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Scattering and pair creation by a constant electric field between two capacitor plates

S. P. Gavrilov[1,3,*] and D. M. Gitman[1,2,4,†]

[1]Department of Physics, Tomsk State University, 634050 Tomsk, Russia
[2]P.N. Lebedev Physical Institute, 53 Leninskiy Prospect, 119991 Moscow, Russia
[3]Department of General and Experimental Physics, Herzen State Pedagogical University of Russia, Moyka Embankment 48, 191186 St. Petersburg, Russia
[4]Institute of Physics, University of São Paulo,CP 66318, CEP 05315-970 São Paulo, São Paulo, Brazil

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Using a quantum field theory approach, we consider particle scattering and vacuum instability in the so-called L-constant electric field, which is a constant electric field confined between two capacitor plates separated by a finite distance L. We obtain and analyze special sets of stationary solutions of the Dirac and Klein-Gordon equations with the L-constant electric field. Then, we represent probabilities of particle scattering and characteristics of the vacuum instability (related to pair creation) in terms of the introduced solutions. From exact formulas, we derive asymptotic expressions for the differential mean numbers, for the total mean number of created particles, and for the vacuum-to-vacuum transition probability. Using the equivalence principle, we demonstrate that the distributions of particles created by the L-constant electric field and the gravitational field of a black hole have a similar thermal structure.

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I. INTRODUCTION

The effect of particle creation by strong electromagnetic and gravitational fields has a pure quantum nature. Depending on the structure of such external backgrounds, different approaches have been proposed for calculating the effect. Initially, the effect of particle creation was considered for time-dependent external electric fields that are switched on and off at the initial and final time instants, respectively. We call such external fields the t-electric potential steps. Scattering, particle creation from the vacuum, and particle annihilation by the t-electric potential steps has been considered in the framework of the relativistic quantum mechanics, see Refs. [1–3]; a more complete list of relevant publications can be found in Refs. [4,5]. At present it is well understood that only an adequate quantum field theory (QFT) with the corresponding external background may consistently describe this effect and possible accompanying processes. In the framework of such a theory, particle creation is related to a violation of the vacuum stability with time. Backgrounds (external fields) that may violate the vacuum stability have to be able to produce nonzero work when interacting with the corresponding particles. In quantum electrodynamics (QED), these are electriclike electromagnetic fields. A general formulation of QED with t-electric potential steps was developed in Refs. [6]. However, there exist many physically interesting situations where external backgrounds are not formally switched off at time infinity, as the corresponding backgrounds are not formally t-electric potential steps. As an example, we may point out time-independent nonuniform electric fields that are concentrated in restricted space areas. The latter fields represent a kind of spatial (or, as we call them, conditionally) x-electric potential steps for charged particles. The x-electric potential steps can also create particles from the vacuum; the Klein paradox is closely related to this process [7–9].

Approaches for treating quantum effects in the explicitly time-dependent external fields are not directly applicable to the x-electric potential steps. Some heuristic calculations of particle creation by x-electric potential steps in the framework of relativistic quantum mechanics, with qualitative discussion from the point of view of QFT, were first presented by Nikishov in Refs. [2,10]. In our recent article [11], we presented a consistent formulation of QED with x-electric potential steps quantizing the Dirac and the Klein-Gordon (scalar) fields in the presence of such steps, in terms of adequate in- and out-particles. We developed a nonperturbative calculation technique for different quantum processes such as scattering, reflection, and electron-positron pair creation. As in the case of QED with t-electric potential steps, this technique essentially uses special sets of exact solutions of the Dirac equation with the corresponding external field of x-electric potential steps. The cases when such solutions can be found explicitly (analytically) are called exactly solvable cases. In QED with t-electric potential steps there exist a few exactly solvable cases. The first (conditionally) is the so-called T-constant electric field, which is a uniform electric field that efficiently acts during a sufficiently large but finite time T. Quantum processes including particle creation in the

*gavrilovsergeyp@yahoo.com
†gitman@if.usp.br

latter field were studied in detail in Refs. [12–16] and used for a number of applications [17,18]. One can also point out other exactly solvable cases, such as the Sauter-like electric field [19] (see also Refs. [13,20]) and an exponentially decreasing electric field [21]. In the recently constructed QED with x-electric potential steps, an important exactly solvable case of the Sauter field $E(x) = E\cosh^{-2}(x/L_{\mathrm{S}})$ was considered in detail in Ref. [11] as an illustration of the general theory. In the present article we consider the second exactly solvable case of x-electric potential steps, namely, the so-called L-constant electric field. Such a step represents a constant electric field situated between two planes, $x = x_{\mathrm{L}} = \mathrm{const}$ and $x = x_{\mathrm{R}} = \mathrm{const}$, where $x_{\mathrm{R}} - x_{\mathrm{L}} = L$ is uniform there and directed along the axis x, see Sec. II. In fact, this is a uniform electric field confined between two capacitor plates separated by a finite distance L. Some heuristic calculations of the particle creation effect by such a field in the framework of the relativistic quantum mechanics were presented in Ref. [22]. The L-constant electric field is an analog of the T-constant field. We stress that the T-constant and L-constant electric fields describe different physical situations in the general case. However, in the limiting case, when both $T \to \infty$ and $L \to \infty$, they represent the same uniform constant electric field, which is obviously an idealization. Nevertheless, such a field allows exact solutions and has been frequently used in various calculations in QED, particularly in the pioneering work of Schwinger [23] (see [24] for a review). These calculations are relatively simple due to the translational symmetry of the field; however, they always contain divergences related to the infinite duration of the field action and to the infinite volume of the consideration. In this respect, calculations in the T-constant and L-constant electric fields can be considered as a kind of different regularization in the case of the constant uniform electric field.

The study of the vacuum instability in the presence of potential steps—in particular, particle creation in the L-constant electric field—is quite important for various applications. For example, it is important in the study of particle emission from black holes and quark and neutron stars, due to a close relation between particle creation by strong electrostatic potentials and the Unruh effect; see, e.g., [4,25] for reviews. The corresponding limiting case of a constant uniform electric field has many similarities with the case of the de Sitter background; see, e.g., Refs. [26,27] and references therein. Recent progress in laser physics allows one to hope that particle creation effect will be experimentally observed in the near future in laboratory conditions (see Refs. [28] for a review). Methods of QFT with strong potential steps are currently being developed in condensed matter physics, and particle creation by external fields has become an observable effect in physics of graphene and similar nanostructures (say, in topological insulators and Weyl semimetals); this area is currently under intense development, see the reviews [29,30]. In particular, the particle creation effect is crucial for understanding the conductivity of graphene, especially in the so-called nonlinear regime. Electron-hole pair creation (which is an analog of the electron-positron pair creation from the vacuum) was recently observed in graphene by its indirect influence on the graphene conductivity [31]. Possible experimental configurations for testing the pair creation by a linear step of finite length were proposed in [32]. The inhomogeneity of the field becomes important in achieving extreme field strengths.

In this article, using the general approach developed in Ref. [11], we consider the particle scattering and the vacuum instability on a quasilinear x-electric potential step corresponding to an L-constant electric field. We use notation from and the final formulas of the latter work. In Sec. II, we obtain and analyze special sets of stationary solutions of the Dirac and Klein-Gordon equations with the L-constant electric field. In Sec. III, we represent probabilities of particle scattering and characteristics of the vacuum instability (related to the pair creation) in terms of the introduced solutions. From exact formulas, we derive asymptotic expressions for differential mean numbers, the total mean number of created particles, and the vacuum-to-vacuum transition probability in the case of a small-gradient field. In the first part of the discussion, Sec. IV A, we consider length and time scales of small-gradient fields and show that quantum effects in an L-constant field are quite representative for a large class of small-gradient electric fields. In the second part of the discussion, Sec. IV B, we show that the distribution of particles created by a strong electrostatic inhomogeneous fields of a small gradient—in particular, by the L-constant field—can be written in a general Hawking-like thermal form, in which the Hawking temperature is reproduced exactly. Section V contains our Conclusion. In Appendix A, we describe briefly basic elements of QED with x-electric potential steps. In Appendix B, we list some useful properties of the Weber parabolic cylinder functions (WPCFs). In what follows we use the system of units where $\hbar = c = 1$.

II. IN- AND OUT-SOLUTIONS IN AN L-CONSTANT ELECTRIC FIELD

A. Dirac equation

Let us consider QED with an x-electric potential step, which is an L-constant electric field. The latter electromagnetic field consists of a pure electric field $\mathbf{E}$ (the corresponding magnetic field $\mathbf{B}$ is zero) of the form[1] $\mathbf{E}(X) = \mathbf{E}(x) = (E_x(x), 0, \ldots, 0)$. The electric field $E(x)$ has the form

[1]We recall that our system is placed in the $d = D + 1$ dimensional Minkowski spacetime parametrized by the coordinates $X = (X^\mu, \mu = 0, 1, \ldots, D) = (t, \mathbf{r})$, $X^0 = t$, $\mathbf{r} = (X^1, \ldots, X^D)$. It consists of a Dirac field $\psi(X)$ interacting with an external electromagnetic field $A^\mu(X)$ in the form of a x-electric potential step.

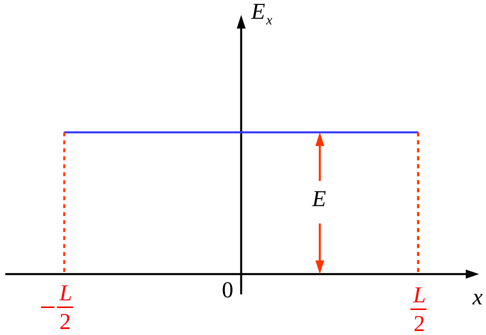

FIG. 1. L-constant electric field.

$$E(x) = E = \text{const} > 0, \quad x \in S_{\text{int}} = (x_{\text{L}}, x_{\text{R}});$$
$$E(x) = 0, \quad x \in S_{\text{L}} = (-\infty, x_{\text{L}}], \quad x \in S_{\text{R}} = [x_{\text{R}}, \infty),$$

and we choose that $x_{\text{L}} = -L/2$ and $x_{\text{R}} = L/2$. The plot of the field is shown in Fig. 1.

We assume that the basic Dirac particle is an electron with the mass m and the charge $-e$, $e > 0$, and the positron is its antiparticle. The electric field under consideration accelerates the electrons along the x axis in the negative direction and the positrons along the x axis in the positive direction.

Potentials of the corresponding electromagnetic field $A^\mu(X)$ can be chosen as

$$A^\mu(X) = (A^0(x), A^j = 0, j = 1, 2, \ldots, D), \quad x = X^1, \qquad (2.1)$$

[so that $E(x) = -A_0'(x)$] with a linearly growing potential $A^0(x)$ on an interval $x \in S_{\text{int}}$ of the length $L = x_{\text{R}} - x_{\text{L}}$, which is constant out of this interval. The potential energy of an electron in the electric field under consideration is $U(x) = -eA_0(x)$,

$$U(x) = \begin{cases} U_{\text{L}} = eEx_{\text{L}}, & x \in S_{\text{L}} \\ eEx, & x \in S_{\text{int}} \\ U_{\text{R}} = eEx_{\text{R}}, & x \in S_{\text{R}} \end{cases}. \qquad (2.2)$$

The magnitude of the x-electric step under consideration is

$$\mathbb{U} = U_{\text{R}} - U_{\text{L}} = eEL > 0. \qquad (2.3)$$

In d dimensions, the Dirac field $\psi(X)$ is a column with $2^{[d/2]}$ components (in what follows, we call it just a spinor), and γ^μ are $2^{[d/2]} \times 2^{[d/2]}$ gamma matrices; see, e.g., Ref. [33],

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \eta^{\mu\nu} = \text{diag}(\underbrace{1, -1, \ldots, -1}_{d}),$$
$$\mu, \nu = 0, 1, \ldots, D.$$

The classical Dirac field $\psi(X)$ satisfies the Dirac equation with the L-constant electric field,

$$i\partial_0\psi(X) = \hat{H}\psi(X), \qquad \hat{H} = \gamma^0(-i\gamma^j\partial_j + m) + U(x), \qquad (2.4)$$

where $\hat{H}$ is the one-particle Dirac Hamiltonian.

