

Gauged twistor formulation of a massive spinning particle in four dimensions

Shinichi Deguchi^{1,2,*} and Satoshi Okano^{2,†}

¹*Institute of Quantum Science, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-8308, Japan*

²*Department of Quantum Science and Technology, Graduate School of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-8308, Japan*

(Received 19 December 2015; published 12 February 2016)

We present a gauged twistor model of a free massive spinning particle in four-dimensional Minkowski space. This model is governed by an action, referred to here as the gauged generalized Shirafuji (GGS) action, that consists of twistor variables, auxiliary variables, and $U(1)$ and $SU(2)$ gauge fields on the one-dimensional parameter space of a particle's worldline. The GGS action remains invariant under reparametrization and the local $U(1)$ and $SU(2)$ transformations of the relevant variables, although the $SU(2)$ symmetry is nonlinearly realized. We consider the canonical Hamiltonian formalism based on the GGS action in the unitary gauge by following Dirac's recipe for constrained Hamiltonian systems. It is shown that just sufficient constraints for the twistor variables are consistently derived by virtue of the gauge symmetries of the GGS action. In the subsequent quantization procedure, these constraints turn into simultaneous differential equations for a twistor function. We perform the Penrose transform of this twistor function to define a massive spinor field of arbitrary rank, demonstrating that the spinor field satisfies generalized Dirac-Fierz-Pauli equations with $SU(2)$ indices. We also investigate the rank-one spinor fields in detail to clarify the physical meanings of the $U(1)$ and $SU(2)$ symmetries.

DOI: [10.1103/PhysRevD.93.045016](https://doi.org/10.1103/PhysRevD.93.045016)

I. INTRODUCTION

Twistor theory is basically appropriate for describing massless systems with conformal symmetry [1–3]. Nevertheless, there have been some approaches to formulating massive particle systems in terms of twistors [4–17]. For describing a massive particle, it is common to use two or more independent twistors. In fact, introducing two twistors has been considered until recently [9–17], and introducing more than two twistors was considered in some earlier studies [4–8]. By virtue of using two or more independent twistors, an extra symmetry between the twistors occurs naturally in the system. Penrose, Perjés, and Hughston proposed the idea of identifying this symmetry with an internal symmetry in particle physics, such as weak isospin or flavor, toward explaining internal symmetries of elementary particles on the basis of twistor theory [4–8]. Although this idea is quite interesting, it seems that its detailed investigations have been made from neither a mechanical point of view nor a dynamical point of view. Therefore we would have to say that the idea is still poorly understood.

Lagrangian mechanics of a massive spinning particle formulated in terms of two twistors has been studied in Refs. [9–17]. Most of these papers begin with generalization of the Shirafuji action that describes a free massless spinning particle in four dimensions in terms of a twistor [18]. In fact,

various generalizations of the Shirafuji action have been presented to specify twistorial models of massive spinning particles. The generalized Shirafuji actions are constructed by incorporating a mass-shell condition of a particle and certain other conditions for the twistor variables. The canonical formalism based on each generalized Shirafuji action and its subsequent quantization were also studied in Refs. [9,11,12,14–17]. It was shown that the canonical quantization of each twistorial model leads to generalized Dirac equations or the Dirac-Fierz-Pauli (DFP) equations for massive spinor fields of arbitrary rank [19–21]. (The supersymmetric Shirafuji action [18] that is written in terms of a supertwistor and describes a massless superparticle in four dimensions has been generalized to the twistorial actions for massive superparticles in four dimensions [13,14,17]. In this paper, however, we are not concerned with the supersymmetric cases.)

In this paper, we consider an alternative generalization of the Shirafuji action to define a new twistor model of a free massive spinning particle in four dimensions by using two twistors. Our formulation is precisely a non-Abelian extension of the gauged twistor formulation of a free massless spinning particle in four dimensions [22–24]. In the gauged twistor formulation, the Shirafuji action is modified in accordance with the gauge principle so that it can become invariant under the local $U(1)$ (phase) transformation of twistor variables. Here “local” means that the transformation parameter depends on a worldline parameter along the particle's worldline. This modification is accomplished by

*deguchi@phys.cst.nihon-u.ac.jp

†okano@phys.cst.nihon-u.ac.jp

gauging the Shirafuji action with the aid of a $U(1)$ gauge field on the one-dimensional (1D) parameter space of the worldline and by adding the 1D Chern-Simons term consisting of the $U(1)$ gauge field. The modified action, named the gauged Shirafuji action, includes a helicity constraint term due to the modification. Hence it follows that this action describes a free massless spinning particle with a fixed value of helicity. Remarkably, the gauged Shirafuji action is equivalent to the action for a massless particle with rigidity, at least at the classical mechanical level [25]. The Shirafuji action can furthermore be modified so as to be invariant under the local scale transformation of twistor variables with the aid of another gauge field on the 1D parameter space. From the point of view of twistor theory, it is desirable that the modified action remains invariant under the combination of the local $U(1)$ and local scale transformations, which is referred to in Refs. [23,24] as the complexified local scale transformation. In actuality, the gauge field for the local scale transformation can be gauged away by a scaling of the twistor variables. Therefore it turns out that only the local $U(1)$ transformation is essential and one does not need to consider the local scale transformation in practice.

In the next section, we begin with setting up a generalized Shirafuji action that consists of two twistors and involves a mass-shell condition. Here, for convenience, we exploit the mass-shell condition with a complexified mass parameter introduced in Refs. [15,16]. The generalized Shirafuji action remains invariant under the global $U(1)$ transformation of twistor variables supplemented with that of auxiliary fields on the 1D parameter space. In addition, the generalized Shirafuji action remains invariant under the global $SU(2)$ transformation defined for a doublet of twistors. In accordance with the gauge principle, we modify the generalized Shirafuji action in such a way that the modified action remains invariant under the local $U(1)$ and $SU(2)$ transformations of twistor variables. The modification is performed by gauging the generalized Shirafuji action with the aid of $U(1)$ and $SU(2)$ gauge fields on the 1D parameter space and by adding the 1D $U(1)$ and $SU(2)$ Chern-Simons terms. The 1D $SU(2)$ Chern-Simons term, however, vanishes owing to the traceless property of the $SU(2)$ gauge field. For this reason, the variation of the modified action with respect to the $SU(2)$ gauge field yields too strong constraints that, after quantizing the model, permit us to have only massive spinless fields in four dimensions. A similar consequence has been found by Fedoruk and Lukierski in their twistorial model of a massive particle [15]. To overcome such an undesirable situation, they modified the model by incorporating the Souriau-Wess-Zumino term, following the successful argument for a twistorial model of a massive spinning particle in three dimensions [14]. In the present paper, we consider an alternative approach based on a nonlinear realization of $SU(2)$ to eventually obtain massive spinor fields of arbitrary rank. This approach makes it possible to define

the 1D $U(1)$ Chern-Simons term consisting of the third (or diagonal) component of the $SU(2)$ gauge field in a particular gauge. In addition, this approach can provide a novel gauge-invariant term consisting of the first and second (or off-diagonal) components of the same $SU(2)$ gauge field. With the new terms, we furthermore modify the generalized Shirafuji action by adding these terms to the modified action mentioned above. The completely modified action is thus the sum of the gauged twistorial part, the two 1D $U(1)$ Chern-Simons terms, and the novel term. This action, hereafter referred to as the gauged generalized Shirafuji (GGS) action, remains invariant under reparametrization of the worldline parameter and under the local $U(1)$ and $SU(2)$ transformations. The GGS action yields just sufficient constraints for the twistor variables in a systematic and consistent manner. All the constraints except for the mass-shell condition are derived on the basis of the gauge symmetry. This is an advantage of our gauged twistor model.

Having obtained the GGS action, we study the canonical Hamiltonian formalism based on it by completely following the Dirac algorithm for Hamiltonian systems with constraints [31–33]. We see that most of the Dirac brackets between the twistor variables take on complicated forms. Fortunately, these Dirac brackets can be reduced to simple Dirac brackets for new twistor variables that are in one-to-one correspondence with the old ones. Also, all the constraints for the (old) twistor variables can be written completely in terms of the new twistor variables. The canonical quantization of the twistor model governed by the GGS action is performed with the commutation relations between the operators that correspond to the new twistor variables or the other canonical variables. Some of the first-class constraints eventually turn into simultaneous differential equations for a holomorphic function of half the new twistor variables. Each solution of the simultaneous differential equations, referred to here as a twistor function, is characterized by the three quantum numbers that originate from the $U(1)$ and $SU(2)$ symmetries inherent in the GGS action.

We also consider the Penrose transform of the twistor function to define a four-dimensional spinor field of arbitrary rank. The spinor field defined in this manner has extra upper and lower $SU(2)$ indices in addition to dotted and undotted spinor indices. Because of the structure of the Penrose transform, the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. We demonstrate that the present spinor field satisfies generalized DFP equations with $SU(2)$ indices. In the simplest case, the generalized DFP equations reduce to the ordinary Dirac equations for particle and antiparticle spinor fields. Investigating properties of these fields, we clarify the physical meanings of the $U(1)$ and $SU(2)$ symmetries; ultimately, we see that the $U(1)$ symmetry is a gauge symmetry concerning the chiralities of the particle and antiparticle spinor fields, while the $SU(2)$ symmetry is a gauge symmetry realized in a doublet

consisting of the particle and antiparticle spinor fields. Therefore it turns out that the idea proposed by Penrose, Perjés, and Hughston, in which the $SU(2)$ symmetry is identified with the weak isospin symmetry, is not valid in our gauged twistor formulation.

This paper is organized as follows. In Sec. II, we elaborate the GGS action, after making some preliminary arrangements. The canonical Hamiltonian formalism based on the GGS action is studied in Sec. III, and the subsequent canonical quantization is performed in Sec. IV. In Sec. V, we define a massive spinor field of arbitrary rank by the Penrose transform of a twistor function and demonstrate that this spinor field satisfies the generalized DFP equations. In Sec. VI, we particularly investigate the rank-one spinor fields to clarify the physical meanings of the $U(1)$ and $SU(2)$ symmetries. Section VII is devoted to a summary and discussion. In the Appendix, we treat the Pauli-Lubanski pseudovector written in terms of the twistor variables.

II. CONSTRUCTION OF THE GGS ACTION

In this section, we construct the GGS action for a free massive spinning particle in four-dimensional Minkowski space.

In order to describe a massive particle in terms of twistors, we introduce two twistors $Z_i^A = (\omega_i^\alpha, \pi_{i\dot{\alpha}})$ ($A = 0, 1, 2, 3; \alpha = 0, 1; \dot{\alpha} = \dot{0}, \dot{1}$) distinguished by the extra index i ($i = 1, 2$) and their dual twistors $\bar{Z}_A^i = (\bar{\pi}_\alpha^i, \bar{\omega}^{i\dot{\alpha}})$. Here, $\bar{\pi}_\alpha^i$ and $\bar{\omega}^{i\dot{\alpha}}$ denote the complex conjugates of $\pi_{i\dot{\alpha}}$ and ω_i^α , respectively: $\bar{\pi}_\alpha^i := \overline{\pi_{i\dot{\alpha}}}$, $\bar{\omega}^{i\dot{\alpha}} := \overline{\omega_i^\alpha}$. It is assumed that Z_1^A and Z_2^A are not proportional to each other: $Z_1^A \neq cZ_2^A$ ($c \in \mathbb{C}$), so that $\bar{Z}_A^1 \neq \bar{c}\bar{Z}_A^2$. The 2-component spinors ω_i^α and $\pi_{i\dot{\alpha}}$ are related by

$$\omega_i^\alpha = iz^{\alpha\dot{\alpha}}\pi_{i\dot{\alpha}}, \quad (2.1)$$

where $z^{\alpha\dot{\alpha}}$ are coordinates of a point in complexified Minkowski space, \mathbb{CM} , with the metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. As can be seen in the literature on twistor theory [4–16], the four-momentum of a massive particle is expressed as $p_{\alpha\dot{\alpha}} = \bar{\pi}_\alpha^1\pi_{1\dot{\alpha}} + \bar{\pi}_\alpha^2\pi_{2\dot{\alpha}} \equiv \bar{\pi}_\alpha^i\pi_{i\dot{\alpha}}$. (For this reason, $\pi_{i\dot{\alpha}}$ and $\bar{\pi}_\alpha^i$ are named as momentum spinors.) The squared norm of $p_{\alpha\dot{\alpha}}$ remains nonvanishing even after using the formula $\pi_{i\dot{\alpha}}\pi_i^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\beta}\pi_{i\dot{\alpha}}\pi_{i\dot{\beta}} = 0$ (no sum with respect to i) and its complex conjugate,¹ because the cross

¹The two-dimensional Levi-Civita symbols $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$, ϵ^{ij} , and ϵ_{ij} are defined as $\epsilon^{01} = \epsilon_{01} = \epsilon^{\dot{0}\dot{1}} = \epsilon_{\dot{0}\dot{1}} = \epsilon^{12} = \epsilon_{12} = 1$ and conform to the rules $\overline{\epsilon^{\alpha\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}$, $\overline{\epsilon_{\alpha\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}$, $\overline{\epsilon^{ij}} = \epsilon_{ij}$, and $\overline{\epsilon_{ij}} = \epsilon^{ij}$. The contravariant spinors $\pi_i^{\dot{\alpha}}$ and $\bar{\pi}^{i\alpha}$ are defined by $\pi_i^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\beta}\pi_{i\dot{\beta}}$ and $\bar{\pi}^{i\alpha} = \epsilon^{\alpha\beta}\bar{\pi}_\beta^i$, respectively. These relations can be expressed as $\pi_{i\dot{\alpha}} = \pi_i^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}}$ and $\bar{\pi}_\alpha^i = \bar{\pi}^{i\beta}\epsilon_{\beta\alpha}$.

terms provided from different twistors still survive: $p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} = \bar{\pi}_\alpha^i\pi_{i\dot{\alpha}}\bar{\pi}^{j\alpha}\pi_j^{\dot{\alpha}} = 2|\pi_{1\dot{\alpha}}\pi_2^{\dot{\alpha}}|^2$. Thus the mass-shell condition $p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} = m^2$ with a mass parameter m can be written as

$$\bar{\pi}_\alpha^i\pi_{i\dot{\alpha}}\bar{\pi}^{j\alpha}\pi_j^{\dot{\alpha}} = m^2. \quad (2.2)$$

It is easy to see that this condition is equivalent to

$$\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi} = 0, \quad (2.3a)$$

$$\epsilon_{ij}\bar{\pi}_\alpha^i\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi} = 0, \quad (2.3b)$$

where φ is a real parameter. These equations have been incorporated in twistorial models of massive spinning particles [15,16], in which $me^{i\varphi}/\sqrt{2}$ is called a complexified mass parameter. In this paper, we also adopt the equation pair Eqs. (2.3a) and (2.3b) as the mass-shell condition because of the convenience for our formulation.

The Shirafuji action of a free massless spinning particle² can be generalized to describe a free spinning particle of mass m propagating in four-dimensional Minkowski space \mathbf{M} . A generalized Shirafuji action is indeed given by

$$S_m = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A^i \dot{Z}_i^A - Z_i^A \dot{\bar{Z}}_A^i) + h(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}) + \bar{h}(\epsilon_{ij}\bar{\pi}_\alpha^i\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}) \right], \quad (2.4)$$

where $Z_i^A = Z_i^A(\tau)$ and $\bar{Z}_A^i = \bar{Z}_A^i(\tau)$ are understood as complex scalar fields on the one-dimensional parameter space $\mathcal{T} := \{\tau|\tau_0 \leq \tau \leq \tau_1\}$ of a particle's worldline, and $h = h(\tau)$ is treated as a complex scalar-density field of weight 1 on \mathcal{T} . [That is, h transforms as $h(\tau) \rightarrow h'(\tau') = (d\tau/d\tau')h(\tau)$ under the proper reparametrization $\tau \rightarrow \tau' = \tau'(\tau)$ ($d\tau'/d\tau > 0$).] The exponent φ is now considered a real scalar field on \mathcal{T} and hence is treated as a real function $\varphi = \varphi(\tau)$. This setting is different from that in Refs. [15,16], in which the complexified mass parameter is regarded as a constant. A dot over a variable denotes its derivative with respect to τ .

²With a twistor Z^A and its dual twistor \bar{Z}_A , the Shirafuji action is defined by [18]

$$S_0 = \int_{\tau_0}^{\tau_1} d\tau \frac{i}{2} (\bar{Z}_A \dot{Z}^A - Z^A \dot{\bar{Z}}_A).$$

The variation of S_m with respect to h and \bar{h} yields the mass-shell condition (2.3).³

The generalized Shirafuji action S_m remains invariant under the reparametrization $\tau \rightarrow \tau' = \tau'(\tau)$. In addition, S_m remains invariant under the global $U(1)$ transformation

$$Z_i^A \rightarrow Z_i'^A = e^{i\theta} Z_i^A, \quad \bar{Z}_A^i \rightarrow \bar{Z}_A'^i = e^{-i\theta} \bar{Z}_A^i, \quad (2.5a)$$

$$h \rightarrow h' = e^{-2i\theta} h, \quad \bar{h} \rightarrow \bar{h}' = e^{2i\theta} \bar{h}, \quad (2.5b)$$

$$\varphi \rightarrow \varphi' = \varphi + 2\theta, \quad (2.5c)$$

with a real constant parameter θ and under the global $SU(2)$ transformation

$$Z_i^A \rightarrow Z_i'^A = U_i^j Z_j^A, \quad \bar{Z}_A^i \rightarrow \bar{Z}_A'^i = \bar{Z}_A^j U^\dagger_j{}^i, \quad (2.6a)$$

$$h \rightarrow h' = h, \quad \bar{h} \rightarrow \bar{h}' = \bar{h}, \quad (2.6b)$$

$$\varphi \rightarrow \varphi' = \varphi, \quad (2.6c)$$

with a constant matrix U belonging to $SU(2)$. The $SU(2)$ invariance of S_m can be verified using $e^{ij} U_i^k U_j^\dagger{}^l = e^{kl}$ and $\epsilon_{ij} U^\dagger_k{}^i U^\dagger_l{}^j = \epsilon_{kl}$ together with the unitarity property of U . We thus see that S_m possesses two global internal symmetries specified by $U(1)$ and $SU(2)$. We also see that the two terms $\bar{Z}_A^i \dot{Z}_i^A$ and $Z_i^A \dot{\bar{Z}}_A^i$ in Eq. (2.4) are invariant under the global $SU(2, 2)$ transformation (or more simply, the global conformal transformation) $Z_i^A \rightarrow Z_i'^A = \mathcal{U}^A_B Z_i^B$, $\bar{Z}_A^i \rightarrow \bar{Z}_A'^i = \bar{Z}_B^i \mathcal{U}^\dagger{}^B_A$, with a constant matrix \mathcal{U} belonging to $SU(2, 2)$. In contrast, the two terms $\epsilon^{ij} \pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}}$ and $\epsilon_{ij} \bar{\pi}_\alpha^i \bar{\pi}^{j\alpha}$ in Eq. (2.4) are invariant only under the global $SL(2, \mathbb{C}) \times \mathbb{R}^{1,3}$ transformation (or more simply, the global Poincaré transformation). Hence it turns out that the symmetry reduction from $SU(2, 2)$ to $SL(2, \mathbb{C}) \times \mathbb{R}^{1,3}$ occurs in S_m as a result of adding the term proportional to h and its complex conjugate term.

