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Derivative expansion and the induced Chern-Simons term in $\mathcal{N}=1$, $d=3$ superspace

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In this paper we apply a supersymmetric generalization of the method of derivative expansion to compute the induced non-Abelian Chern-Simons term in $\mathcal{N}=1$, $d=3$ superspace, for an arbitrary gauge group.

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I. INTRODUCTION

The three-dimensional field theory models represent themselves as a convenient laboratory for the study of many sophisticated aspects of quantum field theory. The main reasons for this are, first, the simplicity of the formulation of these theories, and second, their one-loop finiteness. For the three-dimensional supersymmetric field theories, the most convenient description is the superfield one [1] which essentially simplifies the perturbative calculations. The superfield approach allowed one to obtain many interesting results for quantum corrections in these theories. The most important ones among them are the explicit calculation of the one- and two-loop effective potential in a general scalar superfield model [2] and in other superfield models [3], proof of the explicit all-loop finiteness of the three-dimensional superfield QED [4], and the explicit calculation of the gauge sector of the one-loop effective action in theories with extended ($\mathcal{N}=2$, $\mathcal{N}=3$ and $\mathcal{N}=4$) supersymmetry [5].

At the same time, an interesting problem related to these theories is the problem of an effective (emergent) dynamics. Following this idea, the known field theory models emerge as effective theories, whereas the role of the fundamental theory is played by some simple model involving the coupling of some matter (which is further integrated out thus being unobserved) with the physically interesting fields. For the gauge theories (including the supersymmetric ones), the paradigmatic example of the emergent dynamics is the arising of the Abelian Maxwell term in the CP^{N-1} theory (see for example Ref. [6]). The concept of the emergent gravity, discussed for example in Ref. [7], plays an important role within this concept since it can allow for solving the problem of a consistent description of quantum gravity. In this paper, we apply this methodology to generate the non-Abelian Chern-Simons term. While this has been done in the three-dimensional nonsupersymmetric case in Refs. [8,9], and in the four-dimensional Lorentz-breaking case in Ref. [10], it has not been done in superfield theories up to now.

The structure of the paper looks like follows. In the Sec. II, we describe the derivative expansion methodology for supersymmetric gauge theories. In Sec. III, we explicitly generate the non-Abelian Chern-Simons term. The Summary is devoted to discussing results and perspectives.

II. DERIVATIVE-EXPANSION SCHEME

Let us start with the following three-dimensional matter action:

$$S[\Phi,\bar{\Phi},A_\alpha]=\int d^5z\bar{\Phi}(\nabla^2+m)\Phi, \qquad (1)$$

where the massive complex scalar superfield Φ interacts with the background non-Abelian Lie-algebra-valued gauge superfield $A_\alpha=A_\alpha^aT^a$ (where T^a are Lie algebra generators) through the minimally coupled gauge-covariant derivative $\nabla_\alpha\equiv D_\alpha-iA_\alpha$. Here we are using the scaled gauge superfield $gA_\alpha\to A_\alpha$, where g is the coupling constant.

From $S[\Phi,\bar{\Phi},A_\alpha]$, we can compute the one-loop effective action $\Gamma_{\rm eff}[A_\alpha]$ by formally integrating out the complex scalar superfields. Therefore, we arrive at the following equation:

$$\begin{aligned}\Gamma_{\rm eff}[A_\alpha]&=-\mathrm{Tr}\ln(\nabla^2+m)\\&=-\mathrm{Tr}\ln\Big\{D^2-\frac{i}{2}[2A^\alpha D_\alpha+(D^\alpha A_\alpha)\\&\qquad-iA^\alpha A_\alpha]+m\Big\}.\end{aligned} \qquad (2)$$

Here, we use the usual definition of the trace operation for functional operators, namely

$$\mathrm{Tr}\ln\hat{\mathcal{O}}=\mathrm{tr}\int d^5z\ln\mathcal{O}\delta^5(z-z')|_{z=z'}, \qquad (3)$$

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where "tr" means a trace over group indices, while $\delta^5(z-z') \equiv \delta^3(x-x')\delta^2(\theta-\theta')$.

