Equilibration, particle production, and self-energy

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The discontinuity, or imaginary part of a self-energy at finite temperature, is proportional to the rate at which the corresponding particles are produced when very few of them are present and also to the rate at which their phase space density approaches the thermal one. These relations were suggested by Weldon [Phys. Rev. D 28, 2007 (1983)], who demonstrated them for low orders in perturbation theory. Here we show that they are valid at leading order in a linear coupling of the produced particles, and to all orders in all other interactions of the hot plasma, if there is a separation of the time scales for the production and for the thermalization of the bulk of the plasma.

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I. INTRODUCTION

The phase space density, or occupancy $f_{\mathbf{p}}(t, \mathbf{x})$, plays a central role in statistical mechanics, both for equilibrium and nonequilibrium.[1] It is a sensible concept only for particles which are sufficiently weakly interacting. It is of little use, e.g., for quarks when the temperature is near or below the QCD-temperature $T \sim 160$ MeV. On the other hand, it can make perfect sense for neutrinos at the same temperature.

In thermal equilibrium the phase space density is given by the Bose-Einstein or Fermi-Dirac distribution. For nonequilibrium f has been considered in various situations. Two of them are of interest to us here: (i) f is close to its equilibrium value, $|f - f^{\mathrm{eq}}| \ll 1$, and (ii) f is very small, $f \ll 1$. We restrict ourselves to homogeneous systems, i.e., $\mathbf{x}$-independent f.

We consider the occupancy f of a particle species Φ, and we assume that most physical quantities equilibrate much faster than Φ. This is the case if the coupling of Φ is much weaker than the couplings of the interactions of the other degrees of freedom. There may be other quantities X_a besides f which equilibrate slowly as well. We choose them such that $X_a = 0$ in equilibrium.

In equilibrium the slow variables fluctuate around their thermal expectation values. We will be concerned with nonequilibrium states for which f is much larger than a typical thermal fluctuation. Then we may consider systems out of equilibrium for which the out-of-equilibrium state is specified by the values of f and the X_a, and of the temperature T.[2] This description is valid on time scales much larger than the equilibration time of the fast degrees of freedom. The time derivatives of f and the X_a are fully determined by f, the X_a, and T, because the state of the system is fully specified by these quantities.

First consider a system in which both f and the X_a are close to equilibrium. The interaction with the thermalized plasma will bring f closer to equilibrium. Thus, for sufficiently small $\delta f \equiv f - f^{\mathrm{eq}}$ and X_a, we must have a linear relation,

$$\dot{f}_{\mathbf{p},\lambda} = -\sum_{\mathbf{p}',\lambda'} \tilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'} \delta f_{\mathbf{p}',\lambda'} + \ldots, \tag{1.1}$$

where λ labels possible spins or helicities. Furthermore, "$+\ldots$" denotes terms linear in the other slow variables, as well as higher powers of the deviations from equilibrium. We will see that at leading order in the interaction of Φ

$$\tilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'} = \delta_{\mathbf{p}\mathbf{p}'} \delta_{\lambda\lambda'} \Gamma^{\mathrm{eq}}_{\mathbf{p},\lambda}. \tag{1.2}$$

Therefore, we can write

$$\dot{f}_{\mathbf{p},\lambda} = -\Gamma^{\mathrm{eq}}_{\mathbf{p},\lambda} \delta f_{\mathbf{p},\lambda} + \ldots. \tag{1.3}$$

Now consider an out-of-equilibrium system with very few Φ particles present, *i.e.* $f \ll 1$. The interactions of the otherwise thermalized plasma will then create Φ particles. At the same time the interactions can drive the rest of the plasma slightly away from equilibrium. Thus, in general one would have to consider the evolution not only of f but also of the X_a. However, as long as these are still sufficiently small, we may expand around $f = 0$ and $X_a = 0$, and keep only the lowest-order term in this expansion, which defines the production rate Γ^{pro} of Φ,

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[1] We consider a finite volume V, so that the momenta $\mathbf{p}$ are discrete. As usual, $V^{-1}\sum_{\mathbf{p}} \to (2\pi)^{-3}\int d^3p$ in the infinite volume limit.