Now, we perform a gauging of the global $U(1)$ and $SU(2)$ symmetries in such a way that the gauged action remains invariant under the local $U(1)$ and $SU(2)$ transformations that depend on τ . That is, we consider a $U(1) \times SU(2)$ gauge theory on the parameter space \mathcal{T} . To this end, in accordance with the gauge principle, we introduce a $U(1)$ gauge field, $a = a(\tau)$, and an $SU(2)$ gauge field, $b = b(\tau)$. The field a is assumed to be a real scalar-density

field of weight 1 on \mathcal{T} , while b is assumed to be a 2×2 traceless Hermitian matrix that behaves as a scalar-density field of weight 1 on \mathcal{T} . The field b can be represented as (b_i^j) with its matrix elements b_i^j and can be expanded in terms of the Pauli matrices σ_r ($r = 1, 2, 3$), satisfying $[\sigma_r, \sigma_s] = 2i\epsilon_{rst}\sigma_t$, as $b = b^r \sigma_r$. Here, $b^r = b^r(\tau)$ are real scalar-density fields of weight 1 on \mathcal{T} . The (primitive) gauged action, S_{mg} , can be obtained by replacing $d/d\tau$ in S_m with a covariant derivative operator as follows:

$$S_{\text{mg}} = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A^i D Z_i^A - Z_i^A \bar{D} \bar{Z}_A^i) + h(\epsilon^{ij} \pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi}) + \bar{h}(\epsilon_{ij} \bar{\pi}_\alpha^i \bar{\pi}^{j\alpha} - \sqrt{2} m e^{-i\varphi}) \right], \quad (2.7)$$

where

$$D Z_i^A := \dot{Z}_i^A - ia Z_i^A - ib_i^j Z_j^A, \quad (2.8a)$$

$$\bar{D} \bar{Z}_A^i := \dot{\bar{Z}}_A^i + ia \bar{Z}_A^i + i \bar{Z}_A^j b_j^i. \quad (2.8b)$$

We see that the action S_{mg} is reparametrization invariant. It can easily be verified that S_{mg} remains invariant under the local $U(1)$ transformation

$$Z_i^A \rightarrow Z_i'^A = e^{i\theta(\tau)} Z_i^A, \quad (2.9a)$$

$$\bar{Z}_A^i \rightarrow \bar{Z}_A'^i = e^{-i\theta(\tau)} \bar{Z}_A^i, \quad (2.9b)$$

$$h \rightarrow h' = e^{-2i\theta(\tau)} h, \quad (2.9c)$$

$$\bar{h} \rightarrow \bar{h}' = e^{2i\theta(\tau)} \bar{h}, \quad (2.9d)$$

$$\varphi \rightarrow \varphi' = \varphi + 2\theta(\tau), \quad (2.9e)$$

$$a \rightarrow a' = a + \dot{\theta}, \quad (2.9f)$$

$$b \rightarrow b' = b, \quad (2.9g)$$

with a real gauge function $\theta = \theta(\tau)$ and under the local $SU(2)$ transformation

$$Z_i^A \rightarrow Z_i'^A = U_i^j(\tau) Z_j^A, \quad (2.10a)$$

$$\bar{Z}_A^i \rightarrow \bar{Z}_A'^i = \bar{Z}_A^j U^\dagger_j{}^i(\tau), \quad (2.10b)$$

$$h \rightarrow h' = h, \quad (2.10c)$$

$$\bar{h} \rightarrow \bar{h}' = \bar{h}, \quad (2.10d)$$

$$\varphi \rightarrow \varphi' = \varphi, \quad (2.10e)$$

³Instead of the action S_m , we can consider an alternative action

$$S'_m = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A^i \dot{Z}_i^A - Z_i^A \dot{\bar{Z}}_A^i) + \frac{1}{2} f(\bar{\pi}_\alpha^i \pi_{i\dot{\alpha}} \bar{\pi}^{j\alpha} \pi_j^{\dot{\alpha}} - m^2) \right],$$

where $f = f(\tau)$ is a real scalar-density field of weight 1 on \mathcal{T} . The variation of S'_m with respect to f yields the mass-shell condition (2.2).

$$a \rightarrow a' = a, \quad (2.10f)$$

$$b \rightarrow b' = UbU^\dagger - i\dot{U}U^\dagger, \quad (2.10g)$$

with a gauge function $U = U(\tau)$ taking its value in $SU(2)$. Because each of a and b^r is a single-component gauge field associated with $d/d\tau$, we cannot define their field strengths. For this reason, there exists neither the Maxwell action for a nor the Yang-Mills action for b . As for a , it is possible to define the (nonvanishing) 1D $U(1)$ Chern-Simons term

$$S_a = -2s \int_{\tau_0}^{\tau_1} d\tau a, \quad (2.11)$$

where s is a real constant. The 1D $SU(2)$ Chern-Simons term for b , i.e., $S_b = -2t \int_{\tau_0}^{\tau_1} d\tau \text{Tr} b$ vanishes by the reason of $\text{Tr} b = 0$. Since a is a scalar-density field of weight 1, S_a is reparametrization invariant. Also, S_a remains invariant under the gauge transformation (2.9f), provided that θ satisfies an appropriate boundary condition such as $\theta(\tau_1) = \theta(\tau_0)$. The $SU(2)$ invariance of S_a is evident from Eq. (2.10f). Therefore we can consider the reparametrization-invariant and gauge-invariant action $\tilde{S}_{mg} := S_{mg} + S_a$.⁴ However, \tilde{S}_{mg} eventually turns out to govern only massive *spinless* fields in four dimensions owing to the too strong constraints $\bar{Z}_A \sigma_{ri}{}^j Z_j^A = 0$ ($r = 1, 2, 3$) that are derived by varying \tilde{S}_{mg} with respect to b^r .⁵ [Here, $\sigma_{rj}{}^k$ denotes the (j, k) entry of the Pauli matrix σ_r .] To avoid such an undesirable situation, next we perform a modification of \tilde{S}_{mg} with the aid of a nonlinear realization of $SU(2)$.

⁴The action \tilde{S}_{mg} is a simple and natural generalization of the gauged Shirafuji action (without invariance under the local scale transformation of Z^A and \bar{Z}_A)

$$\tilde{S}_{0g} = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A D Z^A - Z^A \bar{D} \bar{Z}_A) - 2sa \right],$$

where $D := d/d\tau - ia$. This action describes a free massless spinning particle of helicity s [22–24] and is equivalent to the action for a massless particle with rigidity at least at the classical mechanical level [25].

⁵From the action S_{mg} , the Pauli-Lubanski spin vector $W^{a\dot{a}}$ is found to be

$$W^{a\dot{a}} = T_r \sigma_{ri}{}^j \bar{\pi}^{i\dot{a}} \pi_j^{\dot{a}}, \quad T_r := \frac{1}{2} \bar{Z}_A^i \sigma_{ri}{}^j Z_j^A$$

(see Appendix). Using the mass-shell condition (2.3), we can show that $W_{a\dot{a}} W^{a\dot{a}} = -m^2 T_r T_r$. Obviously, $T_r = 0$ ($r = 1, 2, 3$) leads to $W_{a\dot{a}} W^{a\dot{a}} = 0$. Hence, it follows that only massive spinless particles are admissible in the model defined by \tilde{S}_{mg} . Accordingly, it turns out that only massive spinless fields are provided after quantizing the model.

Let us now consider the coset space $SU(2)/U(1)$ ($\cong \mathbb{C}\mathbb{P}^1$) and representative elements, $V(\xi, \bar{\xi})$ ($V \in SU(2)$, $\xi \in \mathbb{C}$), that are chosen one by one from each left coset of $U(1)$ in $SU(2)$. Here, ξ labels the cosets in a way of one-to-one correspondence and can be regarded as an inhomogeneous coordinate of a point on $SU(2)/U(1)$. [To completely coordinatize $SU(2)/U(1)$, it is necessary to use ξ^{-1} in addition to ξ .] The representative elements $V(\xi, \bar{\xi})$ are assumed to constitute a smooth function of ξ and $\bar{\xi}$ so that we can simply treat $V(\xi, \bar{\xi})$ as an $SU(2)$ -valued smooth function. We consider ξ to be a complex scalar field $\xi = \xi(\tau)$ on \mathcal{T} . The left action of U on $V(\xi, \bar{\xi})$ generates a nonlinear transformation $\xi \rightarrow \xi' = \xi'(\xi)$ in accordance with

$$V(\xi, \bar{\xi}) \rightarrow V(\xi', \bar{\xi}') = U(\tau) V(\xi, \bar{\xi}) \Theta^{-1}(\tau), \quad (2.12)$$

where $\Theta(\tau) := \exp\{i\vartheta(\tau)\sigma_3\}$, and $\vartheta = \vartheta(\tau)$ is a real gauge function [26–28]. Note here that ϑ is determined depending on $(\xi, \bar{\xi})$ as well as U . Using $V = V(\xi, \bar{\xi})$, we define the following new fields on \mathcal{T} :

$$Z_i^A := V^\dagger{}_i{}^j Z_j^A, \quad \bar{Z}_A^i := \bar{Z}_A^j V_j^i, \quad (2.13a)$$

$$\mathbf{b} := V^\dagger b V - i\dot{V}^\dagger V. \quad (2.13b)$$

The field \mathbf{b} can be expanded as $\mathbf{b} = \mathbf{b}^r \sigma_r$, where $\mathbf{b}^r = \mathbf{b}^r(\tau)$ are real fields. Clearly, \mathbf{b}^r behave as scalar-density fields of weight 1 on \mathcal{T} . With the new fields, the local $U(1)$ transformation (2.9) reads

$$Z_i^A \rightarrow Z_i'^A = e^{i\theta(\tau)} Z_i^A, \quad (2.14a)$$

$$\bar{Z}_A^i \rightarrow \bar{Z}_A'^i = e^{-i\theta(\tau)} \bar{Z}_A^i, \quad (2.14b)$$

$$h \rightarrow h' = e^{-2i\theta(\tau)} h, \quad (2.14c)$$

$$\bar{h} \rightarrow \bar{h}' = e^{2i\theta(\tau)} \bar{h}, \quad (2.14d)$$

$$\varphi \rightarrow \varphi' = \varphi + 2\theta(\tau), \quad (2.14e)$$

$$a \rightarrow a' = a + \dot{\theta}, \quad (2.14f)$$

$$\mathbf{b} \rightarrow \mathbf{b}' = \mathbf{b}. \quad (2.14g)$$

On the other hand, from Eqs. (2.10) and (2.12), we have

$$Z_i^A \rightarrow Z_i'^A = \Theta_i{}^j(\tau) Z_j^A, \quad (2.15a)$$

$$\bar{Z}_A^i \rightarrow \bar{Z}_A'^i = \bar{Z}_A^j \Theta^i{}_j(\tau), \quad (2.15b)$$

$$h \rightarrow h' = h, \quad (2.15c)$$

$$\bar{h} \rightarrow \bar{h}' = \bar{h}, \quad (2.15d)$$

$$\varphi \rightarrow \varphi' = \varphi, \quad (2.15e)$$

$$a \rightarrow a' = a, \quad (2.15f)$$

$$\mathbf{b} \rightarrow \mathbf{b}' = \Theta \mathbf{b} \Theta^\dagger + \dot{\vartheta} \sigma_3. \quad (2.15g)$$

Equation (2.15) is precisely a local $U(1)$ transformation. Hereafter, we refer to the local $U(1)$ transformation specified by Eq. (2.9), or Eq. (2.14), as the $U(1)_a$ transformation and refer to that specified by Eq. (2.15) as the $U(1)_b$ transformation. Their corresponding gauge groups are simply denoted as $U(1)_a$ and $U(1)_b$. The local $SU(2)$ transformation is not manifestly seen in Eq. (2.15); instead, it is realized as a nonlinear transformation of ξ . We may say that the function V converts the local $SU(2)$ transformation into the $U(1)_b$ transformation while ξ undergoes a nonlinear transformation. Equation (2.15g) defines the transformation rules of the fields \mathbf{b}' ,

$$\mathbf{b}^1 \rightarrow \mathbf{b}'^1 = \mathbf{b}^1 \cos 2\vartheta + \mathbf{b}^2 \sin 2\vartheta, \quad (2.16a)$$

$$\mathbf{b}^2 \rightarrow \mathbf{b}'^2 = -\mathbf{b}^1 \sin 2\vartheta + \mathbf{b}^2 \cos 2\vartheta, \quad (2.16b)$$

$$\mathbf{b}^3 \rightarrow \mathbf{b}'^3 = \mathbf{b}^3 + \dot{\vartheta}. \quad (2.16c)$$

We see that \mathbf{b}^i ($i = 1, 2$) transform homogeneously, obeying together an $SO(2)$ rotation, while \mathbf{b}^3 transforms inhomogeneously as a $U(1)$ gauge field.

Now, we can provide the following two terms:

$$S_{\mathbf{b}12} = -k \int_{\tau_0}^{\tau_1} d\tau \sqrt{\mathbf{b}^i \mathbf{b}^i}, \quad (2.17)$$

with $\mathbf{b}^i \mathbf{b}^i := (\mathbf{b}^1)^2 + (\mathbf{b}^2)^2$, and

$$S_{\mathbf{b}3} = -2t \int_{\tau_0}^{\tau_1} d\tau \mathbf{b}^3. \quad (2.18)$$

Here, k is a positive constant and t is a real constant. Since \mathbf{b}^r are scalar-density fields of weight 1 on \mathcal{T} , both $S_{\mathbf{b}12}$ and $S_{\mathbf{b}3}$ are reparametrization invariant. It is obvious that $S_{\mathbf{b}12}$ remains invariant under the $SO(2)$ rotation defined by Eqs. (2.16a) and (2.16b). Also, $S_{\mathbf{b}3}$, which is the 1D Chern-Simons term for \mathbf{b}^3 , remains invariant under the gauge transformation (2.16c), provided that ϑ satisfies an appropriate boundary condition such as $\vartheta(\tau_1) = \vartheta(\tau_0)$. We thus see that both $S_{\mathbf{b}12}$ and $S_{\mathbf{b}3}$ possess the $U(1)_b$ symmetry. The $U(1)_a$ invariance of $S_{\mathbf{b}12}$ and $S_{\mathbf{b}3}$ is evident from Eq. (2.14g). For our investigation, it is convenient to express $S_{\mathbf{b}12}$ as

$$S_{\mathbf{b}e} = - \int_{\tau_0}^{\tau_1} d\tau \left(\frac{1}{2e} \mathbf{b}^i \mathbf{b}^i + \frac{k^2}{2} e \right) \quad (2.19)$$

with the aid of $e = e(\tau)$ being a positive scalar-density field of weight 1 on \mathcal{T} . It is assumed that e does not change

under the $U(1)_a$ and $U(1)_b$ transformations. (At this stage, we should include the transformation rule $e \rightarrow e' = e$ in each of Eqs. (2.14) and (2.15).) The action S_{mg} can be rewritten in terms of Z_i^A , \bar{Z}_A^i , h , \bar{h} , φ , a , and \mathbf{b} . The resulting rewritten expression of S_{mg} is precisely what is obtained by replacing Z_i^A , \bar{Z}_A^i , and b in Eq. (2.7) with Z_i^A , \bar{Z}_A^i , and \mathbf{b} , respectively. With this expression, we modify $\tilde{S}_{mg} = S_{mg} + S_a$ by adding S_{be} and S_{b3} to it. That is, we consider the modified action $S := S_{mg} + S_a + S_{be} + S_{b3}$, or more precisely,

$$S = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A^i D Z_i^A - Z_i^A \bar{D} \bar{Z}_A^i) - 2sa - 2t\mathbf{b}^3 - \frac{1}{2e} \mathbf{b}^i \mathbf{b}^i - \frac{k^2}{2} e + h(\epsilon^{ij} \varpi_{i\dot{\alpha}} \bar{\varpi}_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi}) + \bar{h}(\epsilon_{ij} \bar{\varpi}_\alpha^i \varpi^{j\alpha} - \sqrt{2} m e^{-i\varphi}) \right], \quad (2.20)$$

with

$$D Z_i^A := \dot{Z}_i^A - ia Z_i^A - ib_i^j Z_j^A, \quad (2.21a)$$

$$\bar{D} \bar{Z}_A^i := \dot{\bar{Z}}_A^i + ia \bar{Z}_A^i + i \bar{Z}_A^j b_j^i. \quad (2.21b)$$