Let us consider stationary solutions of the Dirac equation (2.4) having the following form:

$$\psi_n(X) = \exp(-ip_0t + i\mathbf{p}_\perp\mathbf{r}_\perp)\psi_n(x), \qquad X = (t, x, \mathbf{r}_\perp), \qquad n = (p_0, \mathbf{p}_\perp, \sigma),$$
$$\psi_n(x) = \{\gamma^0[p_0 - U(x)] - \gamma^1\hat{p}_x - \boldsymbol{\gamma}_\perp\mathbf{p}_\perp + m\}\phi_n(x),$$
$$\mathbf{r}_\perp = (X^2, \ldots, X^D), \qquad \mathbf{p}_\perp = (p^2, \ldots, p^D), \qquad \boldsymbol{\gamma}_\perp = (\gamma^2, \ldots, \gamma^D), \qquad \hat{p}_x = -i\partial_x, \qquad (2.5)$$

where $\psi_n(x)$ and $\phi_n(x)$ are spinors that depend on x alone. In fact, these are stationary states with the energy p_0 and with definite momenta $\mathbf{p}_\perp$ in the directions perpendicular to the axis x. Substituting (2.5) into Dirac equation (2.4) (i.e., partially squaring the Dirac equation), we obtain a second-order differential equation for the spinor $\phi_n(x)$,

$$\{\hat{p}_x^2 - i\gamma^0\gamma^1U'(x) - [p_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2\}\phi_n(x) = 0. \qquad (2.6)$$

We separate spinning variables by the substitution

$$\phi_n(x) = \phi_n^{(\chi)}(x) = \varphi_n(x)v_{\chi,\sigma}, \qquad (2.7)$$

where $v_{\chi,\sigma}$ with $\chi = \pm1$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{[d/2]-1})$, $\sigma_s = \pm1$, is a set of constant orthonormalized spinors satisfying the following equations:

$$\gamma^0\gamma^1v_{\chi,\sigma} = \chi v_{\chi,\sigma}, \qquad v_{\chi,\sigma}^\dagger v_{\chi',\sigma'} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'}. \qquad (2.8)$$

The quantum numbers χ and σ_s describe a spin polarization and provide a convenient parametrization of the solutions. Since in $(1+1)$ and $(2+1)$ dimensions ($d = 2, 3$) there are no spin degrees of freedom, the quantum numbers σ are absent. Then, scalar functions $\varphi_n(x)$ have to obey the second-order differential equation

$$\{\hat{p}_x^2 - i\chi U'(x) - [p_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2\}\varphi_n(x) = 0. \qquad (2.9)$$

In d dimensions, for any given set $p_0, \mathbf{p}_\perp$, there exists only $J_{(d)} = 2^{[d/2]-1}$ different spin states. The projection operator, which is situated inside the brackets $\{\cdots\}$ in Eq. (2.5), does not commute with the matrix $\gamma^0\gamma^1$ and, consequently, transforms $\phi_n^{(\chi)}(x)$ with a given χ to a linear superposition of functions $\phi_n^{(+1)}(x)$ and $\phi_{n'}^{(-1)}(x)$ with indices n and n' corresponding to the same $p_0, \mathbf{p}_\perp$. In $d \geq 4$ dimensions this projection operator also does not commute with the matrix $\Xi = i\boldsymbol{\gamma}_\perp \mathbf{p}_\perp / |\mathbf{p}_\perp|$. Assuming that $\sigma_1 = \pm 1$ is an eigenvalue of Ξ, one can see that solutions (2.5), which differ only by values of χ and σ_1, are linearly dependent. Let us denote by $\psi_{n_\pm}^{(\chi)}(x)$ solutions of the Dirac equation with quantum numbers $n_\pm = (p_0, \mathbf{p}_\perp, \sigma_\pm)$, $\sigma_\pm = (\pm 1, \sigma_2, \ldots, \sigma_{[d/2]-1})$. Then, for example, using existing relations between solutions (2.5) for $x \in S_\mathrm{L}$, one can easily verify that

$$[\pi_0(\mathrm{L}) - p^\mathrm{L}]\psi_{n_\pm}^{(+1)}(x) = (\pm i|\mathbf{p}_\perp| + m)\psi_{n_\mp}^{(-1)}(x). \quad (2.10)$$

The case of $(1+1)$ dimensions follows from Eq. (2.10) at $|\mathbf{p}_\perp| = 0$ and $n_\pm = n_\mp = p_0$, assuming $m \neq 0$. Note that in the case of $(2+1)$ dimensions, there are two nonequivalent representations for the γ matrices,

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^2, \qquad \gamma^2 = -i\sigma^1 s, \qquad s = \pm 1,$$

which correspond to different fermion species. In this case, following the same logic, one can see that

$$[\pi_0(\mathrm{L}) - p^\mathrm{L}]\psi_n^{(+1)}(x) = (isp^2 + m)\psi_n^{(-1)}(x),$$
$$n = (p_0, p^2).$$

That is why it is sufficient to work only with solutions corresponding to one of the values of χ. In what follows, we fix the quantum number χ in a certain way.

B. In- and out-solutions

In what follows, we use solutions of the Dirac equation denoted as ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, $\zeta = \pm$, with special left and right asymptotics at $x \in S_\mathrm{L}$ and $x \in S_\mathrm{R}$. Such solutions have the form (2.5) with the functions $\varphi_n(x)$ denoted as ${}_\zeta\varphi_n(x)$ or ${}^\zeta\varphi_n(x)$, respectively. The latter functions satisfy Eq. (2.9) and the following asymptotic conditions:

$$\begin{aligned}
{}_\zeta\varphi_n(x) &= {}_\zeta\mathcal{N} \exp[ip^\mathrm{L}(x - x_\mathrm{L})], \qquad x \in S_\mathrm{L},\\
{}^\zeta\varphi_n(x) &= {}^\zeta\mathcal{N} \exp[ip^\mathrm{R}(x - x_\mathrm{R})], \qquad x \in S_\mathrm{R},\\
p^\mathrm{L} &= \zeta\sqrt{[\pi_0(\mathrm{L})]^2 - \pi_\perp^2}, \qquad p^\mathrm{R} = \zeta\sqrt{[\pi_0(\mathrm{R})]^2 - \pi_\perp^2}, \qquad \zeta = \pm,\\
\pi_0(\mathrm{L/R}) &= p_0 - U_{\mathrm{L/R}}, \qquad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}.
\end{aligned} \quad (2.11)$$

Thus, the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ asymptotically describe particles with given real momenta $p^{\mathrm{L/R}}$ along the x axis. The factors ${}_\zeta\mathcal{N}$ and ${}^\zeta\mathcal{N}$ are normalization constants with respect to conditions (A1) given in Appendix A,

$$\begin{aligned}
&{}_\zeta\mathcal{N} = {}_\zeta CY, \qquad {}^\zeta\mathcal{N} = {}^\zeta CY, \qquad Y = (V_\perp T)^{-1/2},\\
&{}_\zeta C = [2|p^\mathrm{L}||\pi_0(\mathrm{L}) - \chi p^\mathrm{L}|]^{-1/2},\\
&{}^\zeta C = [2|p^\mathrm{R}||\pi_0(\mathrm{R}) - \chi p^\mathrm{R}|]^{-1/2}.
\end{aligned} \quad (2.12)$$

Since p_0 is the total energy of a particle, we interpret $\pi_0(\mathrm{R})$ and $\pi_0(\mathrm{L})$ as sum of its asymptotic kinetic and the rest energies in the regions S_R and S_L, respectively. We call the quantity $\pi_\perp$ the transversal energy.

Nontrivial solutions ${}^\zeta\psi_n(X)$ exist only for quantum numbers n that obey the relation

$$[\pi_0(\mathrm{R})]^2 > \pi_\perp^2 \Leftrightarrow \begin{cases} \pi_0(\mathrm{R}) > \pi_\perp \\ \pi_0(\mathrm{R}) < -\pi_\perp \end{cases}, \quad (2.13)$$

whereas nontrivial solutions ${}_\zeta\psi_n(X)$ exist only for quantum numbers n that obey the relation

$$[\pi_0(\mathrm{L})]^2 > \pi_\perp^2 \Leftrightarrow \begin{cases} \pi_0(\mathrm{L}) > \pi_\perp \\ \pi_0(\mathrm{L}) < -\pi_\perp \end{cases}. \quad (2.14)$$

We distinguish two types of electric steps, noncritical and critical, by their magnitudes as follows:

$$\mathbb{U} = \begin{cases} \mathbb{U} < \mathbb{U}_c = 2m, \text{noncritical step} \\ \mathbb{U} > \mathbb{U}_c, \text{critical step} \end{cases}. \quad (2.15)$$

In the case of noncritical steps, the vacuum is stable; see Ref. [11]. We are interested in the critical steps where there is electron-positron pair production from the vacuum.

In the case of critical steps of any form, and in particular in the step under consideration, there exist five ranges of quantum numbers n, Ω_k, $k = 1, \ldots, 5$, where the introduced solutions have similar forms and physical processes with particles have similar interpretations; see Ref. [11].

(i) In the range Ω_1, where $p_0 \geq U_\mathrm{R} + \pi_\perp$, and the range Ω_5, where $p_0 \leq U_\mathrm{L} - \pi_\perp$, the number of particles is conserved. In the range Ω_1 there exist only incoming $[{}_+\psi_n(X)$ or ${}^-\psi_n(X)]$ and outgoing $[{}_-\psi_n(X)$ or

$^{+}\psi_{n}(X)$] electrons, whereas in the range Ω_5, there exist only incoming [$_{-}\psi_{n}(X)$ and $^{+}\psi_{n}(X)$] and outgoing [$_{+}\psi_{n}(X)$ and $^{-}\psi_{n}(X)$] positrons. In these ranges there exist only scattering and the reflection of the particles. Particle creation is impossible in these ranges.

(ii) In the range Ω_2, where

$$U_{\mathrm{R}} - \pi_{\perp} < p_0 < U_{\mathrm{R}} + \pi_{\perp},$$
$$\pi_0(\mathrm{L}) > \pi_{\perp} \quad \text{if } 2\pi_{\perp} \leq \mathbb{U},$$
$$U_{\mathrm{L}} + \pi_{\perp} < p_0 < U_{\mathrm{R}} + \pi_{\perp} \quad \text{if } 2\pi_{\perp} > \mathbb{U},$$

any solution has zero right asymptotic, which means that we deal with standing waves of the form

$$\psi_n(X) = {}_{+}\psi_n(X)e^{+i\theta_n} + {}_{-}\psi_n(X)e^{-i\theta_n}. \tag{2.16}$$

Here, similar to the range Ω_1, there exist only electrons that are subjected to the total reflection. In the range Ω_4, where

$$U_{\mathrm{L}} - \pi_{\perp} < p_0 < U_{\mathrm{L}} + \pi_{\perp},$$
$$\pi_0(\mathrm{R}) < -\pi_{\perp} \quad \text{if } 2\pi_{\perp} \leq U,$$
$$U_{\mathrm{L}} - \pi_{\perp} < p_0 < U_{\mathrm{R}} - \pi_{\perp} \quad \text{if } 2\pi_{\perp} > U,$$

any solution has zero left asymptotic, which means that we deal with standing waves of the form

$$\psi_n(X) = {}^{+}\psi_n(X)e^{+i\theta_n} + {}^{-}\psi_n(X)e^{-i\theta_n}. \tag{2.17}$$

Here, similar to Ω_5, there exist only positrons that are also subjected to the total reflection. The number of particles in Ω_2 and Ω_4 is conserved.

