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Spacetime curvature in terms of scalar field propagators

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We show how quantum fields can be used to measure the curvature of spacetime. In particular, we find that knowledge of the imprint that spacetime curvature leaves in the correlators of quantum fields suffices, in principle, to reconstruct the metric. We then consider the possibility that the quantum fields obey a natural ultraviolet cutoff, for example, at the Planck scale. We investigate how such a cutoff limits the spatial resolution with which curvature can be deduced from the properties of quantum fields. We find that the metric deduced from the quantum correlator exhibits a peculiar scaling behavior as the scale of the natural UV cutoff is approached.

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I. INTRODUCTION

In general relativity, spacetime measurements are traditionally based on the use of some form of standard rods and clocks. At subatomic scales, there are of course no rods or clocks in Einstein's sense, and the only available tools then are quantum fields. We will, therefore, address here the question of how the curvature of a classical spacetime can be expressed solely through in-principle measurable properties of quantum fields. The ability to express the curvature of a classical spacetime entirely in terms of quantized degrees of freedom of fields could become a useful tool in the quest to then also quantize the spacetime curvature itself; see, e.g., Refs. [1–7]. We will also take into account that quantum fields are likely subject to a natural ultraviolet cutoff at the Planck scale (for a review, see, e.g., Ref. [8]). We will study how such a cutoff limits the spatial resolution with which the spacetime metric can be deduced from in-principle measurable properties of a quantum field.

We begin by recalling that the curvature of a classical spacetime influences not only matter and radiation but also the vacuum, [9]. This is because curvature influences wave operators such as the d'Alembertian and curvature therefore also impacts the normal mode decomposition of quantum fields. This means that curvature affects the vacuum state of quantum fields, affecting, for example, the vacuum entanglement and the correspondingly correlated quantum fluctuations between different locations in spacetime [10–15].

The question that we address here is whether knowledge solely of this imprint that curvature leaves on the quantum vacuum is sufficient to be able to deduce the curvature of the spacetime. This is nontrivial, considering that, for example, knowledge of only the energy-momentum tensor of quantum fields would be insufficient because the energy-momentum tensor determines only the Ricci component but not the Weyl component of the curvature. On the other hand, it is known that spacetime curvature affects interacting quantum fields to the extent that counterterms that include the Einstein action in the leading orders are induced. Spacetime curvature therefore affects quantum fields sufficiently to induce Einsteinian dynamics [16,17].

Our first finding here is that the impact that spacetime curvature has on the statistics of the quantum fluctuations of a scalar field is in fact complete. Concretely, the knowledge of even just the spacetime-dependent propagator of a free scalar field on a curved spacetime suffices to calculate the metric on the spacetime and therefore to obtain the spacetime curvature. The propagator is part of the Feynman rules, and, in principle, in a curved spacetime, the correspondingly spacetime-dependent propagator can be inferred with suitable particle physics experiments. This then replaces standard rods and clocks.

For intuition, let us consider that propagators are correlators. This means that by considering a propagator we are considering the impact that curvature has on the spatial and temporal correlations of vacuum fluctuations of quantum fields. Why then should the correlator yield metric information? Intuitively, the reason is that the strength of the correlations of spatially and temporally separated quantum vacuum fluctuations provides a measure of spacetime distance, and knowing distances is to know the metric, as has been argued in Ref. [18].

For an alternative perspective, let us recall that knowledge of the light cones of a spacetime allows one to deduce the spacetime metric up to a local conformal factor [19]. In effect, our result is that a scalar quantum field's propagator

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does not only indicate the light cones but also the local conformal factor.

Having established a straightforward method to extract the metric from a propagator, we then consider the case where the quantum field is subject to a natural ultraviolet cutoff. In this case, it should not be possible to use quantum fields to probe the curvature at length scales that are smaller than the cutoff scale. To this end, we use a simple model for the natural ultraviolet cutoff, namely, a hard cutoff within the framework of Euclidean-signature quantum field theory. We examine how the metric that is deduced from quantum field correlators behaves as the natural UV cutoff scale is approached. We find characteristic oscillations that are generally unobservable because they are washed out by the cutoff. However, through the fluctuation amplifying effects of cosmic inflation, see, e.g., Ref. [20], such oscillations in the metric may conceivably have left a signature in the cosmic microwave background.

II. DEDUCING THE METRIC FROM THE PROPAGATOR

A. Flat space

We begin with the simple case of quantum field theory in flat Euclidean space. The aim is to determine if the metric tensor can be reconstructed from the correlator of a scalar quantum field. To this end, we recall that in D-dimensional Euclidean space the massive Green's function satisfies

$$(\nabla_x^2 - m^2)G(x,y) = -\delta^{(D)}(x-y) \tag{1}$$

and that $G(x,y)$ is given explicitly by

$$G(x,y) = \frac{(2\pi)^{-D/2}}{r_{xy}^{D-2}}(mr_{xy})^{D/2-1}K_{D/2-1}(mr_{xy}), \tag{2}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and

$$r_{xy}^2 = |x-y|^2 = \sum_{i=1}^{D}(x^i-y^i)^2. \tag{3}$$

In the massless limit, $G(x,y)$ takes the form

$$G(x,y) \xrightarrow{mr_{xy}\to 0} G_0(x,y) = \frac{\Gamma(D/2-1)}{4\pi^{D/2}r_{xy}^{D-2}}. \tag{4}$$

Since $G_0(x,y)$ depends quite simply on the distance, r_{xy}, we can easily use $G_0(x,y)$ to reconstruct the flat metric:

$$\begin{aligned}\delta_{ij} &= -\frac{1}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}r_{xy}^2 \\ &= -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}(G_0(x,y)^{\frac{2}{2-D}}).\end{aligned} \tag{5}$$

Let us now ask if a massive field's Green's function can also directly be used to recover the metric. Intuitively, one expects this to be true because mass mostly affects the infrared and should matter little when $|x-y| \ll m^{-1}$. Indeed, one can verify that, in the ultraviolet limit, $x \to y$:

$$\delta_{ij} = -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}}\lim_{x\to y}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}(G(x,y)^{\frac{2}{2-D}}). \tag{6}$$

Let us consider, for example, the case $D=3$ in which Eq. (2) simplifies to

$$G(x,y) = \frac{e^{-mr_{xy}}}{4\pi r_{xy}}. \tag{7}$$

It can be verified in this case that the rhs of Eq. (6) (without the limit) is given by

$$\begin{aligned}&-\frac{1}{2}\left[\frac{\Gamma(1/2)}{4\pi^{3/2}}\right]^2\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}(G(x,y)^{-2}) \\ &\quad = e^{2mr}\bigg[(1+mr)\delta_{ij} \\ &\qquad + \frac{m}{r}(3+mr)\delta_{ik}\delta_{jl}(x^k-y^k)(x^l-y^l)\bigg].\end{aligned} \tag{8}$$

Taking the $x \to y$ limit, we find δ_{ij}. In Appendix A, we show that Eq. (6) is true also for $D \geq 4$.

