. \quad (15)$$

Clearly this definition is independent of L/E and it guarantees that, in the limit L/E $\to 0$,

$$\sin^2\Delta_{XX} = \cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}. \quad (16)$$

One can then show that

$$|\Delta m_{XX}^2| = |\Delta m_{ee}^2|\left(1 + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right]\right). \quad (17)$$

So $|\Delta m^2_{XX}|$ is essentially equal to $|\Delta m^2_{ee}|$ up to correction on the order of 10^4, including the effects of the solar mixing angle.[5]

A variant of this definition of an effective Δm^2 (here I will used the subscripts "xx"), is defined in terms of the position of the first extremum of $(\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32})$ in L/E. If this extremum occurs at $(\mathrm{L/E})|_1$, then define

$$\Delta m^2_{xx} \equiv \frac{2\pi}{(\mathrm{L/E})|_1}, \tag{18}$$

so that, at this extremum, $\frac{\Delta m^2_{xx}L}{4E} = \frac{\pi}{2}$. With this definition it is again easy to show that,

$$|\Delta m^2_{xx}| = |\Delta m^2_{ee}|\left(1 + \mathcal{O}\left[\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)^2\right]\right). \tag{19}$$

Again, this is essentially equal to Δm^2_{ee}.

In both $|\Delta m^2_{XX}|$ and $|\Delta m^2_{xx}|$, the corrections of order $(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}})^2$ come from the amplitude modulation of the θ_{13} oscillation and the coefficients are $\frac{1}{2}\sin^2\theta_{12}\cos^2\theta_{12}$ and $\sin^2\theta_{12}\cos^2\theta_{12}$ respectively. Note that these corrections are mass ordering independent.

B. Daya Bay's original definition of the effective Δm^2

In Refs. [4] and [5], the Daya Bay experiment used the following definition for an effective Δm^2 (here I will use the symbol Δm^2_{YY}),

$$\sin^2\Delta_{YY} \equiv \cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}, \tag{20}$$

which implies that

$$\Delta m^2_{YY} \equiv \left(\frac{4\mathrm{E}}{\mathrm{L}}\right)\arcsin \times\left[\sqrt{(\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32})}\right]. \tag{21}$$

For $\mathrm{L/E} < 0.3$ km/MeV, so that $\sin^2\Delta_{3i} = \Delta^2_{3i}$ is a good approximation, Δm^2_{YY} is approximately independent of L/E. However, for larger values of L/E, Δm^2_{YY} is L/E dependent, exactly in the L/E region, $0.3 < \mathrm{L/E} < 0.7$ km/MeV, where the bulk of the experimental data from the far detectors of the Daya Bay experiment is obtained. In the center of this L/E region, $\mathrm{L/E} \approx 0.5$ km/MeV, is the position of the oscillation minimum.

Furthermore, the definition given by Eq. (20) is discontinuous at oscillation minimum (OM). This occurs because as you increase L/E, the lhs of Eq. (20) can go to 1, whereas the rhs never reaches 1. So to satisfy Eq. (20), as you increase L/E, your effective Δm^2 must be discontinuous at OM and the size of this discontinuity is given by[6]