(iii) The range Ω_3, where $U_{\mathrm{L}} + \pi_{\perp} \leq p_0 \leq U_{\mathrm{R}} - \pi_{\perp}$, exists only for transversal momenta that satisfy the inequality $2\pi_{\perp} \leq \mathbb{U}$. Here the QFT description of quantum processes is essential. Its brief description is given in Appendix A. In this range, there exist in- and out-electrons that can be situated only to the left of the step, and in- and out-positrons that can be situated only to the right of the step. In the range Ω_3, all the partial vacua are unstable, and particle creation is possible. These pairs consist of out-electrons and out-positrons that appear on the left and on the right of the step and move there to the left and to the right, respectively. At the same time, the in-electrons that move to the step from the left are subjected to the total reflection. After being reflected, they move to the left of the step as out-electrons. Similarly, the in-positrons that move to the step from the right are subjected to the total reflection. After being reflected, they move to the right of the step as out-positrons.

It is assumed that each pair of solutions ${}_{\zeta}\psi_n(X)$ and ${}^{\zeta}\psi_n(X)$, with given quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, is complete in the space of solutions with the corresponding n. Because of Eq. (A1), given in Appendix A, the mutual decompositions of such solutions have the form

$$\eta_{\mathrm{L}}{}^{\zeta}\psi_n(X) = {}_{+}\psi_n(X)g({}_{+}|^{\zeta}) - {}_{-}\psi_n(X)g({}_{-}|^{\zeta}),$$
$$\eta_{\mathrm{R}}{}_{\zeta}\psi_n(X) = {}^{+}\psi_n(X)g({}^{+}|_{\zeta}) - {}^{-}\psi_n(X)g({}^{-}|_{\zeta}), \tag{2.18}$$

where the decomposition coefficients g are

$$({}_{\zeta}\psi_n, {}^{\zeta'}\psi_{n'})_x = \delta_{n,n'}g({}_{\zeta}|^{\zeta'}),$$
$$g({}^{\zeta'}|_{\zeta}) = g({}_{\zeta}|^{\zeta'})^*, \qquad n \in \Omega_1 \cup \Omega_3 \cup \Omega_5. \tag{2.19}$$

These coefficients satisfy the following unitary relations:

$$|g({}_{-}|^{+})|^2 = |g({}_{+}|^{-})|^2, \qquad |g({}_{+}|^{+})|^2 = |g({}_{-}|^{-})|^2, \qquad \frac{g({}_{+}|^{-})}{g({}_{-}|^{-})} = \frac{g({}^{+}|_{-})}{g({}^{+}|_{+})},$$
$$|g({}_{+}|^{-})|^2 - |g({}_{+}|^{+})|^2 = -\eta_{\mathrm{L}}\eta_{\mathrm{R}}. \tag{2.20}$$

For $x \in S_{\mathrm{int}}$, Eq. (2.9) can be written in the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \xi^2 + \mathrm{i}\chi - \lambda\right]\varphi_n(x) = 0, \tag{2.21}$$

where

$$\xi = \frac{eEx - p_0}{\sqrt{eE}}, \qquad \lambda = \frac{\pi_{\perp}^2}{eE}. \tag{2.22}$$

The general solution of Eq. (2.21) is completely determined by an appropriate pair of the linearly independent WPCFs: either $D_{\rho}[(1-\mathrm{i})\xi]$ and $D_{-1-\rho}[(1+\mathrm{i})\xi]$, or $D_{\rho}[-(1-\mathrm{i})\xi]$ and $D_{-1-\rho}[-(1+\mathrm{i})\xi]$, where $\rho = -\mathrm{i}\lambda/2 - (1+\chi)/2$. Then, taking into account Eq. (2.18), the functions ${}_{-}\varphi_n(x)$ and ${}^{+}\varphi_n(x)$ can be presented in the form

$$
{}_{-}\varphi_n(x) = Y\begin{cases} {}_{-}C\exp[-i|p^{\mathrm{L}}|(x-x_{\mathrm{L}})], & x\in S_{\mathrm{L}} \\ {}_{-}C\{a_1 D_\rho[-(1-i)\xi] + a_2 D_{-1-\rho}[-(1+i)\xi]\}, & x\in S_{\mathrm{int}}\,; \\ \eta_{\mathrm{R}}\{g({}^{+}|{}_{-}){}^{+}C\exp[i|p^{\mathrm{R}}|(x-x_{\mathrm{R}})] - g({}^{-}|{}_{-}){}^{-}C\exp[-i|p^{\mathrm{R}}|(x-x_{\mathrm{R}})]\}, & x\in S_{\mathrm{R}} \end{cases}
$$

$$
{}^{+}\varphi_n(x) = Y\begin{cases} \eta_{\mathrm{L}}\{g({}_{+}|{}^{+}){}_{+}C\exp[i|p^{\mathrm{L}}|(x-x_{\mathrm{L}})] - g({}_{-}|{}^{+}){}_{-}C\exp[-i|p^{\mathrm{L}}|(x-x_{\mathrm{L}})]\}, & x\in S_{\mathrm{L}} \\ {}^{+}C\{a_1' D_\rho[(1-i)\xi] + a_2' D_{-1-\rho}[(1+i)\xi]\}, & x\in S_{\mathrm{int}} \\ {}^{+}C\exp[i|p^{\mathrm{R}}|(x-x_{\mathrm{R}})], & x\in S_{\mathrm{R}} \end{cases} \tag{2.23}
$$

on the whole axis x. The functions ${}_{-}\varphi_n(x)$ and ${}^{+}\varphi_n(x)$ and their derivatives satisfy the following gluing conditions:

$$
\begin{aligned} {}^{+}_{-}\varphi_n(x_{\mathrm{L/R}}-0) &= {}^{+}_{-}\varphi_n(x_{\mathrm{L/R}}+0), \\ \partial_x {}^{+}_{-}\varphi_n(x_{\mathrm{L/R}}-0) &= \partial_x {}^{+}_{-}\varphi_n(x_{\mathrm{L/R}}+0). \end{aligned} \tag{2.24}
$$

Note that the following relations hold: $|p^{\mathrm{L}}|/\sqrt{eE} = \sqrt{\xi_1^2-\lambda}$ and $|p^{\mathrm{R}}|/\sqrt{eE} = \sqrt{\xi_2^2-\lambda}$, where $\xi_1 = \xi|_{x=x_{\mathrm{L}}}$, $\xi_2 = \xi|_{x=x_{\mathrm{R}}}$.

A formal transition to the Klein-Gordon case can be done by setting $\chi = 0$ and $\eta_{\mathrm{L}} = \eta_{\mathrm{R}} = 1$ in Eqs. (2.23). In this case $n = (p_0, \mathbf{p}_\perp)$, and normalization factors are

$$
{}_{\zeta}\mathcal{N} = {}_{\zeta}CY, \qquad {}^{\zeta}\mathcal{N} = {}^{\zeta}CY,
$$
$$
{}_{\zeta}C = |2p^{\mathrm{L}}|^{-1/2}, \qquad {}^{\zeta}C = |2p^{\mathrm{R}}|^{-1/2}.
$$

Using Eq. (2.24) and the Wronskian determinant of WPCFs, we find the coefficients a_j and a_j',

$$
\begin{aligned} a_j &= -\frac{(-1)^j}{\sqrt{2}}\exp\left[\frac{i\pi}{2}\left(\rho+\frac{1}{2}\right)\right]\sqrt{\xi_1^2-\lambda}\,f_j^{(-)}(\xi_1), \\ a_j' &= -\frac{(-1)^j}{\sqrt{2}}\exp\left[\frac{i\pi}{2}\left(\rho+\frac{1}{2}\right)\right]\sqrt{\xi_2^2-\lambda}\,f_j^{(+)}(\xi_2), \\ j &= 1,2, \end{aligned} \tag{2.25}
$$

where

$$
\begin{aligned} f_1^{(\pm)}(\xi) &= \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_{-\rho-1}[\pm(1+i)\xi], \\ f_2^{(\pm)}(\xi) &= \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_{\rho}[\pm(1-i)\xi]. \end{aligned} \tag{2.26}
$$

They can be used to determine the coefficients $g({}_{\pm}|{}^{+})$ and $g({}^{\pm}|{}_{-})$. It should be noted that we need to know explicitly only the coefficients $g({}_{-}|{}^{+})$ and $g({}^{+}|{}_{-})$, which are

$$
g({}^{+}|{}_{-}) = \eta_{\mathrm{R}} AB\exp[(\rho+1/2)i\pi/2], \qquad g({}_{-}|{}^{+}) = \eta_{\mathrm{L}} A'B'\exp[(\rho+1/2)i\pi/2],
$$
$$
A = \sqrt{\frac{\sqrt{\xi_2^2-\lambda}\sqrt{\xi_1^2-\lambda}|\xi_2+\chi\sqrt{\xi_2^2-\lambda}|}{8|\xi_1-\chi\sqrt{\xi_1^2-\lambda}|}}, \qquad B = f_1^{(-)}(\xi_1)f_2^{(-)}(\xi_2) - f_2^{(-)}(\xi_1)f_1^{(-)}(\xi_2),
$$
$$
A' = \sqrt{\frac{\sqrt{\xi_2^2-\lambda}\sqrt{\xi_1^2-\lambda}|\xi_1-\chi\sqrt{\xi_1^2-\lambda}|}{8|\xi_2+\chi\sqrt{\xi_2^2-\lambda}|}}, \qquad B' = f_1^{(+)}(\xi_1)f_2^{(+)}(\xi_2) - f_2^{(+)}(\xi_1)f_1^{(+)}(\xi_2). \tag{2.27}
$$

One can see that coefficients (2.27) obey the relations

$$
g({}^{+}|{}_{-})|_{p_0\to -p_0} = -\eta_{\mathrm{L}}\eta_{\mathrm{R}} g({}_{-}|{}^{+}). \tag{2.28}
$$

From these relations, one can conclude that $|g({}_{-}|{}^{+})|$ is an even function of the energy p_0 and transversal momenta $\mathbf{p}_\perp$, and does not depend on a spin polarization.

In the Klein-Gordon case, the coefficients g are

$$
\begin{aligned} g({}^{+}|{}_{-}) &= \exp(\lambda\pi/4)A_{\mathrm{sc}}B|_{\chi=0}, \\ g({}_{-}|{}^{+}) &= \exp(\lambda\pi/4)A_{\mathrm{sc}}B'|_{\chi=0}, \\ A_{\mathrm{sc}} &= \left(\frac{1}{8}\sqrt{\xi_2^2-\lambda}\sqrt{\xi_1^2-\lambda}\right)^{1/2}, \end{aligned} \tag{2.29}
$$

where B and B' are given by Eqs. (2.27). They satisfy the unitary relations (2.20) in which we have to set $\eta_{\mathrm{L}} = \eta_{\mathrm{R}} = 1$.

As follows from Eqs. (2.27) and (2.29), if either $|p^{\mathrm{R}}|$ or $|p^{\mathrm{L}}|$ tends to zero, one of the following limits holds true:

$$
\begin{aligned} |g({}_{-}|{}^{+})|^{-2} &\sim \sqrt{\xi_2^2-\lambda} \to 0, \\ |g({}_{-}|{}^{+})|^{-2} &\sim \sqrt{\xi_1^2-\lambda} \to 0, \quad \forall\ \lambda \neq 0. \end{aligned} \tag{2.30}
$$

These properties are essential for the justification of in- and out-particle interpretation in the general construction [11].

The modulus $|g({}_{-}|^{+})|$ in the form (2.27) and (2.29) was obtained in the course of heuristic calculations in the framework of the relativistic quantum mechanics in Ref. [22].