B. Curved space

Our aim now is to express the metric in terms of the correlator of quantum fluctuations of a scalar field in curved manifolds. Then, the Green's function satisfies the equation

$$(\Delta_x - m^2)G(x,y) = -\frac{\delta^{(D)}(x-y)}{\sqrt{g(x)}}, \tag{9}$$

where $\Delta_x = \frac{1}{\sqrt{g(x)}}\partial_{x^i}(\sqrt{g(x)}g^{ij}(x)\partial_{x^j})$ is the Laplace-Beltrami operator. As we will show, Eq. (6) therefore straightforwardly generalizes to curved manifolds:

$$g_{ij}(y) = -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}}\lim_{x\to y}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}(G(x,y)^{\frac{2}{2-D}}). \tag{10}$$

First, let us confirm that Eq. (10) does not depend on the coordinate system, i.e., that it represents a covariant way to express the metric in quantum terms. To this end, consider two coordinate systems x and $\tilde{x}$ and the Green's function in each coordinate $G(x,y)$ and $\tilde{G}(\tilde{x},\tilde{y})$, respectively. Since G is a biscalar, $\tilde{G}(\tilde{x},\tilde{y}) = G(x,y)$:

$$\begin{aligned}\tilde{g}_{ij}(\tilde{x}) \\ &= -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} \lim_{\tilde{y}\to\tilde{x}} \frac{\partial}{\partial\tilde{x}^i}\frac{\partial}{\partial\tilde{y}^j}\left(\tilde{G}(\tilde{x},\tilde{y})^{\frac{2}{2-D}}\right) \\ &= -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} \lim_{\tilde{y}\to\tilde{x}} \frac{\partial}{\partial\tilde{x}^i}\frac{\partial}{\partial\tilde{y}^j}\left(G(x,y)^{\frac{2}{2-D}}\right) \\ &= -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} \lim_{y\to x} \frac{\partial x^k}{\partial\tilde{x}^i}\frac{\partial y^l}{\partial\tilde{y}^j}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}\left(G(x,y)^{\frac{2}{2-D}}\right) \\ &= \frac{\partial x^k}{\partial\tilde{x}^i}\frac{\partial x^l}{\partial\tilde{x}^j} g_{kl}(x). \end{aligned} \tag{11}$$

Now, to verify Eq. (10), it is instructive to work out a special case in detail, such as the case of the D-sphere where the Green's function is explicitly known. In Appendix B, we show that Eq. (10) holds in this case.

To prove that Eq. (10) holds on all Riemannian and also on all pseudo-Riemannian manifolds, it is fortunately not necessary to know the Green's function explicitly. Because of the presence of the limit $x \to y$ in Eq. (10), it suffices to know the behavior of $G(x, y)$ when x and y are arbitrarily close. In this regime, $G(x, y)$ takes its flat space form plus corrections which arise due to curvature and which are benign in the limit $x \to y$. Based on this idea, we give the detailed proof of Eq. (10) for all (pseudo-)Riemannian manifolds in Appendix C.

Intuitively, the reason why Eq. (10) works, i.e., why the propagator contains the metric information, is that, as mentioned in the Introduction, the strength of the quantum correlations that the propagator expresses is closely related to the biscalar covariant distance function, also called the Synge world function; see Eq. (C13). The fact that the covariant distance function contains the metric information is well known; see, e.g., Eq. (4.2) in Ref. [21].

Regarding the coincidence limit, $x \to y$, we remark that curvature is of course not visible at any single point, nor in the tangent plane to that point. Instead, the curvature is encoded in the metric, which is expressing the distances not in the zeroth or first order but in the second order in the infinitesimal distances (the differentials dx_i^2). This is why, in Eq. (10), we need to take two derivatives in order to recover the metric (and therefore the curvature) in the coincidence limit.

III. INTRODUCTION OF A COVARIANT UV CUTOFF

Equation (10) shows how the metric and therefore the curvature can be reconstructed from its effect on quantum fields. However, what if the quantum fields are subject to a natural ultraviolet cutoff, in which case the matter degrees of freedom cannot be used to resolve any structure that is smaller than, for example, a Planck length? How does the metric that one reconstructs from the matter degrees of freedom behave then, in particular, toward the ultraviolet?

A. Model for a covariant UV cutoff

Because of the lack of experimental evidence, it is not known how spacetime behaves close to the Planck scale. It has been argued, for example, that spacetime is discrete at that scale; see, e.g., Ref. [3]. In this context, notice the interesting related approach in Ref. [4] (and see also the related Ref. [5]) in which the aim is to implement a Planck-scale motivated strict finite lower bound on geodesic distances into metric and propagator. Technically, this could regulate ultraviolet divergences, and it is consistent with the fact that quantization literally and quite often concretely means discretization. But it has also been argued that, as general relativity seems to indicate, spacetime should remain continuous at all scales. This would not help with ultraviolet divergencies, but it would preserve symmetries that lattices break. It would also avoid, for example, potential problems of nonadiabaticity associated with discrete point production during cosmic expansion [22].

But there is also the possibility that spacetime is simultaneously both continuous and discrete, namely, in the same mathematical way that information can be [18,23–25]. This could combine the advantages of both pictures. To see this, let us recall Shannon's sampling theorem from information theory. The theorem establishes the equivalence between continuous and discrete representations of information, and it is in ubiquitous use in digital signal processing and communication engineering. Assume that a signal, f, representing continuous information, is band-limited, i.e., it consists of frequencies only within a finite frequency range $(-\Omega, \Omega)$:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} \tilde{f}(\omega)e^{i\omega t}d\omega. \tag{12}$$

Shannon's theorem holds that it suffices to record the signal at a discrete set of times t_n with spacing $t_{n+1} - t_n = \pi/\Omega$ to capture the signal completely. Namely, $f(t)$ can actually be perfectly reconstructed at all times t from the discrete samples $\{f(t_n)\}$ of the signal:

$$f(t) = \sum_n f(t_n)\frac{\sin((t-t_n)\Omega)}{(t-t_n)\Omega}. \tag{13}$$

In fact, the signal can be perfectly reconstructed from any set of samples, even nonequidistantly chosen samples, if the average density (technically, the Beurling density) of samples is at least Ω/π (although nonequidistant sampling comes at the cost of an increased sensitivity of the reconstruction to inaccuracies in the recording of the samples).

Physical fields could be spatially band-limited in the same way, namely, if there exists a suitable natural ultraviolet cutoff in nature [23–25]. This is a simple model of how quantum fields may behave toward the Planck scale. But it is also the second-quantized manifestation of what is in first quantization the minimum length uncertainty principle which has long been suggested to arise from various approaches to quantum gravity, see, e.g., Ref. [26], including string theory, see, e.g., Ref. [27], and quantum groups [23,28,29] which arise in noncommutative geometry [30].

If we assume this type of natural ultraviolet cutoff, physical fields are defined on a continuous spacetime, as usual, but it suffices to know a field on a sufficiently dense lattice to be able to reconstruct the field everywhere. Crucially, since any sufficiently densely spaced lattice can be chosen, the symmetries of the continuous spacetime are preserved. Physical fields in Euclidean-signature spaces are then considered covariantly band-limited if they are in the span of those eigenfunctions of the Laplacian of which the eigenvalues (playing the role of squared spatial frequencies) are below a cutoff value of Λ, where Λ may be, for example, the square of the Planck momentum.

Recently, it has been shown how the entanglement entropy of quantum fields can be calculated within this framework and that it exhibits the expected scaling laws [31].

Here, let us consider the impact of this type of natural ultraviolet cutoff to the extent to which the metric deduced from the propagator can be spatially resolved. Concretely, consider the eigenvalues ($\lambda_n \in \mathbb{R}$) and eigenfunctions of the Laplacian (with or without the mass term) on an arbitrarily curved Riemannian manifold M,

$$(\Delta - m^2) f_n(x) = -\lambda_n^2 f_n(x), \qquad \langle f_n, f_m \rangle = \delta_{nm}, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product: $\langle f, h \rangle = \int_M f(x) h(x) \sqrt{g} d^D x$. For simplicity, namely, so that the self-adjoint extension of the Laplacian is unique and so that the eigenvalues are discrete, we are assuming here that the Riemannian manifold is compact without boundaries. The associated Green's function, which satisfies (9), can be written expanded in the eigenbasis of the Laplacian:

$$G(x, y) = \sum_n \frac{1}{\lambda_n^2} f_n(x) \bar{f}_n(y). \tag{15}$$

By implementing the ultraviolet cutoff, we now obtain the *band-limited* Green's function $G_\Lambda(x, y)$:

$$G_\Lambda(x, y) = \sum_n^{\lambda_n < \Lambda} \frac{1}{\lambda_n^2} f_n(x) \bar{f}_n(y). \tag{16}$$

Our method above for expressing the metric in terms of the quantum correlator can now be applied to this UV cutoff Green's function; i.e., we apply Eq. (10) to G_Λ to obtain a modified metric g^Λ:

$$g_{ij}^\Lambda(y) \equiv -\frac{1}{2} \left[\frac{\Gamma(D/2 - 1)}{4\pi^{D/2}} \right]^{\frac{2}{D-2}} \lim_{x \to y} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} G_\Lambda(x, y)^{\frac{2}{2-D}}. \tag{17}$$

We can now address the question that we set out to answer, namely, the question of how the UV cutoff impacts the expression of the metric in terms of the correlator.