[2] For simplicity, we assume that there are no chemical potentials present.

$$\dot{f} = \Gamma^{\rm pro} + \dots . \tag{1.4}$$

Now "$+\dots$" denotes terms of linear or higher order in f and X_a. The production rate depends on $|\mathbf{p}|$ and the parameter which characterizes the equilibrated plasma, i.e., the temperature.

If (1.3) were also valid for very small f, where δf is *not* small, but of order one, $\delta f \simeq -f^{\rm eq}$, we would have

$$\Gamma^{\rm pro} = f^{\rm eq}\Gamma^{\rm eq}. \tag{1.5}$$

Weldon [1] has shown that this relation holds at leading order in the couplings, and also for multiparticle processes, assuming the validity of the Boltzmann equation. With the same assumptions he found that the two rates are proportional to the discontinuity of the Φ self-energy, see (2.19). In [2] it was assumed that (1.5) also holds when radiative corrections [3,4] are taken into account. However, for a proper treatment of radiative corrections one should justify (1.5) beyond leading order, where the Boltzmann equation may no longer be valid.

We write the Hamiltonian as

$$H = H_0 + H_{\rm int}, \tag{1.6}$$

where H_0 describes free Φ fields and all other fields including their interactions. The interaction of Φ with the other fields is contained in $H_{\rm int}$. The relation between $\Gamma^{\rm pro}$ and the discontinuity of the self-energy is well known to be valid at leading order in $H_{\rm int}$ and to *all* orders in all other interactions contained in H_0 [5,6]. Recently, a nonlinear evolution equation for the phase space density of sterile neutrino dark matter particles which does not use a perturbative expansion in H_0 has been obtained using an ansatz for the nonequilibrium density matrix [7]. The relation (1.5) can then be obtained by expanding the equation of [7] around small f.

In this note we show that (1.5) holds to all orders in the couplings in H_0 and to leading order in $H_{\rm int}$, if there is a separation of the time scales associated with the interactions in H_0 and those in $H_{\rm int}$, see (2.23). We use the theory of quasistationary fluctuations [8] without making any ansatz for the density matrix, and we treat both bosons and fermions. Our assumptions differ somewhat from [1], where the production of W-bosons was considered; for this process our assumption about the separation of time scales is not satisfied.

II. CHARGED PARTICLE SPECIES

Equilibration rates of slowly evolving quantities can be computed from quantum field theory by defining appropriate number density operators and matching their real time correlation functions with the corresponding one computed from the effective kinetic equations like (1.1) [8,9].

First we need an operator expression for the occupancy. We will treat $H_{\rm int}$ as a small perturbation and work at leading order in $H_{\rm int}$. In the interaction picture with respect to $H_{\rm int}$ the field operator Φ can be written as

$$\Phi_{\rm I}(x) = \sum_{\mathbf{p},\lambda} \frac{1}{\sqrt{2E_{\mathbf{p}}V}} [e^{-ipx} u_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda} + e^{ipx} v_{\mathbf{p},\lambda} d^\dagger_{\mathbf{p},\lambda}]_{p^0=E_{\mathbf{p}}}, \tag{2.1}$$

which defines annihilation operators $c_{\mathbf{p},\lambda}$, and $d_{\mathbf{p},\lambda}$ for particles and antiparticles, normalized such that

$$[c_{\mathbf{p},\lambda}, c^\dagger_{\mathbf{p}',\lambda'}]_{-\sigma} = \delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda\lambda'}, \tag{2.2}$$

and similarly for the antiparticles. Here $\sigma = +1$ for bosons and $\sigma = -1$ for fermions. The occupancy operator for particles in the interaction picture can be defined as

$$(f_{p,\lambda})_{\rm I} \equiv c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda}. \tag{2.3}$$

We will be interested in the Heisenberg picture-f which is related to (2.3) via $f = e^{iHt}e^{-iH_0 t} f_{\rm I} e^{iH_0 t} e^{-iHt}$. For noninteracting Φ the operator $f_{p,\lambda}$ would be conserved and would precisely be the occupancy of free particles. It is, however, also defined for nonvanishing $H_{\rm int}$.