It is convenient to split the effective action into two parts according to their parity under the transformation $m \to -m$,

$$\Gamma_{\rm eff}[A_\alpha] \equiv \Gamma_{\rm even}[A_\alpha] + \Gamma_{\rm odd}[A_\alpha], \quad (4)$$

$$\Gamma_{\rm even}[A_\alpha] \equiv -\frac{1}{2}{\rm Tr}[\ln(\nabla^2+m) + \ln(\nabla^2-m)] = -\frac{1}{2}{\rm Tr}\ln(\Box_A - iW^\alpha\nabla_\alpha - m^2), \quad (5)$$

$$\Gamma_{\rm odd}[A_\alpha] \equiv -\frac{1}{2}{\rm Tr}[\ln(\nabla^2+m) - \ln(\nabla^2-m)]. \quad (6)$$

In a low-energy approximation or, equivalently, in an approximation of slowly varying background superfields, the even-parity part (4) corresponds to the Euler-Heisenberg effective action and the odd-parity part (5) corresponds to the Chern-Simons action [11]. In Ref. [12], the one-loop effective action was calculated in three-dimensional Minkowski space. The supersymmetric generalization of these results was studied in Ref. [5], in the Abelian case, to the $\mathcal{N}=2$ and $\mathcal{N}=4$ superspaces.

In Ref. [8], the authors applied the derivative-expansion scheme proposed in Ref. [13] to compute the induced Chern-Simons term at zero and finite temperatures. The main goal of the present paper is to provide a systematic method to compute the non-Abelian Chern-Simons term induced by radiative corrections in $\mathcal{N}=1$, $d=3$ superspace via a derivative-expansion scheme so that, our results can be seen as a supersymmetric generalization of the result obtained in Ref. [8] at zero temperature. In particular, we will show explicitly that, at low energies, the odd-parity part (5) corresponds to the Chern-Simons action, namely [1]

$$\Gamma_{\rm odd}[A_\alpha] = C{\rm tr}\int d^5z\left(A^\alpha W_\alpha + \frac{i}{6}\{A^\alpha, A^\beta\}D_\beta A_\alpha + \frac{1}{12}\{A^\alpha, A^\beta\}\{A_\alpha, A_\beta\}\right), \quad (7)$$

where C is a dimensionless constant to be determined, and

$$W_\alpha = \frac{1}{2}D^\beta D_\alpha A_\beta - \frac{i}{2}[A^\beta, D_\beta A_\alpha] - \frac{1}{6}[A^\beta, \{A_\beta, A_\alpha\}]. \quad (8)$$

In order to apply the method of derivative expansion, let us substitute Eq. (2) into Eq. (6), so that we can rewrite the result as

$$\Gamma_{\rm odd}[A_\alpha] = -\frac{1}{2}{\rm Tr}\left\{\ln\left[1-\frac{i}{2}(2A^\alpha D_\alpha + (D^\alpha A_\alpha) - iA^\alpha A_\alpha)\frac{D^2-m}{\Box-m^2}\right] - \ln\left[1-\frac{i}{2}(2A^\alpha D_\alpha + (D^\alpha A_\alpha) - iA^\alpha A_\alpha)\frac{D^2+m}{\Box-m^2}\right]\right\} + \Gamma_{\rm odd}[A_\alpha=0]. \quad (9)$$

Since the inverse of the A_α propagator, namely $\Gamma_{\rm odd}[A_\alpha=0]$, does not depend on the background superfield, it follows that we can drop it out by means of the normalization of the effective action.

Expanding the logarithmic terms of Eq. (9), we get

$$\Gamma_{\rm odd}[A_\alpha] = \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n\frac{S^{(n)}}{n}, \quad (10)$$

where

$$S^{(n)} \equiv \frac{1}{2}{\rm Tr}\left\{\left[(2A^\alpha D_\alpha + (D^\alpha A_\alpha) - iA^\alpha A_\alpha)\frac{D^2-m}{\Box-m^2}\right]^n - \left[(2A^\alpha D_\alpha + (D^\alpha A_\alpha) - iA^\alpha A_\alpha)\frac{D^2+m}{\Box-m^2}\right]^n\right\}. \quad (11)$$