B. Impact of the covariant UV cutoff on the reconstructed metric: Flat space

First, let us consider again the case of a massless scalar field in flat $\mathbb{R}^D$. To this end, we start with the Green's function with the ultraviolet cutoff implemented:

$$G_\Lambda(x, y) = \int_{|p| < \Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} e^{ip\cdot(x-y)}. \tag{18}$$

Substituting Eq. (18) in Eq. (17), we obtain the following metric:

$$g_{ij}^\Lambda = \frac{4}{D^2} \Gamma[D/2]^{\frac{4}{D-2}} \delta_{ij}. \tag{19}$$

The details of the calculation are in Appendix D. This result shows that one recovers the flat metric, δ_{ij}, but only up to a constant prefactor $\nu(D) = \frac{4}{D^2} \Gamma[D/2]^{\frac{4}{D-2}}$. Notice that the prefactor $\nu(D)$ is independent of the UV cutof;, i.e., it persists even when the UV cutoff, Λ, is sent to infinity, $\Lambda \to \infty$. Interestingly, this means that, in Eq. (17), the UV limit $\Lambda \to \infty$ does not commute with the UV limit $x \to y$. This is made possible by the fact that the Green's function is UV divergent as $x \to y$ without the cutoff but becomes a regular function in x and y with the UV cutoff implemented.

Given that the prefactor $\nu(D)$ is a UV phenomenon, we expect that it also appears on all curved spacetimes, so long as there is no significant curvature close to the UV cutoff scale. We present concrete evidence for this expectation in Appendix E, where we show that applying Eq. (17) to the 3-sphere yields the correct metric with the same prefactor $\nu(D)$, which in three dimensions reads $\nu = \frac{\pi^2}{36}$. Because of this feature, we shall refer to $\nu(D)$ as the *universal prefactor*.

Universality of $\nu(D)$ suggests that Eq. (17), once corrected by the overall scaling $\nu(D)$, yields a methodology for "smoothening out" a Riemannian metric on the length scale $1/\Lambda$. This may prove to be useful as a mathematical tool in quantum gravity, where integrating out the metric degrees of freedom is of interest. Note that smoothening out the metric on a given length scale is nontrivial because the metric is what *defines* length scales. Here, we arrived at a smoothening method for the metric by using two key properties of the Green's function: it encodes distances, and it is straightforward to implement the UV cutoff in the Green's function. In Sec. III D, we apply our methodology to explicitly demonstrate how a wiggly manifold's metric is indeed smoothened out by adopting the metric deduced from the propagator in which the UV cutoff has been implemented.

Before moving on to Sec. III D, let us discuss the possible physical origin and consequences of the universal prefactor.

C. Oscillations in the reconstructed metric without performing the coincidence limit

Let us recall that our method for expressing the metric in terms of the two-point function of a scalar quantum field works accurately when there is no ultraviolet cutoff. However, we also found that in the presence of a natural UV cutof, our method (17) recovers the metric up to a prefactor.

In fact, as we will now show, our method recovers the metric correctly, i.e., without any need for a corrective prefactor, once we properly take into account all implications of the presence of a natural UV cutoff. Namely, if there is a natural UV cutof,f then distances smaller than the cutoff scale cannot be resolved. This means that in Eq. (17) the limit $x \to y$ should not be taken, given that it has no operational meaning in terms of measurable quantities.

Let us, therefore, consider the right-hand side of Eq. (17) but without taking the limit. Instead, let us view the right-hand side as a function of the distance between x and y.

In addition, to be fully consistent with the presence of the natural ultraviolet cutoff, we should also not take the Newton Leibniz limit that is implicit in the taking of the two derivatives in Eq. (17). It will be instructive, however, to first study the case where the Newton Leibniz limits are taken.

We will first consider the case of three-dimensional Euclidean space. This case is representative for all those situations in which curvature is significantly present only at length scales that are significantly larger than the length scale of the natural ultraviolet cutoff.

To this end, we recall that in the case of three-dimensional flat space the band-limited Green's function is given by

$$G(x,y) = \frac{1}{2\pi^2}\frac{Si(\Lambda r_{xy})}{r_{xy}}, \tag{20}$$

where Λ is the cutoff, $r_{xy} = |x-y|$, and $Si(x) = \int_0^x dt\,\frac{\sin(t)}{t}$.

Our Green's function-to-metric method, now without performing the coincidence limit but still performing the Newton Leibniz limits that are part of the derivatives, yields

$$g_{\alpha\beta}(x,y) = \delta_{\alpha\beta} f_1(\Lambda r_{xy}) + \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{r_{xy}^2} f_2(\Lambda r_{xy}), \tag{21}$$

$$f_1(x) = \frac{\pi^2}{4}\left(\frac{1}{Si^2(x)} - \frac{\sin(x)}{Si^3(x)}\right), \tag{22}$$

$$f_2(x) = \frac{\pi^2}{4}\left(3\frac{\sin^2(x)}{Si^4(x)} - 2\frac{\sin(x)}{Si^3(x)} - \frac{\Lambda r\cos(x)}{Si^3(x)}\right). \tag{23}$$

If we perform the coincidence limit, we obtain the expected prefactor ν:

$$g_{\alpha\beta}(x) = \lim_{y\to x} g_{\alpha\beta}(x,y) = \frac{\pi^2}{36}\delta_{\alpha\beta}. \tag{24}$$

However, as is shown in Fig. 1, the "metric" $g_{\alpha\beta}(x,y)$ oscillates as y approaches x from larger distances. The oscillations have wavelengths at the cutoff scale. This could only happen because we did perform the Newton Leibniz limit inside the derivatives in Eq. (17), as if there were no ultraviolet cutoff.

In fact, of course, these oscillations cannot actually be resolved, due to the fact that differences of distance as small as the Planck scale cannot be resolved in the presence of the natural UV cutoff. With the precision that is accessible, these oscillations are washed out, and only their average value matters. That average value is unity, which means that the metric is in fact recovered from the Green's function with a prefactor of 1 via Eq. (17), when working with only the precision that is available in the presence of the ultraviolet cutoff.