Here, $\varpi_{i\dot{\alpha}}$ and $\bar{\varpi}_\alpha^i$ are momentum-spinor components of the twistors $Z_i^A = (q_i^\alpha, \varpi_{i\dot{\alpha}})$ and $\bar{Z}_A^i = (\bar{\varpi}_\alpha^i, \bar{q}^{i\dot{\alpha}})$, respectively. We refer to S as the GGS action. From Eq. (2.13a), it follows that $\varpi_{i\dot{\alpha}} = V^\dagger_i{}^j \pi_{j\dot{\alpha}}$ and $\bar{\varpi}_\alpha^i = \bar{\pi}_\alpha^j V_j^i$. The other components are given by $q_i^\alpha = V^\dagger_i{}^j \omega_j^\alpha$ and $\bar{q}^{i\dot{\alpha}} = \bar{\omega}^{j\dot{\alpha}} V_j^i$. It is now obvious that $\bar{\varpi}_{i\dot{\alpha}} = \bar{\varpi}_\alpha^i$ and $\bar{q}_i^\alpha = \bar{q}^{i\dot{\alpha}}$. In terms of Z_i^A , Eq. (2.1) can be written as

$$q_i^\alpha = iz^{\dot{\alpha}\alpha} \varpi_{i\dot{\alpha}}. \quad (2.22)$$

It is clear from (2.20) that the GGS action S remains invariant under the reparametrization and the $U(1)_a$ and $U(1)_b$ transformations. However, in actuality, S remains invariant under the reparametrization and the $U(1)_a$ and local $SU(2)$ transformations, because the $U(1)_b$ transformation is induced by the local $SU(2)$ transformation in accordance with Eq. (2.12). In fact, we can express S in a manifestly $SU(2)$ invariant form as follows:

$$S = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A^i D Z_i^A - Z_i^A \bar{D} \bar{Z}_A^i) - 2sa - 2t(b^r \mathcal{V}_r^3 - \dot{\xi} e_\xi^3 - \dot{\bar{\xi}} e_{\bar{\xi}}^3) - \frac{1}{e} g_{\xi\bar{\xi}} D \xi D \bar{\xi} - \frac{k^2}{2} e + h(\epsilon^{ij} \pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi}) + \bar{h}(\epsilon_{ij} \bar{\pi}_\alpha^i \pi^{j\alpha} - \sqrt{2} m e^{-i\varphi}) \right], \quad (2.23)$$

with $g_{\xi\bar{\xi}} := e_{\xi}^{\hat{i}} e_{\bar{\xi}}^{\hat{i}}$, $D\xi := \dot{\xi} - b^r K_r^{\xi}$, and $D\bar{\xi} := \dot{\bar{\xi}} - b^r K_r^{\bar{\xi}}$. Here, $g_{\xi\bar{\xi}}$ is a metric on $SU(2)/U(1)$, $(K_r^{\xi}, K_r^{\bar{\xi}})$ ($r = 1, 2, 3$) are the $SU(2)$ Killing vectors on this coset space, and e_{ξ}^r and $e_{\bar{\xi}}^r$ ($r = \hat{i}, 3$) are defined by $e_{\xi}^r \sigma_r = -iV^{\dagger}(\partial V/\partial \xi)$ and $e_{\bar{\xi}}^r \sigma_r = -iV^{\dagger}(\partial V/\partial \bar{\xi})$, respectively. Also, \mathcal{V}_r^3 is defined according to $V^{\dagger} \sigma_r V = \mathcal{V}_r^{\hat{i}} \sigma_{\hat{i}} + \mathcal{V}_r^3 \sigma_3$. Using the transformation rule (2.12), we can show that $\mathcal{V}_r^{\hat{i}} = K_r^{\xi} e_{\xi}^{\hat{i}} + K_r^{\bar{\xi}} e_{\bar{\xi}}^{\hat{i}}$. In addition, it can be verified that $K_r := K_r^{\xi} \partial/\partial \xi + K_r^{\bar{\xi}} \partial/\partial \bar{\xi}$ satisfy the $SU(2)$ commutation relations. In the expression (2.20), we should understand that the local $SU(2)$ symmetry of S is hidden rather than is broken, because no symmetry breaking mechanisms are incorporated in the model. The action (2.20) can be regarded as the action (2.23) in a particular gauge $\xi(\tau) = \xi_0$, where ξ_0 is a constant such that $V(\xi_0, \bar{\xi}_0) = 1$. We term this gauge the unitary gauge, because it corresponds to the so-called unitary gauge in massive Yang-Mills theory [29,30]. Then \mathbf{b} can be said to be the $SU(2)$ gauge field in the unitary gauge. The action (2.20) can be written as

$$S = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{\mathbf{Z}}_A^i \dot{\mathbf{Z}}_i^A - \mathbf{Z}_i^A \dot{\bar{\mathbf{Z}}}_A^i) + a(\bar{\mathbf{Z}}_A^i \mathbf{Z}_i^A - 2s) + \mathbf{b}^3 (\bar{\mathbf{Z}}_A^j \sigma_{3j}^k \mathbf{Z}_k^A - 2t) + \mathbf{b}^i \bar{\mathbf{Z}}_A^j \sigma_{ij}^k \mathbf{Z}_k^A - \frac{1}{2\mathbf{e}} \mathbf{b}^i \mathbf{b}^i - \frac{k^2}{2} \mathbf{e} + h(\epsilon^{ij} \omega_{i\alpha} \omega_j^{\alpha} - \sqrt{2} m e^{i\varphi}) + \bar{h}(\epsilon_{ij} \bar{\omega}_{\alpha}^i \bar{\omega}^{j\alpha} - \sqrt{2} m e^{-i\varphi}) \right]. \quad (2.24)$$

III. CANONICAL FORMALISM

In this section, we study the canonical Hamiltonian formalism of the model governed by the GGS action in the unitary gauge.

Let L be the Lagrangian defined in Eq. (2.24) as the integrand of the GGS action S . We treat the variables $(\mathbf{Z}_i^A, \bar{\mathbf{Z}}_A^i, a, \mathbf{b}^r, \mathbf{e}, h, \bar{h}, \varphi)$ as canonical coordinates. Their canonical conjugate momenta are found to be

$$P_A^i := \frac{\partial L}{\partial \dot{\mathbf{Z}}_i^A} = \frac{i}{2} \bar{\mathbf{Z}}_A^i, \quad (3.1a)$$

$$\bar{P}_i^A := \frac{\partial L}{\partial \dot{\bar{\mathbf{Z}}}_A^i} = -\frac{i}{2} \mathbf{Z}_i^A, \quad (3.1b)$$

$$P^{(a)} := \frac{\partial L}{\partial \dot{a}} = 0, \quad (3.1c)$$

$$P_r^{(b)} := \frac{\partial L}{\partial \dot{\mathbf{b}}^r} = 0, \quad (3.1d)$$

$$P^{(e)} := \frac{\partial L}{\partial \dot{\mathbf{e}}} = 0, \quad (3.1e)$$

$$P^{(h)} := \frac{\partial L}{\partial \dot{h}} = 0, \quad (3.1f)$$

$$P^{(\bar{h})} := \frac{\partial L}{\partial \dot{\bar{h}}} = 0, \quad (3.1g)$$

$$P^{(\varphi)} := \frac{\partial L}{\partial \dot{\varphi}} = 0. \quad (3.1h)$$

The canonical Hamiltonian corresponding to L is defined by the Legendre transform of L ,

$$H_C := \dot{\mathbf{Z}}_i^A P_A^i + \dot{\bar{\mathbf{Z}}}_A^i \bar{P}_i^A + \dot{a} P^{(a)} + \dot{\mathbf{b}}^r P_r^{(b)} + \dot{\mathbf{e}} P^{(e)} + \dot{h} P^{(h)} + \dot{\bar{h}} P^{(\bar{h})} + \dot{\varphi} P^{(\varphi)} - L = -a(\bar{\mathbf{Z}}_A^i \mathbf{Z}_i^A - 2s) - \mathbf{b}^3 (\bar{\mathbf{Z}}_A^j \sigma_{3j}^k \mathbf{Z}_k^A - 2t) - \mathbf{b}^i \bar{\mathbf{Z}}_A^j \sigma_{ij}^k \mathbf{Z}_k^A + \frac{1}{2\mathbf{e}} \mathbf{b}^i \mathbf{b}^i + \frac{k^2}{2} \mathbf{e} - h(\epsilon^{ij} \omega_{i\alpha} \omega_j^{\alpha} - \sqrt{2} m e^{i\varphi}) - \bar{h}(\epsilon_{ij} \bar{\omega}_{\alpha}^i \bar{\omega}^{j\alpha} - \sqrt{2} m e^{-i\varphi}). \quad (3.2)$$

The equal-time Poisson brackets between the canonical variables are given by

$$\begin{aligned} \{\mathbf{Z}_i^A, P_B^j\} &= \delta_i^j \delta_B^A, & \{\bar{\mathbf{Z}}_A^i, \bar{P}_j^B\} &= \delta_j^i \delta_A^B, \\ \{a, P^{(a)}\} &= 1, & \{\mathbf{b}^r, P_s^{(b)}\} &= \delta_s^r, \\ \{\mathbf{e}, P^{(e)}\} &= 1, & \{h, P^{(h)}\} &= 1, \\ \{\bar{h}, P^{(\bar{h})}\} &= 1, & \{\varphi, P^{(\varphi)}\} &= 1, \\ \text{all others} &= 0, \end{aligned} \quad (3.3)$$

which can be used for calculating the Poisson bracket between two arbitrary analytic functions of the canonical variables.

Equations (3.1a)–(3.1h) are read as the primary constraints

$$\phi_A^i := P_A^i - \frac{i}{2} \bar{\mathbf{Z}}_A^i \approx 0, \quad (3.4a)$$

$$\bar{\phi}_i^A := \bar{P}_i^A + \frac{i}{2} \mathbf{Z}_i^A \approx 0, \quad (3.4b)$$

$$\phi^{(a)} := P^{(a)} \approx 0, \quad (3.4c)$$

$$\phi_r^{(b)} := P_r^{(b)} \approx 0, \quad (3.4d)$$

$$\phi^{(e)} := P^{(e)} \approx 0, \quad (3.4e)$$

$$\phi^{(h)} := P^{(h)} \approx 0, \quad (3.4f)$$

$$\phi^{(\bar{h})} := P^{(\bar{h})} \approx 0, \quad (3.4g)$$

$$\phi^{(\varphi)} := P^{(\varphi)} \approx 0, \quad (3.4h)$$

where the symbol “ \approx ” denotes the weak equality. Now, we follow the Dirac algorithm for constrained Hamiltonian systems [31–33] to establish the canonical formalism of the present model. We see that the Poisson brackets between the primary constraint functions ϕ 's are summarized in

$$\{\phi_A^i, \bar{\phi}_j^B\} = -i\delta_j^i \delta_A^B \quad \text{all others} = 0. \quad (3.5)$$

The Poisson brackets between H_C and the primary constraint functions are found to be

$$\{\phi_A^i, H_C\} = a\bar{Z}_A^i + \mathbf{b}^r \sigma_{rj}^i \bar{Z}_A^j, \quad (3.6a)$$

$$\{\bar{\phi}_i^A, H_C\} = aZ_i^A + \mathbf{b}^r \sigma_{ri}^j Z_j^A, \quad (3.6b)$$

$$\{\phi^{(a)}, H_C\} = \bar{Z}_A^i Z_i^A - 2s, \quad (3.6c)$$

$$\{\phi_i^{(b)}, H_C\} = \bar{Z}_A^j \sigma_{ij}^k Z_k^A - \frac{1}{\mathbf{e}} \mathbf{b}_i, \quad (3.6d)$$

$$\{\phi_3^{(b)}, H_C\} = \bar{Z}_A^j \sigma_{3j}^k Z_k^A - 2t, \quad (3.6e)$$

$$\{\phi^{(e)}, H_C\} = \frac{1}{2\mathbf{e}^2} (\mathbf{b}^i \mathbf{b}^i - k^2 \mathbf{e}^2), \quad (3.6f)$$

$$\{\phi^{(h)}, H_C\} = \epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi}, \quad (3.6g)$$

$$\{\phi^{(\bar{h})}, H_C\} = \epsilon_{ij} \bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2} m e^{-i\varphi}, \quad (3.6h)$$

$$\{\phi^{(\varphi)}, H_C\} = -i\sqrt{2} m (h e^{i\varphi} - \bar{h} e^{-i\varphi}). \quad (3.6i)$$

With H_C and the primary constraint functions, we define the total Hamiltonian

$$H_T := H_C + u_i^A \phi_A^i + \bar{u}_A^i \bar{\phi}_i^A + u_{(a)} \phi^{(a)} + u_{(b)}^r \phi_r^{(b)} + u_{(e)} \phi^{(e)} + u_{(h)} \phi^{(h)} + u_{(\bar{h})} \phi^{(\bar{h})} + u_{(\varphi)} \phi^{(\varphi)}, \quad (3.7)$$

where u_i^A , \bar{u}_A^i , $u_{(a)}$, $u_{(b)}^r$, $u_{(e)}$, $u_{(h)}$, $u_{(\bar{h})}$, and $u_{(\varphi)}$ are Lagrange multipliers. The time evolution of a function f of the canonical variables is governed by the canonical equation

$$\dot{f} = \{f, H_T\}. \quad (3.8)$$

Using this equation together with Eqs. (3.4)–(3.7), we can evaluate the time evolution of the primary constraint functions. Because the primary constraints (3.4a)–(3.4h)

are valid at any time, they must be preserved in time. This fact leads to the consistency conditions

$$\dot{\phi}_A^i = \{\phi_A^i, H_T\} \approx a\bar{Z}_A^i + \mathbf{b}^r \sigma_{rj}^i \bar{Z}_A^j - i\bar{u}_A^i \approx 0, \quad (3.9a)$$

$$\dot{\bar{\phi}}_i^A = \{\bar{\phi}_i^A, H_T\} \approx aZ_i^A + \mathbf{b}^r \sigma_{ri}^j Z_j^A + iu_i^A \approx 0, \quad (3.9b)$$

$$\dot{\phi}^{(a)} = \{\phi^{(a)}, H_T\} \approx \bar{Z}_A^i Z_i^A - 2s \approx 0, \quad (3.9c)$$

$$\dot{\phi}_i^{(b)} = \{\phi_i^{(b)}, H_T\} \approx \bar{Z}_A^j \sigma_{ij}^k Z_k^A - \frac{1}{\mathbf{e}} \mathbf{b}_i \approx 0, \quad (3.9d)$$

$$\dot{\phi}_3^{(b)} = \{\phi_3^{(b)}, H_T\} \approx \bar{Z}_A^j \sigma_{3j}^k Z_k^A - 2t \approx 0, \quad (3.9e)$$

$$\dot{\phi}^{(e)} = \{\phi^{(e)}, H_T\} \approx \frac{1}{2\mathbf{e}^2} (\mathbf{b}^i \mathbf{b}^i - k^2 \mathbf{e}^2) \approx 0, \quad (3.9f)$$

$$\dot{\phi}^{(h)} = \{\phi^{(h)}, H_T\} \approx \epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi} \approx 0, \quad (3.9g)$$

$$\dot{\phi}^{(\bar{h})} = \{\phi^{(\bar{h})}, H_T\} \approx \epsilon_{ij} \bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2} m e^{-i\varphi} \approx 0, \quad (3.9h)$$

$$\dot{\phi}^{(\varphi)} = \{\phi^{(\varphi)}, H_T\} \approx -i\sqrt{2} m (h e^{i\varphi} - \bar{h} e^{-i\varphi}) \approx 0. \quad (3.9i)$$

Equations (3.9a) and (3.9b) determine \bar{u}_A^i and u_i^A , respectively, as follows:

$$\bar{u}_A^i = -ia\bar{Z}_A^i - i\mathbf{b}^r \sigma_{rj}^i \bar{Z}_A^j, \quad (3.10a)$$

$$u_i^A = iaZ_i^A + i\mathbf{b}^r \sigma_{ri}^j Z_j^A. \quad (3.10b)$$

In contrast, Eqs. (3.9c)–(3.9i) give rise to the secondary constraints

$$\chi^{(a)} := \bar{Z}_A^i Z_i^A - 2s \approx 0, \quad (3.11a)$$

$$\chi_i^{(b)} := \bar{Z}_A^j \sigma_{ij}^k Z_k^A - \frac{1}{\mathbf{e}} \mathbf{b}_i \approx 0, \quad (3.11b)$$

$$\chi_3^{(b)} := \bar{Z}_A^j \sigma_{3j}^k Z_k^A - 2t \approx 0, \quad (3.11c)$$

$$\chi^{(e)} := \frac{1}{2} (\mathbf{b}^i \mathbf{b}^i - k^2 \mathbf{e}^2) \approx 0, \quad (3.11d)$$

$$\chi^{(h)} := \epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi} \approx 0, \quad (3.11e)$$

$$\chi^{(\bar{h})} := \epsilon_{ij} \bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2} m e^{-i\varphi} \approx 0, \quad (3.11f)$$

$$\chi^{(\varphi)} := i(h e^{i\varphi} - \bar{h} e^{-i\varphi}) \approx 0. \quad (3.11g)$$

All the Poisson brackets between H_C and the secondary constraint functions χ 's vanish. The Poisson brackets between the primary and secondary constraint functions are found to be

$$\begin{aligned}
\{\chi^{(a)}, \phi_A^i\} &= \bar{Z}_A^i, & \{\chi^{(a)}, \bar{\phi}_i^A\} &= Z_i^A, \\
\{\chi_r^{(b)}, \phi_A^i\} &= \sigma_{rj}^i \bar{Z}_A^j, & \{\chi_r^{(b)}, \bar{\phi}_i^A\} &= \sigma_{ri}^j Z_j^A, \\
\{\chi_i^{(b)}, \phi_j^{(b)}\} &= -\frac{1}{e} \delta_{ij}, & \{\chi_i^{(b)}, \phi^{(e)}\} &= \frac{1}{e^2} \mathbf{b}_i, \\
\{\chi^{(e)}, \phi_j^{(b)}\} &= \mathbf{b}_j, & \{\chi^{(e)}, \phi^{(e)}\} &= -k^2 \mathbf{e}, \\
\{\chi^{(h)}, \phi^{i\dot{\alpha}}\} &= 2\epsilon^{ij} \bar{\omega}_j^{\dot{\alpha}}, & \{\chi^{(h)}, \phi^{(\varphi)}\} &= -i\sqrt{2} m e^{i\varphi}, \\
\{\chi^{(\bar{h})}, \bar{\phi}_i^{\dot{\alpha}}\} &= 2\epsilon_{ij} \bar{\omega}^{j\dot{\alpha}}, & \{\chi^{(\bar{h})}, \phi^{(\varphi)}\} &= i\sqrt{2} m e^{-i\varphi}, \\
\{\chi^{(\varphi)}, \phi^{(h)}\} &= i e^{i\varphi}, & \{\chi^{(\varphi)}, \phi^{(\bar{h})}\} &= -i e^{-i\varphi}, \\
\{\chi^{(\varphi)}, \phi^{(\varphi)}\} &= -(h e^{i\varphi} + \bar{h} e^{-i\varphi}), \\
\text{all others} &= 0.
\end{aligned} \tag{3.12}$$

All the Poisson brackets between the secondary constraint functions vanish.