Notice that the derivatives D_α and $\partial_{\alpha\beta}$ act on all functions on their right side, including the delta function (3). For the purpose of calculating the functional trace in Eq. (11), we need to move all derivatives D_α and $\partial_{\alpha\beta}$ to the right and all functions of A_α and their derivatives to the left so that, at the end of the calculation, we can calculate the functional trace by using the Fourier representation of the delta function. In order to perform this task, we need to use the following identities:

$$[D_\alpha, f(z)\} = (D_\alpha f) \Leftrightarrow D_\alpha f(z) = (D_\alpha f) + (-)^{\epsilon_f} f D_\alpha, \quad (12)$$

$$\frac{1}{\Box-m^2}f(z) = f\frac{1}{\Box-m^2} - [\Box, f]\frac{1}{(\Box-m^2)^2} + [\Box,[\Box,f]]\frac{1}{(\Box-m^2)^3} + \cdots, \quad (13)$$

where $[,\}$ is the graded commutator, while $f(z)$ is a function of the background superfield and their derivatives. Notice that Eq. (12) is the graded Leibniz rule.

Our calculation is also based on an extensive use of the following identities satisfied by the covariant derivatives D_α:

$$D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\beta\alpha}D^2, \qquad \{D^2, D_\alpha\} = 0,$$
$$[D_\alpha, D_\beta] = -2C_{\alpha\beta}D^2, \qquad (D^2)^2 = \Box. \quad (14)$$

Now, let us calculate the terms $S^{(n)}$ in Eq. (11). However, taking into account that Eq. (7) is a functional which depends only on quadratic, cubic, and quartic terms in the background superfield, it follows that we only need to calculate the lowest-order terms in the expansion of Γ_{odd} in Eq. (10), namely

$$\Gamma_{\mathrm{odd}}[A_\alpha] = \frac{i}{2}S^{(1)}_{AA} - \frac{1}{8}(S^{(2)}_{AA} + S^{(2)}_{AAA} + S^{(2)}_{AAAA}) - \frac{i}{24}(S^{(3)}_{AAA} + S^{(3)}_{AAAA}) + \frac{1}{64}S^{(4)}_{AAAA}. \tag{15}$$

In the next section, we perform explicit calculations of all the terms in the expansion above.

III. INDUCED CHERN-SIMONS TERM

Let us start the calculation of the contributions $S^{(n)}$ in Eq. (15). At first order, $n = 1$, it is only necessary to calculate $S^{(1)}_{AA}$. Hence, it follows from Eq. (11) that

$$S^{(1)}_{AA} = im\mathrm{Tr}\left[A^\alpha A_\alpha \frac{1}{\Box - m^2}\right] = im\,\mathrm{tr}\int d^5z\left[A^\alpha A_\alpha \frac{1}{\Box - m^2}\right]\delta^5(z - z')|_{z=z'}. \tag{16}$$

There is no covariant derivative D_α in Eq. (16) and the Grassmann delta function satisfies the following identities:

$$\delta^2(\theta - \theta')|_{\theta=\theta'} = 0, \qquad D_\alpha\delta^2(\theta - \theta')|_{\theta=\theta'} = 0, \qquad D^2\delta^2(\theta - \theta')|_{\theta=\theta'} = 1. \tag{17}$$

It follows that Eq. (16) vanishes identically:

$$S^{(1)}_{AA} = 0. \tag{18}$$

Let us move on and calculate the second-order contributions in Eq. (15), namely $S^{(2)}_{AA}$, $S^{(2)}_{AAA}$, and $S^{(2)}_{AAAA}$. First, let us begin with the quadratic contribution in the background superfield:

$$S^{(2)}_{AA} = -m\mathrm{Tr}\left[X_A \frac{D^2}{\Box - m^2} X_A \frac{1}{\Box - m^2} + X_A \frac{1}{\Box - m^2} X_A \frac{D^2}{\Box - m^2}\right], \tag{19}$$

$$X_A \equiv 2A^\alpha D_\alpha + (D^\alpha A_\alpha), \tag{20}$$

where again we have used Eq. (11). By assuming that the background superfield varies slowly in superspace, we can discard terms involving spinor derivatives higher than the second order acting on A_α in the calculation of Eq. (19). Moreover, in this approximation, we can move the operator $(\Box - m^2)^{-1}$ to the right using the identity (13), so that only the first two terms on the right of Eq. (13) need be retained. Therefore, it follows from Eq. (13) that

$$\frac{1}{\Box - m^2}X_A \approx X_A \frac{1}{\Box - m^2} - [4(D^2A^\alpha)D^2D_\alpha - 2(D^\lambda D^\gamma A^\alpha)D_\lambda D_\gamma D_\alpha]\frac{1}{(\Box - m^2)^2}, \tag{21}$$