Now, interestingly, we arrived at this conclusion under the assumption that there is curvature only at length scales that are significantly larger than the length scale of the natural ultraviolet cutoff. However, for example, in inflationary cosmology, the Hubble radius during inflation is thought to have been only about 5 or 6 orders of magnitude larger than the Planck length. It is conceivable, therefore,

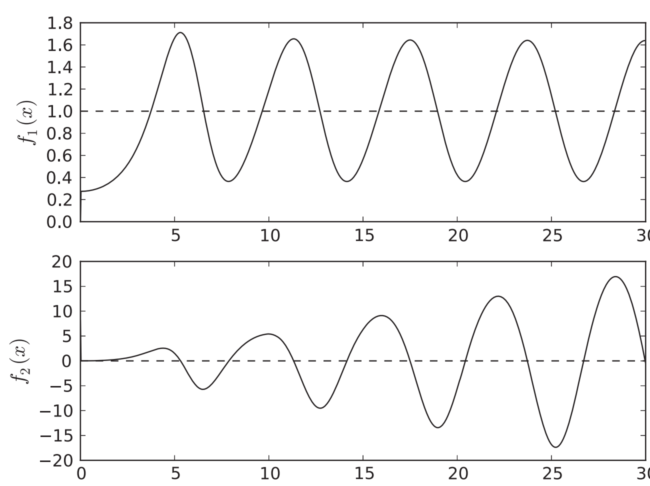

FIG. 1. Behavior of $f_1(x)$ and $f_2(x)$, which are defined in Eqs. (22)–(23).

that Planck scale physics could impact to some extent the predictions for the cosmic microwave background (CMB). Inflation may have acted as a magnifying glass to make the above-discussed oscillations visible in the CMB.

D. Example of the curvature scale reaching the cutoff scale

Let us investigate the implications for constructing the metric from the Green's function when there exists significant curvature down to scales close to the cutoff scale. How is this curvature smoothed out? We choose a simple example in three dimensions,

$$ds^2 = a^2(\eta)(d\eta^2 + dx^i dx^i), \tag{25}$$

where $a(\eta) = 1 + \epsilon(\eta)$ and $\epsilon(\eta) \ll 1$. Let us call η a "time" coordinate just to distinguish it from x^i coordinates. We use Greek letters for all coordinates and Latin ones only for "spatial" coordiantes. The aim is to investigate the effect of the UV cutoff when the UV cutoff and the curvature scales are not well separated. We further assume that the perturbation ϵ exists only in a finite interval of η and that

$$\int \epsilon(\eta) d\eta = 0. \tag{26}$$

The Laplace operator is given by

$$\Delta = a^{-2}\partial_\eta^2 + \frac{a'}{a^3}\partial_\eta + a^{-2}\nabla^2, \tag{27}$$

where $' = \frac{d}{d\eta}$ and $\nabla^2 = \partial_{x^i}\partial_{x^i}$. One can check that $\psi(\eta, \vec{x}) = f(\eta)e^{i\vec{k}\cdot\vec{x}}$ is the eigenfunction of Δ with the corresponding eigenvalue λ provided that $f(\eta)$ satisfies

$$f'' + \frac{a'}{a}f' - (\lambda a^2 + \vec{k}^2)f = 0. \tag{28}$$

Working up to first order in ϵ and performing the substitutions

$$f(\eta) = e^{i\omega\eta}(1 + \chi(\eta)) \tag{29}$$

$$\lambda = -(\vec{k}^2 + \omega^2) + \delta\lambda \tag{30}$$

in Eq. (28), we arrive at

$$\chi'' + 2i\omega\chi' = \delta\lambda - 2(\vec{k}^2 + \omega^2)\epsilon - i\omega\epsilon'. \tag{31}$$

Note that to zeroth order in ϵ, $\delta\lambda = \chi = 0$. Integrating Eq. (31) from $\eta = -\infty$ to $\eta = +\infty$ and using Eq. (26), we get $\delta\lambda = 0$ even at first order in ϵ. Taking the Fourier transform of Eq. (31), we get

$$\tilde{\chi}(\Omega) = \frac{2(\vec{k}^2 + \omega^2) - \omega\Omega}{\Omega(\Omega + 2\omega)}\tilde{\epsilon}(\Omega), \tag{32}$$

where the Fourier transform is defined as $A(\eta) \equiv \int d\Omega \tilde{A}(\Omega)e^{i\Omega\eta}$.

Performing the Fourier transform to get $\chi(\eta)$ back, we need to choose how the contour passes through the poles of $\tilde{\chi}(\Omega)$. Here, we add a small imaginary number to each term in the denominator of Eq. (32),

$$\tilde{\chi}(\Omega) = \frac{2(\vec{k}^2 + \omega^2) - \omega\Omega}{(\Omega + ic)(\Omega + 2\omega + ic)}\tilde{\epsilon}(\Omega). \tag{33}$$

Note that at the end of calculation c must be taken to zero. The massless Green's function is given as

$$G(\eta, \vec{x}; \eta', \vec{x}') = G_0(\eta, \vec{x}; \eta', \vec{x}') + G_1(\eta, \vec{x}; \eta', \vec{x}'), \tag{34}$$

where

$$G_0(\eta, \vec{x}; \eta', \vec{x}') \equiv \int \frac{d\omega d^2k}{(2\pi)^3}\frac{1}{\omega^2 + \vec{k}^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(\eta-\eta')}, \tag{35}$$

$$G_1(\eta, \vec{x}; \eta', \vec{x}') \equiv \int \frac{d\omega d^2k}{(2\pi)^3}\frac{1}{\omega^2 + \vec{k}^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(\eta-\eta')} \times (\chi_{\omega,\vec{k}}(\eta) + \chi^*_{\omega,\vec{k}}(\eta')). \tag{36}$$

We first put a cutoff on the eigenvalues of the Laplace operator as follows:

$$\omega^2 + \vec{k}^2 \leq \Lambda^2. \tag{37}$$

Then, we substitute Eq. (34) with the cutoff Λ into Eq. (17). After some manipulations, we arrive at

$$g^\Lambda_{\alpha\beta}(\eta) = \frac{\pi^2}{36}\delta_{\alpha\beta} + h^\Lambda_{\alpha\beta}(\eta), \tag{38}$$

where

$$h^\Lambda_{\alpha\beta} = \frac{1}{(4\pi)^2}\left[\frac{G_{1;\alpha\beta}(\eta)}{G_0^3} - \frac{4\pi^4}{3}\frac{G_1(\eta)}{G_0}\delta_{\alpha\beta}\right] \tag{39}$$

and

$$G_0 \equiv G_0(\eta, \vec{x}; \eta, \vec{x}) = \frac{4\pi\Lambda}{(2\pi)^3} \tag{40}$$

$$G_1(\eta) \equiv G_1(\eta, \vec{x}; \eta, \vec{x}) \tag{41}$$

$$G_{1;\alpha\beta}(\eta) \equiv \partial_{x^\alpha}\partial_{x'^\beta}G_1(\eta, \vec{x}; \eta', \vec{x}')|_{x'^\mu = x^\mu}. \tag{42}$$

Equation (38) shows that the band-limited metric is the flat metric (with the universal prefactor) with additional perturbations. Let us now investigate how these perturbations are related to the original metric perturbations $2\epsilon(\eta)\delta_{\alpha\beta}$. Does one recover the original metric perturbations with the universal prefactor in the limit $\Lambda \to \infty$?

1. h_{ij}^{Λ} components

There is no spatiotemporal component to the metric perturbations, since $G_{1;0i} = 0$. Spatial components are given by

$$h_{ij}^{\Lambda}(\eta) = \frac{\pi}{16}\int^{\Lambda} \frac{d\omega d^2k}{\Lambda^3} \frac{1}{\omega^2 + \vec{k}^2} (\chi_{\omega,\vec{k}}(\eta) + \chi^*_{\omega,\vec{k}}(\eta)) \times \left(k_i k_j - \frac{\Lambda^2}{3}\delta_{ij}\right), \tag{43}$$

or in Fourier space

$$\begin{aligned}\tilde{h}_{ij}^{\Lambda}(\Omega) &= \frac{\pi}{4}\tilde{\epsilon}(\Omega)\int^{\Lambda} \frac{d\omega d^2k}{\Lambda^3} \frac{1}{\omega^2 + \vec{k}^2} \frac{\vec{k}^2 + 2\omega^2}{\Omega^2 - 4(\omega + ic)^2} \times \left(k_i k_j - \frac{\Lambda^2}{3}\delta_{ij}\right) \\ &= \frac{\pi}{4}\tilde{\epsilon}(\Omega)\int^{S(1)} d\omega d^2k \frac{1}{\omega^2 + \vec{k}^2} \frac{\vec{k}^2 + 2\omega^2}{\frac{\Omega^2}{\Lambda^2} - 4(\omega + ic)^2} \times \left(\frac{\vec{k}^2}{2} - \frac{1}{3}\right)\delta_{ij},\end{aligned} \tag{44}$$

where the last integral is over the region $\omega^2 + \vec{k}^2 \leq 1$. This means that the spatial part of the original metric perturbation $(2\epsilon(\eta)\delta_{ij})$ in Fourier space is multiplied by the following window function:

$$W_s^{\Lambda}(\Omega) = \frac{\pi}{8}\int^{S(1)} d\omega d^2k \frac{1}{\omega^2 + \vec{k}^2} \frac{\vec{k}^2 + 2\omega^2}{\frac{\Omega^2}{\Lambda^2} - 4(\omega + ic)^2} \times \left(\frac{\vec{k}^2}{2} - \frac{1}{3}\right). \tag{45}$$