Next we investigate the time evolution of the secondary constraint functions using Eqs. (3.8) and (3.12). The time evolution of $\chi^{(a)}$ is evaluated as

$$\dot{\chi}^{(a)} = \{\chi^{(a)}, H_T\} \approx u_i^A \bar{Z}_A^i + \bar{u}_A^i Z_i^A. \tag{3.13}$$

The condition $\dot{\chi}^{(a)} \approx 0$ is identically fulfilled with the aid of Eqs. (3.10a) and (3.10b), and hence no new constraints are obtained from $\dot{\chi}^{(a)} \approx 0$. The time evolution of $\chi_r^{(b)}$ is evaluated as

$$\begin{aligned}
\dot{\chi}_r^{(b)} &= \{\chi_r^{(b)}, H_T\} \\
&\approx u_i^A \sigma_{rj}^i \bar{Z}_A^j + \bar{u}_A^i \sigma_{ri}^j Z_j^A \\
&\quad + u_{(b)}^s \{\chi_r^{(b)}, \phi_s^{(b)}\} + u_{(e)} \{\chi_r^{(b)}, \phi^{(e)}\} \\
&= -2\epsilon_{rst} \mathbf{b}^s \bar{Z}_A^j \sigma_{ij}^k Z_k^A \\
&\quad + u_{(b)}^s \{\chi_r^{(b)}, \phi_s^{(b)}\} + u_{(e)} \{\chi_r^{(b)}, \phi^{(e)}\} \\
&\approx -\frac{2}{e} \epsilon_{rsj} \mathbf{b}^s \mathbf{b}^j - 4t \epsilon_{rs3} \mathbf{b}^s \\
&\quad + u_{(b)}^s \{\chi_r^{(b)}, \phi_s^{(b)}\} + u_{(e)} \{\chi_r^{(b)}, \phi^{(e)}\}
\end{aligned} \tag{3.14}$$

by using Eqs. (3.10a), (3.10b), (3.11b), and (3.11c). Then we see that the condition $\dot{\chi}_i^{(b)} \approx 0$ determines $u_{(b)}^i$ as follows:

$$u_{(b)}^i = 2\epsilon^{ij} \mathbf{b}_j (\mathbf{b}^3 - 2t\mathbf{e}) + \frac{1}{e} \mathbf{b}^i u_{(e)}, \tag{3.15}$$

while $\dot{\chi}_3^{(b)} \approx 0$ is identically satisfied. The time evolution of $\chi^{(e)}$ is calculated as

$$\begin{aligned}
\dot{\chi}^{(e)} &= \{\chi^{(e)}, H_T\} \\
&\approx \mathbf{b}_i u_{(b)}^i - k^2 \mathbf{e} u_{(e)} = \frac{2}{e} \chi^{(e)} u_{(e)} \approx 0
\end{aligned} \tag{3.16}$$

by using Eqs. (3.15) and (3.11d). Hence $\dot{\chi}^{(e)} \approx 0$ is identically satisfied. The time evolution of $\chi^{(h)}$ is evaluated as

$$\begin{aligned}
\dot{\chi}^{(h)} &= \{\chi^{(h)}, H_T\} \\
&\approx 2\epsilon^{ij} u_{i\dot{\alpha}} \bar{\omega}_j^{\dot{\alpha}} - i\sqrt{2} m e^{i\varphi} u_{(\varphi)} \\
&= 2ia\epsilon^{ij} \bar{\omega}_{i\dot{\alpha}} \bar{\omega}_j^{\dot{\alpha}} - i\sqrt{2} m e^{i\varphi} u_{(\varphi)} \\
&\approx i\sqrt{2} m e^{i\varphi} (2a - u_{(\varphi)})
\end{aligned} \tag{3.17}$$

by using Eqs. (3.10b), (3.11e) and the formula $\sigma_{rk}^i \epsilon^{kj} = \sigma_{rk}^j \epsilon^{ki}$. (Its associated formula $\sigma_{ri}^k \epsilon_{kj} = \sigma_{rj}^k \epsilon_{ki}$ is also valid.) From the condition $\dot{\chi}^{(h)} \approx 0$, the Lagrange multiplier $u_{(\varphi)}$ is determined to be $u_{(\varphi)} = 2a$. Similarly, $\dot{\chi}^{(\bar{h})} \approx -i\sqrt{2} m e^{-i\varphi} (2a - u_{(\varphi)}) \approx 0$ leads to $u_{(\varphi)} = 2a$. The time evolution of $\chi^{(\varphi)}$ is found to be

$$\begin{aligned}
\dot{\chi}^{(\varphi)} &= \{\chi^{(\varphi)}, H_T\} \\
&\approx i(u_{(h)} - u_{(\bar{h})}) - 2a(h e^{i\varphi} + \bar{h} e^{-i\varphi}),
\end{aligned} \tag{3.18}$$

so that the condition $\dot{\chi}^{(\varphi)} \approx 0$ gives $u_{(h)} - u_{(\bar{h})} = -2ia(h e^{i\varphi} + \bar{h} e^{-i\varphi})$. From the above analysis, we see that no further constraints can be derived; thus, the procedure for deriving constraints is now completed. We also see that u_i^A , \bar{u}_A^i , $u_{(b)}^i$, $u_{(h)} - u_{(\bar{h})}$, and $u_{(\varphi)}$ are determined to be what are written in terms of other variables such as the canonical coordinates, while $u_{(a)}$, $u_{(b)}^3$, $u_{(e)}$, and $u_{(h)} + u_{(\bar{h})}$ still remain as arbitrary functions of τ .

We have obtained all the Poisson brackets between the constraint functions, as in Eqs. (3.5) and (3.12). However, it is difficult to classify the constraints in Eqs. (3.4) and (3.11) into first and second classes on the basis of Eqs. (3.5) and (3.12) together with the vanishing Poisson brackets between the secondary constraint functions. To find simpler forms of the relevant Poisson brackets, we first define

$$\tilde{\phi}^{(e)} := \phi^{(e)} + \frac{1}{e} \mathbf{b}^i \phi_i^{(b)}, \tag{3.19a}$$

$$\tilde{\phi}^{(\varphi)} := \phi^{(\varphi)} - ih\phi^{(h)} + i\bar{h}\phi^{(\bar{h})}, \tag{3.19b}$$

$$\tilde{\chi}^{(a)} := \chi^{(a)} + i\bar{Z}_A^j \bar{\phi}_j^A - i\phi_A^j Z_j^A, \tag{3.19c}$$

$$\tilde{\chi}_i^{(b)} := \chi_i^{(b)} + i\bar{Z}_A^j \sigma_{ij}^k \bar{\phi}_k^A - i\phi_A^j \sigma_{ij}^k Z_k^A - 2t\mathbf{e}\epsilon_{ij} \mathbf{b}^j \phi_j^{(b)}, \tag{3.19d}$$

$$\tilde{\chi}_3^{(b)} := \chi_3^{(b)} + i\bar{Z}_A^j \sigma_{3j}^k \bar{\phi}_k^A - i\phi_A^j \sigma_{3j}^k Z_k^A + 2\epsilon_{ij} \mathbf{b}^i \phi_j^{(b)}, \tag{3.19e}$$

$$\tilde{\chi}^{(e)} := \chi^{(e)} + \mathbf{e}^i (\tilde{\chi}_i^{(b)} - 2t\epsilon_{ij}\phi_j^{(b)}), \quad (3.19f)$$

$$\tilde{\chi}^{(h)} := \chi^{(h)} + 2i\epsilon^{jk}\bar{\phi}_{j\dot{\alpha}}\bar{\omega}_k^{\dot{\alpha}}, \quad (3.19g)$$

$$\tilde{\chi}^{(\bar{h})} := \chi^{(\bar{h})} - 2i\epsilon_{jk}\phi_\alpha^j\bar{\omega}^{k\alpha}, \quad (3.19h)$$

where $\bar{\phi}_{i\dot{\alpha}}$ and ϕ_α^i are spinor components of $\bar{\phi}_i^A = (\bar{\phi}_i^\alpha, \bar{\phi}_{i\dot{\alpha}})$ and $\phi_A^i = (\phi_\alpha^i, \phi^{i\dot{\alpha}})$, respectively. Furthermore, it is convenient to define

$$v^{(\pm)} := \frac{1}{2\sqrt{2}m} \left(\tilde{\phi}^{(\varphi)} \pm \frac{1}{2}\tilde{\chi}^{(a)} \right), \quad (3.20a)$$

$$\phi^{(+)} := \frac{1}{2}(e^{-i\varphi}\phi^{(h)} + e^{i\varphi}\phi^{(\bar{h})}), \quad (3.20b)$$

$$\phi^{(-)} := \frac{1}{2i}(e^{-i\varphi}\phi^{(h)} - e^{i\varphi}\phi^{(\bar{h})}), \quad (3.20c)$$

$$\tilde{\chi}^{(+)} := \frac{1}{2}(e^{-i\varphi}\tilde{\chi}^{(h)} + e^{i\varphi}\tilde{\chi}^{(\bar{h})}), \quad (3.20d)$$

$$\tilde{\chi}^{(-)} := \frac{1}{2i}(e^{-i\varphi}\tilde{\chi}^{(h)} - e^{i\varphi}\tilde{\chi}^{(\bar{h})}). \quad (3.20e)$$

It can readily be seen that the set of all the constraints given in Eqs. (3.4) and (3.11), i.e.,

$$(\phi_A^i, \bar{\phi}_i^A, \phi^{(a)}, \phi_r^{(b)}, \phi^{(e)}, \phi^{(h)}, \phi^{(\bar{h})}, \phi^{(\varphi)}, \chi^{(a)}, \chi_i^{(b)}, \chi_3^{(b)}, \chi^{(e)}, \chi^{(h)}, \chi^{(\bar{h})}, \chi^{(\varphi)}) \approx 0, \quad (3.21)$$

is equivalent to the new set of constraints

$$(\phi_A^i, \bar{\phi}_i^A, \phi^{(a)}, \phi_r^{(b)}, \tilde{\phi}^{(e)}, \phi^{(+)}, \phi^{(-)}, v^{(+)}, v^{(-)}, \tilde{\chi}_i^{(b)}, \tilde{\chi}_3^{(b)}, \tilde{\chi}^{(e)}, \tilde{\chi}^{(+)}, \tilde{\chi}^{(-)}, \chi^{(\varphi)}) \approx 0. \quad (3.22)$$

We can show that except for

$$\{\phi_A^i, \bar{\phi}_j^B\} = -i\delta_j^B \delta_A^i, \quad \{\phi_i^{(b)}, \tilde{\chi}_j^{(b)}\} = \frac{1}{\mathbf{e}} \delta_{ij}, \quad \{v^{(+)}, \tilde{\chi}^{(-)}\} = 1, \quad \{\chi^{(\varphi)}, \phi^{(-)}\} = 1, \quad (3.23)$$

all other Poisson brackets between the constraint functions in Eq. (3.22) vanish. In this way, the relevant Poisson brackets are simplified with the aid of the new constraint functions. The Poisson brackets between the constraint functions are summarized in a matrix form as

$$\begin{array}{c} \phi_A^i \\ \bar{\phi}_i^A \\ \phi^{(a)} \\ \phi_i^{(b)} \\ \phi_3^{(b)} \\ \tilde{\phi}^{(e)} \\ \phi^{(+)} \\ \phi^{(-)} \\ v^{(+)} \\ v^{(-)} \\ \tilde{\chi}_i^{(b)} \\ \tilde{\chi}_3^{(b)} \\ \tilde{\chi}^{(e)} \\ \tilde{\chi}^{(+)} \\ \tilde{\chi}^{(-)} \\ \chi^{(\varphi)} \end{array} \left(\begin{array}{cccccccccccccccc} \phi_B^j & \bar{\phi}_j^B & \phi^{(a)} & \phi_j^{(b)} & \phi_3^{(b)} & \tilde{\phi}^{(e)} & \phi^{(+)} & \phi^{(-)} & v^{(+)} & v^{(-)} & \tilde{\chi}_j^{(b)} & \tilde{\chi}_3^{(b)} & \tilde{\chi}^{(e)} & \tilde{\chi}^{(+)} & \tilde{\chi}^{(-)} & \chi^{(\varphi)} \\ 0 & -i\delta_j^B \delta_A^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\delta_i^j \delta_B^A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mathbf{e}} \delta_{ij} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\mathbf{e}} \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (3.24)$$

We can immediately see from this matrix that $\phi^{(a)} \approx 0$, $\phi_3^{(b)} \approx 0$, $\tilde{\phi}^{(e)} \approx 0$, $\phi^{(+)} \approx 0$, $v^{(-)} \approx 0$, $\tilde{\chi}_3^{(b)} \approx 0$, $\tilde{\chi}^{(e)} \approx 0$, and $\tilde{\chi}^{(+)} \approx 0$ are first-class constraints, while $\phi_A^i \approx 0$, $\bar{\phi}_i^A \approx 0$, $\phi_i^{(b)} \approx 0$, $\phi^{(-)} \approx 0$, $v^{(+)} \approx 0$, $\tilde{\chi}_i^{(b)} \approx 0$, $\tilde{\chi}^{(-)} \approx 0$, and $\chi^{(\varphi)} \approx 0$ are second-class constraints. Following Dirac's approach to second-class constraints, we define the Dirac bracket by using the largest invertible submatrix of the matrix (3.24). For arbitrary smooth functions f and g of the canonical variables, the Dirac bracket is defined by

$$\{f, g\}_D \qquad \chi^{(e)} \approx 0, \quad (3.28g)$$

$$\begin{aligned} &:= \{f, g\} + i\{f, \phi_A^i\}\{\bar{\phi}_i^A, g\} - i\{f, \bar{\phi}_i^A\}\{\phi_A^i, g\} \\ &\quad - \mathbf{e}\{f, \tilde{\chi}_i^{(b)}\}\{\phi_i^{(b)}, g\} + \mathbf{e}\{f, \phi_i^{(b)}\}\{\tilde{\chi}_i^{(b)}, g\} \end{aligned} \qquad \chi^{(h)} \approx 0, \quad (3.28h)$$

$$\begin{aligned} &+ \{f, \chi^{(\varphi)}\}\{\phi^{(-)}, g\} - \{f, \phi^{(-)}\}\{\chi^{(\varphi)}, g\} \\ &+ \{f, v^{(+)}\}\{\tilde{\chi}^{(-)}, g\} - \{f, \tilde{\chi}^{(-)}\}\{v^{(+)}, g\}. \end{aligned} \qquad \chi^{(\bar{h})} \approx 0. \quad (3.28i)$$

The Dirac bracket between f and each of the constraint functions $\phi_A^i, \bar{\phi}_i^A, \phi_i^{(b)}, \phi^{(-)}, v^{(+)}, \tilde{\chi}_i^{(b)}, \tilde{\chi}^{(-)}$, and $\chi^{(\varphi)}$ vanishes identically. For this reason, the second-class constraints can be set strongly equal to zero and may be expressed as $\phi_A^i = 0, \bar{\phi}_i^A = 0, \phi_i^{(b)} = 0, \phi^{(-)} = 0, v^{(+)} = 0, \tilde{\chi}_i^{(b)} = 0, \tilde{\chi}^{(-)} = 0$, and $\chi^{(\varphi)} = 0$, as long as the Dirac bracket $\{f, g\}_D$ is adopted. We see that the second-class constraints lead to

$$P_A^i = \frac{i}{2}\bar{Z}_i^A, \quad \bar{P}_i^A = -\frac{i}{2}Z_i^A, \quad (3.26a)$$

$$\mathbf{b}_i = \mathbf{e}\bar{Z}_i^j \sigma_{ij}^k Z_k^A, \quad P_i^{(b)} = 0, \quad (3.26b)$$

$$h = \mathbf{h}e^{-i\varphi}, \quad P^{(h)} = e^{i\varphi}P^{(h)}, \quad (3.26c)$$

$$\bar{h} = \mathbf{h}e^{i\varphi}, \quad P^{(\bar{h})} = e^{-i\varphi}P^{(h)}, \quad (3.26d)$$

$$\varphi = -\frac{i}{2}\ln\left(\frac{\epsilon^{ij}\varpi_{i\dot{\alpha}}\bar{\varpi}_j^{\dot{\alpha}}}{\epsilon_{ij}\bar{\varpi}_i^{\dot{\alpha}}\varpi_j^{\dot{\alpha}}}\right), \quad P^{(\varphi)} = -\frac{1}{2}\chi^{(a)}, \quad (3.26e)$$

where $\mathbf{h} = \mathbf{h}(\tau)$ is a real scalar-density field of weight 1 on \mathcal{T} , and $P^{(h)}$ its associated momentum variable. At this stage, $P_A^i, \bar{P}_i^A, \mathbf{b}^i, P_i^{(b)}, h, P^{(h)}, \bar{h}, P^{(\bar{h})}, \varphi$, and $P^{(\varphi)}$ are treated as dependent variables specified by Eq. (3.26), while the other canonical variables $Z_i^A, \bar{Z}_i^A, a, P^{(a)}, \mathbf{b}^3, P_3^{(b)}, \mathbf{e}, P^{(e)}, \mathbf{h}$, and $P^{(h)}$ are treated as independent variables. By virtue of the strong equalities of the second-class constraints, the set of all the first-class constraints, i.e.,

$$\{\phi^{(a)}, \phi_3^{(b)}, \bar{\phi}^{(e)}, \phi^{(+)}, v^{(-)}, \tilde{\chi}_3^{(b)}, \tilde{\chi}^{(e)}, \tilde{\chi}^{(+)}\} \approx 0, \quad (3.27)$$

turns out to be equivalent to the set consisting of

$$\phi^{(a)} \approx 0, \quad (3.28a)$$

$$\phi_3^{(b)} \approx 0, \quad (3.28b)$$

$$\phi^{(e)} \approx 0, \quad (3.28c)$$

$$\phi^{(h)} := P^{(h)} \approx 0, \quad (3.28d)$$

$$\chi^{(a)} \approx 0, \quad (3.28e)$$

$$\chi_3^{(b)} \approx 0, \quad (3.28f)$$

Here we have taken into account both Eqs. (3.28h) and (3.28i) for later convenience, although it is sufficient to consider one of them in actuality.

The Dirac brackets between the spinor components of Z_i^A and \bar{Z}_i^A are found from Eq. (3.25) to be

$$\{Q_i^\alpha, Q_j^\beta\}_D = \frac{i}{4\sqrt{2}m} e^{i\varphi} (Q_i^\alpha \epsilon_{jk} \bar{\omega}^{k\beta} - Q_j^\beta \epsilon_{ik} \bar{\omega}^{k\alpha}),$$

$$\{Q_i^\alpha, \varpi_{j\dot{\beta}}\}_D = -\frac{i}{4\sqrt{2}m} e^{i\varphi} \epsilon_{ik} \bar{\omega}^{k\alpha} \varpi_{j\dot{\beta}},$$

$$\{\varpi_{i\dot{\alpha}}, \varpi_{j\dot{\beta}}\}_D = 0,$$

$$\{Q_i^\alpha, \bar{Q}^{j\dot{\beta}}\}_D = \frac{i}{4\sqrt{2}m} (e^{i\varphi} \epsilon_{ik} \bar{\omega}^{k\alpha} \bar{Q}^{j\dot{\beta}} + e^{-i\varphi} Q_i^\alpha \epsilon^{jk} \omega_k^{\dot{\beta}}),$$

$$\{Q_i^\alpha, \bar{\varpi}_\beta^j\}_D = -i\delta_i^j \delta_\beta^\alpha + \frac{i}{4\sqrt{2}m} e^{i\varphi} \epsilon_{ik} \bar{\omega}^{k\alpha} \bar{\varpi}_\beta^j,$$

$$\{\varpi_{i\dot{\alpha}}, \bar{Q}^{j\dot{\beta}}\}_D = -i\delta_i^j \delta_{\dot{\alpha}}^{\dot{\beta}} + \frac{i}{4\sqrt{2}m} e^{-i\varphi} \varpi_{i\dot{\alpha}} \epsilon^{jk} \omega_k^{\dot{\beta}},$$

$$\{\varpi_{i\dot{\alpha}}, \bar{\varpi}_\beta^j\}_D = 0,$$

$$\{\bar{Q}^{i\dot{\alpha}}, \bar{Q}^{j\dot{\beta}}\}_D = -\frac{i}{4\sqrt{2}m} e^{-i\varphi} (\bar{Q}^{i\dot{\alpha}} \epsilon^{jk} \omega_k^{\dot{\beta}} - \bar{Q}^{j\dot{\beta}} \epsilon^{ik} \omega_k^{\dot{\alpha}}),$$

$$\{\bar{Q}^{i\dot{\alpha}}, \bar{\varpi}_\beta^j\}_D = \frac{i}{4\sqrt{2}m} e^{-i\varphi} \epsilon^{ik} \omega_k^{\dot{\alpha}} \bar{\varpi}_\beta^j,$$

$$\{\bar{\varpi}_i^{\dot{\alpha}}, \bar{\varpi}_\beta^j\}_D = 0. \quad (3.29)$$

Using Eq. (3.29) and taking into account Eqs. (3.28h) and (3.28i), we can show that

$$\{\chi^{(a)}, Q_i^\alpha\}_D = \frac{i}{2} Q_i^\alpha, \quad \{\chi^{(a)}, \varpi_{i\dot{\alpha}}\}_D = \frac{i}{2} \varpi_{i\dot{\alpha}},$$

$$\{\chi^{(a)}, \bar{\varpi}_\alpha^i\}_D = -\frac{i}{2} \bar{\varpi}_\alpha^i, \quad \{\chi^{(a)}, \bar{Q}^{i\dot{\alpha}}\}_D = -\frac{i}{2} \bar{Q}^{i\dot{\alpha}}. \quad (3.30)$$