Following [9] one can then compute the coefficients in (1.1) via

$$\tilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'} = \frac{1}{2V}\sum_{\mathbf{p}'',\lambda''} \lim_{\omega\to 0} \frac{\rho_{\mathbf{p}\mathbf{p}'',\lambda\lambda''}(\omega)}{\omega} (\Xi^{-1})_{\mathbf{p}''\mathbf{p}',\lambda''\lambda'} \tag{2.4}$$

with the spectral function

$$\rho_{\mathbf{p}\mathbf{p}',\lambda\lambda'}(\omega) = \int dt e^{i\omega t} \langle [\dot{f}_{\mathbf{p},\lambda}(t), \dot{f}_{\mathbf{p}',\lambda'}(0)] \rangle, \tag{2.5}$$

and the matrix

$$\Xi_{\mathbf{p}\mathbf{p}',\lambda\lambda'} \equiv \frac{1}{TV} \langle \delta f_{\mathbf{p},\lambda} \delta f_{\mathbf{p}',\lambda'} \rangle. \tag{2.6}$$

Here and below the pointy brackets denote the thermal average with Hamiltonian H_0 and vanishing chemical potentials, $\langle\cdots\rangle \equiv {\rm tr}[\cdots \exp(-H_0/T)]/{\rm tr}\exp(-H_0/T)$. The right-hand side of (2.6) is determined by the free theory which yields

$$\langle \delta f_{\mathbf{p},\lambda} \delta f_{\mathbf{p}',\lambda'} \rangle = \delta_{\lambda\lambda'}\delta_{\mathbf{p}\mathbf{p}'} f_\sigma(E_{\mathbf{p}})[1 + \sigma f_\sigma(E_{\mathbf{p}})]. \tag{2.7}$$

Here f_σ is the Bose-Einstein or Fermi-Dirac distribution for $\sigma = +1$ or $\sigma = -1$ respectively, and $E_{\mathbf{p}} = (\mathbf{p}^2 + m_\Phi^2)^{1/2}$.

To compute the time derivatives in (2.5) we need to specify the interaction of Φ. We consider a linear coupling

$$\mathcal{L}_{\rm int} = -\bar{J}\Phi - \bar{\Phi}J, \tag{2.8}$$

where the operator J can be elementary or composite and does not contain Φ. Furthermore, $\bar{\Phi} \equiv \Phi^\dagger$ for integer spin, and $\bar{\Phi} \equiv \Phi^\dagger \gamma^0$ for spin $1/2$.

In the following we will obtain a master-formula (2.15) for the coefficients $\tilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'}$ at leading order in $H_{\rm int}$. For this purpose we now determine the spectral function (2.5) to this order. The time derivatives of the occupancy operators in (2.5) are given by

$$\dot{f}_{\mathbf{p},\lambda} = i[H, f_{\mathbf{p},\lambda}]. \tag{2.9}$$

To simplify (2.9) we rewrite the right-hand side in terms of interaction picture operators, use (2.1) and (2.2) and finally express the result in terms of Heisenberg operators. At leading order in $H_{\rm int}$, this yields

$$\dot{f}_{\mathbf{p},\lambda} = \frac{i}{\sqrt{2E_p V}} \int d^3x [\bar{J}(x) e^{-ipx} u_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda} - \text{H.c.}]. \tag{2.10}$$

Since we work at leading order in $H_{\rm int}$, we neglect it in the computation of the expectation value in (2.5). Now we insert (2.10) into (2.5), use (2.2) and treat Φ as a free field, after which we obtain Wightman functions,

$$\Delta^>_{AB}(x) \equiv \langle A(x)B(0)\rangle, \qquad \Delta^<_{AB}(x) \equiv \sigma\langle B(0)A(x)\rangle, \tag{2.11}$$

of $A = \bar{u}_{\mathbf{p},\lambda} J$ and $B = A^\dagger$. Using

$$\begin{aligned} \Delta^>_{AB}(\omega) &= [1 + \sigma f_\sigma(\omega)]\tilde{\rho}_{AB}(\omega), \\ \Delta^<_{AB}(\omega) &= \sigma f_\sigma(\omega)\tilde{\rho}_{AB}(\omega) \end{aligned} \tag{2.12}$$