Figure 2 shows how this window function dampens high-frequency modes of metric perturbation and in effect makes the metric more smooth. We can also check that for large values of the UV cutoff, this window function approaches the value $\nu = \frac{\pi^2}{36}$, which is in agreement with our earlier observations.

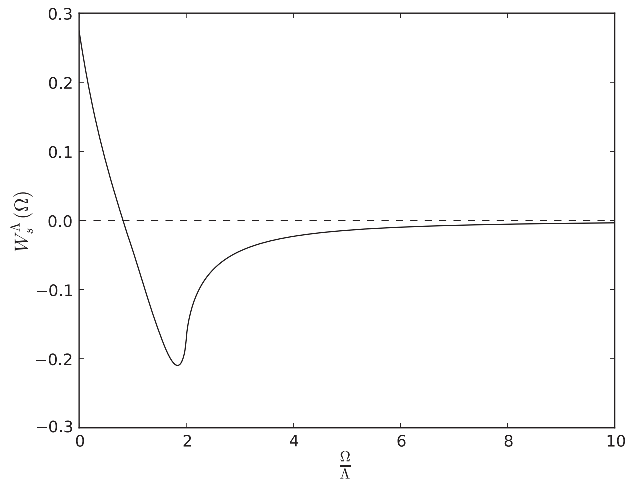

FIG. 2. High-frequency modes compared to the cutoff Λ have been damped. As a result, the band-limited metric is becoming smoother.

2. $h_{\eta\eta}^{\Lambda}$ component

We can use Eq. (39) to find the $\eta\eta$ component of the metric perturbation. Since $h_{\eta\eta}$ is only a function of η, however, with a time redefinition, we can absorb this term in the definition of the time coordinate. So, there is no physical significance to explicitly calculate this metric component. One can directly check that for large values of the cutof, one obtains

$$h_{\eta\eta}^{\Lambda\to\infty} = \frac{\pi^2}{36} 2\epsilon(\eta), \tag{46}$$

in agreement with the universality of prefactor ν.

IV. CONCLUSIONS AND OUTLOOK

We showed how, in the absence of rods and clocks at subatomic scales, quantum fields can be used, in principle, to measure the curvature of spacetime. Indeed, the imprint that curvature leaves in a scalar propagator, i.e., in the vacuum correlators of a scalar quantum field, suffices to reconstruct the metric and consequently the Riemann tensor. In this sense, the measurement of the Green's function, i.e., of a correlator of quantum fluctuations of fields, can replace rods and clocks.

That it should be possible at all to deduce the metric from the propagator was conjectured in Ref. [18], and here we confirm this by giving a constructive method. As a subject for further study, we remark here only that, in Refs. [18,32], it was also argued that, at least in the case of compact Riemannian manifolds, the mere spectra of the quantum noise on manifolds should suffice to deduce their metric, although in dimensions higher than 2, the spectra also of certain tensorial fields should be needed.

Here, we continued within the framework of Euclidean quantum field theory, where we investigated how a hard natural ultraviolet cutoff limits the maximal spatial resolution with which one can reconstruct the metric from the propagator. We found that the metric, expressed in terms of the propagator, exhibits characteristic oscillations as the natural UV cutoff scale is approached. These oscillations are generally unobservable in the sense that they should be washed out by the natural ultraviolet cutoff. However, it is conceivable that, through the amplifying effect of cosmic inflation, such oscillations in the metric may have left a signature in the cosmic microwave background.

To this end, it will be necessary and very interesting to study the covariant natural hard ultraviolet cutoff also in the case of the Lorentzian signature. This is nontrivial because, for example, while a hard cutoff makes the Green's function in the Euclidean case finite even in the coincidence limit, the corresponding hard cutoff on the Lorentzian Green's function is still divergent in the coincidence limit and on the light cone. This is currently being investigated; see Ref. [33].

Finally, since the tools of Shannon sampling that we applied here to implement an ultraviolet cutoff originate in information theory, it should be very interesting to explore the information-theoretic implications of our findings here. In this context, see, e.g., Refs. [34,35].

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APPENDIX A: GREEN'S FUNCTION TO METRIC: *D*-DIMENSIONAL EUCLIDEAN SPACE

Here, we will prove Eq. (6) for $D \geq 4$. (Proof for $D = 3$ is contained in the main text.) Let us start with some notation:

$$G(x,y) = f(r_{xy}) \tag{A1}$$

$$f(r) = \frac{(2\pi)^{-\frac{D}{2}}}{r^{D-2}}(mr)^{\frac{D}{2}-1}K_{\frac{D}{2}-1}(mr) \tag{A2}$$

$$r_{xy}^2 = \sum_{i=1}^{D}(x^i - y^i)^2. \tag{A3}$$

Let

$$\tilde{g}_{ij}(x,y) \equiv -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}(G(x,y)^{\frac{2}{2-D}}). \tag{A4}$$

Then, proving Eq. (6) is equivalent to showing

$$\lim_{x\to y}\tilde{g}_{ij}(x,y) = \delta_{ij}. \tag{A5}$$

It can be checked that

$$\tilde{g}_{ij}(x,y) = h_1(r_{xy})\delta_{ij} + h_2(r_{xy})\delta_{ik}\delta_{jl}(x^k - y^k)(x^l - y^l), \tag{A6}$$

where

$$h_1(r) = \frac{1}{2-D}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} r^{-1} f(r)^{\frac{D}{2-D}} f'(r), \tag{A7}$$

$$h_2(r) = \frac{1}{2-D}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} r^{-2} f(r)^{\frac{2D-2}{2-D}} \times\left[\frac{D}{2-D}f'(r)^2 + f(r)f''(r) - r^{-1}f(r)f'(r)\right]. \tag{A8}$$

Since we are interested in the $x \to y$ limit of Eq. (A6), it suffices to know the behavior of $f(r)$ for small r (see, e.g., Eqs. (10.31.1), (10.25.2), and (10.27.4) of Ref. [36]):

$$f(r) \xrightarrow[D>4]{mr\to 0} \frac{\Gamma(D/2-1)}{4\pi^{D/2}r^{D-2}}\left[1 + \frac{(mr)^2}{2(4-D)} + \ldots\right], \tag{A9}$$

$$f(r) \xrightarrow[D=4]{mr\to 0} \frac{1}{4\pi^2 r^2}\left[1 + \frac{(mr)^2}{2}\ln(mr) + \ldots\right]. \tag{A10}$$

Substituting this back into the definition of $h_1(r)$ and $h_2(r)$, we find

$$h_1(r) \xrightarrow{mr\to 0} 1 + \ldots \tag{A11}$$

$$h_2(r) \xrightarrow[D>4]{mr\to 0} \frac{4m^2}{(4-D)(2-D)} + \ldots \tag{A12}$$

$$h_2(r) \xrightarrow[D=4]{mr\to 0} -2m^2\ln(mr) + \ldots, \tag{A13}$$

where $\cdots$ corresponds to subleading terms in the expansion. It then follows directly from Eqs. (A11)–(A13) that

$$\lim_{x\to y} h_1(r_{xy}) = 1,$$
$$\lim_{x\to y} h_2(r_{xy})\delta_{ik}\delta_{jl}(x^k - y^k)(x^l - y^l) = 0. \tag{A14}$$

Our desired result (A5) then follows from combining Eqs. (A14) and (A6).