Many of the Dirac brackets in Eq. (3.29) are rather complicated. Fortunately, however, Eq. (3.29) can be expressed in the form of simple canonical brackets as

$$\{\rho_i^\alpha, \bar{\varpi}_\beta^j\}_D = -i\delta_i^j \delta_\beta^\alpha, \quad \{\varpi_{i\dot{\alpha}}, \bar{\rho}^{j\dot{\beta}}\}_D = -i\delta_i^j \delta_{\dot{\alpha}}^{\dot{\beta}},$$

$$\text{all others} = 0, \quad (3.31)$$

in terms of $\varpi_{i\dot{\alpha}}, \bar{\varpi}_\alpha^i$, and

$$\rho_i^\alpha := Q_i^\alpha + \frac{1}{2\sqrt{2}m} e^{i\varphi} \epsilon_{ij} \bar{\varpi}^{j\alpha} \chi^{(a)}, \quad (3.32a)$$

$$\bar{\rho}^{i\dot{\alpha}} := \bar{q}^{i\dot{\alpha}} + \frac{1}{2\sqrt{2}m} e^{-i\varphi} \epsilon^{ij} \bar{\omega}_j^{\dot{\alpha}} \check{\chi}^{(a)}. \quad (3.32b)$$

In showing this fact, it is convenient to use Eqs. (3.28e) and (3.30). Note here that the weak equalities $\rho_i^\alpha \approx q_i^\alpha$, $\bar{\rho}^{i\dot{\alpha}} \approx \bar{q}^{i\dot{\alpha}}$ hold owing to Eq. (3.28e). Now we define the new twistors $\mathbf{W}_i^A := (\rho_i^\alpha, \bar{\omega}_{i\dot{\alpha}})$ and $\bar{\mathbf{W}}_A^i := (\bar{\omega}_i^{\dot{\alpha}}, \bar{\rho}^{i\dot{\alpha}})$, with which Eq. (3.31) can concisely be written as

$$\begin{aligned} \{\mathbf{W}_i^A, \bar{\mathbf{W}}_B^j\}_{\text{D}} &= -i\delta_i^j \delta_B^A, \\ \{\mathbf{W}_i^A, \mathbf{W}_j^B\}_{\text{D}} &= 0, \quad \{\bar{\mathbf{W}}_A^i, \bar{\mathbf{W}}_B^j\}_{\text{D}} = 0. \end{aligned} \quad (3.33)$$

Using Eqs. (3.28h), (3.28i), and the formulas given under Eq. (3.17), we can show for

$$\check{\chi}^{(a)} := \bar{\mathbf{W}}_A \mathbf{W}_i^A - 2s, \quad (3.34a)$$

$$\check{\chi}_3^{(b)} := \bar{\mathbf{W}}_A^j \sigma_{3j}^k \mathbf{W}_k^A - 2t \quad (3.34b)$$

that

$$\check{\chi}^{(a)} = 2\chi^{(a)}, \quad (3.35a)$$

$$\check{\chi}_3^{(b)} = \chi_3^{(b)}. \quad (3.35b)$$

Accordingly, the first-class constraints (3.28e) and (3.28f) read

$$\check{\chi}^{(a)} \approx 0, \quad (3.36a)$$

$$\check{\chi}_3^{(b)} \approx 0. \quad (3.36b)$$

With Eq. (3.35a), Eqs. (3.32a) and (3.32b) can be solved inversely as

$$q_i^\alpha = \rho_i^\alpha - \frac{1}{4\sqrt{2}m} e^{i\varphi} \epsilon_{ij} \bar{\omega}^{j\alpha} \check{\chi}^{(a)}, \quad (3.37a)$$

$$\bar{q}^{i\dot{\alpha}} = \bar{\rho}^{i\dot{\alpha}} - \frac{1}{4\sqrt{2}m} e^{-i\varphi} \epsilon^{ij} \bar{\omega}_j^{\dot{\alpha}} \check{\chi}^{(a)}. \quad (3.37b)$$

Hence it follows that there is a one-to-one correspondence between $(\mathbf{Z}_i^A, \bar{\mathbf{Z}}_A^i)$ and $(\mathbf{W}_i^A, \bar{\mathbf{W}}_A^i)$. Taking into account this fact, we hereafter adopt \mathbf{W}_i^A and $\bar{\mathbf{W}}_A^i$ as canonical variables instead of \mathbf{Z}_i^A and $\bar{\mathbf{Z}}_A^i$. The first equation in Eq. (3.26b) can be written as $\mathbf{b}_i = \mathbf{e} \bar{\mathbf{W}}_A^j \sigma_{ij}^k \mathbf{W}_k^A$. Substituting this into Eq. (3.11d), we see that the first-class constraint $\chi^{(e)} \approx 0$ can be expressed as

$$\check{\chi}^{(e)} := \mathbf{T}_i \mathbf{T}_i - \frac{1}{4} k^2 \approx 0, \quad (3.38)$$

where \mathbf{T}_i ($\hat{i} = 1, 2$) are defined in

$$\mathbf{T}_0 := \frac{1}{2} \bar{\mathbf{W}}_A^i \mathbf{W}_i^A, \quad \mathbf{T}_r := \frac{1}{2} \bar{\mathbf{W}}_A^j \sigma_{rj}^k \mathbf{W}_k^A. \quad (3.39)$$

Using Eq. (3.33), we can readily verify that \mathbf{T}_0 and \mathbf{T}_r constitute a bases of the $U(1)_a \times SU(2)$ Lie algebra in the following sense:

$$\{\mathbf{T}_0, \mathbf{T}_r\}_{\text{D}} = 0, \quad \{\mathbf{T}_r, \mathbf{T}_s\}_{\text{D}} = \epsilon_{rst} \mathbf{T}_t. \quad (3.40)$$

The canonical variables that we need to consider at the present stage are $\mathbf{W}_i^A, \bar{\mathbf{W}}_A^i, a, P^{(a)}, \mathbf{b}^3, P_3^{(b)}, \mathbf{e}, P^{(e)}, \mathbf{h}$, and $P^{(h)}$. All the Dirac brackets between these variables are given in Eq. (3.33) and

$$\begin{aligned} \{a, P^{(a)}\}_{\text{D}} &= 1, & \{\mathbf{b}^3, P_3^{(b)}\}_{\text{D}} &= 1, \\ \{\mathbf{e}, P^{(e)}\}_{\text{D}} &= 1, & \{\mathbf{h}, P^{(h)}\}_{\text{D}} &= \frac{1}{2}, \\ \text{all others} &= 0. \end{aligned} \quad (3.41)$$

We also need to consider the first class constraints (3.28a)–(3.28d), (3.36a), (3.36b), (3.38), (3.28h), and (3.28i).

IV. CANONICAL QUANTIZATION

In this section, we perform the canonical quantization of the Hamiltonian system studied in Sec. III. To this end, in accordance with Dirac's method of quantization, we introduce the operators \hat{f} and \hat{g} corresponding to the functions f and g , respectively, and set the commutation relation

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}_{\text{D}}} \quad (4.1)$$

in units such that $\hbar = 1$. Here, $\widehat{\{f, g\}_{\text{D}}}$ denotes the operator corresponding to the Dirac bracket $\{f, g\}_{\text{D}}$. From Eqs. (3.33), (3.41), and (4.1), we have the canonical commutation relations

$$[\hat{\mathbf{W}}_i^A, \hat{\bar{\mathbf{W}}}_B^j] = \delta_i^j \delta_B^A, \quad (4.2a)$$

$$[\hat{\mathbf{W}}_i^A, \hat{\mathbf{W}}_j^B] = 0, \quad [\hat{\bar{\mathbf{W}}}_A^i, \hat{\bar{\mathbf{W}}}_B^j] = 0, \quad (4.2b)$$

$$[\hat{a}, \hat{P}^{(a)}] = i, \quad [\hat{\mathbf{b}}^3, \hat{P}_3^{(b)}] = i, \quad (4.2c)$$

$$[\hat{\mathbf{e}}, \hat{P}^{(e)}] = i, \quad [\hat{\mathbf{h}}, \hat{P}^{(h)}] = \frac{i}{2}, \quad (4.2d)$$

$$\text{all others} = 0. \quad (4.2e)$$

The commutation relations (4.2a) and (4.2b) govern together so-called twistor quantization [1,2].

In the procedure of canonical quantization, the first-class constraints are treated as conditions imposed on the physical states, after the replacement of the first-class

constraint functions by the corresponding operators. In the present model, the physical state conditions are found from Eqs. (3.28a)–(3.28d), (3.36a), (3.36b), (3.38), (3.28h), and (3.28i) to be

$$\hat{\phi}^{(a)}|F\rangle = \hat{P}^{(a)}|F\rangle = 0, \quad (4.3a)$$

$$\hat{\phi}_3^{(b)}|F\rangle = \hat{P}_3^{(b)}|F\rangle = 0, \quad (4.3b)$$

$$\hat{\phi}^{(e)}|F\rangle = \hat{P}^{(e)}|F\rangle = 0, \quad (4.3c)$$

$$\hat{\phi}^{(h)}|F\rangle = \hat{P}^{(h)}|F\rangle = 0, \quad (4.3d)$$

$$\begin{aligned} \hat{\chi}^{(a)}|F\rangle &= \left[\frac{1}{2}(\hat{W}_A^i \hat{W}_i^A + \hat{W}_i^A \hat{W}_A^i) - 2s \right] |F\rangle \\ &= 2(\hat{T}_0 - s - 2)|F\rangle = 0, \end{aligned} \quad (4.3e)$$

$$\begin{aligned} \hat{\chi}_3^{(b)}|F\rangle &= \left[\frac{1}{2}(\hat{W}_A^j \sigma_{3j}^k \hat{W}_k^A + \hat{W}_k^A \sigma_{3j}^k \hat{W}_A^j) - 2t \right] |F\rangle \\ &= 2(\hat{T}_3 - t)|F\rangle = 0, \end{aligned} \quad (4.3f)$$

$$\hat{\chi}^{(e)}|F\rangle = \left(\hat{T}_i \hat{T}_i - \frac{1}{4}k^2 \right) |F\rangle = 0, \quad (4.3g)$$

$$\hat{\chi}^{(h)}|F\rangle = (\epsilon^{ij} \hat{\omega}_{i\dot{\alpha}} \hat{\omega}_j^{\dot{\alpha}} - \sqrt{2}me^{i\hat{\phi}}) |F\rangle = 0, \quad (4.3h)$$

$$\hat{\chi}^{(\bar{h})}|F\rangle = (\epsilon_{ij} \hat{\omega}_i^{\dot{\alpha}} \hat{\omega}_j^{\dot{\alpha}} - \sqrt{2}me^{-i\hat{\phi}}) |F\rangle = 0. \quad (4.3i)$$

Here, $|F\rangle$ denotes a physical state, \hat{T}_0 and \hat{T}_r ($r = \hat{i}, 3$) are defined by

$$\hat{T}_0 := \frac{1}{2} \hat{W}_i^A \hat{W}_A^i, \quad \hat{T}_r := \frac{1}{2} \sigma_{rj}^k \hat{W}_k^A \hat{W}_A^j, \quad (4.4)$$

and $\hat{\phi}$ is defined according to the first equation in Eq. (3.26e) as

$$\hat{\phi} := -\frac{i}{2} [\ln(\epsilon^{ij} \hat{\omega}_{i\dot{\alpha}} \hat{\omega}_j^{\dot{\alpha}}) - \ln(\epsilon_{ij} \hat{\omega}_i^{\dot{\alpha}} \hat{\omega}_j^{\dot{\alpha}})]. \quad (4.5)$$

In defining the operators $\hat{\chi}^{(a)}$, $\hat{\chi}_3^{(b)}$, and $\hat{\chi}^{(e)}$, we have obeyed the Weyl ordering rule and have used the commutation relation (4.2a) to simplify the Weyl ordered operators. Using Eqs. (4.2a) and (4.2b), we can easily show that

$$[\hat{T}_0, \hat{T}_r] = 0, \quad [\hat{T}_r, \hat{T}_s] = i\epsilon_{rst} \hat{T}_t, \quad (4.6)$$

which is precisely the quantum mechanical counterpart of Eq. (3.40). It is evident that \hat{T}_0 is the generator of $U(1)_a$ and \hat{T}_r ($r = 1, 2, 3$) are the generators of $SU(2)$. In particular, \hat{T}_3 is the generator of $U(1)_b$.

Now we introduce the bra-vector

$$\begin{aligned} \langle W, a, b^3, e, h | := \langle 0 | \exp(-W_i^A \hat{W}_A^i + ia\hat{P}^{(a)} + ib^3\hat{P}_3^{(b)} \\ + ie\hat{P}^{(e)} + 2ih\hat{P}^{(h)}) \end{aligned} \quad (4.7)$$

with a reference bra-vector $\langle 0 |$ satisfying

$$\langle 0 | \hat{W}_i^A = \langle 0 | \hat{a} = \langle 0 | \hat{b}^3 = \langle 0 | \hat{e} = \langle 0 | \hat{h} = 0. \quad (4.8)$$

Using the commutation relations (4.2a)–(4.2e), we can show that

$$\langle W, a, b^3, e, h | \hat{W}_i^A = W_i^A \langle W, a, b^3, e, h |, \quad (4.9a)$$

$$\langle W, a, b^3, e, h | \hat{a} = a \langle W, a, b^3, e, h |, \quad (4.9b)$$

$$\langle W, a, b^3, e, h | \hat{b}^3 = b^3 \langle W, a, b^3, e, h |, \quad (4.9c)$$

$$\langle W, a, b^3, e, h | \hat{e} = e \langle W, a, b^3, e, h |, \quad (4.9d)$$

$$\langle W, a, b^3, e, h | \hat{h} = h \langle W, a, b^3, e, h |. \quad (4.9e)$$

Equation (4.9a) can be decomposed into two parts,

$$\langle W, a, b^3, e, h | \hat{\rho}_i^\alpha = \rho_i^\alpha \langle W, a, b^3, e, h |, \quad (4.10a)$$

$$\langle W, a, b^3, e, h | \hat{\omega}_{i\dot{\alpha}} = \omega_{i\dot{\alpha}} \langle W, a, b^3, e, h |. \quad (4.10b)$$

Also, it is easy to see that

$$\langle W, a, b^3, e, h | \hat{W}_A^i = -\frac{\partial}{\partial W_A^i} \langle W, a, b^3, e, h |, \quad (4.11a)$$

$$\langle W, a, b^3, e, h | \hat{P}^{(a)} = -i \frac{\partial}{\partial a} \langle W, a, b^3, e, h |, \quad (4.11b)$$

$$\langle W, a, b^3, e, h | \hat{P}_3^{(b)} = -i \frac{\partial}{\partial b^3} \langle W, a, b^3, e, h |, \quad (4.11c)$$

$$\langle W, a, b^3, e, h | \hat{P}^{(e)} = -i \frac{\partial}{\partial e} \langle W, a, b^3, e, h |, \quad (4.11d)$$

$$\langle W, a, b^3, e, h | \hat{P}^{(h)} = -\frac{i}{2} \frac{\partial}{\partial h} \langle W, a, b^3, e, h |. \quad (4.11e)$$

Equation (4.11a) can be decomposed into two parts,

$$\langle W, a, b^3, e, h | \hat{\omega}_\alpha^i = -\frac{\partial}{\partial \rho_\alpha^i} \langle W, a, b^3, e, h |, \quad (4.12a)$$

$$\langle W, a, b^3, e, h | \hat{\rho}^{i\dot{\alpha}} = -\frac{\partial}{\partial \omega_{i\dot{\alpha}}} \langle W, a, b^3, e, h |. \quad (4.12b)$$

Multiplying each of Eqs. (4.3a)–(4.3i) by $\langle W, a, b^3, e, h |$ on the left and using Eqs. (4.9)–(4.12), we obtain a set of

simultaneous differential equations for $F(\mathbf{W}, a, \mathbf{b}^3, \mathbf{e}, \mathbf{h}) := \langle \mathbf{W}, a, \mathbf{b}^3, \mathbf{e}, \mathbf{h} | F \rangle$ as follows:

$$\frac{\partial}{\partial a} F = 0, \quad (4.13a)$$

$$\frac{\partial}{\partial \mathbf{b}^3} F = 0, \quad (4.13b)$$

$$\frac{\partial}{\partial \mathbf{e}} F = 0, \quad (4.13c)$$

$$\frac{\partial}{\partial \mathbf{h}} F = 0, \quad (4.13d)$$

$$\check{T}_0 F = (s + 2)F, \quad (4.13e)$$

$$\check{T}_3 F = tF, \quad (4.13f)$$

$$\check{T}_i \check{T}_i F = \frac{1}{4} k^2 F, \quad (4.13g)$$

$$\epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_{j\dot{\beta}} F = \sqrt{2} m e^{i\dot{\varphi}} F, \quad (4.13h)$$

$$\epsilon_{ij} e^{\alpha\beta} \frac{\partial}{\partial \rho_i^\alpha} \frac{\partial}{\partial \rho_j^\beta} F = \sqrt{2} m e^{-i\dot{\varphi}} F. \quad (4.13i)$$

Here, \check{T}_0 and \check{T}_r ($r = \hat{i}, 3$) are defined by

$$\check{T}_0 := -\frac{1}{2} \mathbf{W}_i^A \frac{\partial}{\partial \mathbf{W}_i^A}, \quad \check{T}_r := -\frac{1}{2} \sigma_{rj}{}^k \mathbf{W}_k^A \frac{\partial}{\partial \mathbf{W}_j^A}, \quad (4.14)$$

and $\dot{\varphi}$ is defined by

$$\dot{\varphi} := -\frac{i}{2} \left[\ln(\epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_{j\dot{\beta}}) - \ln \left(\epsilon_{ij} e^{\alpha\beta} \frac{\partial}{\partial \rho_i^\alpha} \frac{\partial}{\partial \rho_j^\beta} \right) \right]. \quad (4.15)$$

Equations (4.13a)–(4.13d) imply that F is actually independent of a , \mathbf{b}^3 , \mathbf{e} , and \mathbf{h} . Hence it follows that F is a function of the twistors \mathbf{W}_i^A only. The holomorphic functions of \mathbf{W}_i^A , such as F , are often referred to as the twistor functions. As can be seen immediately, Eqs. (4.13h) and (4.13i) are respectively equivalent to

$$\varpi_{i\dot{\alpha}} \varpi_{j\dot{\beta}} F = \frac{m}{\sqrt{2}} \epsilon_{ij} e^{i\dot{\varphi}} F, \quad (4.16a)$$

$$\epsilon^{\alpha\beta} \frac{\partial}{\partial \rho_i^\alpha} \frac{\partial}{\partial \rho_j^\beta} F = \frac{m}{\sqrt{2}} \epsilon^{ij} e^{-i\dot{\varphi}} F. \quad (4.16b)$$

Combining Eqs. (4.13e) and (4.13f), we have

$$\mathbf{W}_1^A \frac{\partial}{\partial \mathbf{W}_1^A} F = -2(s_1 + 1)F, \quad (4.17a)$$

$$\mathbf{W}_2^A \frac{\partial}{\partial \mathbf{W}_2^A} F = -2(s_2 + 1)F, \quad (4.17b)$$

where

$$s_1 := \frac{1}{2}(s + t), \quad s_2 := \frac{1}{2}(s - t). \quad (4.18)$$

The pair of Eqs. (4.13e) and (4.13f) is equivalent to the pair of Eqs. (4.17a) and (4.17b). Obviously, Eqs. (4.17a) and (4.17b) are simultaneously satisfied by a homogeneous twistor function of degree $-2s_1 - 2$ with respect to \mathbf{W}_1^A and degree $-2s_2 - 2$ with respect to \mathbf{W}_2^A . These degrees must be integers so that F can be a single-valued function of \mathbf{W}_i^A . In this way, the allowed values of s_1 and s_2 are restricted to arbitrary integer or half-integer values, and accordingly s and t are also restricted to arbitrary integer or half-integer values. We thus see that the Chern-Simons coefficients $2s$ and $2t$, which are coefficients of the one-dimensional Chern-Simons terms S_a and S_{b3} , respectively, are quantized to be arbitrary integer values.