with the spectral function

$$\tilde{\rho}_{AB}(p) = \int d^4x e^{ipx} \langle [A(x), B(0)]_{-\sigma}\rangle, \tag{2.13}$$

we obtain after some work

$$\begin{aligned} \lim_{\omega\to 0} \frac{\rho_{\mathbf{p}\mathbf{p}',\lambda\lambda'}(\omega)}{\omega} = \frac{1}{TE_{\mathbf{p}}} \delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda\lambda'} f_\sigma(E_{\mathbf{p}})[1 \\ + \sigma f_\sigma(E_{\mathbf{p}})]\tilde{\rho}_{\bar{u}J,\bar{J}u}(E_{\mathbf{p}}, \mathbf{p}). \end{aligned} \tag{2.14}$$

Using (2.4) in combination with (2.7) and (2.14), we find that the equilibration rate indeed takes the form (1.2) with

$$\Gamma^{\rm eq}_{\mathbf{p},\lambda} = \frac{1}{2E_{\mathbf{p}}} \tilde{\rho}_{\bar{u}J,\bar{J}u}(E_{\mathbf{p}}, \mathbf{p}). \tag{2.15}$$

Like in [1] we can relate this expression to the Φ self-energy. A spectral function such as (2.13) can be obtained by starting with an imaginary time, or Euclidean correlator,

$$\Delta_{AB}(i\omega_n, \mathbf{p}) \equiv \int_0^\beta d\tau \int d^3x e^{i(\omega_n\tau - \mathbf{p}\cdot\mathbf{x})} \langle A(-i\tau, \mathbf{x})B(0)\rangle, \tag{2.16}$$

with the discrete Matsubara frequencies $\omega_n = n\pi T$ with even and odd integer n for bosons and fermions, respectively. (2.16) can be continued $i\omega_n \to p^0$ into the complex p^0 plane. The resulting function has cuts and poles on the real p^0 axis. The spectral function is then proportional to its discontinuity $\text{Disc}\Delta(p^0) \equiv \Delta(p^0 + i0^+) - \Delta(p^0 - i0^+)$ across the real axis,

$$\tilde{\rho}_{AB}(p^0, \mathbf{p}) = \frac{1}{i} \text{Disc}\Delta_{AB}(p^0, \mathbf{p}). \tag{2.17}$$

At leading order in $H_{\rm int}$ the Euclidean Φ self-energy is given by

$$\Sigma(i\omega_n, \mathbf{p}) = \int_0^\beta d\tau \int d^3x e^{i(\omega_n\tau - \mathbf{p}\cdot\mathbf{x})} \langle J(-i\tau, \mathbf{x})\bar{J}(0)\rangle. \tag{2.18}$$

Therefore,

$$\Gamma^{\rm eq}_{\mathbf{p},\lambda} = \frac{1}{2iE_{\mathbf{p}}} \bar{u}_{\mathbf{p},\lambda} \text{Disc}\Sigma(E_{\mathbf{p}}, \mathbf{p}) u_{\mathbf{p},\lambda}. \tag{2.19}$$

The same relation has been obtained by Weldon [1]: the equilibration rate of particles is proportional to the discontinuity of their self-energy. We differ from Weldon in the following two respects: (i) We have shown that this result is valid at leading order in $H_{\rm int}$ and to *all* orders in the other interactions. In particular, we have, unlike [1], not made use of the Boltzmann equation which is not valid beyond leading order. (ii) We have assumed a separation of the time scales on which Φ thermalizes and on the other thermalization times in the system. Our derivation would not be valid for the equilibration of electroweak gauge bosons, which was considered by Weldon.