III. SCATTERING AND CREATION OF PARTICLES

A. Ranges of stable vacuum

We know that in the ranges Ω_i, $i = 1, 2, 4, 5$ the partial vacua are stable. Let us start with discussion of formulas obtained for these ranges. In the ranges Ω_2 and Ω_4, a particle is subjected to the total reflection. In the adjacent ranges, Ω_1 and Ω_5, a particle can be reflected and transmitted. For example, in the range Ω_1, total $\tilde{R}$ and relative R amplitudes of an electron reflection, and total $\tilde{T}$ and relative T amplitudes of an electron transmission, can be presented as the following matrix elements:

$$\begin{aligned} R_{+,n} &= \tilde{R}_{+,n} c_v^{-1}, \qquad \tilde{R}_{+,n} = \langle 0,\mathrm{out}|{}_{-}a_n(\mathrm{out})_{+}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle, \\ T_{+,n} &= \tilde{T}_{+,n} c_v^{-1}, \qquad \tilde{T}_{+,n} = \langle 0,\mathrm{out}|{}^{+}a_n(\mathrm{out})_{+}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle, \\ R_{-,n} &= \tilde{R}_{-,n} c_v^{-1}, \qquad \tilde{R}_{-,n} = \langle 0,\mathrm{out}|{}^{+}a_n(\mathrm{out})^{-}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle, \\ T_{-,n} &= \tilde{T}_{-,n} c_v^{-1}, \qquad \tilde{T}_{-,n} = \langle 0,\mathrm{out}|{}_{-}a_n(\mathrm{out})^{-}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle, \end{aligned} \tag{3.1}$$

where state vectors in the corresponding Fock space and the vacuum-to-vacuum transition amplitude c_v are defined in Appendix A. The relative reflection $|R_{\zeta,n}|^2$ and transmission $|T_{\zeta,n}|^2$ probabilities satisfy the relation

$$\begin{aligned} &|T_{\zeta,n}|^2 = 1 - |R_{\zeta,n}|^2, \\ &|R_{\zeta,n}|^2 = [1 + |g({}_{-}|^{+})|^{-2}]^{-1}, \zeta = \pm. \end{aligned} \tag{3.2}$$

Similar expressions can be derived for positron amplitudes in the range Ω_5. In particular, relation (3.2) holds true literally for the positrons in the range Ω_5.

It is clear that $|R_{\zeta,n}|^2 \le 1$. This result may be interpreted as QFT justification of the rules of time-independent potential scattering theory in the ranges Ω_1 and Ω_5. Amplitudes of Klein-Gordon particle reflection and transmission in the ranges Ω_i, $i = 1, 2, 4, 5$ have the same form as in the Dirac particle case with coefficients g given by the corresponding inner product. Substituting the coefficients g given by Eqs. (2.27) or (2.29) into relations (3.2), one can find explicitly reflection and transmission probabilities in the L-constant field.

The limits (2.30) imply the following properties of the coefficients $|g({}_{-}|^{+})|$:

(i) $|g({}_{-}|^{+})|^{-2} \to 0$ in the range Ω_1 if n tends to the boundary with the range Ω_2 ($|p^{\mathrm{R}}| \to 0$);

(ii) $|g({}_{-}|^{+})|^{-2} \to 0$ in the range Ω_5 if n tends to the boundary with the range Ω_4 ($|p^{\mathrm{L}}| \to 0$).

Thus, in the above cases the relative reflection probabilities $|R_{\zeta,n}|^2$ tend to the unity; i.e., they are continuous functions of the quantum numbers n on the boundaries. In addition, it follows from Eqs. (2.27) and (2.29) that $|g({}_{-}|^{+})|^2 \to 0$ and, therefore, $|R_{\zeta,n}|^2 \to 0$ as $p_0 \to \pm\infty$, as of course it must be.

B. Klein zone

The range Ω_3, which is called the Klein zone, is of special interest due to the vacuum instability. We recall, as it follows from the general consideration [11], that if in the range Ω_3 there exists an in-particle, it will be subjected to the total reflection. For example, it follows from Eq. (A16), given in Appendix A, that the probability of reflection of a particle with given quantum numbers n, under the condition that all other partial vacua remain vacua, is $P(+|+)_{n,n} P_v^{-1} p_v^n = 1$. In the Dirac case, the presence of an in-particle with a given $n \in \Omega_3$ disallows the pair creation from the vacuum in this state due to the Pauli principle. Of course, pairs of bosons can be created from the vacuum in any already-occupied states.

The differential mean numbers of created pairs have the form $N_n^{\mathrm{cr}} = |g({}_{-}|^{+})|^{-2}$; see Eq. (A14), where $g({}_{-}|^{+})$ is given by Eq. (2.27) for Dirac particles and by Eq. (2.29) for Klein-Gordon particles. Note that dimensionless parameters λ and ξ entering these expressions satisfy the condition

$$\sqrt{\lambda} \le \xi_2, \qquad \xi_1 \le -\sqrt{\lambda}, \tag{3.3}$$

which are, in fact, consequences of the definition of the range Ω_3.

From properties (2.30), one finds that $N_n^{\mathrm{cr}} \to 0$ if n tends to the boundary with either the range Ω_2 ($\sqrt{\xi_2^2 - \lambda} \to 0$) or the range Ω_4 ($\sqrt{\xi_1^2 - \lambda} \to 0$); in the latter ranges, the vacuum is stable.

One can see that absolute values of $\sqrt{\xi_1^2 - \lambda}$ and $\sqrt{\xi_2^2 - \lambda}$ are related as follows:

$$\begin{aligned} &0 \le \left|\sqrt{\xi_2^2 - \lambda} - \sqrt{\xi_1^2 - \lambda}\right| \le \omega, \\ &\omega = [\sqrt{eE}L(\sqrt{eE}L - 2\sqrt{\lambda})]^{1/2}. \end{aligned} \tag{3.4}$$

Then, for any p_0 and $\mathbf{p}_\perp$ the numbers N_n^{cr} are negligible if the range Ω_3 is small enough,

$$N_n^{\mathrm{cr}} \sim \sqrt{\xi_1^2 - \lambda}\sqrt{\xi_2^2 - \lambda} \to 0 \quad \text{if } \omega \to 0. \tag{3.5}$$

The L-constant field is one of a regularization for a constant uniform electric field, and it is suitable for imitating a small-gradient field. That is reason why the L-constant field with a sufficiently large length L,

$$\sqrt{eE}L \gg \max\{1, m^2/eE\}, \tag{3.6}$$

and $\omega \gg 1$ is of interest. In what follows, we suppose that these conditions hold true and additionally assume that

$$\sqrt{\lambda} < K_\perp, \tag{3.7}$$

where $K_\perp$ is any given number satisfying the condition $\sqrt{eE}L/2 \gg K_\perp^2 \gg \max\{1, m^2/eE\}$.

Let us analyze how the numbers $N_n^{\rm cr}$ depend on the parameters $\xi_{1,2}$ and λ. Here we assume that $\chi = 1$. Since $N_n^{\rm cr}$ are even functions of p_0, we can consider only the case of $p_0 \le 0$. In this case $\xi_2 \ge \sqrt{eE}L/2$ is large, $\xi_2 \gg \max\{1, \lambda\}$, and the asymptotic expansions of the WPCFs with respect to ξ_2 are valid. As for the parameter ξ_1, the whole interval $-\sqrt{eE}L/2 \le \xi_1 \le -\sqrt{\lambda}$ can be divided in two parts,

$$\begin{aligned}&(\mathrm{a})\ -\sqrt{eE}L/2 \le \xi_1 \le -K,\\ &(\mathrm{b})\ -K < \xi_1 \le -\sqrt{\lambda},\end{aligned} \tag{3.8}$$

where K is any given number satisfying the condition $\sqrt{eE}L/2 \gg K \gg K_\perp^2$.

In the case (a), using asymptotic expansions of the WPCFs given by Eq. (B3) in Appendix B, we obtain

$$\begin{aligned}N_n^{\rm cr} &= e^{-\pi\lambda}\left[1-(1-e^{-\pi\lambda})^{1/2}\frac{\sqrt{\lambda}}{2}\left(\frac{\sin\phi_1}{|\xi_1|^3}+\frac{\sin\phi_2}{|\xi_2|^3}\right)+O(|\xi_1|^{-4})+O(|\xi_2|^{-4})\right],\\ \phi_{1,2} &= |\xi_{1,2}|^2-\lambda\ln(\sqrt{2}|\xi_{1,2}|)+\arg\Gamma(i\lambda/2)-\pi/4.\end{aligned} \tag{3.9}$$

Consequently, the quantity (3.9) is almost constant over the wide range of energy p_0 for any given λ satisfying Eq. (3.7). One finds the same leading asymptotic term for scalar particles,

$$N_n^{\rm cr} = e^{-\pi\lambda}[1 + O(|\xi_1|^{-2}) + O(|\xi_2|^{-2})]. \tag{3.10}$$

It should be noted that asymptotic forms of $N_n^{\rm cr}$ for fermions and bosons were calculated in Ref. [22] only at $p_0 = 0$. Equations (3.9) and (3.10) contain these results as a particular case if one restores the factor $2\sqrt{2}$ omitted for oscillating term in the case of fermions in Ref. [22]. When $\sqrt{eE}L \to \infty$, one obtains the well-known result for fermions and bosons in a constant uniform electric field [1,2,10],

$$N_n^{\rm cr} \to N_n^{\rm uni} = e^{-\pi\lambda}, \tag{3.11}$$

setting $K \to \infty$ in Eqs. (3.9) and (3.10), respectively.

In the range (b), using the only asymptotic expansions with respect to ξ_2 given by Eq. (B3) in Appendix B and the exact form of $f_2^{(+)}(\xi_1)$ given by Eq. (2.26), we find

$$\begin{aligned}N_n^{\rm cr} &= 4e^{-\pi\lambda/4}\left[\left(\sqrt{\xi_1^2-\lambda}+|\xi_1|\right)\sqrt{\xi_1^2-\lambda}+O(|\xi_2|^{-3})\right]^{-1}\\ &\quad\times|f_2^{(+)}(\xi_1)|^{-2}\end{aligned} \tag{3.12}$$

exactly with respect to ξ_1. The dependence on $\lambda < \xi_1^2$ of $N_n^{\rm cr}$ given by Eq. (3.12) vs the function $e^{-\pi\lambda}$ is found numerically for different $\xi_1 = -1, -2, -3, -4$ and presented in Fig. 2. One can see that the asymptotic behavior of $e^{-\pi\lambda}$, which is typical for large $|\xi_1| \gg \max\{1, \lambda\}$, appears if $\sqrt{\xi_1^2-\lambda} \gtrsim 1$. We have $N_n^{\rm cr} \to 0$ as $\sqrt{\xi_1^2-\lambda} \to 0$ in accordance with the general property (2.30). In particular, it is clear that the value $K = 2$ is sufficiently large for the case of small $\lambda \lesssim 1$. That is the reason why we have retained explicitly oscillating terms in Eq. (3.9). We see that $N_n^{\rm cr} \lesssim e^{-\pi\lambda}$ in the range (b). The same conclusion can be made for bosons.