The operators \check{T}_r fulfill the $SU(2)$ commutation relation

$$[\check{T}_r, \check{T}_s] = i\epsilon_{rst} \check{T}_t. \quad (4.19)$$

Following the general method for solving the eigenvalue problem in the $SU(2)$ Lie algebra [34], we can simultaneously solve the eigenvalue equation for the Casimir operator $\check{T}_r \check{T}_r = \check{T}_i \check{T}_i + \check{T}_3 \check{T}_3$, i.e.,

$$\check{T}_r \check{T}_r F = \Lambda F, \quad (4.20)$$

and Eq. (4.13f) to obtain

$$\Lambda = I(I + 1), \quad I = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (4.21a)$$

$$t = -I, -I + 1, \dots, I - 1, I. \quad (4.21b)$$

In deriving Eqs. (4.21a) and (4.21b), we assume the existence of a positive-definite inner product in the function space consisting of twistor functions. (As for the twistor formulation of a massless system, a twistor-function space with a positive-definite inner product has been established [35].) Since t takes integer or half-integer values as explained above, I also takes integer or half-integer values accordingly. From Eqs. (4.13f), (4.13g), (4.20), and (4.21a), the allowed values of the positive constant k are determined to be

$$k = 2\sqrt{I(I + 1) - t^2}. \quad (4.22)$$

In this way, the coefficient of S_{b12} is also quantized in addition to the Chern-Simons coefficients. It is now clear

that the twistor function F is characterized by the set of three quantum numbers (s, I, t) , or equivalently, by (I, s_1, s_2) ; for this reason, it is convenient to label F as $F_{s,I,t}$ or F_{I,s_1,s_2} .

V. PENROSE TRANSFORM AND A MASSIVE SPINOR FIELD OF ARBITRARY RANK

In this section, we define a spinor field of arbitrary rank by the Penrose transform of F_{I,s_1,s_2} . We also demonstrate that this spinor field satisfies generalized DFP equations with $SU(2)$ indices.

Let us consider the Penrose transform of F_{I,s_1,s_2} specified by

$$\begin{aligned} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) \\ = \frac{1}{(2\pi i)^4} \oint_{\Sigma} e^{ip\bar{\varphi}} \varpi_{j_1 \dot{\alpha}_1} \dots \varpi_{j_q \dot{\alpha}_q} \\ \times \frac{\partial}{\partial \rho_{i_1}^{\alpha_1}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} F_{I,s_1,s_2}(\mathbf{W}) d^4 \varpi \end{aligned} \quad (5.1)$$

with

$$d^4 \varpi := d\varpi_{1\dot{0}} \wedge d\varpi_{1\dot{1}} \wedge d\varpi_{2\dot{0}} \wedge d\varpi_{2\dot{1}} \quad (5.2)$$

to define the rank- $(p+q)$ spinor field $\Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}$ (occasionally abbreviated as Ψ) on complexified Minkowski space \mathbb{CM} . Here, Σ denotes a suitable four-dimensional contour. Equation (5.1) is identified as a nonprojective form of the Penrose transform in the massive case [2].⁶ It should be noted that Ψ has the upper and lower $SU(2)$ indices in addition to the dotted and undotted spinor indices. Because of the structure of Eq. (5.1), the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. It is obvious that Ψ has the symmetric properties

$$\begin{aligned} \Psi_{\alpha_1 \dots \alpha_m \dots \alpha_n \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_m \dots i_n \dots i_p} \\ = \Psi_{\alpha_1 \dots \alpha_n \dots \alpha_m \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_n \dots i_m \dots i_p}, \end{aligned} \quad (5.3a)$$

⁶The two-dimensional projective form of the Penrose transform (5.1) is given by

$$\begin{aligned} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) \\ = \frac{1}{(2\pi i)^2} \oint_{\Gamma} e^{ip\bar{\varphi}} \varpi_{j_1 \dot{\alpha}_1} \dots \varpi_{j_q \dot{\alpha}_q} \\ \times \frac{\partial}{\partial \rho_{i_1}^{\alpha_1}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} F_{I,s_1,s_2}(\mathbf{W}) \varpi_{1\dot{\beta}} d\varpi_1^{\dot{\beta}} \wedge \varpi_{2\dot{\gamma}} d\varpi_2^{\dot{\gamma}}, \end{aligned}$$

where Γ denotes a suitable two-dimensional contour [3]. We can also find the three-dimensional projective form of the Penrose transform (5.1) [8].

$$\begin{aligned} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p} \\ = \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}. \end{aligned} \quad (5.3b)$$

Suppose that among i_1, \dots, i_p , the number of 1's is p_1 and the number of 2's is $p_2 (= p - p_1)$. Similarly, suppose that among j_1, \dots, j_q , the number of 1's is q_1 and the number of 2's is $q_2 (= q - q_1)$. The integral in Eq. (5.1) can remain nonvanishing if

$$s_1 = \frac{1}{2}(q_1 - p_1), \quad s_2 = \frac{1}{2}(q_2 - p_2). \quad (5.4)$$

Combining Eqs. (4.18) and (5.4), we have

$$s = \frac{1}{2}(q_1 - p_1 + q_2 - p_2), \quad (5.5a)$$

$$t = \frac{1}{2}(q_1 - p_1 - q_2 + p_2). \quad (5.5b)$$

Now we can show that

$$\begin{aligned} \frac{\partial}{\partial z_{\beta\dot{\beta}}} F(\mathbf{W}) &= \frac{\partial \rho_k^{\dot{\gamma}}}{\partial z_{\beta\dot{\beta}}} \frac{\partial}{\partial \rho_k^{\dot{\gamma}}} F(\mathbf{W}) = \frac{\partial \rho_k^{\dot{\gamma}}}{\partial z_{\beta\dot{\beta}}} \frac{\partial}{\partial \rho_k^{\dot{\gamma}}} F(\mathbf{W}) \\ &= \frac{\partial (iz^{\dot{\gamma}} \varpi_{k\dot{\gamma}})}{\partial z_{\beta\dot{\beta}}} \frac{\partial}{\partial \rho_k^{\dot{\gamma}}} F(\mathbf{W}) = i \varpi_k^{\dot{\beta}} \epsilon^{\beta\dot{\gamma}} \frac{\partial}{\partial \rho_k^{\dot{\gamma}}} F(\mathbf{W}). \end{aligned} \quad (5.6)$$

Here the weak equality $\rho_k^{\dot{\gamma}} \approx \rho_k^{\dot{\gamma}}$, Eq. (2.22), and the formula $\partial/\partial z_{\beta\dot{\beta}} = e^{\beta\alpha} e^{\dot{\beta}\dot{\alpha}} \partial/\partial z^{\alpha\dot{\alpha}}$ have been used. The derivative of Ψ with respect to $z_{\beta\dot{\beta}}$ can be calculated by using Eq. (5.6) as follows:

$$\begin{aligned} \frac{\partial}{\partial z_{\beta\dot{\beta}}} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) \\ = \frac{1}{(2\pi i)^4} \oint_{\Sigma} e^{ip\bar{\varphi}} \varpi_{j_1 \dot{\alpha}_1} \dots \varpi_{j_q \dot{\alpha}_q} \\ \times \frac{\partial}{\partial \rho_{i_1}^{\alpha_1}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} \frac{\partial}{\partial z_{\beta\dot{\beta}}} F_{I,s_1,s_2}(\mathbf{W}) d^4 \varpi \\ = \frac{i}{(2\pi i)^4} \oint_{\Sigma} e^{ip\bar{\varphi}} \varpi_{j_1 \dot{\alpha}_1} \varpi_k^{\dot{\beta}} \varpi_{j_2 \dot{\alpha}_2} \dots \varpi_{j_q \dot{\alpha}_q} \\ \times \frac{\partial}{\partial \rho_{i_2}^{\alpha_2}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} e^{\beta\dot{\gamma}} \frac{\partial}{\partial \rho_{i_1}^{\alpha_1}} \frac{\partial}{\partial \rho_k^{\dot{\gamma}}} F_{I,s_1,s_2}(\mathbf{W}) d^4 \varpi. \end{aligned} \quad (5.7)$$

Contracting over the indices $\dot{\beta}$ and $\dot{\alpha}_1$ in Eq. (5.7) and using Eq. (4.16a), we obtain

$$\begin{aligned}
& \frac{\partial}{\partial z_{\beta\dot{\beta}}} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\beta}\dot{\alpha}_2 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) \\
&= \frac{m}{\sqrt{2}} \epsilon^{\beta\gamma} \epsilon_{j_1 k} \frac{i}{(2\pi i)^4} \oint_{\Sigma} e^{i(p+1)\varphi} \varpi_{j_2 \dot{\alpha}_2} \dots \varpi_{j_q \dot{\alpha}_q} \\
&\quad \times \frac{\partial}{\partial \rho_k^\gamma} \frac{\partial}{\partial \rho_{i_1}^{\alpha_1}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} F_{I, s_1, s_2}(\mathbf{W}) d^A \varpi \\
&= \frac{im}{\sqrt{2}} \epsilon^{\beta\gamma} \epsilon_{j_1 k} \Psi_{\gamma \alpha_1 \dots \alpha_p; j_2 \dots j_q, \dot{\alpha}_2 \dots \dot{\alpha}_q}^{ki_1 \dots i_p}(z). \tag{5.8}
\end{aligned}$$

Similarly, contracting over the indices β and α_1 in Eq. (5.7) and using Eq. (4.16b), we obtain

$$\begin{aligned}
& \frac{\partial}{\partial z_{\beta\dot{\beta}}} \Psi_{\beta \alpha_2 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) \\
&= \frac{m}{\sqrt{2}} \epsilon^{\beta\dot{\gamma}} \epsilon^{i_1 k} \frac{i}{(2\pi i)^4} \oint_{\Sigma} e^{i(p-1)\varphi} \varpi_{k\dot{\gamma}} \varpi_{j_1 \dot{\alpha}_1} \dots \varpi_{j_q \dot{\alpha}_q} \\
&\quad \times \frac{\partial}{\partial \rho_{i_2}^{\alpha_2}} \dots \frac{\partial}{\partial \rho_{i_p}^{\alpha_p}} F_{I, s_1, s_2}(\mathbf{W}) d^A \varpi \\
&= \frac{im}{\sqrt{2}} \epsilon^{\beta\dot{\gamma}} \epsilon^{i_1 k} \Psi_{\alpha_2 \dots \alpha_p; k j_1 \dots j_q, \dot{\gamma} \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_2 \dots i_p}(z). \tag{5.9}
\end{aligned}$$

In this way, it has been shown that the spinor field Ψ satisfies the generalized DFP equations with $SU(2)$ indices

$$\begin{aligned}
& i\sqrt{2} \frac{\partial}{\partial z_{\beta\dot{\beta}}} \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\beta}\dot{\alpha}_2 \dots \dot{\alpha}_q}^{i_1 \dots i_p} \\
& \quad + m \epsilon^{\beta\gamma} \epsilon_{j_1 k} \Psi_{\gamma \alpha_1 \dots \alpha_p; j_2 \dots j_q, \dot{\alpha}_2 \dots \dot{\alpha}_q}^{ki_1 \dots i_p} = 0, \tag{5.10a}
\end{aligned}$$

$$\begin{aligned}
& i\sqrt{2} \frac{\partial}{\partial z_{\beta\dot{\beta}}} \Psi_{\beta \alpha_2 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p} \\
& \quad + m \epsilon^{\beta\dot{\gamma}} \epsilon^{i_1 k} \Psi_{\alpha_2 \dots \alpha_p; k j_1 \dots j_q, \dot{\gamma} \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_2 \dots i_p} = 0. \tag{5.10b}
\end{aligned}$$

Using Eqs. (5.10a) and (5.10b) and noting

$$\frac{\partial}{\partial z^{\alpha\dot{\beta}}} \frac{\partial}{\partial z_{\beta\dot{\beta}}} = \frac{1}{2} \delta_{\alpha}^{\beta} \frac{\partial}{\partial z^{\gamma\dot{\gamma}}} \frac{\partial}{\partial z_{\gamma\dot{\gamma}}}, \tag{5.11}$$

we can derive the Klein-Gordon equation

$$\left(\frac{\partial}{\partial z^{\beta\dot{\beta}}} \frac{\partial}{\partial z_{\beta\dot{\beta}}} + m^2 \right) \Psi_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p} = 0. \tag{5.12}$$

This makes it clear that Ψ is a field of mass m . Thus, we obtain a spinor field of arbitrary rank with mass m by means of the Penrose transform (5.1).

VI. RANK-ONE SPINOR FIELDS AND PHYSICAL MEANINGS OF THE GAUGE SYMMETRIES

In this section, we investigate the rank-one spinor fields in detail to clarify the physical meanings of the $U(1)_a$, $U(1)_b$, and $SU(2)$ symmetries as well as those of the constants s and t .

Now we particularly consider Eq. (5.10a) in the case $(p, q) = (0, 1)$ and Eq. (5.10b) in the case $(p, q) = (1, 0)$, which respectively read

$$i\sqrt{2} \frac{\partial}{\partial z^{\alpha\dot{\beta}}} \Psi_i^{\dot{\beta}}(z) - m \epsilon_{ij} \Psi_{\alpha}^j(z) = 0, \tag{6.1a}$$

$$i\sqrt{2} \frac{\partial}{\partial z_{\beta\dot{\alpha}}} \Psi_{\beta}^i(z) + m \epsilon^{ij} \Psi_j^{\dot{\alpha}}(z) = 0, \tag{6.1b}$$

with $\Psi_i^{\dot{\beta}} := \epsilon^{\dot{\beta}\dot{\gamma}} \Psi_{i\dot{\gamma}}$. Equation (6.1a) with $i = 1$ and Eq. (6.1b) with $i = 2$ can be combined in the form of the ordinary Dirac equation

$$D\psi_1(z) = 0, \quad \psi_1(z) := \begin{pmatrix} \Psi_{\beta}^2(z) \\ \Psi_1^{\dot{\beta}}(z) \end{pmatrix}, \tag{6.2}$$

while Eq. (6.1a) with $i = 2$ and Eq. (6.1b) with $i = 1$ can be combined, after replacing $z^{\alpha\dot{\alpha}}$ by $-z^{\alpha\dot{\alpha}}$, as

$$D\psi_2(z) = 0, \quad \psi_2(z) := \begin{pmatrix} \Psi_{\beta}^1(-z) \\ \Psi_2^{\dot{\beta}}(-z) \end{pmatrix}. \tag{6.3}$$

In Eqs. (6.2) and (6.3), D denotes the Dirac operator

$$D := \begin{pmatrix} -m\delta_{\alpha}^{\beta} & i\sqrt{2} \frac{\partial}{\partial z^{\alpha\dot{\beta}}} \\ i\sqrt{2} \frac{\partial}{\partial z_{\beta\dot{\alpha}}} & -m\delta_{\beta}^{\dot{\alpha}} \end{pmatrix}. \tag{6.4}$$

The charge conjugate of $\psi_1(z)$ is found to be

$$\begin{aligned}
\psi_1^c(z) &:= \begin{pmatrix} 0 & -\epsilon_{\beta\dot{\gamma}} \\ \epsilon^{\beta\dot{\gamma}} & 0 \end{pmatrix} \overline{\psi_1(\bar{z})} \\
&= \begin{pmatrix} 0 & -\epsilon_{\beta\dot{\gamma}} \\ \epsilon^{\beta\dot{\gamma}} & 0 \end{pmatrix} \begin{pmatrix} \overline{\Psi_{2\dot{\gamma}}(z)} \\ \overline{\Psi_1^{\dot{\gamma}}(z)} \end{pmatrix} = \begin{pmatrix} \overline{\Psi_{\beta}^1(z)} \\ \overline{\Psi_2^{\dot{\beta}}(z)} \end{pmatrix}, \tag{6.5}
\end{aligned}$$

where the arguments of ψ_1 , namely $z^{\alpha\dot{\alpha}}$, have been replaced by their complex conjugates $\bar{z}^{\alpha\dot{\alpha}} := z^{\alpha\dot{\alpha}}$ so that ψ_1^c can be a holomorphic function of $z^{\alpha\dot{\alpha}}$. Using the complex conjugates of Eqs. (6.1a) and (6.1b), we can show that $D\psi_1^c(z) = 0$. Since ψ_2 and ψ_1^c satisfy the same Dirac equation and have the same spinor and $SU(2)$ indices, they can be identified

TABLE I. A classification of the rank-one spinor fields.

	Particle	Antiparticle
Left-handed	Ψ_α^2	Ψ_α^1
Right-handed	Ψ_1^α	Ψ_2^α

with each other up to an overall constant.⁷ (This identification may be confirmed by the *CPT* symmetry.) If $\psi_1(z)$ is a spinor field of a particle with four-momentum (E, \mathbf{p}) , then $\psi_1^\alpha(z) (\simeq \psi_2(z))$ is regarded as a spinor field of a corresponding antiparticle with four-momentum $(-E, -\mathbf{p})$. Accordingly, $\psi_2(-z) = (\Psi_\alpha^1(z), \Psi_2^\alpha(z))^T$ is considered a spinor field of the antiparticle with four-momentum (E, \mathbf{p}) . In the light of this fact, it is clear that $\Psi_\alpha^2(z)$ and $\Psi_\alpha^1(z)$ represent a left-handed particle and a corresponding left-handed antiparticle, respectively, while $\Psi_1^\alpha(z)$ and $\Psi_2^\alpha(z)$ represent a right-handed particle and a corresponding right-handed antiparticle, respectively, as summarized in Table I. We thus see that the index i of $\Psi_\alpha^i(z)$ and $\Psi_i^\alpha(z)$ distinguishes between a particle and its antiparticle.