APPENDIX B: GREEN'S FUNCTION TO METRIC: THE *D*-SPHERE

Here, we check that Eq. (10) is true for the *D*-sphere ($D > 2$). Let us start by establishing some notation.

The D-sphere is defined as the surface

$$\delta_{ij}x^i x^j + (x^{D+1})^2 = 1 \tag{B1}$$

embedded in $D+1$-dimensional Euclidean space with metric $ds^2 = \delta_{ab}dx^a dx^b + (dx^{D+1})^2$, where $a, b = 1, \ldots, D$. The induced metric on the D-sphere is given by

$$ds^2 = g_{ab}dx^a dx^b \tag{B2}$$

$$g_{ab} = \delta_{ab} + \frac{\delta_{ac}\delta_{bd}x^c x^d}{1 - x\cdot x}, \tag{B3}$$

where $x\cdot y \equiv \delta_{ab}x^a y^b$. The Green's function on the D-sphere is given by[1]

$$G(x,y) = f(Z(x,y)) \tag{B4}$$

$$f(Z) = \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{D/2}\Gamma(D/2)} F\left(h_+, h_-; \frac{D}{2}; \frac{1+Z}{2}\right) \tag{B5}$$

$$Z(x,y) = x\cdot y + \sqrt{(1 - x\cdot x)(1 - y\cdot y)}, \tag{B6}$$

where F is the hypergeometric function ${}_2F_1$ and

$$h_\pm = \frac{D-1}{2} \pm \nu, \qquad \nu^2 = \frac{(D-1)^2}{4} - m^2. \tag{B7}$$

Let

$$\tilde{g}_{ab}(x,y) \equiv -\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} \frac{\partial}{\partial x^a}\frac{\partial}{\partial y^b}(G(x,y)^{\frac{2}{2-D}}). \tag{B8}$$

Then, confirming Eq. (10) is equivalent to showing

$$\lim_{x\to y}\tilde{g}_{ab}(x,y) = g_{ab}(y). \tag{B9}$$

It can be shown using straightforward algebra that

$$\tilde{g}_{ab}(x,y) = h_1(Z(x,y))\frac{\partial Z}{\partial x^a}\frac{\partial Z}{\partial y^b} + h_2(Z(x,y))\frac{\partial^2 Z}{\partial x^a \partial y^b}, \tag{B10}$$

$$h_1(Z) = \frac{1}{D-2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} f(Z)^{\frac{2-2D}{D-2}} \times \left[\frac{D}{2-D}f'(Z)^2 + f(Z)f''(Z)\right], \tag{B11}$$

$$h_2(Z) = \frac{1}{D-2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}} f(Z)^{\frac{D}{2-D}} f'(Z). \tag{B12}$$

Also,

$$\frac{\partial Z}{\partial x^a} = \delta_{ab}\left(y^b - x^b\sqrt{\frac{1-y\cdot y}{1-x\cdot x}}\right), \tag{B13}$$

$$\frac{\partial Z}{\partial y^b} = \delta_{ba}\left(x^a - y^a\sqrt{\frac{1-x\cdot x}{1-y\cdot y}}\right), \tag{B14}$$

$$\frac{\partial^2 Z}{\partial x^a \partial y^b} = \delta_{ab} + \frac{\delta_{ac}\delta_{bd}x^c y^d}{\sqrt{(1-x\cdot x)(1-y\cdot y)}}. \tag{B15}$$

Note that in the $x\to y$ limit $Z\to 1^-$. Therefore, we have to investigate the leading behavior of $h_1(Z)$ and $h_2(Z)$ when $Z\to 1^-$, which in turn depends on the behavior of $f(Z)$. It can be shown from asymptotic properties of the hypergeometric function F that

$$f(Z) \xrightarrow{Z\to 1^-} \frac{\Gamma(D/2-1)}{(4\pi)^{D/2}}\left(\frac{1-Z}{2}\right)^{-\frac{D}{2}+1}[1 + C(Z) + \ldots], \tag{B16}$$

where

$$C(Z) = \begin{cases} -\frac{\sqrt{2}\Gamma(1-\sqrt{1-m^2})\Gamma(1+\sqrt{1-m^2})}{\Gamma(1/2-\sqrt{1-m^2})\Gamma(1/2+\sqrt{1-m^2})}\sqrt{1-Z} & \text{if } D=3 \\ \frac{m^2-2}{2}(1-Z)\ln(1-Z) & \text{if } D=4 \\ \frac{D^2-2D-4m^2}{4(D-4)}(1-Z) & \text{if } D>4. \end{cases} \tag{B17}$$

Plugging this back into the definition of $h_1(Z)$ and $h_2(Z)$, we find for $Z\to 1^-$

$$h_1(Z) \to \begin{cases} \frac{-3\Gamma(1-\sqrt{1-m^2})\Gamma(1+\sqrt{1-m^2})}{2\Gamma(1/2-\sqrt{1-m^2})\Gamma(1/2+\sqrt{1-m^2})}\left(\frac{1-Z}{2}\right)^{-1/2}, & D=3 \\ (m^2-2)(1-Z)\ln(1-Z), & D=4 \\ \frac{D^2-2D-4m^2}{(D-2)(D-4)}, & D>4, \end{cases} \tag{B18}$$

$$h_2(Z) \to 1 + \ldots. \tag{B19}$$

It then follows from Eqs. (B18), (B19), and (B13)–(B15) that

$$\lim_{x\to y} h_1(Z(x,y))\frac{\partial Z}{\partial x^a}\frac{\partial Z}{\partial y^b} = 0 \tag{B20}$$

$$\lim_{x\to y} h_2(Z(x,y))\frac{\partial^2 Z}{\partial x^a \partial y^b} = \delta_{ab} + \frac{\delta_{ac}\delta_{bd}y^c y^d}{1-y\cdot y} = g_{ab}(y). \tag{B21}$$

[1]In the embedding $D+1$-dimensional Euclidean space, $Z(x,y) = \delta_{AB}X^A Y^B$ $(A, B = 1, \ldots, D+1)$, where X and Y are the Euclidean coordinates of x and y.

Our desired result (B9) then follows from combining Eqs. (B20), (B21), and (B10).

APPENDIX C: GREEN'S FUNCTION TO METRIC: CURVED MANIFOLDS

Here, we will prove Eq. (10) for all curved manifolds ($D > 2$). The main idea of the proof is as follows: because of the presence of the limit $x \to y$ in Eq. (10), it suffices to know the behavior of $G(x, y)$ when x and y are arbitrarily close. We will derive first-order deviations of $G(x, y)$ from flatness in Riemann normal coordinates (RNCs) (for convenience) and show that corrections due to curvature do not spoil the Green's function → metric prescription (10) when the limit $x \to y$ is performed.

We start with a brief review of RNCs, for the sake of establishing our notation.