Let us consider the total number $N^{\rm cr}$ of pairs created by the L-constant field, which is defined by Eq. (A15) in Appendix A. Calculating this number in the fermionic case, one has to sum the corresponding differential mean numbers $N_n^{\rm cr}$ over the spin projections and over the transversal momenta $\mathbf{p}_\perp$ and energy p_0. Since the $N_n^{\rm cr}$ do not depend on the spin polarization parameters σ_s, the sum over the spin projections produces only the factor $J_{(d)} = 2^{[\frac{d}{2}]-1}$. The sum over the momenta and the energy can be easily transformed into an integral in the following way:

$$N^{\rm cr} = \sum_{\mathbf{p}_\perp, p_0\in\Omega_3}\sum_\sigma N_n^{\rm cr} = \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}}\int_{\Omega_3} dp_0 d\mathbf{p}_\perp N_n^{\rm cr}, \tag{3.13}$$

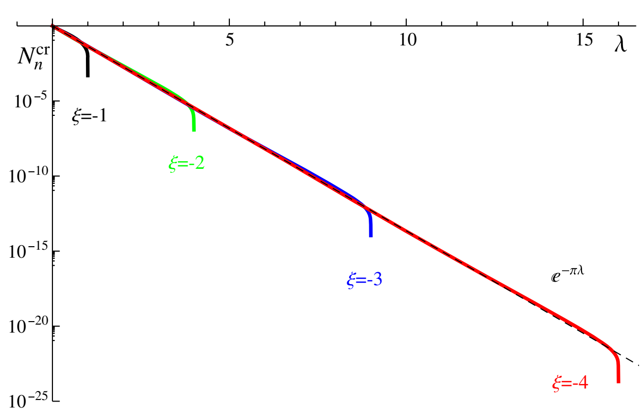

FIG. 2. The dependence on λ of $N_n^{\rm cr}$ for different $\xi = \xi_1 = -1, -2, -3, -4$.

where $V_\perp$ is the spatial volume of the $(d-1)$-dimensional hypersurface orthogonal to the electric field direction and T is the time duration of the electric field. The total number of bosonic pairs created in all possible states follows from Eq. (3.13) at $J_{(d)} = 1$. Taking into account that $N_n^{\mathrm{cr}} \lesssim e^{-\pi\lambda}$ in the range (b), one obtains a rough estimation that the contribution from the range (b) to the integral in the right-hand side of Eq. (3.13) is relatively small,

$$\int_{|\xi_1|<K} N_n^{\mathrm{cr}} dp_0 \sim e^{-\pi\lambda}\sqrt{eE}K.$$

Therefore, the main contribution to integral (3.13) is due to an inner subrange $D \subset \Omega_3$, which is defined by Eq. (3.7), and to the range (a) given by Eq. (3.8) for $p_0 \le 0$. Taking into account that N_n^{cr} is an even function of p_0, we find the complete subrange D as

$$D{:}\, \sqrt{\lambda} < K_\perp, \qquad |p_0|/\sqrt{eE} < \sqrt{eE}L/2 - K,$$
$$\sqrt{eE}L/2 \gg K \gg K_\perp^2 \gg \max\{1, m^2/eE\}. \quad (3.14)$$

In this subrange $N_n^{\mathrm{cr}} \approx e^{-\pi\lambda}$ for both fermions and bosons. Then, one can find the total number of created particles with given transversal momentum and spin polarization but with all possible energies,

$$N_{\mathbf{p}_\perp,\sigma} = \frac{T}{2\pi}\int_{\Omega_3} dp_0 N_n^{\mathrm{cr}} = \Delta e^{-\pi\lambda},$$
$$\Delta = \frac{\sqrt{eE}T}{2\pi}[\sqrt{eE}L + O(K)]. \quad (3.15)$$

The factor Δ can be interpreted as a number of quantum states with a given energy in which the particles can be created. If $\sqrt{eE}L$ is big enough, the dependence on K and $K_\perp$ can be ignored; that is, the form of N_n^{cr} is unchanged in the inner subrange D. Thus, the definition of the subrange D (3.14) can be also treated as the stabilization condition for N_n^{cr}.

Substituting Eq. (3.15) into integral (3.13), performing the integration over $\mathbf{p}_\perp$, and neglecting the exponentially small contribution from the range $\sqrt{\lambda} \to K_\perp$, we finally obtain

$$N^{\mathrm{cr}} = V_\perp T n^{\mathrm{cr}}, \qquad n^{\mathrm{cr}} = r^{\mathrm{cr}}\left[L + \frac{O(K)}{\sqrt{eE}}\right],$$
$$r^{\mathrm{cr}} = \frac{J_{(d)}(eE)^{d/2}}{(2\pi)^{d-1}}\exp\left\{-\pi\frac{m^2}{eE}\right\}. \quad (3.16)$$

Here n^{cr} presents the total number density of pairs created per unit time and unit surface orthogonal to the electric field direction on an interval of the length L. The density $r^{\mathrm{cr}} = n^{\mathrm{cr}}/L$ is known in the theory of constant uniform electric field as the pair-production rate (see the d-dimensional case in Ref. [13]). Note that unlike the model of pair creation by the T-constant field [13], where N^{cr} is a function of the time duration of the field, Eq. (3.16) represents N^{cr} as a function of the field length L. The T-constant and L-constant fields are physically distinct; only in the asymptotic case, when $T \to \infty$ and $L \to \infty$, can one consider these fields as regularizations of a constant uniform electric field given by two distinct gauge conditions on the electromagnetic potentials $A^\mu(X)$.

Using expressions for the vacuum-to-vacuum transition probability P_v [Eq. (A16) for fermions and a similar form in Ref. [11] for bosons], we find

$$P_v = \exp(-\mu N^{\mathrm{cr}}),$$
$$\mu = \sum_{j=0}^{\infty}\frac{(-1)^{(1-\kappa)j/2}}{(j+1)^{d/2}}\exp\left(-j\pi\frac{m^2}{eE}\right), \quad (3.17)$$

where $\kappa = +1$ for fermions and $\kappa = -1$ for bosons. In the asymptotic case when $T \to \infty$ and $L \to \infty$, the vacuum-to-vacuum transition probability (3.17) coincides with the result obtained for the T-constant field [13] and represents the d-dimensional analog of the well-known Schwinger formula [23].

IV. DISCUSSION

A. Length and time scales of small-gradient fields

It should be noted that the subrange $D \subset \Omega_3$, where a stabilization condition for N_n^{cr} holds, is derived for an arbitrary external field E satisfying the uniform inequality (3.6). We stress that the dimensionless parameter $\sqrt{eE}L$ plays an important role in inequality (3.6). Studying the particle creation in the L-constant field, one can see that the stabilization of N_n^{cr} (3.11) for the energies $|p_0| \ll eEL/2$ comes at sufficiently large L,[2]

$$L \gg \Delta l_0, \qquad \Delta l_0 = \Delta l_{\mathrm{st}}\max\{1, \lambda\},$$
$$\Delta l_{\mathrm{st}} = (\hbar c/eE)^{1/2}. \quad (4.1)$$

The characteristic length Δl_0 can be called the stabilization length for a given λ. We note that in the latter equation there appears a uniform length scale Δl_{st}. In addition, we can define another specific uniform length scale Δl_{st}^m,

$$\Delta l_{\mathrm{st}}^m = \Delta l_{\mathrm{st}}\max\{1, c^3m^2/\hbar eE\}, \quad (4.2)$$

which appears under the stabilization condition. The length scale Δl_{st}^m plays the role of the characteristic length to form the distribution (3.11) for all λ from the subrange D. The latter distribution is typical for a uniform $(L \to \infty)$ electric field. Note that the exponential form (3.11) plays the role of a cutoff factor over $p_\perp$ such that only contributions from

[2] We have restored $\hbar$ and c here for convenience of the reader.

relatively small transversal momenta, $p_\perp^2/eE \lesssim \max\{1, m^2/eE\}$, are essential. The length $\Delta l_{\rm st}^m$ can be called the uniform stabilization length. Note that the characteristic value m^2/eE can be represented as the ratio of two characteristic lengths, $c^3 m^2/\hbar eE = (\Delta l_{\rm st}/\Lambda_C)^2$, where $\Lambda_C = \hbar/mc$ is the Compton wavelength. We are primarily interested in strong electric fields, $(\Delta l_{\rm st}/\Lambda_C)^2 \lesssim 1$. In this case, inequality (3.6) is simplified to the form $L/\Delta l_{\rm st} \gg 1$, in which the Compton wavelength is absent. We see that the scale $\Delta l_{\rm st}$ plays the role of the uniform stabilization length for a strong electric field. This means that such a strong field has a macroscopic length for the problem under consideration, and that $\Delta l_{\rm st}$ is the characteristic length that differentiates between fields that have microscopical or macroscopic inhomogeneity; i.e., it plays the role that the Compton wavelength plays in the case of a weak field.

We assume that our constant external electric field exists during a macroscopically large time period T, which means that T is sufficiently large compared to the time scale $\Delta t_{\rm st}^m = \Delta l_{\rm st}^m/c$, $\Delta t_{\rm st}^m \ll T$. On the other hand, the pair creation is a transient process and the applicability of the constant field approximation is limited by the smallness of the backreaction. For example, in $d = 3 + 1$ and $L/\Delta l_{\rm st} \gg 1$, depletion of an electric field due to the backreaction implies a restriction

$$1 \ll (cT/\Delta l_{\rm st})^2 \ll \frac{\pi^2}{J\alpha} \exp\left(\pi \frac{c^3 m^2}{\hbar eE}\right) \tag{4.3}$$

on time T for a given electric field strength. Here α is the fine structure constant and J is the number of the spin degrees of freedom ($J = 1$ for scalar particles, $J = 2$ for spin-1/2 particles, and $J = 3$ for vector particles), see [15]. Thus, there is a window in the parameter range of E and T where the approximation of the constant external field is consistent. For QCD with a constant $SU(3)$ chromoelectric field E^a ($a = 1, \ldots, 8$), during the period when the produced partons can be treated as weakly coupled (due to the property of asymptotic freedom in QCD), and at low temperatures $\theta \ll q\sqrt{C_1}T$, the consistency restriction for the dimensionless parameter $q\sqrt{C_1}T^2$ has the form

$$1 \ll q\sqrt{C_1}T^2 \ll \pi^2/3q^2, \tag{4.4}$$

where q is the coupling constant and $C_1 = E^a E^a$ is a Casimir invariant for $SU(3)$.

In order to estimate the boundary effects of quasiuniform electric fields, one can study another example of a small-gradient field. We take as our example the Sauter field [9],

$$E(x) = E\cosh^{-2}(x/L_{\rm S}), \qquad L_{\rm S} > 0, \tag{4.5}$$

which can be considered as another regularization for the constant uniform electric field when $eEL_{\rm S}^2 \gg 1$. It can be shown (see, e.g., Ref. [11]) that in the wide range of energies where $|p_0| \ll eEL_{\rm S}$ and

$$L_{\rm S} \gg \Delta l_{\rm f}, \qquad \Delta l_{\rm f} = \Delta l_{\rm st} \max\{1, \sqrt{\lambda}\},$$

the numbers $N_n^{\rm cr}$ do not, in fact, depend on the parameter $L_{\rm S}$; they have the form (3.11), which coincides with the differential number of created particles in a uniform electric field [2,10]. For large $L_{\rm S}$ the Sauter field varies slowly and nearly coincides with the uniform field on the distance $|x| < L_{\rm S}$. Then, $\Delta l_{\rm f}$ is a characteristic length of the stabilization for a given λ in this field. For any given $\lambda > 1$ the stabilization length of the Sauter field, $\Delta l_{\rm f}$, is less than the stabilization length of the L-constant field, Δl_0. In the case of a strong field when there exists $\lambda < 1$, both stabilization lengths have the same value, $\Delta l_{\rm st}$. We conclude that the stabilization process for $\lambda > 1$ depends on the boundary effects. In addition, one can see that a smooth asymptotic decrease of the Sauter field at $|x| \gtrsim L_{\rm S}$ affects the quantum system less than a sharp disappearance of the L-constant field at $|x| = L/2$. Thus, a stabilization length for a given $\lambda > 1$ is not a universal characteristic; it depends on the field form. Since the Sauter field for $\lambda > 1$ is nearly uniform on the interval $\sim\Delta l_{\rm f}$ and is strong enough for the stabilization there, one can interpret this interval as a universal length of pair formation, which does not depend on the field behavior in sufficiently remote regions. One can extrapolate this interpretation of $\Delta l_{\rm f}$ for any field. A semiclassical consideration is in agreement with this interpretation. Thus, the uniform electric field produces a work $eE\Delta l_{\rm f} = \pi_\perp$ acting on a charge e on the distance $\Delta l_{\rm f}$, such that a virtual electron-positron pair obtains the energy $2\pi_\perp$; this is the sum of the kinetic and the rest energy, and can be materialized.