where we have used the commutators

$$[\Box, X_A] = [(D^2)^2, X_A] \approx 4(D^2A^\alpha)D^2D_\alpha - 2(D^\lambda D^\gamma A^\alpha)D_\lambda D_\gamma D_\alpha, \tag{22}$$

$$[\Box, [\Box, X_A]] = [(D^2)^2, [(D^2)^2, X_A]] \approx 0. \tag{23}$$

Moreover, neglecting terms involving derivatives higher than the second, we notice that

$$\{D^2, 4(D^2A^\alpha)D^2D_\alpha - 2(D^\lambda D^\gamma A^\alpha)D_\lambda D_\gamma D_\alpha\} \approx 0. \tag{24}$$

Substituting Eq. (21) into Eq. (19) and using Eq. (24), we obtain

$$S^{(2)}_{AA} = -m\mathrm{Tr}\left[(X_A D^2 X_A + X_A X_A D^2)\frac{1}{(\Box - m^2)^2}\right]. \tag{25}$$

As a next step, we need to push all derivatives D_α to the right by means of Eq. (12). This task can be carried out by using the expressions

$$X_A D^2 X_A \approx -[4A^\alpha(D^2A_\alpha) + 4A^\alpha(D_\alpha D^\beta A_\beta) + (D^\alpha A_\alpha)(D^\beta A_\beta) - 4A^\alpha A^\beta i\partial_{\alpha\beta}]D^2, \tag{26}$$

$$X_A X_A D^2 \approx [2A^\alpha(D_\alpha D^\beta A_\beta) + (D^\alpha A_\alpha)(D^\beta A_\beta) - 4A^\alpha A^\beta i\partial_{\alpha\beta}]D^2 \tag{27}$$

where we have kept only terms which give a nonvanishing contribution [see Eq. (17)].

Substituting Eqs. (26) and (27) into Eq. (25), and noticing that

$$\frac{D^2}{(\Box - m^2)^2}\delta^5(z - z')|_{z=z'} = \frac{1}{(\Box - m^2)^2}\delta^3(x - x')|_{x=x'} = \frac{1}{8\pi|m|}, \tag{28}$$

we obtain

$$S^{(2)}_{AA} = \frac{2m}{8\pi|m|}\mathrm{tr}\int d^5z A^\alpha(D^\beta D_\alpha A_\beta), \tag{29}$$

where we have used the Fourier representation of the delta function in Eq. (28) and the identities (14). It is worth noting that the expression (29) is, in its functional structure, similar to the quadratic term in Eq. (7). In particular, if A_α is an Abelian superfield, then Eq. (29) is invariant under the gauge transformation $\delta A_\alpha = D_\alpha K$. Hence, in this particular case, if we substituted Eqs. (18) and (29) into Eq. (15), we should obtain the induced Abelian Chern-Simons action.

Next, let us calculate the cubic contribution in the background superfield, namely $S^{(2)}_{AAA}$. Due to the low-energy approximation, the terms involving spinor derivatives higher than the first order acting on A_α can be neglected. It follows that, at this approximation,

$$[\Box, X_A] = [(D^2)^2, X_A] \approx 0, \\ [\Box, A^\alpha A_\alpha] = [(D^2)^2, A^\alpha A_\alpha] \approx 0. \tag{30}$$

Therefore, it follows from Eq. (11) that

$$S^{(2)}_{AAA} = \mathrm{imTr}\Big[(X_A A^\beta A_\beta D^2 + A^\alpha A_\alpha X_A D^2 \\ + X_A D^2 A^\beta A_\beta + A^\alpha A_\alpha D^2 X_A)\frac{1}{(\Box - m^2)^2}\Big], \tag{31}$$

where we have pushed the operator $(\Box - m^2)^{-1}$ to the right by using Eqs. (13) and (30). Again, each term in Eq. (31) can be calculated by moving all derivatives D_α to the right by means of Eq. (12). Hence, the terms that give a nonvanishing contribution are

$$X_A A^\beta A_\beta D^2 \approx [2A^\alpha(D_\alpha A^\beta)A_\beta - 2A^\alpha A^\beta(D_\alpha A_\beta) \\ + (D^\alpha A_\alpha)A^\beta A_\beta]D^2, \tag{32}$$

$$A^\alpha A_\alpha X_A D^2 \approx A^\alpha A_\alpha(D^\beta A_\beta)D^2; \\ X_A D^2 A^\beta A_\beta \approx (D^\alpha A_\alpha)A^\beta A_\beta D^2, \tag{33}$$

$$A^\alpha A_\alpha D^2 X_A \approx -A^\alpha A_\alpha(D^\beta A_\beta)D^2. \tag{34}$$