1. Riemann normal coordinates

Consider a generic coordinate system $\tilde{x}$ on a curved manifold. Starting with this coordinate system, we can construct RNCs—which we shall denote by x—about point P which has coordinate $\tilde{y}$ using the transformation

$$\tilde{x}^i - \tilde{y}^i = x^i - \frac{1}{2}\tilde{\Gamma}^i_{jk}(\tilde{y})x^j x^k + \ldots, \tag{C1}$$

where $\tilde{\Gamma}^i_{jk}(\tilde{y})$ denote the Christoffel symbols at point P in our original coordinate system. By construction, the Christoffel symbols and all first derivatives of the metric vanish at the origin (i.e., $x = 0$) of Riemann normal coordinates:

$$\Gamma^i_{jk}(0) = 0, \qquad g_{ij,k}(0) = 0. \tag{C2}$$

It can also be shown that the metric in Riemann normal coordinates takes the form

$$g_{ij}(x) = \tilde{g}_{ij}(\tilde{y}) + \frac{1}{3}\tilde{R}_{iklj}(\tilde{y})x^k x^l + \ldots, \tag{C3}$$

where $\tilde{R}_{iklj}(\tilde{y})$ are the components of the Riemann tensor in the original coordinate system at point P. We can always pick the coordinate system $\tilde{x}$ so that at point P the metric is flat: $\tilde{g}_{ij}(\tilde{y}) = \delta_{ij}$ for the Riemannian manifold (in the following steps, replace δ_{ij} with η_{ij} for the Lorentzian manifold). Furthermore, it can be checked using Eq. (C1) that $\tilde{R}_{iklj}(\tilde{y}) = R_{iklj}(0)$, where $R_{iklj}(0)$ are the components of the Riemann tensor in RNCs at point P. Therefore,

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{iklj}x^k x^l + \ldots, \tag{C4}$$

where for simplicity of notation we have let $R_{iklj} \equiv R_{iklj}(0)$. Below, we list some more useful relations which we will later make use of:

$$g^{ij}(x) = \delta^{ij} + \delta g^{ij}(x), \qquad \delta g^{ij}(x) = \frac{1}{3}R^i{}_k{}^j{}_l x^k x^l + \cdots \tag{C5}$$

as well as

$$\sqrt{g(x)} = 1 + \delta\sqrt{g(x)}, \qquad \delta\sqrt{g(x)} = -\frac{1}{6}R_{ij}x^i x^j + \ldots. \tag{C6}$$

It is also useful to note that

$$\partial_i \delta g^{ij}(x) = -\frac{1}{3}R^j{}_i x^i + \ldots,$$
$$\partial_i \delta\sqrt{g(x)} = -\frac{1}{3}R_{ij}x^j + \ldots. \tag{C7}$$

2. Singularity structure of Green's function

The Green's function satisfies the equation

$$\left[\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right) - m^2\sqrt{g}\right]G(x, y) = -\delta^{(D)}(x - y). \tag{C8}$$

In Riemannian geometry, Eq. (C8) has a unique solution. However, in Lorentzian geometry, there are different solutions to Eq. (C8), corresponding to different boundary conditions, and we have to specify which Green's function we are considering. In the Lorentzian case, we choose the solution that corresponds to the Feynman propagator. For the purpose of our proof, it suffices that the solution to Eq. (C8) asymptotes to the flat space Feynman Green's function in the coincidence limit where the effect of curvature is negligible (for more discussion on this, see Ref. [9]). A large class of states satisfies this condition. For example, all the Hadamard states, considered to be physically reasonable states, satisfy this condition (see Ref. [37] and references therein for more details on Hadamard states and their importance.)

We will solve this equation in Riemann normal coordinates (to first order) where $y = 0$ is the origin. To do so, let

$$G(x) = G^E(x)(1 + \delta G(x)), \tag{C9}$$

where we have used the notation $G(x, 0) = G(x)$ and $G^E(x)$ is the massive flat space Green's function which satisfies

$$\left(\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - m^2\right)G^E(x) = -\delta^{(D)}(x). \tag{C10}$$

It can be checked that to first order Eq. (C8) becomes

$$\begin{aligned}&(1+\delta G+\delta\sqrt{g})(\nabla^2 G^E-m^2G^E)+2\delta^{ij}\partial_i\delta G\partial_j G^E\\&\quad+G^E\nabla^2\delta G+\partial_i\delta g^{ij}\partial_j G^E\\&\quad+\delta g^{ij}\partial_i\partial_j G^E+\delta^{ij}\partial_i\delta\sqrt{g}\partial_j G^E\\&=-\delta^{(D)}(x),\end{aligned}\tag{C11}$$

where δg^{ij} and $\delta\sqrt{g}$ are defined in Eqs. (C5)–(C6). Using Eqs. (C10) and (C7), Eq. (C12) reduces to

$$\begin{aligned}&2\delta^{ij}\partial_i\delta G\partial_j G^E+G^E\nabla^2\delta G+\partial_i\delta g^{ij}\partial_j G^E\\&\quad+\delta g^{ij}\partial_i\partial_j G^E+\delta^{ij}\partial_i\delta\sqrt{g}\partial_j G^E=0.\end{aligned}\tag{C12}$$

Let $\sigma(x)$ denote half the geodesic distance from the origin[2]:

$$\sigma(x)=\frac{1}{2}\delta_{ij}x^ix^j.\tag{C13}$$

Noting that G^E is only a function of σ [see Eq. (2) for the Riemannian case solution)] and using Eq. (C6), Eq. (C12) takes the simpler form:

$$G^E\nabla^2\delta G+2\frac{dG^E}{d\sigma}x^i\partial_i\delta G-\frac{1}{3}\frac{dG^E}{d\sigma}R_{ij}x^ix^j=0.\tag{C14}$$

Using the ansatz for δG,

$$\delta G(x)=\chi(\sigma(x))+\frac{1}{12}R_{ij}x^ix^j,\tag{C15}$$

Eq. (C14) reduces to the following equation for χ:

$$2G^E\frac{d^2\chi}{d\sigma^2}+\left(DG^E+4\sigma\frac{dG^E}{d\sigma}\right)\frac{d\chi}{d\sigma}+\frac{R}{6}G^E=0.\tag{C16}$$

Since we are interested in the small σ limit, we can substitute $G^E(\sigma)$ in Eq. (C16) with its $m^2\sigma\to 0$ behavior

$$G^E(\sigma)\xrightarrow{m^2\sigma\to 0}\frac{\Gamma(D/2-1)}{2(2\pi)^{D/2}\sigma^{D/2-1}}.\tag{C17}$$

In this case, Eq. (C16) simplifies to

$$2\sigma\frac{d^2\chi}{d\sigma^2}+(4-D)\frac{d\chi}{d\sigma}+\frac{R}{6}=0.\tag{C18}$$

The general solution of Eq. (C18) is

$$\chi=\begin{cases}\frac{R}{6(D-4)}\sigma+A\sigma^{D/2-1}+B & \text{if } D\neq 4\\ -\frac{R}{12}\sigma\ln(R\sigma) & \text{if } D=4,\end{cases}\tag{C19}$$

where A and B are constants. Requiring $\chi\to 0$ as $\sigma\to 0$, we find $B=0$. Therefore, the leading behavior of χ for small σ is

$$\chi(\sigma)\xrightarrow{R\sigma\to 0}\begin{cases}A(R\sigma)^{1/2}+\ldots & \text{if } D=3\\ -\frac{R}{12}\sigma\ln(R\sigma)+\ldots & \text{if } D=4\\ \frac{R}{6(D-4)}\sigma+\ldots & \text{if } D>4.\end{cases}\tag{C20}$$