One may ask the question: For which maximal potential difference some of total effects of particle creation are the same by the Sauter field and by the L-constant field? The potential differences are $2eEL_{\rm S}$ and eEL, respectively, for these fields. For example, by comparing the number of states with the given energy in which particles can be created, one sees that at any finite λ the effect of the Sauter field at large $L_{\rm S}$ [11] is equivalent to the effect of the L-constant field at the large L, given by Eq. (3.15), with the identification $L_{\rm S} = \sqrt{\lambda}L$. Performing the integration over $\mathbf{p}_\perp$, one can compare the results for the total number of created particles $N^{\rm cr}$ for both cases and can see the equivalence of these total numbers with the identification $L_{\rm S} = L/\delta$, where

$$\delta = \sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; \pi\frac{m^2}{eE}\right)$$

and $\Psi(a, b; x)$ is the confluent hypergeometric function [34]. If the field is weak (i.e., $m^2/eE \gg 1$), using the asymptotic expression for the Ψ function, one obtains that

$\delta \approx \sqrt{eE}/m$. This can be treated as the above identification, $L_S = \sqrt{\lambda}L$ at $p_\perp \to 0$, and means that only small $p_\perp \to 0$ are essential. In the case of a very strong field (i.e., $m^2/eE \ll 1$), one obtains from the Ref. [34] that the leading term for δ does not depend on the parameter m^2/eE,

$$\delta \approx \sqrt{\pi}\Gamma(d/2)/\Gamma(d/2 + 1/2). \tag{4.6}$$

For example, $\delta \approx \pi/2$ if $d = 3$ and $\delta \approx 4/3$ if $d = 4$.

Another total quantity is the vacuum-to-vacuum transition probability P_v. The probability P_v obtained for the Sauter field reads [11]

$$\begin{aligned} P_v &= \exp(-\mu^S N^{cr}), \\ \mu^S &= \sum_{j=0}^{\infty} \frac{(-1)^{(1-\kappa)j/2}\epsilon_{j+1}}{(j+1)^{d/2}} \exp\left(-j\pi \frac{m^2}{eE}\right), \\ \epsilon_j &= \delta^{-1}\sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; j\pi\frac{m^2}{eE}\right). \end{aligned} \tag{4.7}$$

Comparing the probability (4.7) and (3.17), obtained for the L-constant field, one can establish a relation between parameters L_S and L. If the field is weak ($m^2/eE \gg 1$), then $\epsilon_j \approx j^{-1/2}$ and $\mu^S \approx \mu \approx 1$, and the identification $L_S = L/\delta \approx Lm/\sqrt{eE}$ is the same as the one extracted from the comparison of total numbers N^{cr}. In the case of a strong field, all the terms with different ϵ_j contribute significantly to the sum in Eq. (4.7) if $j\pi m^2/eE \sim 1$, and the expression for P_v in (4.7) differs essentially from the one in (3.17). However, for the very strong field if $j\pi m^2/eE \ll 1$, the leading contribution of ϵ_j has a quite simple form, $\epsilon_j \approx 1$. In this case

$$\mu^S \approx \mu \approx \sum_{j=0}^{\infty} \frac{(-1)^{(1-\kappa)j/2}}{(j+1)^{d/2}},$$

and the identification $L_S = L/\delta$ is the same as the one extracted from the comparison of the total numbers N^{cr}, where δ is given by Eq. (4.6).

It should be noted that total contributions to vacuum mean values, e.g., to the mean electric current and the mean energy-momentum tensor, are usually of interest in small-gradient fields. These quantum effects are proportional to corresponding sums of differential numbers of created particles, e.g., to the number of particles with given transversal momentum and to the total number of created particles. Consequently, it is useful to derive a relation between these total numbers and parameters L_S and L. Such a relation derived from the vacuum-to-vacuum transition probability P_v is interesting for semiclassical approaches based on Schwinger's technique [23]. For the weak field, $m^2/eE \gg 1$, the identification $L_S \approx Lm/\sqrt{eE}$ follows from the comparison of N^{cr} and P_v in L-constant and Sauter fields.

The above consideration allows us to conclude that for both Sauter and L-constant fields, the differential and total effects of pair creation in sufficiently large regions (where the fields are nearly uniform) and for finite energies of particles are not signicantly affected by the field behavior far away from the region. Extrapolating these results, one may believe that in any quasiuniform electric field $\approx E$ on the region $L \gg \Delta l_{st}^m$, that electric field that vanishes out the region, particle-creation effects must not depend on the details of the switching off. Therefore, calculations in an L-constant field are quite representative for a large class of small-gradient electric fields.

It is well known that at certain conditions (the so-called charge neutrality point), electronic excitations in a graphene monolayer behave as relativistic Dirac massless fermions in $2+1$ dimensions, with the Fermi velocity $v_F \simeq 10^6$ m/s playing the role of the speed of light in relativistic particle dynamics; see details in recent reviews [29,30]. Then, in the range of applicability of the Dirac model to the graphene physics, any electric field is strong. There appears a length scale specific to graphene (and to similar nanostructures with the Dirac fermions),

$$\Delta l_{st}^g = (\hbar v_F/eE)^{1/2}, \tag{4.8}$$

which plays the role of the stabilization length. The generation of a mass gap in the graphene band structure is an important fundamental and practical problem under current research; see, e.g., the recent report [35] on the fabrication of a large band gap, 0.5 eV, in epitaxially grown graphene samples. In the presence of the mass gap $\Delta\varepsilon = mv_F^2$, the stabilization condition has general form (3.6) that involves a length scale $\Delta l_{st}^{gm} = \Delta l_{st}^g \max\{1, (\Delta\varepsilon)^2/\hbar v_F eE\}$. In this case the strong field condition reads $(\Delta\varepsilon)^2/v_F\hbar eE \ll 1$. It is shown in [36,37] that the time scale $\Delta t_{st}^g = \Delta l_{st}^g/v_F$ appears, for the tight-binding model, as the time scale when the perturbation theory with respect to electric field breaks down ($\Delta t_{st}^g \gg t_\gamma$, where the microscopic time scale is $t_\gamma = \hbar/\gamma \simeq 0.24$ fs, with $\gamma = 2.7$ eV being the hopping energy), and the dc response changes from the linear-in-E duration-independent regime to a nonlinear-in-E and duration-dependent regime. The length between two electrodes L is less than the length of a graphene flake L^g, $L < L^g$. In the experimental situation described in Ref. [31], a constant voltage between two electrodes connected to the graphene was applied, and current-voltage characteristics $(I - V)$ are measured within exposition time $T_{ex} \sim 1$ s, which is a very large time scale compared with the ballistic flight time $T_{bal} = L^g/v_F$ (the time that the electron spends to cross the length L^g). The time dimension T is macroscopically large, $\Delta t_{st}^g \ll T$, and less than the time T_{ex}. The external constant electric field can be considered as a good approximation of the effective mean field as long as the field produced by the induced current of created particles is negligible compared to the applied field. This gives the consistency restriction $T \ll \Delta t_{\rm br} = \Delta t^g_{\rm st}\pi/4\alpha$ [16], where α is the fine structure constant. Thus, there is a window in the parameter range of E and T where the model with constant external field is consistent,

$$\Delta t^g_{\rm st} \ll T \ll \Delta t_{\rm br}. \tag{4.9}$$

For example, let us assume that $T = T_{\rm bal}$. In typical experiments, $L^g \sim 1\ \mu$m, so that $T_{\rm bal} \sim 10^{-12}$ s. Then, we obtain from Eq. (4.9) the following restrictions on the external electric field:

$$7 \times 10^2\ \mathrm{V/m} \ll E \ll 8 \times 10^6\ \mathrm{V/m}.$$

Since the voltage is $V = EL$, and assuming that $L \approx L^g$, one finds the inequalities

$$7 \times 10^{-4}\ \mathrm{V} \ll V \ll 8\ \mathrm{V}.$$

These voltages are in the range typically used in experiments with graphene. In this electric field, we find that the range of length scale $\Delta l^g_{\rm st}$ satisfies the inequalities

$$0.01\ \mu\mathrm{m} \ll \Delta l^g_{\rm st} \ll 1\ \mu\mathrm{m}.$$

This shows that QED with an L-constant field is a good model to describe the quantum effects in graphene placed in a constant external electric field.

B. Connection between the vacuum instability in external electromagnetic and gravitational fields

To gain insight into universal features of particle creation from the vacuum, it is useful to compare effects caused by external fields of a different nature. The situation with a uniform electric field confined between two capacitor plates has many similarities with both the chromoelectric flux tube and the de Sitter case; see, e.g., Refs. [5,17,26,27] for reviews. The idea that particle creation by an electric field has similarities with the particle emission from black holes calculated by Hawking [38] was tested for the first time by Frolov and Gitman in Refs. [39]. Since the Hawking radiation was considered to be a component of created particles (particles out of the horizon), the authors of the latter works derived for their comparison a reduced density matrix of electrons created by a quasiconstant electric field. Using the equivalence principle, they obtained an almost Hawking distribution (up to a factor of 2 away from the Hawking temperature). Then, taking vacuum polarization effects into account, we showed that the distribution of electrons created by a slowly varying uniform electric field —in particular, by the T-constant field—can be written in a general Hawking-like thermal form, in which the Hawking temperature is reproduced exactly [13]. One can establish a similar connection for the case of strong electrostatic inhomogeneous fields with a small gradient. To do this, we will use the model of the L-constant field.

Note that the T-constant and L-constant electric fields produce the same quantum effects (coinciding with ones caused by a constant uniform electric field) in the limiting case, $T \to \infty$ and $L \to \infty$, if these limits exist (see discussions of the applicability of the model of a constant uniform electric field in Refs. [14–16]). However, the T-constant and L-constant fields describe different physical situations in the general case. This is the reason why we cannot follow the method used in Ref. [13] to study the consequences of the equivalence principle.

The phenomenon of particle emission from black holes was first considered by Hawking [38], who calculated the mean numbers N_n of particles created by static gravitational field of a black hole in a specific thermal environment,

$$N_n = \left\{\exp\left[2\pi\frac{\omega}{g_{(H)}}\right] + \kappa\right\}^{-1}. \tag{4.10}$$

Here ω is the energy of a created particle, which we suppose to be dependent on quantum numbers n, $g_{(H)} = GM/r_g^2$, where r_g is the gravitational radius of mass M, so that $g_{(H)}$ is free-falling acceleration at this radius. This spectrum was interpreted as a Planck distribution with the temperature $\theta_{(H)} = g_{(H)}(2\pi k_B)^{-1}$ (k_B is the Boltzmann constant). As before, $\kappa = +1$ for fermions and $\kappa = -1$ for bosons. It is also known [40] that an observer moving with a constant acceleration $g_{(R)}$ (with respect to its proper time) will register some particles (called Rindler particles) in the Minkowski vacuum. The mean numbers of Rindler bosons have the same Planck form as (4.10) (with $\kappa = -1$), where one has to replace $g_{(H)}$ by $g_{(R)}$, so that the corresponding temperature is $\theta_{(R)} = g_{(R)}(2\pi k_B)^{-1}$. One can find many other examples when particle creation in external gravitation fields (and due to a nontrivial topology) can be described by means of an effective temperature; see Refs. [41–44] for reviews.

The distribution (3.11) obtained for the L-constant field does not have a thermal form at first blush. Nevertheless, in the framework of a semiclassical consideration, and using some results of the present paper, one can find a close connection to a thermal-like form. As established, both electrons and positrons with given transversal momentum $\mathbf{p}_\perp$ and zero longitudinal momentum are created in a subregion of the region $S_{\rm int}$ with the kinetic and rest energy $\pi_\perp$ per particle. At the same time, the created electrons and positrons are accelerated by the electric field along the x axis to the left and to the right, respectively. Note that in the subrange $D \subset \Omega_3$, where a stabilization condition for $N^{\rm cr}_n$ holds, the width $\Delta l_{\rm f}$ of the pair formation subregion is small compared to the distance L. Finally, the particles appear on the left and the right of the step already having ultrarelativistic velocities and longitudinal momenta $|p^{\rm L}|$ and $|p^{\rm R}|$, respectively, given by Eq. (2.11). Using classical equations of motion in a constant uniform electric field $d\mathbf{P}/dt = e\mathbf{E}$, one finds final accelerations for both kinds of particles as

$$g(\mathrm{L/R}) = eE/|\pi_0(\mathrm{L/R})| \approx 2/L, \qquad (4.11)$$

respectively.