Substituting Eqs. (32)–(34) into Eq. (31), and calculating the functional trace, we get

$$S^{(2)}_{AAA} = \frac{2\mathrm{im}}{8\pi|m|}\mathrm{tr}\int d^5z[(D^\alpha A_\alpha)A^\beta A_\beta + A^\alpha(D_\alpha A^\beta)A_\beta \\ - A^\alpha A^\beta(D_\alpha A_\beta)]. \tag{35}$$

Notice that Eq. (35) is a surface term, which for present purposes, we may neglect. Therefore,

$$S^{(2)}_{AAA} = 0. \tag{36}$$

Finally, regarding the second-order contributions, let us move on and calculate $S^{(2)}_{AAAA}$. In this case, we discard terms involving covariant derivatives D_α of any order acting on the background superfield. Hence, it follows from Eq. (11) that

$$S^{(2)}_{AAAA} = m\mathrm{Tr}\Big[A^\alpha A_\alpha \frac{D^2}{\Box - m^2}A^\beta A_\beta \frac{1}{\Box - m^2} \\ + A^\alpha A_\alpha \frac{1}{\Box - m^2}A^\beta A_\beta \frac{D^2}{\Box - m^2}\Big] \\ = 2m\mathrm{Tr}\Big[A^\alpha A_\alpha A^\beta A_\beta \frac{D^2}{(\Box - m^2)^2}\Big]. \tag{37}$$

Therefore, using Eqs. (3) and (28), we obtain

$$S^{(2)}_{AAAA} = \frac{2m}{8\pi|m|}\mathrm{tr}\int d^5z A^\alpha A_\alpha A^\beta A_\beta. \tag{38}$$

Let us now consider the calculation of the third-order contributions in Eq. (15): $S^{(3)}_{AAA}$ and $S^{(3)}_{AAAA}$. First, let us start with the cubic contribution in the background superfield $S^{(3)}_{AAA}$. In particular, by means of the same reasoning used to obtain Eq. (31), we can use Eq. (11) and write $S^{(3)}_{AAA}$ as

$$S^{(3)}_{AAA} = -m\mathrm{Tr}\Big[(X_A D^2 X_A X_A D^2 + X_A X_A D^2 X_A D^2 \\ + X_A D^2 X_A D^2 X_A + m^2 X_A X_A X_A)\frac{1}{(\Box - m^2)^3}\Big]. \tag{39}$$

As previously described, we need to push all derivatives D_α to the right using the identity (12) and keep only terms which give a nonvanishing contribution. After some tedious algebraic work, we find that

$$X_A D^2 X_A X_A D^2 \approx 8[A^\alpha(D^\beta A^\gamma)A^\lambda - A^\alpha A^\gamma(D^\beta A^\lambda)](-\partial_{\alpha\beta}\partial_{\gamma\lambda} + C_{\beta\alpha}C_{\lambda\gamma}\Box)D^2 \\ + 4[2\{A^\alpha, A^\beta\}(D_\beta A_\alpha) - 2A^\alpha(D_\alpha A^\beta)A_\beta + A^\alpha A_\alpha(D^\beta A_\beta) \\ + A^\alpha(D^\beta A_\beta)A_\alpha - (D^\alpha A_\alpha)A^\beta A_\beta]\Box D^2, \tag{40}$$

$$X_A X_A D^2 X_A D^2 \approx 8A^\alpha A^\beta(D^\gamma A^\lambda)(-\partial_{\alpha\beta}\partial_{\gamma\lambda} + C_{\beta\alpha}C_{\lambda\gamma}\Box)D^2 + 4[2A^\alpha(D_\alpha A^\beta)A_\beta \\ + 2[A^\alpha, A^\beta](D_\beta A_\alpha) - A^\alpha A_\alpha(D^\beta A_\beta) + A^\alpha(D^\beta A_\beta)A_\alpha \\ + (D^\alpha A_\alpha)A^\beta A_\beta]\Box D^2, \tag{41}$$