For $D=3$, A is a constant which depends on global properties of the manifold (e.g., topology). Plugging this result back into Eq. (C15) and then Eq. (C9), and also using the $m^2\sigma\to 0$ behavior of G^E, we find

$$G(x)\underset{m^2\sigma\to 0}{\xrightarrow{R\sigma\to 0}}\frac{\Gamma(D/2-1)}{2(2\pi)^{D/2}\sigma^{D/2-1}}(1+C(x)+\ldots),\tag{C21}$$

where

$$C(x)=\begin{cases}A(R\sigma)^{1/2}+(2m^2\sigma)^{1/2} & \text{if } D=3\\ \frac{m^2\sigma}{2}\ln(m^2\sigma)-\frac{R}{12}\sigma\ln(R\sigma) & \text{if } D=4\\ \frac{m^2\sigma}{4-D}+\frac{R\sigma}{6(D-4)}+\frac{1}{12}R_{ij}x^ix^j & \text{if } D>4.\end{cases}\tag{C22}$$

Going back to our original coordinate system [using the coordinate transformation (C1)] and computing the rhs of Eq. (10), we find

$$-\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}}\lim_{\tilde{x}\to\tilde{y}}\frac{\partial}{\partial\tilde{x}^i}\frac{\partial}{\partial\tilde{y}^j}\left(G(\tilde{x},\tilde{y})^{\frac{2}{2-D}}\right)=\delta_{ij}.\tag{C23}$$

Therefore, we have verified that Eq. (10) is true in the coordinate system $\tilde{x}$, where the metric is chosen to be flat at point $\tilde{y}$. Since Eq. (10) is a tensorial equality, however, it follows that it is true in all coordinate systems.

APPENDIX D: BAND-LIMITED FLAT METRIC

Here, we want to find the band-limited metric associated to band-limited Green's function (18) of flat space in D dimensions. If we perform a derivative in Eq. (17), we obtain

$$\begin{aligned}g^{\Lambda}_{ij}(x)=&-\frac{1}{2}\left[\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\right]^{\frac{2}{D-2}}\\&\times\left(\frac{2\partial_{x^i}\partial_{y^j}G_\Lambda(x,y)|_{y=x}}{(2-D)G_\Lambda(x,x)^{\frac{D}{D-2}}}\right.\\&\left.+\frac{2D\partial_{x^i}G_\Lambda(x,y)\partial_{y^j}G_\Lambda(x,y)|_{y=x}}{(D-2)^2G_\Lambda(x,x)^{\frac{2D-2}{D-2}}}\right).\end{aligned}\tag{D1}$$

Here, we calculate each term separately. If we use Eq. (18), we get

$$\partial_{x^i}\partial_{y^j}G_\Lambda(x,y)|_{y=x}=\int^\Lambda\frac{d^Dp}{(2\pi)^D}\frac{p_ip_j}{p^2}=\int^\Lambda\frac{d^Dp}{(2\pi)^D}\frac{\delta_{ij}}{D},\tag{D2}$$

[2]For Lorentzian manifolds, we use the $(-,+,+,\cdots)$ signature.

where $p_i p_j$ in the integrand is substituted by $\frac{p^2}{D}\delta_{ij}$. Performing the integral, we end up with

$$\partial_{x^i}\partial_{y^j}G_\Lambda(x,y)|_{y=x} = \frac{S_{D-1}}{D^2(2\pi)^D}\Lambda^D\delta_{ij}, \tag{D3}$$

where $S_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the area of $D-1$-dimensional unit sphere.

The Green's function at the coincidence point is given by

$$G_\Lambda(x,x) = \int^\Lambda \frac{p^{D-1}dpd\Omega_{D-1}}{(2\pi)^D}\frac{1}{p^2} = \frac{S_{D-1}}{(D-2)(2\pi)^D}\Lambda^{D-2}. \tag{D4}$$

Finally, since the first derivative of band-limited Green's function at the coincidence point results in an integral over an odd function, the last term in Eq. (D1) is zero. Substituting these values back in Eq. (D1), we get Eq. (19).

APPENDIX E: UNIVERSAL PREFACTOR FOR 3-SPHERE

In this section, we show that the band-limited metric of the 3-sphere is the original 3-sphere metric up to the universal constant $\nu = \frac{\pi^2}{36}$ when the UV cutoff is taken to infinity. To this end, let us choose a convenient coordinate system for S^3. Recall that our method is diffeomorphism invariant and independent of the chosen coordinate system. We choose the toroidal coordinate system (χ,θ,ϕ), defined as

$$x^0 = \cos(\chi)\cos(\theta) \tag{E1}$$

$$x^1 = \sin(\chi)\cos(\phi) \tag{E2}$$

$$x^2 = \sin(\chi)\sin(\phi) \tag{E3}$$

$$x^3 = \cos(\chi)\sin(\theta), \tag{E4}$$

where x^μ is a Cartesian coordinate of a point at R^4 on a unit sphere. The line element on S^3 then reads

$$ds^2 = d\chi^2 + \cos^2(\chi)d\theta^2 + \sin^2(\chi)d\phi^2. \tag{E5}$$

We also need the eigenvalues and normalized eigenfunctions of the Laplacian on S^3 [38],

$$\Delta T_{k,m_1,m_2} = -k(k+2)T_{k,m_1,m_2}, \tag{E6}$$

where $k\in\{0,1,2,\cdots\}$, $m_1, m_2 \in \{-k/2,\ldots,k/2\}$ and

$$T_{k,m_1,m_2}(X) = C_{k,m_1,m_2}(\cos(\chi)e^{i\theta})^l(\sin(\chi)e^{i\phi})^m \times P^{(m,l)}_{k/2-m_2}[\cos(2\chi)] \tag{E7}$$

with $l = m_1 + m_2$, $m = m_2 - m_1$, $C_{k,m_1,m_2} = \frac{\sqrt{k+1}}{\sqrt{2}\pi}\sqrt{\frac{(k/2+m_2)!(k/2-m_2)!}{(k/2+m_1)!(k/2-m_1)!}}$, and X denotes $\{\chi,\theta,\phi\}$ collectively. For the purpose of our calculations, we only need the following identities:

$$\sum_{m_i=-k/2}^{k/2}(m_1+m_2)^2|T_{k,m_1,m_2}|^2 = \frac{k(k+1)^2(k+2)}{6\pi^2}\cos^2(\chi), \tag{E8}$$

$$\sum_{m_i=-k/2}^{k/2}|T_{k,m_1,m_2}|^2 = \frac{(k+1)^2}{2\pi^2}, \tag{E9}$$

$$\sum_{m_i=-k/2}^{k/2}(m_1\pm m_2)|T_{k,m_1,m_2}|^2 = 0. \tag{E10}$$

Then, the band-limited Green's function (with mass μ) is given by

$$G_L(X,Y) = \sum_{k=0}^{L}\sum_{m_i}\frac{1}{k(k+2)+\mu^2}T_{k,m_1,m_2}(X)T^*_{k,m_1,m_2}(Y). \tag{E11}$$

Let us find the $\theta\theta$ component of band-limited metric. If we substitute G_L in Eq. (17) and use Eq. (E8)–(E10), we get

$$g^L_{\theta\theta} = \frac{\pi^2}{12}\frac{N}{M^3}\cos^2(\chi), \tag{E12}$$

where

$$N = \sum_{k=0}^{L}\frac{k(k+1)^2(k+2)}{k(k+2)+\mu^2}$$

$$M = \sum_{k=0}^{L}\frac{(k+1)^2}{k(k+2)+\mu^2}.$$

For large values of L (cutoff), N diverges as $L^3/3$, while M diverges as L. So, we get

$$g^{L\to\infty}_{\theta\theta} = \frac{\pi^2}{36}\cos^2(\chi) = \frac{\pi^2}{36}g_{\theta\theta}. \tag{E13}$$

The same manipulation can be done for the other components of the band-limited metric, which confirms the universality of the prefactor $\nu = \frac{\pi^2}{36}$. Fortunately, however, we do not need to work out the other components of g^L. Since the band-limited metric is rotationally invariant (because the cutoff is a rotationally invariant cutoff), g^L can only be the metric of S^3 up to an overall constant. Therefore, one component of g^L already fixes the prefactor.

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