The relation between the production rate (1.4) and the self-energy or the correlation functions of J, which is valid at leading order in $H_{\rm int}$ and to all orders in H_0, is well known [5,6]. It can be obtained by considering the probability $P(t)$ to find one Φ particle with momentum $\mathbf{p}$ and spin λ at time t when there was none at time $t = 0$. In the interaction picture, it can be written as

$$P(t) = |\langle f; \mathbf{p}, \lambda | U_{\rm I}(t, 0) | i\rangle|^2, \tag{2.20}$$

where i and f label states which contain no Φ particles. Now expand the time evolution operator $U_{\rm I}$ in powers of $H_{\rm int}$, and choose t small enough, $t \ll t_{\rm int}$ with some time scale $t_{\rm int}$, so that it is sufficient to keep only the term linear in $H_{\rm int}$. Using (2.8), summing over f, and thermally averaging over i turns (2.20) into

$$\langle P(t)\rangle = \frac{1}{2E_{\mathbf{p}}V}\int_0^t dt_1 \int_0^t dt_2 \int d^3x_1 \times \int d^3x_2 e^{ip(x_1-x_2)}\langle \bar{J}u(x_2)\bar{u}J(x_1)\rangle. \tag{2.21}$$

Now consider the time derivative of $\langle P(t)\rangle$.[3] Using translational invariance of the thermal average and shifting integration variables it can be written as

$$\frac{d\langle P\rangle}{dt} = \frac{1}{2E_{\mathbf{p}}}\int_{-t}^{t} dt' \int d^3x' e^{ipx'}\langle \bar{J}u(0)\bar{u}J(x')\rangle. \tag{2.22}$$

Note that (2.22) was obtained assuming $t \ll t_{\rm int}$. However, if the Wightman function on the right-hand side has a finite correlation time $t_{\rm corr}$, so that it practically vanishes for times $|t'| > t_{\rm corr}$, and if furthermore

$$t_{\rm corr} \ll t_{\rm int}, \tag{2.23}$$

we can choose t in the range $t_{\rm corr} \ll t \ll t_{\rm int}$. Then we have

$$\frac{d\langle P\rangle}{dt} = \frac{1}{2E_{\mathbf{p}}}\sigma\Delta^{<}_{\bar{u}J,\bar{J}u}(E_{\mathbf{p}},\mathbf{p}). \tag{2.24}$$

The production rate is

$$\Gamma^{\rm pro}_{\mathbf{p},\lambda} = \frac{(2\pi)^3}{Vd^3p}\frac{d\langle P\rangle}{dt}, \tag{2.25}$$

which in finite volume equals $d\langle P(t)\rangle/dt$. Therefore,

$$\Gamma^{\rm pro}_{\mathbf{p},\lambda} = \frac{\sigma}{2E_{\mathbf{p}}}\Delta^{<}_{\bar{u}J,\bar{J}u}(E_{\mathbf{p}},\mathbf{p}). \tag{2.26}$$

Recall that, since we worked at first order in $H_{\rm int}$, the thermal average in (2.21), and thus the Wightman function in (2.26) does not contain the Φ field. Combining (2.26) and (2.12) and comparison with (2.15) shows that the relation (1.5) indeed holds, again at leading order in $H_{\rm int}$ and to all orders in H_0.

III. UNCHARGED PARTICLES

Now consider particles which are their own antiparticles. Important examples could be sterile neutrinos which may be responsible for the baryon asymmetry of the Universe [10] or constitute the dark matter [11], or other interesting dark matter candidates, or photons produced in a quark-gluon plasma. In this case one can write Φ as a field which is equal to its charge conjugate,

$$\Phi = \Phi^C, \tag{3.1}$$

where $\Phi^C \equiv S(\bar{\Phi})^T$ with an appropriate matrix S. Then one can write

$$\mathcal{L}_{\rm int} = -\bar{I}\Phi = -\bar{\Phi}I. \tag{3.2}$$

In this case, we find

$$\dot{f}_{\mathbf{p},\lambda,\rm uncharged} = \frac{i}{\sqrt{2E_pV}} \times \int d^3x\bar{I}(x)[e^{-ipx}u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda} - e^{ipx}v_{\mathbf{p},\lambda}c^{\dagger}_{\mathbf{p},\lambda}]. \tag{3.3}$$

Now we insert this into (2.5) and follow the same steps as in Sec. II. We obtain correlation functions of $\bar{u}I$ and of $\bar{v}I$. Using $v = u^C$ and the fact that (3.2) is Hermitian, we obtain

$$\Gamma^{\rm eq}_{\mathbf{p},\lambda} = \frac{1}{2E_{\mathbf{p}}}\tilde{\rho}_{\bar{u}I,\bar{I}u}(E_{\mathbf{p}},\mathbf{p}). \tag{3.4}$$

The Φ self-energy is the same as (2.18) but with J replaced by I. Thus we obtain precisely the same relation (2.19) between the discontinuity of the self-energy and the equilibration rate as we did for charged particles. Also the production rate is simply obtained by replacing J by I in (2.26) which shows that (1.5) holds under the same conditions as in the charged particle case.