We can improve the classical consideration by taking into account properties of the physical vacuum in the L-constant field. Within the general construction [11], it is assumed that electrons and positrons in one of corresponding asymptotic regions, $S_{\rm L}$ and $S_{\rm R}$, occupy quasistationary states; i.e., they are described by wave packets that maintain their form on a sufficiently large distance in one of the corresponding asymptotic regions. Such wave packets are superpositions of the corresponding plane waves from some subrange of energies $p_0 \subset D$. In the semiclassical approximation, we assume that the energy of a particle $\bar{p}_0$ is an average value of these energies, $p_0 \subset D$. Then, the total energy of a pair created with a given $\bar{p}_0$, $\mathbf{p}_\perp$ is a sum $\bar{\pi}_0(\mathrm{L}) + |\bar{\pi}_0(\mathrm{R})|$, where $\bar{\pi}_0(\mathrm{L/R}) = \bar{p}_0 - U_{\mathrm{L/R}}$; i.e., this energy is equal to the total field work $\bar{\pi}_0(\mathrm{L}) + |\bar{\pi}_0(\mathrm{R})| = \mathbb{U}$. We note that this field work was partially used for the pair creation and partially for their further acceleration, such that before leaving the region $S_{\rm int}$ they have gained the following average longitudinal momenta:

$$|\bar{p}^{\mathrm{L/R}}| = \sqrt{[\bar{\pi}_0(\mathrm{L/R})]^2 - \pi_\perp^2}.$$

At the same time, the corresponding part of energy was lost for the field restricted in the region $S_{\rm int}$. Let us estimate this energy considering wave packets ${}^{+}\psi_{x_{\rm F}}(X)$ and ${}_{+}\psi_{x_{\rm F}}(X)$, which consist of partial waves of an out-electron ${}^{+}\psi_n(X)$ and an out-positron ${}_{+}\psi_n(X)$, and which have focal planes $x = x_{\rm F}$ somewhere in $S_{\rm L}$ and $S_{\rm R}$, respectively. One can construct such wave packets using a procedure described in Appendix D of Ref. [11]. The energy flux of the field $\psi_{x_{\rm F}}$ through a surface $x = x^{\rm out}$ is defined as

$$F(x) = \int_{x=x^{\rm out}} \psi^\dagger_{x_{\rm F}}(X)\hat{p}_x\psi_{x_{\rm F}}(X)d\mathbf{r}_\perp.$$

Then, through any plane $x = x^{\rm out}_{\rm R} \in S_{\rm R}$ a positron carries away the energy $|\bar{p}^{\rm R}|$ for the time T, whereas through any plane $x = x^{\rm out}_{\rm L} \in S_{\rm L}$ an electron carries away the energy $|\bar{p}^{\rm L}|$ for the time T. Thus, the field work spent for a pair acceleration is $|\bar{p}^{\rm L}| + |\bar{p}^{\rm R}|$. Consequently, the work spent for the creation of a pair in a given state is

$$2\omega = U - |\bar{p}^{\rm L}| - |\bar{p}^{\rm R}| \approx \lambda g, \qquad g = [g(\mathrm{L}) + g(\mathrm{R})]/2, \qquad (4.12)$$

where ω is the work spent for the creation of a particle from a pair, λ is given by Eq. (2.22), and $g(\mathrm{L/R})$ are given by Eq. (4.11).

Then the distribution (3.11) can be rewritten in the following form:

$$N_n^{\rm uni} = \exp\left\{-2\pi\frac{\omega}{g}\right\}. \qquad (4.13)$$

The energy of a particle in the Hawking formula (4.10) can be treated as the work, ω, that a gravitational field has spent for the creation of a particle. Then, Eq. (4.13) is, in fact, the Boltzmann distribution with the temperature $\theta = g(2\pi k_B)^{-1}$ having literally the Hawking form. Thus, once again we see that the distributions of particles created by electromagnetic and gravitational fields have similar thermal structures.

It is a direct consequence of the equivalence principle that the effective temperature θ of distribution (4.13) has literally the Hawking form. Regarding the distinction between the Planck and the Boltzmann distributions, we believe that the Planck distribution for the Hawking case necessarily arises due to the appearance of an event horizon (there is a boundary of the domain of the Hamiltonian); that is, it is due to the condition for which the space domains of particle and antiparticle vacua are not the same. In contrast to this, in an electric field, we deal with both the particle vacuum and the antiparticle vacuum defined over the entire space, see Ref. [11]; that is why these space domains coincide.

V. CONCLUSION

In this Conclusion we would like to try to characterize the place of the present article among the numerous works devoted to the effect of pair creation from a vacuum by an external electromagnetic field. First, we recall that in the area under discussion the possibility of obtaining any nonperturbative result is, as a rule, based on the existence of special exact solutions of the Dirac equation with external electromagnetic fields that are able to create pairs from the vacuum. However, it is well known that there exist few such field configurations and corresponding exact solutions. We believe that, among these, the T-constant and the L-constant electric fields with sufficiently large parameters T and L have *a priori*ty, because studying the effect in such relatively simple field configurations allows one to understand typical physical characteristics of the effect in wide classes of external fields. The study of a pair creation effect in the T-constant electric fields already has a long story, which we cited in the Introduction. Here almost all local and global characteristics of the effect were calculated in detail using the well-developed formulation of QED with t-electric potential steps. A similar study for L-constant electric fields did not exist until the present, as a formulation of QED with x-electric potential steps, which is sufficient for this problem, was developed by us a short time ago in Ref. [11]. Our present article, then, contains for the first time a consistent QED treatment of the pair creation effect in the L-constant electric field, a treatment that is free from misunderstandings of a naive one-particle consideration. Moreover, the presented sets of stationary solutions, as well as their interpretation, probabilities, vacuum mean values, and analysis of length and time scales, constitute a possible basis for future research in strong-field QFT of small-gradient fields.

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APPENDIX A: BASIC ELEMENTS OF QED WITH x-ELECTRIC CRITICAL POTENTIAL STEPS

In this appendix, we briefly present some basic constructions of QED with an L-constant electric field that follow from the general formulation of QED with x-electric potential step [11] and the results of Sect. II.

Solutions of the Dirac equation, ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, can be subjected to the following orthonormality conditions on the $x=\mathrm{const}$ hyperplane:

$$\left({}_\zeta\psi_n, {}_{\zeta'}\psi_{n'}\right)_x = \zeta\eta_{\mathrm{L}}\delta_{\zeta,\zeta'}\delta_{n,n'}, \qquad \eta_{\mathrm{L}} = \mathrm{sgn}\pi_0(\mathrm{L}),$$
$$\left({}^\zeta\psi_n, {}^{\zeta'}\psi_{n'}\right)_x = \zeta\eta_{\mathrm{R}}\delta_{\zeta,\zeta'}\delta_{n,n'}, \qquad \eta_{\mathrm{R}} = \mathrm{sgn}\pi_0(\mathrm{R}),$$
$$(\psi,\psi')_x = \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X)dtd\mathbf{r}_\perp. \tag{A1}$$

We consider our theory in a large spacetime box that has a spatial volume $V_\perp = \prod_{j=2}^D K_j$ and the time dimension T, where all K_j and T are macroscopically large. The integration over the transverse coordinates is fulfilled from $-K_j/2$ to $+K_j/2$, and over the time t from $-T/2$ to $+T/2$. The limits $K_j\to\infty$ and $T\to\infty$ are assumed in final expressions.

The time-independent inner product for any pair of solutions of the Dirac equation, $\psi_n(X)$ and $\psi'_{n'}(X)$, is defined on the $t=\mathrm{const}$ hyperplane as follows:

$$(\psi_n,\psi'_{n'}) = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}} \psi_n^\dagger(X)\psi'_{n'}(X)dx, \tag{A2}$$

where the improper integral over x in the right-hand side of Eq. (A2) is reduced to its special principal value to provide a certain additional property important for us, and the limits $K^{(\mathrm{L/R})}\to\infty$ are assumed in final expressions. The following orthonormality relations are on the $t=\mathrm{const}$ hyperplane:

$$\left({}_\zeta\psi_n, {}_\zeta\psi_{n'}\right) = \left({}^\zeta\psi_n, {}^\zeta\psi_{n'}\right) = \delta_{n,n'}\mathcal{M}_n, \qquad n\in\Omega_1\cup\Omega_3\cup\Omega_5,$$
$$(\psi_n,\psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, \qquad n\in\Omega_2\cup\Omega_4,$$
$$\mathcal{M}_n = 2\frac{K^{(\mathrm{R})}}{T}\left|\frac{\pi_0(\mathrm{R})}{p^{\mathrm{R}}}\right||g({}_+|{}^+)|^2, \qquad n\in\Omega_1\cup\Omega_5,$$
$$\mathcal{M}_n = 2\frac{K^{(\mathrm{R})}}{T}\left|\frac{\pi_0(\mathrm{R})}{p^{\mathrm{R}}}\right||g({}_+|{}^-)|^2, \qquad n\in\Omega_3,$$
$$\mathcal{M}_n = 2\frac{K^{(\mathrm{L})}}{T}\left|\frac{\pi_0(\mathrm{L})}{p^{\mathrm{L}}}\right|, \qquad n\in\Omega_2, \qquad \mathcal{M}_n = 2\frac{K^{(\mathrm{R})}}{T}\left|\frac{\pi_0(\mathrm{R})}{p^{\mathrm{R}}}\right|, \qquad n\in\Omega_4. \tag{A3}$$

All the wave functions having different quantum numbers n are orthogonal, and

$$\left({}_\zeta\psi_n, {}^{-\zeta}\psi_n\right) = 0, \qquad n\in\Omega_1\cup\Omega_5, \qquad {}_\zeta\psi_n \text{ and } {}^{-\zeta}\psi_n \text{ independent},$$
$$\left({}_\zeta\psi_n, {}^{\zeta}\psi_n\right) = 0, \qquad n\in\Omega_3, \qquad {}_\zeta\psi_n \text{ and } {}^{\zeta}\psi_n \text{ independent}. \tag{A4}$$

We denote the corresponding quantum numbers by n_k, so that $n_k\in\Omega_k$. Then we identify

$$\text{in solutions: } {}_+\psi_{n_1}, {}^-\psi_{n_1}; \qquad {}_-\psi_{n_5}, {}^+\psi_{n_5}; \qquad {}_-\psi_{n_3}, {}^-\psi_{n_3},$$
$$\text{out solutions: } {}_-\psi_{n_1}, {}^+\psi_{n_1}; \qquad {}_+\psi_{n_5}, {}^-\psi_{n_5}; \qquad {}_+\psi_{n_3}, {}^+\psi_{n_3}. \tag{A5}$$

We decompose the Heisenberg operator $\hat\Psi(X)$ in two sets of solutions $\{{}_\zeta\psi_n(X)\}$ and $\{{}^\zeta\psi_n(X)\}$ of the Dirac equation (2.4) complete on the $t=\mathrm{const}$ hyperplane. Operator-valued coefficients in such decompositions do not depend on coordinates. Our division of the quantum numbers n in five ranges implies the representation for $\hat\Psi(X)$ as a sum of five operators $\hat\Psi_i(X)$, $i=1,2,3,4,5$,

$$\hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X). \tag{A6}$$

For each of three operators $\hat{\Psi}_i(X)$, $i = 1, 3, 5$, there exist two possible decompositions according to the existence of two different complete sets of solutions with the same quantum numbers n in the ranges Ω_1, Ω_3, and Ω_5,

$$\begin{aligned}
\hat{\Psi}_1(X) &= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2} [{}_{+}a_{n_1}(\mathrm{in}){}_{+}\psi_{n_1}(X) + {}^{-}a_{n_1}(\mathrm{in}){}^{-}\psi_{n_1}(X)] \\
&= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2} [{}^{+}a_{n_1}(\mathrm{out}){}^{+}\psi_{n_1}(X) + {}_{-}a_{n_1}(\mathrm{out}){}_{-}\psi_{n_1}(X)], \\
\hat{\Psi}_3(X) &= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2} [{}^{-}a_{n_3}(\mathrm{in}){}^{-}\psi_{n_3}(X) + {}_{-}b^{\dagger}_{n_3}(\mathrm{in}){}_{-}\psi_{n_3}(X)] \\
&= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2} [{}^{+}a_{n_3}(\mathrm{out}){}^{+}\psi_{n_3}(X) + {}_{+}b^{\dagger}_{n_3}(\mathrm{out}){}_{+}\psi_{n_3}(X)], \\
\hat{\Psi}_5(X) &= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} [{}^{+}b^{\dagger}_{n_5}(\mathrm{in}){}^{+}\psi_{n_5}(X) + {}_{-}b^{\dagger}_{n_5}(\mathrm{in}){}_{-}\psi_{n_5}(X)] \\
&= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} [{}_{+}b^{\dagger}_{n_5}(\mathrm{out}){}_{+}\psi_{n_5}(X) + {}^{-}b^{\dagger}_{n_5}(\mathrm{out}){}^{-}\psi_{n_5}(X)].
\end{aligned} \tag{A7}$$