$$X_A D^2 X_A D^2 X_A \approx -8[A^\alpha (D^\beta A^\gamma) A^\lambda + A^\alpha A^\beta (D^\gamma A^\lambda) - A^\alpha A^\gamma (D^\beta A^\lambda)](-\partial_{\alpha\beta}\partial_{\gamma\lambda} + C_{\beta\alpha} C_{\lambda\gamma} \Box) D^2 + 4[2A^\alpha (D_\alpha A^\beta) A_\beta - 2\{A^\alpha, A^\beta\}(D_\beta A_\alpha) + A^\alpha A_\alpha (D^\beta A_\beta) - A^\alpha (D^\beta A_\beta) A_\alpha + (D^\alpha A_\alpha) A^\beta A_\beta] \Box D^2, \tag{42}$$

$$X_A X_A X_A \approx -4[2A^\alpha (D_\alpha A^\beta) A_\beta + 2[A^\alpha, A^\beta](D_\beta A_\alpha) + A^\alpha A_\alpha (D^\beta A_\beta) + A^\alpha (D^\beta A_\beta) A_\alpha + (D^\alpha A_\alpha) A^\beta A_\beta] D^2. \tag{43}$$

Hence, by inserting Eqs. (40)–(43) into Eq. (39) and carrying out some algebraic manipulation, we obtain a very simple result:

$$S^{(3)}_{AAA} = -\frac{4m}{8\pi|m|} \mathrm{tr} \int d^5 z \{A^\alpha, A^\beta\}(D_\beta A_\alpha). \tag{44}$$

Next, let us calculate the quartic contribution in the background superfield, namely $S^{(3)}_{AAAA}$. In an approximation of slowly varying background superfields, we neglect terms involving derivatives acting on the background superfield, just as was done in Eq. (37). Therefore, from Eq. (11), we obtain

$$\begin{aligned} S^{(3)}_{AAAA} = 4\mathrm{imTr}\Big[& (A^\alpha D_\alpha D^2 A^\beta D_\beta A^\gamma A_\gamma D^2 + A^\alpha D_\alpha D^2 A^\beta A_\beta A^\gamma D_\gamma D^2 + A^\alpha A_\alpha D^2 \\ & \times A^\beta D_\beta A^\gamma D_\gamma D^2 + A^\alpha D_\alpha A^\beta D_\beta D^2 A^\gamma A_\gamma D^2 + A^\alpha D_\alpha A^\beta A_\beta D^2 A^\gamma D_\gamma D^2 \\ & + A^\alpha A_\alpha A^\beta D_\beta D^2 A^\gamma D_\gamma D^2 + A^\alpha D_\alpha D^2 A^\beta D_\beta D^2 A^\gamma A_\gamma + A^\alpha D_\alpha D^2 A^\beta A_\beta D^2 \\ & \times A^\gamma D_\gamma + A^\alpha A_\alpha D^2 A^\beta D_\beta D^2 A^\gamma D_\gamma + m^2 A^\alpha D_\alpha A^\beta D_\beta A^\gamma A_\gamma + m^2 A^\alpha D_\alpha A^\beta A_\beta \\ & \times A^\gamma D_\gamma + m^2 A^\alpha A_\alpha A^\beta D_\beta A^\gamma D_\gamma) \frac{1}{(\Box - m^2)^3} \Big] \\ = 12\mathrm{imTr}\Big[& A^\alpha A_\alpha A^\beta A_\beta \frac{D^2}{(\Box - m^2)^2} \Big]. \end{aligned} \tag{45}$$

Hence, by calculating the trace of Eq. (45), we get

$$S^{(3)}_{AAAA} = \frac{12\mathrm{im}}{8\pi|m|} \mathrm{tr} \int d^5 z A^\alpha A_\alpha A^\beta A_\beta. \tag{46}$$