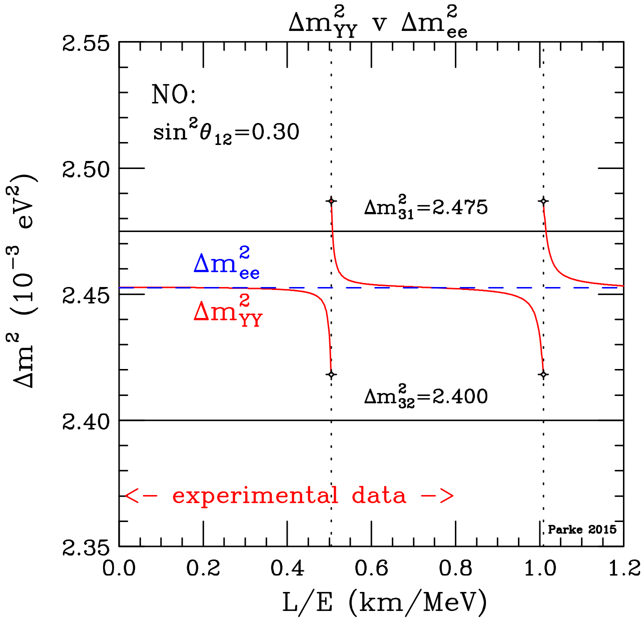

FIG. 4. Daya Bay's original definition (see [4] and [5]) for an effective Δm^2, Δm^2_{YY}, is given by the solid red line. Notice the sizable L/E dependence near oscillation minimum and maximum (vertical black dotted lines). At all oscillation extrema, this definition is discontinuous and the size of the discontinuity is $\sin 2\theta_{12}\Delta m^2_{21} \sim 3\%$. The first discontinuity occurs in the middle of the experimental data of the Daya Bay, RENO and Double Chooz experiments. For the L/E independent lines, $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$ is the blue dashed, and Δm^2_{31} and Δm^2_{32} are the labeled black lines. This figure is for normal mass ordering with $\sin^2\theta_{12} = 0.30$ and $\Delta m^2_{ee} = 2.453 \times 10^{-3}$ eV^2.

$$\delta\Delta m^2_{\mathrm{EE}}|_{\mathrm{OM}} = \sin 2\theta_{12}\Delta m^2_{21} \tag{22}$$

which is of the order of 3%. In Fig. 4, the various Δm^2's are plotted as a function of L/E.

The relationship between Daya Bay's Δm^2_{YY} and that of the previous section is as follows:

$$\Delta m^2_{YY}|_{\mathrm{L/E}\to 0} = \Delta m^2_{ee}\sqrt{\left(1 + \sin^2\theta_{12}\cos^2\theta_{12}\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)^2\right)}. \tag{23}$$

Therefore they are identical up to corrections of $\mathcal{O}(10^{-4})$ as $\mathrm{L/E} \to 0$.

[5]The following, useful identity is easy to prove by writing $\Delta m^2_{21} = \Delta m^2_{31} - \Delta m^2_{32}$: $(\cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32})^2 = [\cos^2\theta_{12}(\Delta m^2_{31})^2 + \sin^2\theta_{12}(\Delta m^2_{32})^2] - \cos^2\theta_{12}\sin^2\theta_{12}(\Delta m^2_{21})^2$.

[6]The following identity is useful to understand this point, $\sin^2(\frac{\pi}{2} \pm \epsilon) \approx 1 - \epsilon^2$ where here $\epsilon = s_{12}c_{12}\Delta_{21}$. Similarly at oscillation maximum, $\sin^2(\pi \pm \epsilon) \approx \epsilon^2$.

Given that Δm^2_{YY} is L/E dependent one should take the average of Δm^2_{YY} over the L/E range of the experiment

$$\langle \Delta m^2_{YY} \rangle = \frac{\int_{(\mathrm{L/E})_{\mathrm{min}}}^{(\mathrm{L/E})_{\mathrm{max}}} d(\mathrm{L/E}) \Delta m^2_{YY}}{[(\mathrm{L/E})_{\mathrm{max}} - (\mathrm{L/E})_{\mathrm{min}}]}. \tag{24}$$

For the current experiments this range is from $[0, 0.8]$ km/MeV and then from Fig. 4 it is clear that

$$\langle \Delta m^2_{YY} \rangle \approx \Delta m^2_{ee}, \tag{25}$$

if the discontinuity at OM is averaged over in a symmetric way. In practice, of course, one needs to weight the average over the L/E range by the experimental L/E sensitivity. This is something that can only be performed by the experiment. This was not performed in Ref. [4] or [5].

C. Daya Bay's new definition of the effective Δm^2

After the issue with Δm^2_{YY} was pointed out to the Daya Bay Collaboration [11], the Daya Bay Collaboration defined a new effective Δm^2 in the supplemental material of Ref. [6]. Here I will use the symbol Δm^2_{ZZ} for this new definition which is defined in terms of the kinematic phase, Ω, given Eq. (3), as

$$\Delta m^2_{ZZ} \equiv \frac{2\mathrm{E}}{\mathrm{L}}\Omega, = |\Delta m^2_{ee}| \pm \frac{2\mathrm{E}}{\mathrm{L}}\phi. \tag{26}$$

Unfortunately, since ϕ is not a linear function in L/E, Δm^2_{ZZ} is also L/E dependent. In contrast remember that, from Eq. (4), $\Delta m^2_{ee} \equiv \frac{\partial \Omega}{\partial (L/2E)}|_{\frac{L}{E}\to 0}$.