In the case of sterile neutrinos which are uncharged particles, the interaction is usually written in the form (2.8). In the symmetric phase of the Standard Model $J = \tilde{\varphi}^\dagger h_\nu \ell$, where $\tilde{\varphi} = i\sigma^2\varphi^*$ is the SU(2) conjugate Higgs doublet, ℓ are the (left-handed) SU(2) lepton doublets and h_ν is the Yukawa coupling matrix. In the symmetry-broken phase the dominant contribution is obtained by replacing $\tilde{\varphi}$ by its expectation value which gives $J = vh_\nu\nu/\sqrt{2}$ with $v = 246$ GeV and the neutrino field ν. Writing Φ as a Majorana spinor, $\Phi^C = \Phi$, one can also write the interaction in the form (3.2), where now $I = J + J^C$. To compute the equilibration rate, one can use either (2.8) or (3.2). In the latter case one can directly use the result (3.4), which shows that (2.19) again holds.

For practical calculations it may be more convenient to write the rates in terms of correlation functions of J. To achieve this, one may either use (3.4) and re-express I in terms of J, or directly start from the form (2.8). Using that correlators like $\langle JJ\rangle$ in the Standard Model vanish due to $B - L$ conservation, one obtains[4]

[3]We would like to thank the referee for this suggestion.

[4]It is interesting to note that the second term in the square bracket on the right-hand side would give precisely the equilibration rate for antifermions in the charged particle case (see Sec. II).

$$\Gamma^{\mathrm{eq}}_{\mathbf{p},\lambda} = \frac{1}{2E_{\mathbf{p}}}[\tilde{\rho}_{\bar{u}J,\bar{J}u}(E_{\mathbf{p}},\mathbf{p}) + \tilde{\rho}_{\bar{v}J,\bar{J}v}(-E_{\mathbf{p}},-\mathbf{p})]. \quad (3.5)$$

Averaging over spins/helicities, keeping in mind that J is left-handed, one obtains

$$\Gamma^{\mathrm{eq}}_{\mathbf{p}} = \frac{1}{4E_{\mathbf{p}}}\mathrm{Tr}\{\not{p}[\tilde{\rho}_{J\bar{J}}(E_{\mathbf{p}},\mathbf{p}) + \tilde{\rho}_{J\bar{J}}(-E_{\mathbf{p}},-\mathbf{p})]\}. \quad (3.6)$$

Combined with (1.5) this gives the result for the production rate which was obtained in [6] at leading order in H_{int} and to all orders in H_0.

IV. SUMMARY AND DISCUSSION

We have revisited Weldon's relations [1] between the equilibration rate of a particle species Φ, its thermal production rate, and the discontinuity of its self-energy. To obtain these relations we had to assume, in contrast to [1], that Φ equilibrates much more slowly than the bulk of the plasma. We find that these relations are valid to leading order in the interaction of Φ and to *all* orders in the other interactions. Unlike [1], we did not make any use of a Boltzmann equation, which is only valid at leading order. Our results imply that radiative corrections to the production rate of sterile neutrinos [3,4] can also be used for the equilibration rate, which has been implicitly assumed previously when incorporating radiative corrections into leptogenesis computations [2].

In Ref. [7], a nonlinear evolution equation for the phase space density of sterile neutrino dark matter particles has been obtained using an *ansatz* for the nonequilibrium density matrix. If this equation is expanded around $f = 0$, one also finds the relation (1.5) between equilibration and production rates. It would be interesting to see whether also higher-order terms in the nonlinear evolution equation for the phase space density of sterile neutrino dark matter particles of [7] can be obtained with the methods we used here.

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