There may exist only one complete set of solutions with the same quantum numbers n_2 and n_4. Therefore, we have only one possible decomposition for each of the two operators $\hat{\Psi}_i(X)$, $i = 2, 4$,

$$\begin{aligned}
\hat{\Psi}_2(X) &= \sum_{n_2} \mathcal{M}_{n_2}^{-1/2} a_{n_2}\psi_{n_2}(X), \\
\hat{\Psi}_4(X) &= \sum_{n_4} \mathcal{M}_{n_4}^{-1/2} b^{\dagger}_{n_4}\psi_{n_4}(X).
\end{aligned} \tag{A8}$$

We interpret all a and b as annihilation and all $a^{\dagger}$ and $b^{\dagger}$ as creation operators. All a and $a^{\dagger}$ are interpreted as describing electrons and all b and $b^{\dagger}$ as describing positrons. All the operators labeled by the argument in are interpreted as in-operators, whereas all the operators labeled by the argument out as out-operators. This identification is confirmed by a detailed mathematical and physical analysis of solutions of the Dirac equation, with subsequent QFT analysis of correctness of such an identification, in Ref. [11].

Taking into account the orthogonality and orthonormalization relations, we find that the standard anticommutation relations for the Heisenberg operator (A6) yield the standard anticommutation rules for the introduced creation and annihilation in- or out-operators. Note that commutation relations between sets of in- and out-operators follow from the linear canonical transformation that relates in- and out-operators.

We define two vacuum vectors $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$, one of which is the zero-vector for all in-annihilation operators and the other is zero-vector for all out-annihilation operators. Besides, both vacua are zero-vectors for the annihilation operators a_{n_2} and b_{n_4}. Thus, we have

$$\begin{aligned}
{}_{+}a_{n_1}(\mathrm{in})|0,\mathrm{in}\rangle &= {}^{-}a_{n_1}(\mathrm{in})|0,\mathrm{in}\rangle = 0, \\
{}_{-}b_{n_5}(\mathrm{in})|0,\mathrm{in}\rangle &= {}^{+}b_{n_5}(\mathrm{in})|0,\mathrm{in}\rangle = 0, \\
{}^{-}a_{n_3}(\mathrm{in})|0,\mathrm{in}\rangle &= {}_{-}b_{n_3}(\mathrm{in})|0,\mathrm{in}\rangle = 0, \\
a_{n_2}|0,\mathrm{in}\rangle &= b_{n_4}|0,\mathrm{in}\rangle = 0,
\end{aligned} \tag{A9}$$

and

$$\begin{aligned}
{}_{-}a_{n_1}(\mathrm{out})|0,\mathrm{out}\rangle &= {}^{+}a_{n_1}(\mathrm{out})|0,\mathrm{out}\rangle = 0, \\
{}_{+}b_{n_5}(\mathrm{out})|0,\mathrm{out}\rangle &= {}^{-}b_{n_5}(\mathrm{out})|0,\mathrm{out}\rangle = 0, \\
{}_{+}b_{n_3}(\mathrm{out})|0,\mathrm{out}\rangle &= {}^{+}a_{n_3}(\mathrm{out}|0,\mathrm{out}\rangle = 0, \\
a_{n_2}|0,\mathrm{out}\rangle &= b_{n_4}|0,\mathrm{out}\rangle = 0.
\end{aligned} \tag{A10}$$

One can verify that the introduced vacua have minimum (zero by definition) kinetic energy and zero electric charge, and all the excitations above the vacuum have positive energies. Then, we postulate that the state space of the system under consideration is the Fock space constructed, say, with the help of the vacuum $|0,\mathrm{in}\rangle$ and the corresponding creation operators. This Fock space is unitarily equivalent to the other Fock space constructed with the help of the vacuum $|0,\mathrm{out}\rangle$ and the corresponding creation operators if the total number of particles created by the external field is finite.

Because any annihilation operators with quantum numbers n_i corresponding to different i anticommute between themselves, we can represent the introduced vacua as tensor products of the corresponding vacua in the five ranges,

$$|0,\mathrm{in}\rangle = \prod_{i=1}^{5}{}^{\otimes}|0,\mathrm{in}\rangle^{(i)}, \qquad |0,\mathrm{out}\rangle = \prod_{i=1}^{5}{}^{\otimes}|0,\mathrm{out}\rangle^{(i)}, \tag{A11}$$

where the partial vacua $|0,\mathrm{in}\rangle^{(i)}$ and $|0,\mathrm{out}\rangle^{(i)}$ obey relations (A9) and (A10) for any n_i and ζ.

It follows from relations (A9) and (A10) that the partial vacua are stable in Ω_i, $i = 1, 2, 4, 5$,

$$|0,\mathrm{in}\rangle^{(i)} = |0,\mathrm{out}\rangle^{(i)}, \qquad i = 1,2,4,5. \tag{A12}$$

Then, the total vacuum-to-vacuum transition amplitude c_v is due to the partial vacuum-to-vacuum transition amplitude formed in Ω_3,

$$c_v = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle = {}^{(3)}\langle 0,\mathrm{out}|0,\mathrm{in}\rangle^{(3)}. \tag{A13}$$

The differential mean numbers of electrons and positrons from electron-positron pairs created are equal,

$$\begin{aligned} N_n^a(\mathrm{out}) &= \langle 0,\mathrm{in}|^{+}a_n^{\dagger}(\mathrm{out})^{+}a_n(\mathrm{out})|0,\mathrm{in}\rangle = |g({}_{-}|{}^{+})|^{-2}, \\ N_n^b(\mathrm{out}) &= \langle 0,\mathrm{in}|_{+}b_n^{\dagger}(\mathrm{out})_{+}b_n(\mathrm{out})|0,\mathrm{in}\rangle = |g({}_{+}|{}^{-})|^{-2}, \\ N_n^{\mathrm{cr}} &= N_n^b(\mathrm{out}) = N_n^a(\mathrm{out}), \qquad n \in \Omega_3, \end{aligned} \tag{A14}$$

and they present the number of pairs created, N_n^{cr}. The total number of pairs created from the vacuum is the sum over the range Ω_3 of the differential mean numbers N_n^{cr},

$$N = \sum_{n\in\Omega_3} N_n^{\mathrm{cr}} = \sum_{n\in\Omega_3} |g({}_{-}|{}^{+})|^{-2}. \tag{A15}$$

Considering the range Ω_3, we see that the probabilities of a particle reflection, a pair creation, and the probability for a vacuum to remain a vacuum can be expressed via differential mean numbers of created pairs N_n^{cr},

$$\begin{aligned} P(+|+)_{n',n} &= |\langle 0,\mathrm{out}|^{+}a_{n'}(\mathrm{out})^{-}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle|^2 = \delta_{n,n'}\frac{1}{1-N_n^{\mathrm{cr}}}P_v, \\ P(+-|0)_{n',n} &= |\langle 0,\mathrm{out}|^{+}a_{n'}(\mathrm{out})_{+}b_n(\mathrm{out})|0,\mathrm{in}\rangle|^2 = \delta_{n,n'}\frac{N_n^{\mathrm{cr}}}{1-N_n^{\mathrm{cr}}}P_v, \\ P_v &= |c_v|^2 = \prod_n p_v^n, \qquad p_v^n = (1-N_n^{\mathrm{cr}}). \end{aligned} \tag{A16}$$

The probabilities for a positron scattering $P(-|-)_{n,n'}$ and a pair annihilation $P(0|-+)_{n,n'}$ coincide with the expressions $P(+|+)$ and $P(+-|0)$, respectively.

APPENDIX B: ASYMPTOTIC EXPANSIONS OF WPCFS

Here we list some properties of the WPCFs used in the present work.[3]

Asymptotic expansions of WPCFs that correspond to large absolute values of the argument $|\xi|$ have the following form:

$$D_\nu[(1\pm\mathrm{i})\xi] = e^{\mp\mathrm{i}\xi^2/2}(\sqrt{2}e^{\pm\mathrm{i}\pi/4}\xi)^{\nu}\left[1\mp\mathrm{i}\frac{\nu(1-\nu)}{4\xi^2}+\ldots\right] \quad \text{if } \xi \geq K, \tag{B1}$$

where $K \gg \max\{1,|\nu|\}$. If $\xi < 0$, we have that

$$\begin{aligned} D_\nu[(1-\mathrm{i})\xi] &= e^{\mathrm{i}\pi\nu}D_\nu[(1-\mathrm{i})|\xi|] + \mathrm{i}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\mathrm{i}\pi\nu/2}D_{-\nu-1}[(1+\mathrm{i})|\xi|], \\ D_{-\nu-1}[(1+\mathrm{i})\xi] &= e^{\mathrm{i}\pi(\nu+1)}D_{-\nu-1}[(1+\mathrm{i})|\xi|] - \mathrm{i}\frac{\sqrt{2\pi}}{\Gamma(\nu+1)}e^{\mathrm{i}\pi(\nu+1)/2}D_\nu[(1-\mathrm{i})|\xi|], \end{aligned} \tag{B2}$$

[3] Note that a more detailed description of the properties of the WPCFs can be found, e.g., in Ref. [34].

where $\Gamma(z)$ is the Euler gamma function.

Let $\chi = 1$ and $\nu = -1 - \rho = \mathrm{i}\lambda/2$. Then, using Eqs. (B1) and (B2), we obtain the following expansions of the coefficients $f_k^{(+)}(\xi_l)$, given by Eqs. (2.26):

$$\begin{aligned}
f_1^{(+)}(\xi) &\approx e^{-\mathrm{i}\xi^2/2}(\sqrt{2}e^{\mathrm{i}\pi/4}\xi)^{\nu} 2\left[1 - i\frac{\nu(1-\nu)}{4\xi^2} + O(\xi^{-4})\right],\\
f_2^{(+)}(\xi) &\approx e^{\mathrm{i}\xi^2/2}(\sqrt{2}e^{-\mathrm{i}\pi/4}\xi)^{-\nu-1}\left[-\frac{i}{\xi^2} + O(\xi^{-4})\right] \quad \text{if } \xi \geq K;\\
f_1^{(+)}(\xi) &\approx -ie^{\mathrm{i}\xi^2/2}(\sqrt{2}e^{-\mathrm{i}\pi/4}|\xi|)^{-\nu-1}e^{-i\pi\nu/2}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}[2 + O(\xi^{-2})],\\
f_2^{(+)}(\xi) &\approx -e^{\mathrm{i}\xi^2/2}(\sqrt{2}e^{-\mathrm{i}\pi/4}|\xi|)^{-\nu-1}e^{-i\pi\nu}\left[2 - i\left(\frac{3}{2}\nu + \nu^2\right)\xi^{-2} + O(\xi^{-4})\right]\\
&\quad + e^{-\mathrm{i}\xi^2/2}(\sqrt{2}e^{\mathrm{i}\pi/4}|\xi|)^{\nu}e^{-i\pi\nu/2}\frac{\sqrt{2\pi}}{2\Gamma(\nu)\xi^4} \quad \text{if } \xi < 0, \qquad |\xi| \geq K.
\end{aligned} \tag{B3}$$

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