Using Eq. (5.5), we can obtain the possible values of s and t for each of the rank-one spinor fields as in Table II. We observe that the left-handed spinor fields $\Psi_\alpha^i(z)$ ($i = 1, 2$) have $s = -1/2$, while the right-handed spinor fields $\Psi_i^\alpha(z)$ ($i = 1, 2$) have $s = 1/2$. Hence, s turns out to be a quantum number specifying the chirality of a spinor field. Since s is an eigenvalue of \check{T}_0 up to the additive constant 2, as can be seen from (4.13e), \check{T}_0 can be interpreted as the operator of chirality. Accordingly, $U(1)_a$ can be identified as the gauge group of chirality, and the $U(1)_a$ symmetry is physically understood as a gauge symmetry leading to chirality conservation. We also observe that the particle spinor fields $\Psi_\alpha^2(z)$ and $\Psi_1^\alpha(z)$ have $t = 1/2$, while the antiparticle spinor fields $\Psi_\alpha^1(z)$ and $\Psi_2^\alpha(z)$ have $t = -1/2$. Hence, t turns out to be a quantum number distinguishing between a particle and its antiparticle. Then it follows that t is proportional to the electric charge of the particle/antiparticle. Since t is an eigenvalue of \check{T}_3 as can be seen

⁷The plane wave solution of Eq. (6.1) given by

$$\begin{aligned}\Psi_\alpha^i(z) &= -C e^{i\varphi/2} \bar{\omega}_\alpha^i \exp(-iz^{\dot{\gamma}} \bar{\omega}_{\dot{\gamma}}^k \omega_{k\dot{\gamma}}), \\ \Psi_i^\alpha(z) &= C e^{-i\varphi/2} \omega_i^\alpha \exp(-iz^{\dot{\gamma}} \bar{\omega}_{\dot{\gamma}}^k \omega_{k\dot{\gamma}})\end{aligned}$$

fulfills the conditions $\Psi_\alpha^i(-z) = -(C/\bar{C})\bar{\Psi}_\alpha^i(z)$ and $\Psi_i^\alpha(-z) = -(C/\bar{C})\bar{\Psi}_i^\alpha(z)$. Here, C is a complex constant and φ is given in Eq. (3.26e). These conditions lead to $\psi_2(z) = -(C/\bar{C})\psi_1^\alpha(z)$, and hence, in this case, ψ_2 and ψ_1^α can indeed be identified with each other. For verifying that the plane wave solution satisfies Eq. (6.1), it is convenient to use the classical counterparts of Eqs. (4.16a) and (4.16b),

$$\omega_{i\dot{\alpha}} \bar{\omega}_{\dot{\gamma}}^i = \frac{m}{\sqrt{2}} \epsilon_{ij} e^{i\varphi}, \quad \bar{\omega}_\alpha^i \bar{\omega}^{j\alpha} = \frac{m}{\sqrt{2}} \epsilon^{ij} e^{-i\varphi}.$$

TABLE II. The values of s and t of the rank-one spinor fields.

	s	t		s	t
Ψ_α^2	$-1/2$	$1/2$	Ψ_α^1	$-1/2$	$-1/2$
Ψ_1^α	$1/2$	$1/2$	Ψ_2^α	$1/2$	$-1/2$

from (4.13f), \check{T}_3 can be interpreted as the operator of electric charge up to a constant of proportionality. Accordingly, $U(1)_b$ can be identified with the gauge group of electric charge, and the $U(1)_b$ symmetry is physically understood as a gauge symmetry leading to electric charge conservation.

Now we recall that our study has been performed in the unitary gauge in which the GGS action takes the form of Eq. (2.20) or Eq. (2.24). In the unitary gauge, the local $SU(2)$ symmetry is hidden and the $U(1)_b$ symmetry is linearly realized in accordance with Eq. (2.12). The manifestly $SU(2)$ covariant formulation can be developed on the basis of the action (2.23). The rank-one spinor fields found in this formulation, denoted by Ω_i^α and Ω_α^i , are related to Ψ_i^α and Ψ_α^i by⁸

$$\Omega_i^\alpha(z) = V_i^j \Psi_j^\alpha(z), \quad \Omega_\alpha^i(z) = \Psi_\alpha^j(z) V_j^\dagger{}^i. \quad (6.6)$$

Because V is independent of $z^{\dot{\alpha}\alpha}$, we can readily verify by using Eqs. (6.1a) and (6.1b) that

$$i\sqrt{2} \frac{\partial}{\partial z^{\dot{\alpha}\beta}} \Omega_i^{\dot{\beta}}(z) - m \epsilon_{ij} \Omega_\alpha^j(z) = 0, \quad (6.7a)$$

$$i\sqrt{2} \frac{\partial}{\partial z_{\beta\dot{\alpha}}} \Omega_\beta^i(z) + m \epsilon^{ij} \Omega_j^\alpha(z) = 0. \quad (6.7b)$$

Following the above consideration for $\Psi_i^\alpha(z)$ and $\Psi_\alpha^i(z)$, we see that $\Omega_\alpha^2(z)$ and $\Omega_\alpha^1(z)$ constitute a doublet of left-handed particle and antiparticle spinor fields, while $\Omega_1^\alpha(z)$ and $\Omega_2^\alpha(z)$ constitute a doublet of right-handed particle and antiparticle spinor fields. Under the $SU(2)$ transformation, Ω_i^α and Ω_α^i transform linearly as

$$\Omega_i^\alpha \rightarrow \Omega_i^\alpha = U_i^j \Omega_j^\alpha, \quad \Omega_\alpha^i \rightarrow \Omega_\alpha^i = \Omega_\alpha^j U_j^\dagger{}^i, \quad (6.8)$$

whereas Ψ_i^α and Ψ_α^i transform according to the $U(1)_b$ transformation

⁸The rank- $(p+q)$ spinor field in the manifestly $SU(2)$ covariant formulation is given by

$$\begin{aligned}\Omega_{\alpha_1 \dots \alpha_p; j_1 \dots j_q; \dot{\alpha}_1 \dots \dot{\alpha}_q}^{i_1 \dots i_p}(z) &= \frac{1}{(2\pi i)^4} \oint_{\Sigma} e^{ip\varphi} \pi_{j_1 \dot{\alpha}_1} \dots \pi_{j_q \dot{\alpha}_q} \\ &\times \frac{\partial}{\partial \mu_{i_1}^{\alpha_1}} \dots \frac{\partial}{\partial \mu_{i_p}^{\alpha_p}} F_{I, s_1, s_2}(\mu, \pi) d^4 \pi,\end{aligned}$$

where μ_i^α is a spinor related to ω_i^α by the weak equality $\mu_i^\alpha \approx \omega_i^\alpha$.

$$\Psi_i^{\dot{\alpha}} \rightarrow \Psi_i^{\prime\dot{\alpha}} = \Theta_i^j \Psi_j^{\dot{\alpha}}, \quad \Psi_\alpha^i \rightarrow \Psi_\alpha^{\prime i} = \Psi_\alpha^j \Theta_j^{\dagger i}. \quad (6.9)$$

As seen from Eq. (6.8), the $SU(2)$ transformation causes a continuous transformation between the particle spinor field $\Omega_\alpha^2 (\Omega_1^{\dot{\alpha}})$ and the antiparticle spinor field $\Omega_\alpha^1 (\Omega_2^{\dot{\alpha}})$. The $SU(2)$ symmetry therefore turns out to be a gauge symmetry realized in the particle-antiparticle doublets $(\Omega_\alpha^2, \Omega_\alpha^1)$ and $(\Omega_1^{\dot{\alpha}}, \Omega_2^{\dot{\alpha}})$. Such a symmetry, however, is not observed in nature; hence, it should be considered that the $SU(2)$ symmetry is hidden or broken. The formulation in the unitary gauge is appropriate for this situation, because, in the unitary gauge, the $SU(2)$ symmetry is hidden and the $U(1)_b$ symmetry is manifestly exhibited instead.

VII. SUMMARY AND DISCUSSION

We have presented a gauged twistor model of a free massive spinning particle in four dimensions. This model is a non-Abelian extension of the gauged twistor model of a free massless spinning particle in four dimensions, presented in Refs. [22–24]. The extended model is governed by the GGS action that was elaborated by adding the 1D Chen-Simons terms S_a and S_{b_3} and the novel term S_{b_e} to the gauged twistorial action S_{mg} [see Eq. (2.20)]. The GGS action remains invariant under the reparametrization and the $U(1)_a$ and local $SU(2)$ transformations, although the $SU(2)$ symmetry is nonlinearly realized in the action. In the unitary gauge, the $U(1)_b$ symmetry is manifestly exhibited, while the $SU(2)$ symmetry is hidden.

We have studied the canonical Hamiltonian formalism based on the GGS action in the unitary gauge by following Dirac’s recipe for constrained Hamiltonian systems. The classification of the constraints into first and second classes was carried out strictly, and the Dirac brackets between the canonical variables were obtained concretely. It was demonstrated that just sufficient constraints for the twistor variables are consistently derived as the secondary first-class constraints [see Eqs. (3.28e)–(3.28i)].

The subsequent canonical quantization of the system was performed in terms of the new twistor variables W_i^A and \bar{W}_A^i , because they satisfy the simple Dirac brackets given in Eq. (3.33). We have shown that the Chern-Simons coefficients $2s$ and $2t$ are quantized to be arbitrary integer values as a result of the canonical quantization based on the commutation relations (4.2a)–(4.2e). In general, the quantization of Chern-Simons coefficient is a common consequence in certain theories in which the Chern-Simons terms play crucial roles (see e.g. Refs. [36–39]). Our gauged twistor model can be regarded as a specific example of such theories. Intriguingly, the coefficient k of $S_{b_{12}}$ is also quantized via solving the eigenvalue problem of the $SU(2)$ Lie algebra. We found that the twistor functions in our model are eigenfunctions of the relevant differential operators governed by the $U(1)_a \times SU(2)$ Lie algebra [see Eqs. (4.13e)–(4.13g)]. Each twistor function F is then

labeled by a set of three quantum numbers associated with the $U(1)_a \times SU(2)$ Lie algebra.

We have carried out the Penrose transform of the twistor function F to obtain a massive spinor field of arbitrary rank defined on complexified Minkowski space [see Eq. (5.1)]. As emphasized earlier, this spinor field has the upper and lower $SU(2)$ indices in addition to the dotted and undotted spinor indices. In fact, we observed that the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. We also demonstrated that the spinor field satisfies the generalized DFP equations with $SU(2)$ indices, given in Eq. (5.10).

We have investigated the rank-one spinor fields in detail to clarify the physical meanings of the gauge symmetries as well as those of the constants s and t . It turned out that s is a quantum number specifying the chirality of a spinor field and that the $U(1)_a$ symmetry is a gauge symmetry leading to chirality conservation. It also turned out that t is a quantum number proportional to the electric charge of a spinor field and that the $U(1)_b$ symmetry is a gauge symmetry leading to electric charge conservation. The $SU(2)$ symmetry was shown to be a gauge symmetry realized in the particle-antiparticle doublets. Such a symmetry, however, is not observed in nature, so that it should be considered to be hidden or broken. Fortunately our twistor formulation in the unitary gauge is appropriate for describing this situation. Since the $SU(2)$ symmetry is a symmetry realized in the particle-antiparticle doublets, it cannot be identified with the weak isospin symmetry. We thus conclude that the idea proposed by Penrose, Perjés, and Hughston [4–8] is not valid in our gauged twistor model.

The observation that s is a quantum number specifying the chirality of a spinor field is supported for the following reason: The gauged Shirafuji action for a massless spinning particle enjoys the $U(1)_a$ symmetry and contains its associated constant s [22–24]. This constant is indeed shown to be the helicity of a massless spinning particle. As is well known, the chirality is an analog of the helicity, while the chirality is a Lorentz invariant quantity valid for massive particles as well as massless particles. (For massless particles, chirality is the same as helicity.) For this reason, in the present twistor model, it is quite natural to identify the Lorentz invariant quantity s as the chirality quantum number.

We have seen that each eigenstate of \check{T}_3 corresponds (via the Penrose transform) to a particle or antiparticle state represented by its own spinor field. Remarkably, we encounter a similar situation in studying the rigid body model [40,41]. In this model, the rigid body rotation leads to an intrinsic $SU(2)$ symmetry in addition to the spin $SU(2)$ symmetry. Hara *et al.* showed that the eigenstates of the third generator of the intrinsic $SU(2)$ group are assigned to particle and antiparticle spinor fields. They also pointed out that this generator cannot be identified with the third component of the isospin generators.

(Accordingly, it turns out that the intrinsic $SU(2)$ symmetry cannot be regarded as the isospin symmetry. This result contradicts the earlier idea concerning isospin proposed in Refs. [42,43].) We thus see that the gauged twistor model and the rigid body model share common aspects.

Now we recall that the secondary first-class constraints (3.28e)–(3.28g), or equivalently, Eqs. (3.36a), (3.36b), and (3.38), have been derived systematically on the basis of the $U(1)_a$, $U(1)_b$, and reparametrization symmetries of the GGS action. By contrast, the remaining secondary first-class constraints (3.28h) and (3.28i) have been derived as a result of incorporating the mass-shell condition (2.3) into the GGS action by hand. Considering this fact, we can never say that the present approach for constructing the GGS action is satisfactory from the gauge-theoretical point of view. To make our gauged twistor formulation complete, we need to establish an approach in which the mass-shell condition (2.3) is supplied as an inevitable outcome of an extra gauge symmetry.

In this paper, we have not presented precise definitions of the chirality and charge conjugation for a massive spinor field of arbitrary rank. The chirality may be defined on the basis of the type of spinor indices of the field. For clarifying the definition of charge conjugation and its associated concept of particle-antiparticle, it is necessary to examine coupling of a massive spinor field of arbitrary rank to the electromagnetic field. The precise definitions of chirality and charge conjugation should confirm our observation on the physical meanings of the constants s and t . We hope to address the aforementioned issues in the near future.

ACKNOWLEDGMENTS

We would like to thank Shigefumi Naka for useful comments. The work of S.D. is supported in part by a Grant-in-Aid for Fundamental Scientific Research from College of Science and Technology, Nihon University.

APPENDIX: POINCARÉ SYMMETRY AND PAULI-LUBANSKI PSEUDOVECTOR

In this appendix, we consider the Poincaré symmetry and the Pauli-Lubanski pseudovector within the framework of the gauged twistor formulation.

We can easily show that the GGS action (2.20) remains invariant under the infinitesimal Poincaré transformation [or more accurately, the infinitesimal $SL(2, \mathbb{C}) \times \mathbb{R}^{1,3}$ transformation],

$$Q_i^\alpha \rightarrow Q_i'^\alpha = Q_i^\alpha - \varepsilon^\alpha{}_\beta Q_i^\beta - i\varepsilon^{\alpha\dot{\beta}} \varpi_{i\dot{\beta}}, \quad (\text{A1a})$$

$$\bar{Q}^{i\dot{\alpha}} \rightarrow \bar{Q}'^{i\dot{\alpha}} = \bar{Q}^{i\dot{\alpha}} - \bar{\varepsilon}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{i\dot{\beta}} + i\varepsilon^{\beta\dot{\alpha}} \bar{\varpi}_\beta^i, \quad (\text{A1b})$$

$$\varpi_{i\dot{\alpha}} \rightarrow \varpi'_{i\dot{\alpha}} = \varpi_{i\dot{\alpha}} + \bar{\varepsilon}^{\dot{\alpha}}{}_{\dot{\beta}} \varpi_{i\dot{\beta}}, \quad (\text{A1c})$$

$$\bar{\varpi}_\alpha^i \rightarrow \bar{\varpi}'_\alpha^i = \bar{\varpi}_\alpha^i + \varepsilon_\alpha{}^\beta \bar{\varpi}_\beta^i. \quad (\text{A1d})$$

Here, $\varepsilon^{\alpha\dot{\beta}}$ and $\bar{\varepsilon}^{\dot{\alpha}\beta} (= \overline{\varepsilon^{\alpha\dot{\beta}}})$ are parameters of the infinitesimal Lorentz transformation [or more accurately, the infinitesimal $SL(2, \mathbb{C})$ transformation], satisfying the symmetric properties $\varepsilon^{\alpha\dot{\beta}} = \varepsilon^{\beta\dot{\alpha}}$ and $\bar{\varepsilon}^{\dot{\alpha}\beta} = \bar{\varepsilon}^{\beta\dot{\alpha}}$, while $\varepsilon^{\alpha\dot{\beta}}$ is a parameter of the infinitesimal translation, satisfying the Hermiticity $\overline{\varepsilon^{\alpha\dot{\beta}}} = \varepsilon^{\beta\dot{\alpha}}$. The fields h , \bar{h} , a , and b^r are assumed to be Poincaré invariant. Since the GGS action is Poincaré invariant, we can derive conserved quantities by applying Noether's theorem. The conserved quantities corresponding to $\varepsilon^{\alpha\dot{\beta}}$, $\bar{\varepsilon}^{\dot{\alpha}\beta}$, and $\varepsilon^{\alpha\dot{\beta}}$ are found to be

$$\mu_{\alpha\dot{\beta}} := \frac{i}{2} (Q_{i\dot{\alpha}} \bar{\varpi}_\beta^i + Q_{i\dot{\beta}} \bar{\varpi}_\alpha^i), \quad (\text{A2a})$$

$$\bar{\mu}_{\dot{\alpha}\beta} := -\frac{i}{2} (\bar{Q}_{\dot{\alpha}}^i \varpi_{i\dot{\beta}} + \bar{Q}_{\dot{\beta}}^i \varpi_{i\dot{\alpha}}), \quad (\text{A2b})$$

$$p_{\alpha\dot{\beta}} := \bar{\varpi}_\alpha^i \varpi_{i\dot{\beta}}. \quad (\text{A2c})$$

Substituting Eqs. (3.37a) and (3.37b) into Eqs. (A2a) and (A2b), respectively, we can rewrite $\mu_{\alpha\dot{\beta}}$ and $\bar{\mu}_{\dot{\alpha}\beta}$ as

$$\mu_{\alpha\dot{\beta}} = \frac{i}{2} (\rho_{i\dot{\alpha}} \bar{\varpi}_\beta^i + \rho_{i\dot{\beta}} \bar{\varpi}_\alpha^i), \quad (\text{A3a})$$

$$\bar{\mu}_{\dot{\alpha}\beta} = -\frac{i}{2} (\bar{\rho}_{\dot{\alpha}}^i \varpi_{i\dot{\beta}} + \bar{\rho}_{\dot{\beta}}^i \varpi_{i\dot{\alpha}}). \quad (\text{A3b})$$

The angular momentum tensor is given by

$$M_{\alpha\dot{\alpha}\beta\dot{\beta}} := \mu_{\alpha\dot{\beta}} \varepsilon_{\dot{\alpha}\beta} + \bar{\mu}_{\dot{\alpha}\beta} \varepsilon_{\alpha\dot{\beta}}, \quad (\text{A4})$$

and the four-momentum vector is given by Eq. (A2c).