The last contribution which needs to be calculated is $S^{(4)}_{AAAA}$. However, we will not calculate explicitly $S^{(4)}_{AAAA}$, because the calculation proceeds in the same way as the results obtained in Eqs. (38) and (46). Therefore, it is given by

$$S^{(4)}_{AAAA} = 0. \tag{47}$$

Finally, by substituting Eqs. (18), (29), (36), (38), (44), (46) and (47) into Eq. (15), we obtain the induced non-Abelian Chern-Simons term in its most simple form

$$\Gamma_{\mathrm{odd}}[A_\alpha] = -\frac{1}{8\pi|m|} \mathrm{tr} \int d^5 z \frac{m}{2} \left[\frac{1}{2} A^\alpha (D^\beta D_\alpha A_\beta) - \frac{i}{3} \{A^\alpha, A^\beta\}(D_\beta A_\alpha) - \frac{1}{2} A^\alpha A_\alpha A^\beta A_\beta \right]. \tag{48}$$

In order to put Eq. (48) into the functional form of Eq. (7), we note that, on the one hand, the trace over the generators T^a of the Lie algebra is invariant under cyclic permutations. Therefore, by using this property, we are able to prove that

$$\mathrm{tr}(\{A^\alpha, A^\beta\} D_\beta A_\alpha) = \mathrm{tr}\left(\frac{3}{2} A^\alpha [A^\beta, D_\beta A_\alpha] - \frac{1}{2} \{A^\alpha, A^\beta\} D_\beta A_\alpha \right), \tag{49}$$

$$\frac{1}{3} \mathrm{tr}(A^\alpha A^\beta A_\alpha A_\beta + A^\alpha A_\alpha A^\beta A_\beta) = \mathrm{tr}\left(\frac{1}{3} A^\alpha [A^\beta, \{A_\beta, A_\alpha\}] - \frac{1}{6} \{A^\alpha, A^\beta\}\{A_\alpha, A_\beta\} \right). \tag{50}$$

On the other hand, in three dimensions, any object with three spinor indices satisfies the identity $O_{[\alpha\beta\gamma]} = 0$, due to the fact that spinor indices can take only two values $\alpha = 1, 2$. Therefore, by contracting $A_{[\alpha} A_\beta A_{\gamma]} = 0$ with $C^{\beta\gamma}$, we are able to show that

$$A^\beta A_\alpha A_\beta = A_\alpha A^\beta A_\beta + A^\beta A_\beta A_\alpha \Rightarrow \mathrm{tr}(A^\alpha A^\beta A_\alpha A_\beta) = 2\mathrm{tr}(A^\alpha A_\alpha A^\beta A_\beta). \tag{51}$$

By means of the identities (49)–(51), it is trivial to rewrite Eq. (48) as

$$\Gamma_{\rm odd}[A_\alpha] = -\frac{g^2}{8\pi|m|}{\rm tr}\int d^5z\frac{m}{2g^2}\times\left(A^\alpha W_\alpha+\frac{i}{6}\{A^\alpha,A^\beta\}D_\beta A_\alpha+\frac{1}{12}\{A^\alpha,A^\beta\}\{A_\alpha,A_\beta\}\right), \quad (52)$$

where the Yang-Mills field strength superfield W_α is given by Eq. (8). Of course, Eq. (52) is the non-Abelian Chern-Simons action [1], up to the constant factor $-\frac{g^2}{8\pi|m|}$. As a result, the gauge superfield A_α being nondynamical on the classical level acquires a nontrivial dynamics due to quantum corrections. It is worth pointing out that the methodology presented here can be easily adapted and applied to the noncommutative and Lorentz-breaking aether superspaces [14].

IV. SUMMARY

We succeeded in generating the non-Abelian supersymmetric Chern-Simons term. Within our calculations we (i) used the superfield formalism at all steps, and (ii) did not impose any restrictions on structure of the gauge group, which makes our results universal, and applicable to an arbitrary Lie algebra. As a consequence, we generalized the results of Refs. [8,9], where the non-Abelian Chern-Simons term has been generated from the fermionic determinant, in a nonsupersymmetric theory. The finiteness (and hence independence of any renormalization scheme) of our result is natural due to the well-known one-loop finiteness of three-dimensional field theory models.

The most important consequence of our result is its natural interpretation within the context of the concept of the emergent dynamics. Essentially, we have showed that the integration over scalar matter coupled to the external non-Abelian gauge superfield will yield the correct form of the non-Abelian supersymmetric Chern-Simons term. Therefore, we hope, that the approach we use in this paper can serve for more sophisticated supersymmetric field theory models, in particular, to generate the complete non-Abelian super-Yang-Mills action whose expression will involve Feynman supergraphs up to sixth order.

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