For short baseline experiments, such as Daya Bay, RENO and Double Chooz, this dependence is small, and can be calculated analytically from Eq. (11):

$$\begin{aligned} \Delta m^2_{ZZ} &= |\Delta m^2_{ee}| \left[1 \pm \frac{1}{6} \cos 2\theta_{12} \sin^2 2\theta_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \right) \Delta^2_{21} \right. \\ &\qquad \left. + \mathcal{O}\left(\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \right) \Delta^4_{21} \right) \right] \\ &\approx |\Delta m^2_{ee}| \left[1 \pm 6 \times 10^{-6} \left(\frac{\mathrm{L/E}}{0.5\ \mathrm{km/MeV}} \right)^2 \right]. \end{aligned} \tag{27}$$

Given the current and expected future accuracy of the current short baseline experiments, the L/E dependence in Δm^2_{ZZ} can be ignored.

However for future experiments such as JUNO [12] and RENO-50 [13], the L/E dependence of Δm^2_{ZZ} is significant; see Fig. 5. These experiments explore an L/E range from 6 to 25 km/MeV. In this range, Δm^2_{ZZ} changes by $\sim 1\%$ whereas the expected accuracy of the measurement is better than 0.5%; see [12]. So this definition of Δm^2_{ZZ} is not appropriate for these experiments unless the experimenters want to do the L/E averaging as discussed in the previous section.

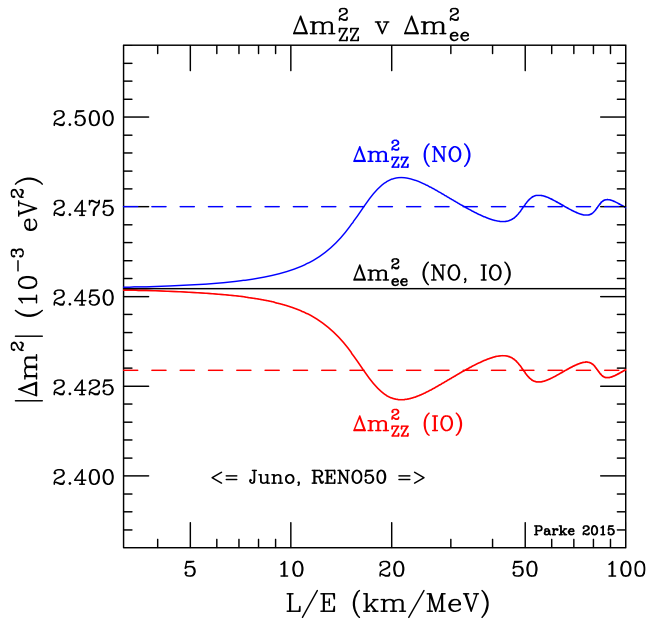

FIG. 5. Daya Bay's new definition (see [6]) of an effective Δm^2, Δm^2_{ZZ}, for $\bar{\nu}_e$ disappearance compared to $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$. The L/E range appropriate for JUNO and RENO-50 is 6 to 25 km/MeV, exactly the range in which Δm^2_{ZZ} changes by $\pm 1\%$. Yet, the expected accuracy of these two experiments is better than 0.5%. The sign of the variation of Δm^2_{ZZ} is mass ordering dependent. The blue and red dashed lines are Δm^2_{31} for NO and IO respectively.

IV. CONCLUSIONS

Having a single, L/E independent effective Δm^2 which can be used for reactor experiments of any L/E is highly desirable. Δm^2_{ee}, defined in Eq. (4), is the best effective Δm^2 for ν_e disappearance in the literature for the following reasons:

(i) It is independent of L/E for all values of L/E.
(ii) It is a simple combination of fundamental parameters:

$$\begin{aligned} \Delta m^2_{ee} &\equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32} \\ &= \Delta m^2_{31} - \sin^2\theta_{12}\Delta m^2_{21} \\ &= \Delta m^2_{32} + \cos^2\theta_{12}\Delta m^2_{21} \\ &= m^2_3 - (\cos^2\theta_{12} m^2_1 + \sin^2\theta_{12} m^2_2). \end{aligned}$$

(iii) It has a direct, simple, physical interpretation: Δm^2_{ee} is "the ν_e weighted average of Δm^2_{31} and Δm^2_{32}," since the ratio of the ν_e fraction in $\nu_1 : \nu_2$ is $\cos^2\theta_{12} : \sin^2\theta_{12}$.
(iv) It can be used in the future medium baseline reactor experiments, $\mathrm{L/E} > 6$ and <25 km/MeV, using the exact oscillation probability,

$$P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \frac{1}{2}\sin^2 2\theta_{13}\Big(1 - \sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} \times \cos(2|\Delta_{ee}| \pm \phi)\Big),$$

where $\phi \equiv \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}$. This probability can be used to determine solar parameters $\sin^2\theta_{12}$ and Δm^2_{21} as well as $|\Delta m^2_{ee}|$ with unprecedented precision and may be able to determine the atmospheric mass ordering, if the sign in front of ϕ can be determined at a high enough confidence level.