The Pauli-Lubanski pseudovector is defined by [2,44,45]

$$W^{\alpha\dot{\alpha}} := \frac{1}{2} \varepsilon^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} p_{\beta\dot{\beta}} M_{\gamma\dot{\gamma}\delta\dot{\delta}}, \quad (\text{A5})$$

which can be written as

$$W^{\alpha\dot{\alpha}} = -i\mu^{\alpha\dot{\beta}} p_{\beta\dot{\alpha}} + i\bar{\mu}^{\dot{\alpha}\beta} p_{\dot{\alpha}\beta}, \quad (\text{A6})$$

by using the formula

$$\varepsilon^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = i(\varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \varepsilon^{\dot{\alpha}\dot{\delta}} \varepsilon^{\dot{\beta}\dot{\gamma}} - \varepsilon^{\alpha\delta} \varepsilon^{\beta\gamma} \varepsilon^{\dot{\alpha}\dot{\gamma}} \varepsilon^{\dot{\beta}\dot{\delta}}). \quad (\text{A7})$$

Using the identity

$$\varepsilon^{\alpha\dot{\beta}} \rho_i^\gamma + \varepsilon^{\beta\dot{\gamma}} \rho_i^\alpha + \varepsilon^{\gamma\alpha} \rho_i^\beta = 0 \quad (\text{A8})$$

and its complex conjugate, we can express Eq. (A6) as

$$W^{\alpha\dot{\alpha}} = (\rho_i^\beta \bar{\omega}_\beta^j + \omega_{i\dot{\beta}} \bar{\rho}^{j\dot{\beta}}) \bar{\omega}^{i\alpha} \omega_j^{\dot{\alpha}} - \frac{1}{2} (\rho_i^\beta \bar{\omega}_\beta^i + \omega_{i\dot{\beta}} \bar{\rho}^{i\dot{\beta}}) \bar{\omega}^{j\alpha} \omega_j^{\dot{\alpha}}, \quad (\text{A9})$$

or more concisely,

$$W^{\alpha\dot{\alpha}} = \left(\delta_i^l \delta_k^j - \frac{1}{2} \delta_i^j \delta_k^l \right) \bar{W}_B^k W_l^B \bar{\omega}^{i\alpha} \omega_j^{\dot{\alpha}}. \quad (\text{A10})$$

Here, W_k^B and \bar{W}_B^k are the twistors defined by $W_k^B := (\rho_k^\beta, \omega_{k\dot{\beta}})$ and $\bar{W}_B^k := (\bar{\omega}_\beta^k, \bar{\rho}^{k\dot{\beta}})$ [see the text above Eq. (3.33)]. Applying the formula

$$\frac{1}{2} \sigma_{ri}^j \sigma_{rk}^l = \delta_i^j \delta_k^l - \frac{1}{2} \delta_i^l \delta_k^j \quad (\text{A11})$$

valid for the Pauli matrices σ_r to Eq. (A10), we obtain

$$W^{\alpha\dot{\alpha}} = T_r \sigma_{ri}^j \bar{\omega}^{i\alpha} \omega_j^{\dot{\alpha}}, \quad (\text{A12})$$

with

$$T_r := \frac{1}{2} \bar{W}_B^k \sigma_{rk}^l W_l^B \quad (\text{A13})$$

[see Eq. (3.39)]. Equation (A12) can be written in terms of the (original) twistors Z_k^B and \bar{Z}_B^k as

$$W^{\alpha\dot{\alpha}} = T_r \sigma_{ri}^j \bar{\pi}^{i\alpha} \pi_j^{\dot{\alpha}}, \quad (\text{A14})$$

with

$$T_r := \frac{1}{2} \bar{Z}_B^k \sigma_{rk}^l Z_l^B. \quad (\text{A15})$$

Using the mass-shell constraints

$$\omega_{i\dot{\alpha}} \omega_j^{\dot{\alpha}} \approx \frac{m}{\sqrt{2}} \epsilon_{ij} e^{i\varphi}, \quad (\text{A16a})$$

$$\bar{\omega}_\alpha^i \bar{\omega}^{j\alpha} \approx \frac{m}{\sqrt{2}} \epsilon^{ij} e^{-i\varphi} \quad (\text{A16b})$$

equivalent, respectively, to Eqs. (3.11e) and (3.11f), and utilizing the formula $\sigma_2 \sigma_r \sigma_2 = -\sigma_r^T$, we can show for Eq. (A12) that

$$W_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}} \approx -m^2 T_r T_r. \quad (\text{A17})$$

In our model, twistor quantization is performed with the commutation relations (4.2a) and (4.2b), or equivalently,

$$[\hat{\rho}_{i\alpha}, \hat{\omega}_\beta^j] = -\delta_i^j \epsilon_{\alpha\beta}, \quad [\hat{\rho}_{\dot{\alpha}}^i, \hat{\omega}_{j\dot{\beta}}] = \delta_j^i \epsilon_{\dot{\alpha}\dot{\beta}}, \quad (\text{A18})$$

all others = 0.

The operators corresponding to $\mu_{\alpha\beta}$ and $\bar{\mu}_{\dot{\alpha}\dot{\beta}}$ are defined by replacing the twistor variables in Eq. (A3) with their corresponding operators and by obeying the Weyl ordering rule. After using the commutation relations in Eq. (A18), we have

$$\hat{\mu}_{\alpha\beta} = \frac{i}{2} (\hat{\rho}_{i\alpha} \hat{\omega}_\beta^i + \hat{\rho}_{i\beta} \hat{\omega}_\alpha^i), \quad (\text{A19a})$$

$$\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}} = -\frac{i}{2} (\hat{\rho}_{\dot{\alpha}}^i \hat{\omega}_{i\dot{\beta}} + \hat{\rho}_{i\dot{\beta}} \hat{\omega}_{i\dot{\alpha}}). \quad (\text{A19b})$$

The operator corresponding to $p_{\alpha\dot{\beta}}$ is found immediately from Eq. (A2c) to be

$$\hat{p}_{\alpha\dot{\beta}} = \hat{\omega}_\alpha^i \hat{\omega}_{i\dot{\beta}}. \quad (\text{A20})$$

Using Eq. (A18), we can calculate the commutation relations between $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ to obtain

$$\begin{aligned} [\hat{\mu}_{\alpha\beta}, \hat{\mu}_{\gamma\delta}] &= -\frac{i}{2} (\epsilon_{\alpha\gamma} \hat{\mu}_{\beta\delta} + \epsilon_{\alpha\delta} \hat{\mu}_{\beta\gamma} + \epsilon_{\beta\gamma} \hat{\mu}_{\alpha\delta} + \epsilon_{\beta\delta} \hat{\mu}_{\alpha\gamma}), \\ [\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}, \hat{\bar{\mu}}_{\dot{\gamma}\dot{\delta}}] &= -\frac{i}{2} (\epsilon_{\dot{\alpha}\dot{\gamma}} \hat{\bar{\mu}}_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}} \hat{\bar{\mu}}_{\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\beta}\dot{\gamma}} \hat{\bar{\mu}}_{\dot{\alpha}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}} \hat{\bar{\mu}}_{\dot{\alpha}\dot{\gamma}}), \\ [\hat{\mu}_{\alpha\beta}, \hat{p}_{\gamma\dot{\delta}}] &= -\frac{i}{2} (\epsilon_{\alpha\gamma} \hat{p}_{\beta\dot{\delta}} + \epsilon_{\beta\gamma} \hat{p}_{\alpha\dot{\delta}}), \\ [\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}, \hat{p}_{\gamma\dot{\delta}}] &= -\frac{i}{2} (\epsilon_{\dot{\alpha}\dot{\delta}} \hat{p}_{\gamma\dot{\beta}} + \epsilon_{\dot{\beta}\dot{\delta}} \hat{p}_{\gamma\dot{\alpha}}), \\ \text{all others} &= 0. \end{aligned} \quad (\text{A21})$$

These commutation relations specify together a spinor representation of the Poincaré algebra. The operators $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ are thus established as the generators of $SL(2, \mathbb{C}) \times \mathbb{R}^{1,3}$. We can verify that $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ commute with the generators \hat{T}_0 and \hat{T}_r defined in Eq. (4.4). This implies that the Poincaré symmetry and the $U(1)_a \times SU(2)$ internal symmetry are not combined, so that the result is consistent with the Coleman-Mandula theorem [46,47].

The Weyl ordered operator corresponding to the Pauli-Lubanski pseudovector $W^{\alpha\dot{\alpha}}$ can be simplified as

$$\hat{W}^{\alpha\dot{\alpha}} = \hat{T}_r \sigma_{ri}^j \hat{\omega}^{i\alpha} \hat{\omega}_j^{\dot{\alpha}} \quad (\text{A22})$$

by using the commutation relation

$$[\hat{T}_r, \sigma_{si}^j \hat{\omega}^{i\alpha} \hat{\omega}_j^{\dot{\alpha}}] = i \epsilon_{rst} \sigma_{ti}^j \hat{\omega}^{i\alpha} \hat{\omega}_j^{\dot{\alpha}}. \quad (\text{A23})$$

Then, using the physical state conditions

$$\hat{\omega}_{i\dot{\alpha}} \hat{\omega}_j^{\dot{\alpha}} |F\rangle = \frac{m}{\sqrt{2}} \epsilon_{ij} e^{i\varphi} |F\rangle, \quad (\text{A24a})$$

$$\hat{\omega}_\alpha^i \hat{\omega}^{j\alpha} |F\rangle = \frac{m}{\sqrt{2}} e^{ij} e^{-i\hat{\varphi}} |F\rangle \quad (\text{A24b})$$

equivalent, respectively, to Eqs. (4.3h) and (4.3i), we can show that

$$\hat{W}_{\dot{\alpha}\dot{\alpha}} \hat{W}^{\dot{\alpha}\dot{\alpha}} |F\rangle = -m^2 \hat{T}_r \hat{T}_r |F\rangle. \quad (\text{A25})$$

This is precisely a quantum mechanical counterpart of Eq. (A17). The Casimir operators of the Poincaré algebra are given by $\hat{p}_{\dot{\alpha}\dot{\beta}} \hat{p}^{\dot{\alpha}\dot{\beta}}$ and $\hat{W}_{\dot{\alpha}\dot{\alpha}} \hat{W}^{\dot{\alpha}\dot{\alpha}}$. From Eq. (A24), it follows that

$$\hat{p}_{\dot{\alpha}\dot{\beta}} \hat{p}^{\dot{\alpha}\dot{\beta}} |F\rangle = m^2 |F\rangle. \quad (\text{A26})$$

Then it can be shown that [44,45]

$$\hat{W}_{\dot{\alpha}\dot{\alpha}} \hat{W}^{\dot{\alpha}\dot{\alpha}} |F\rangle = -m^2 J(J+1) |F\rangle, \quad (\text{A27})$$

where J denotes the spin quantum number taking the values

$$J = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (\text{A28})$$

Here, $|F\rangle$ is assumed to be a simultaneous eigenvector of $\hat{W}_{\dot{\alpha}\dot{\alpha}} \hat{W}^{\dot{\alpha}\dot{\alpha}}$ and the other relevant operators $\hat{T}_0, \hat{T}_3, \hat{T}_i \hat{T}_i$, and $\hat{p}_{\dot{\alpha}\dot{\beta}} \hat{p}^{\dot{\alpha}\dot{\beta}}$ [see Eqs. (4.3e), (4.3f), (4.3g), and (A26)]. This assumption holds true, because the generators of $SL(2, \mathbb{C}) \times \mathbb{R}^{1,3}$ commute with those of $U(1)_a \times SU(2)$. The vector $|F\rangle$ turns out to be characterized by the set of quantum numbers $(s, I, t; m, J)$. In terms of $|F\rangle$, Eq. (4.20) reads

$$\hat{T}_r \hat{T}_r |F\rangle = \Lambda |F\rangle, \quad (\text{A29})$$

where Λ is determined to be

$$\Lambda = I(I+1), \quad I = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (\text{A30})$$

Applying Eqs. (A27) and (A29) to Eq. (A25), we eventually have

$$I = J. \quad (\text{A31})$$

This result is consistent with the fact that the number of $SU(2)$ indices of the spinor field Ψ given in Eq. (5.1) is equal to the number of its spinor indices.

-
- [1] R. Penrose and M. A. H. MacCallum, Twistor theory: An approach to the quantisation of fields and space-time, *Phys. Rep.* **6**, 241 (1973).
- [2] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vol. 2, Spinor and Twistor Methods in Space-Time Geometry, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1986).
- [3] S. A. Huggett and K. P. Tod, *An Introduction to Twistor Theory*, 2nd ed., London Mathematical Society, Student Texts Vol. 4 (Cambridge University Press, Cambridge, England, 1994).
- [4] R. Penrose, The twistor programme, *Rep. Math. Phys.* **12**, 65 (1977).
- [5] Z. Perjés, Twistor variables of relativistic mechanics, *Phys. Rev. D* **11**, 2031 (1975).
- [6] Z. Perjés, Unitary space of particle internal states, *Phys. Rev. D* **20**, 1857 (1979).
- [7] Z. Perjés, Perspectives of Penrose theory in particle physics, *Rep. Math. Phys.* **12**, 193 (1977).
- [8] L. P. Hughston, *Twistors and Particles*, Lecture Notes in Physics Vol. 97 (Springer-Verlag, Berlin, 1979).
- [9] S. Fedoruk and V. G. Zima, Bitwistor formulation of massive spinning particle, *J. Kharkiv Univ.* **585**, 39 (2003).
- [10] A. Bette, J. A. de Azcárraga, J. Lukierski, and C. Miquel-Espanya, Massive relativistic free fields with Lorentz spins and electric charges, *Phys. Lett. B* **595**, 491 (2004).
- [11] J. A. de Azcárraga, A. Frydryszak, J. Lukierski, and C. Miquel-Espanya, Massive relativistic particle model with spin from free two-twistor dynamics and its quantization, *Phys. Rev. D* **73**, 105011 (2006).
- [12] S. Fedoruk, A. Frydryszak, J. Lukierski, and C. Miquel-Espanya, Extension of the Shirafuji model for massive particles with spin, *Int. J. Mod. Phys. A* **21**, 4137 (2006).
- [13] J. A. de Azcárraga, J. M. Izquierdo, and J. Lukierski, Supertwistors, massive superparticles and κ -symmetry, *J. High Energy Phys.* **01** (2009) 041.
- [14] L. Mezincescu, A. J. Routh, and P. K. Townsend, Supertwistors and massive particles, *Ann. Phys. (Amsterdam)* **346**, 66 (2014).
- [15] S. Fedoruk and J. Lukierski, Massive twistor particle with spin generated by Souriau-Wess-Zumino term and its quantization, *Phys. Lett. B* **733**, 309 (2014).
- [16] J. A. de Azcárraga, S. Fedoruk, J. M. Izquierdo, and J. Lukierski, Two-twistor particle models and free massive higher spin fields, *J. High Energy Phys.* **04** (2015) 010.
- [17] L. Mezincescu, A. J. Routh, and P. K. Townsend, Twistors and the massive spinning particle, *J. Phys. A* **49**, 025401 (2016).
- [18] T. Shirafuji, Lagrangian mechanics of massless particles with spin, *Prog. Theor. Phys.* **70**, 18 (1983).
- [19] P. A. M. Dirac, Relativistic wave equations, *Proc. R. Soc. A* **155**, 447 (1936).
- [20] M. Fierz, Über die relativistische theorie Kräftefreier teilchen mit beliebigem spin, *Helv. Phys. Acta* **12**, 3 (1939).

- [21] M. Fierz and W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, *Proc. R. Soc. A* **173**, 211 (1939).
- [22] I. Bars and M. Picón, Single twistor description of massless, massive, AdS, and other interacting particles, *Phys. Rev. D* **73**, 064002 (2006).
- [23] S. Deguchi, T. Egami, and J. Note, Spinor and twistor formulations of tensionless bosonic strings in four dimensions, *Prog. Theor. Phys.* **124**, 969 (2010).
- [24] S. Deguchi, S. Negishi, S. Okano, and T. Suzuki, Canonical formalism and quantization of a massless spinning bosonic particle in four dimensions, *Int. J. Mod. Phys. A* **29**, 1450044 (2014).
- [25] S. Deguchi and T. Suzuki, Spinor and twistor formulations of massless particles with rigidity, *Phys. Lett. B* **731**, 337 (2014).
- [26] S. Coleman, J. Wess, and B. Zumino, Structure of phenomenological Lagrangians I, *Phys. Rev.* **177**, 2239 (1969).
- [27] A. Salam and J. Strathdee, Nonlinear realizations I: The role of goldstone bosons, *Phys. Rev.* **184**, 1750 (1969).
- [28] P. van Nieuwenhuizen, in *Supersymmetry and Supergravity '84*, edited by B. de Wit *et al.* (World Scientific, Singapore, 1984), p. 239.
- [29] G. 't Hooft, Renormalizable Lagrangians for massive Yang-Mills fields, *Nucl. Phys.* **B35**, 167 (1971).
- [30] S. Weinberg, General theory of broken local symmetries, *Phys. Rev. D* **7**, 1068 (1973).
- [31] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).
- [32] A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).
- [33] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
- [34] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Redwood City, CA, 1985).
- [35] S. Deguchi and J. Note, (Pre-)Hilbert spaces in twistor quantization, *J. Math. Phys. (N.Y.)* **54**, 072304 (2013).
- [36] S. Deser, R. Jackiw, and S. Templeton, Three-Dimensional Massive Gauge Theories, *Phys. Rev. Lett.* **48**, 975 (1982).
- [37] A. P. Polychronakos, Topological mass quantization and parity violation in $2 + 1$ dimensional QED, *Nucl. Phys.* **B281**, 241 (1987).
- [38] E. Witten, Quantum field theory and the Jones polynomial, *Commun. Math. Phys.* **121**, 351 (1989).
- [39] D. Tong and K. Wong, Monopoles and Wilson lines, *J. High Energy Phys.* **06** (2014) 048.
- [40] O. Hara, S. Y. Tsai, and H. Yabuki, Transformation properties of the rigid sphere wave functions and their relation to the Dirac spinor, *Prog. Theor. Phys. Suppl.* **37–38**, 495 (1966).
- [41] O. Hara and T. Goto, Extended particle model of elementary particles, *Prog. Theor. Phys. Suppl.* **41**, 56 (1968).
- [42] T. Nakano, A relativistic field theory of an extended particle, I, *Prog. Theor. Phys.* **15**, 333 (1956).
- [43] T. Takabayasi, Theory of relativistic rotators and elementary particles. I, *Prog. Theor. Phys.* **23**, 915 (1960).
- [44] V. Bargmann and E. P. Wigner, Group theoretical discussion of relativistic wave equations, *Proc. Natl. Acad. Sci. U.S.A.* **34**, 211 (1948).
- [45] L. H. Ryder, *Quantum Field Theory*, 2nd ed. (Cambridge University Press, Cambridge, England, 1996).
- [46] S. Coleman and J. Mandula, All possible symmetries of the S matrix, *Phys. Rev.* **159**, 1251 (1967).
- [47] S. Weinberg, *The Quantum Theory of Fields*, Vol. III, Supersymmetry (Cambridge University Press, Cambridge, England, 2000).