(v) It can be used in the current short baseline reactor experiments, $L/E < 1$ km/MeV, using the approximate oscillation probability,

$$P(\bar{\nu}_e \to \bar{\nu}_e) \approx 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{ee}.$$

This is trivially obtained from the exact expression, Eq. (2), by setting both the amplitude modulation to one and the phase advancement or retardation to zero,

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} \to 1 \quad \text{and} \quad \phi \to 0$$

as these are higher order effects. This approximates the exact oscillation probability to better than one part in 10^4 and can be improved in a systematic way; see Eq. (A2). This probability, using the current experimental data, allows for an accurate determination of mixing angle θ_{13} and the atmospheric mass splitting $|\Delta m^2_{ee}|$, independent of the atmospheric mass ordering, and it is only very weakly dependent on our current knowledge of the solar parameters, through the solar term. From a measured value of $|\Delta m^2_{ee}|$, using short baseline reactor experiments, it is simple to calculate Δm^2_{31} for both mass orderings. However, the uncertainties on Δm^2_{31} will be more dependent on solar parameters, measured by other experiments, than $|\Delta m^2_{ee}|$.

Furthermore, Δm^2_{ee}, defined by Eq. (4), naturally appears as the renormalized atmospheric Δm^2 in neutrino propagation in matter (see [14]), as using this renormalized Δm^2 significantly reduces the complexity of the oscillation probabilities.

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APPENDIX A: ALTERNATIVE DERIVATION OF Δm^2_{ee}

In this appendix, an alternative derivation of why $\sin^2\Delta_{ee}$ is the most accurate approximation for $\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}$ is given. Starting with the following linear combination of Δm^2_{31} and Δm^2_{32}, given by

$$\Delta_{rr} \equiv (1-r)\Delta_{31} + r\Delta_{32} \quad \text{then}$$
$$\Delta_{31} = \Delta_{rr} + r\Delta_{21},$$
$$\Delta_{32} = \Delta_{rr} - (1-r)\Delta_{21},$$

since $\Delta_{21} = \Delta_{31} - \Delta_{32}$ and r is a number between [0,1]. The relevant range of kinematic phases is $0 \le |\Delta_{31}| \sim |\Delta_{32}| < \pi$ and $0 \le \Delta_{21} < \pi/30 \approx 0.1$. So it is a simple exercise to perform a Taylor series expansion about Δ_{rr} using expansion parameter Δ_{21}, and obtain (using $c^2_{12} \equiv \cos^2\theta_{12}$ and $s^2_{12} \equiv \sin^2\theta_{12}$)

$$\begin{aligned} c^2_{12}\sin^2\Delta_{31} + s^2_{12}\sin^2\Delta_{32} &= \sin^2\Delta_{rr} + [c^2_{12}r - s^2_{12}(1-r)]\Delta_{21}\sin(2\Delta_{rr}) \\ &\quad + [c^2_{12}r^2 + s^2_{12}(1-r)^2]\Delta^2_{21}\cos(2\Delta_{rr}) \\ &\quad - \frac{2}{3}[c^2_{12}r^3 - s^2_{21}(1-r)^3]\Delta^3_{21}\sin(2\Delta_{rr}) \\ &\quad - \frac{1}{3}[c^2_{12}r^4 + s^2_{12}(1-r)^4]\Delta^4_{21}\cos(2\Delta_{rr}) + \mathcal{O}(\Delta^5_{21}). \end{aligned} \tag{A1}$$

The choice of $r = s^2_{12}$, making $\Delta_{rr} = \Delta_{ee}$, does two great things for this Taylor series expansion:

(1) the coefficient of Δ_{21} vanishes, since $[c^2_{12}r - s^2_{12}(1-r)] = 0$, and

(2) the coefficient of Δ^2_{21} is minimized, since

$$\frac{\partial}{\partial r}[c^2_{12}r^2 + s^2_{12}(1-r)^2]|_{r=s^2_{12}} = 0 \quad \text{and}$$
$$\frac{\partial^2}{\partial^2 r}[c^2_{12}r^2 + s^2_{12}(1-r)^2] > 0.$$

No other value of r satisfies either of these requirements. Thus, using $r = s^2_{12}$ makes $\sin^2\Delta_{ee}$ the best possible approximation to $c^2_{12}\sin^2\Delta_{31} + s^2_{12}\sin^2\Delta_{32}$ for a constant Δm^2 and the corrections are tiny, of $\mathcal{O}(10^{-3})$ for $L/E < 1$ km/ MeV.

Using this expansion the ν_e survival probability can be written as

$$
\begin{aligned}
&P_{\mathrm{xshort}}(\bar{\nu}_e \to \bar{\nu}_e)\\
&= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21}\\
&\quad - \sin^2 2\theta_{13}\Big[\sin^2|\Delta_{ee}| + \sin^2\theta_{12}\cos^2\theta_{12}\Delta_{21}^2\cos(2|\Delta_{ee}|)]\\
&\quad \mp \frac{1}{6}\cos 2\theta_{12}\sin^2 2\theta_{12}\Delta_{21}^3\sin(2|\Delta_{ee}|)\\
&\quad - \frac{1}{48}\sin^2 2\theta_{12}[4 - 3\sin^2 2\theta_{12}]\Delta_{21}^4\cos(2|\Delta_{ee}|) + \mathcal{O}(\Delta_{21}^5)\Big].
\end{aligned} \tag{A2}
$$

APPENDIX B: COMBINING Δm_{31}^2 AND Δm_{32}^2 INTO Δm_{ee}^2 PLUS A PHASE

The simplest way to show that

$$
\begin{aligned}
&\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}\\
&\quad = \frac{1}{2}\left(1 - \sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}\cos\Omega\right)
\end{aligned} \tag{B1}
$$

with

$$\Omega = 2\Delta_{ee} + \phi \tag{B2}$$

$$
\begin{aligned}
\text{where } \Delta m_{ee}^2 &\equiv \left.\frac{\partial\Omega}{\partial(\mathrm{L}/2\mathrm{E})}\right|_{\frac{\mathrm{L}}{\mathrm{E}}\to 0}\\
&= \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2
\end{aligned} \tag{B3}
$$

$$
\begin{aligned}
\text{and}\quad \phi &\equiv \Omega - 2\Delta_{ee}\\
&= \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12},
\end{aligned} \tag{B4}
$$

is to write

$$
\begin{aligned}
&c_{12}^2\sin^2\Delta_{31} + s_{12}^2\sin^2\Delta_{32}\\
&\quad = \frac{1}{2}(1 - (c_{12}^2\cos 2\Delta_{31} + s_{12}^2\cos 2\Delta_{32})),
\end{aligned} \tag{B5}
$$

using $c_{12}^2 \equiv \cos^2\theta_{12}$ and $s_{12}^2 \equiv \sin^2\theta_{12}$.

Then, if we rewrite $2\Delta_{31}$ and $2\Delta_{32}$ in terms of $(\Delta_{31} + \Delta_{32})$ and Δ_{21}, we have

$$
\begin{aligned}
&c_{12}^2\cos 2\Delta_{31} + s_{12}^2\cos 2\Delta_{32}\\
&= c_{12}^2\cos(\Delta_{31} + \Delta_{32} + \Delta_{21}) + s_{12}^2\cos(\Delta_{31} + \Delta_{32} - \Delta_{21})\\
&= \cos(\Delta_{31} + \Delta_{32})\cos\Delta_{21} - \sin(\Delta_{31} + \Delta_{32})\cos 2\theta_{12}\sin\Delta_{21}.
\end{aligned}
$$

Since

$$\cos^2\Delta_{21} + \cos^2 2\theta_{12}\sin^2\Delta_{21} = 1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}$$

we can then write

$$
\begin{aligned}
&c_{12}^2\cos 2\Delta_{31} + s_{12}^2\cos 2\Delta_{32}\\
&\quad = \sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}\cos\Omega,
\end{aligned} \tag{B6}
$$

where

$$\Omega = \Delta_{31} + \Delta_{32} + \arctan(\cos 2\theta_{12}\tan\Delta_{21}).$$

Applying the prescription given in Sec. II to separate Ω into an effective 2Δ and a phase, ϕ, we find

$$
\begin{aligned}
\left.\frac{\partial\Omega}{\partial\mathrm{L}/2\mathrm{E}}\right|_{\frac{\mathrm{L}}{\mathrm{E}}\to 0} &= \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2 = \Delta m_{ee}^2\\
\text{and}\quad \phi &= \Omega - 2\Delta_{ee}\\
&= \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}
\end{aligned}
$$

and thus

$$\Omega = 2\Delta_{ee} + (\arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}), \tag{B7}$$

Q.E.D.

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