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Some considerations on duality concerning κ-Minkowski spacetime theories

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In this paper we have analyzed the κ-deformed Minkowski spacetime through the light of the interference phenomena in quantum field theory where two opposite chiral fields are put together in the same multiplet and its consequences are discussed. The chiral models analyzed here are the chiral Schwinger model, its generalized version, and its gauge invariant version, where a Wess-Zumino term was added. We will see that the final actions obtained here are, in fact, related to the original ones via duality transformations.

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I. INTRODUCTION

The fermion-boson mapping has been one of the most investigated topics in theoretical physics during the past three decades due to its importance in the quantization of strings and also the Hall quantum effect. The possibility of mapping a complicated fermionic model into a scalar bosonic one was really attractive. This mapping is called bosonization, and the chiral bosons can be obtained from the restriction of a scalar field to move in one direction only, as done by Siegel [1], or by a first-order Lagrangian theory, as proposed by Floreanini and Jackiw [2]. A few years later, Tseytlin [3] implemented the Floreanini-Jackiw (FJ) construction on the world sheet.

In two dimensions (2D), scalar fields can be viewed as bosonized versions of Dirac fermions, and chiral bosons can be seen to correspond to two-dimensional versions of Weyl fermions. As a generalization, in supergravity models, the extension of the chiral boson to higher dimensions has naturally introduced the concept of chiral p-forms. Harada [4] investigated the chiral Schwinger model via chiral bosonization and analyzed its spectrum. He has shown how to obtain a consistent coupling of FJ chiral bosons with a U(1) gauge field, starting from the chiral Schwinger model and discarding the right-handed degrees of freedom by means of a restriction in the phase space implemented by imposing the chiral constraint $\pi = \phi'$. In [5], Bellucci, Golterman, and Petcher have introduced an O(N) generalization of Siegel's model for chiral bosons coupled with Abelian and non-Abelian gauge fields. The physical spectrum of the resulting Abelian theory is that of a (massless) chiral boson and a free massive scalar field.

Initially, several models were suggested for chiral bosons, but later it was shown that there are some relations between these models [6]. For instance, the FJ model is the chiral dynamical sector of the more general model proposed by Siegel. The Siegel modes (rightons and leftons) carry not only chiral dynamics but also symmetry information. The symmetry content of the theory can be described by the Siegel algebra, a truncate diffeomorphism, that disappears at the quantum level. As another application, chiral bosons appear in the analysis of the quantum Hall effect [7]. The introduction of a soliton field as a charge-creating field obeying one additional equation of motion leads to a bosonization rule [8].

The direct sum of two chiral fermions in 2D gives rise to a full Dirac fermion; however, this is not true for their bosonized versions as noticed in [9] (see also [10]). Besides, the fermionic determinant of a Dirac fermion interacting with a vector gauge field in $D = 1 + 1$ factorizes into the product of two chiral determinants but the full bosonic effective action is not the naive direct sum of both chiral effective actions as discussed in [11]. Stated differently, the action of a bosonized Dirac fermion is not simply the sum of the actions of both bosonized Weyl fermions or chiral bosons. Physically, this is connected to the necessity to abandon the separated right and left symmetries, and to accept that the vector gauge symmetry should be preserved at all times. This restriction will force both independent chiral bosons to belong to the same *multiplet*, effectively soldering them together. In both cases it turns out that an interference term between the opposite chiral bosonic actions is needed to achieve the expected result; such a term is provided by the so-called soldering procedure.

The concept of soldering has proved extremely useful in different contexts [12,13]. This formalism essentially combines two distinct Lagrangians carrying dual aspects

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of some symmetry to yield a new Lagrangian which is exposed of, or rather hides, that symmetry. These so-called quantum interference effects, whether constructive or destructive, among the dual aspects of symmetry, are thereby captured through this mechanism [11]. The formalism introduced by Stone [9] could actually be interpreted as a new method of dynamical mass generation through the result obtained in [11]. This is possible by considering the interference of right and left gauged FJ chiral bosons. The result of the chiral interference shows the presence of a massive vectorial mode for the special case where the Bose symmetry fixed the Jackiw-Rajaraman regularization parameter as $a = 1$ [14], which is the value where the chiral theories have only one massless excitation in their spectra. This clearly shows that the massive vector mode results from the interference between two massless modes.

It was shown recently [15] that in the soldering process of two opposite chiral fields, a lefton and a righton, coupled with a gauge field, the gauge field decouples from the physical field. The final action describes a nonmover field (a noton) at the classical level. The noton acquires dynamics upon quantization. This field was introduced by Hull [16] to cancel out the Siegel anomaly. It carries a representation of the full diffeomorphism group, while its chiral components carry the representation of the chiral diffeomorphism.

The same procedure works in $D = 2 + 1$ if we substitute chirality by helicity. For instance, by fusing together two topologically massive modes generated by the bosonization of two massive Thirring models with opposite mass signatures in the long wavelength limit. The bosonized modes, which are described by self- and anti-self-dual Chern-Simons models [17,18], were then soldered into two massive modes of the three-dimensional (3D) Proca model [19]. More generally, the ± 1 helicity modes may have different masses that lead after soldering to a Maxwell-Chern-Simons-Proca theory. In this case, technical problems [20] regarding a full off-shell soldering can be resolved by defining a generalized soldering procedure [21].

The basic idea of the generalized soldering is the introduction of a free parameter α with a freedom sign which plays this role whenever interactions are present. In the soldering of two chiral Schwinger models that results either in an axial ($\alpha = -1$) or in a vector ($\alpha = +1$) Schwinger model, which are dual do each other. In the case of two Maxwell-Chern-Simons theories, the choice of the α parameter with opposite sign leads to dual interaction terms. We can have either a derivative coupling or a minimal coupling plus a Thirring term. After integration over the soldering field the dependence on the sign of α disappears, which proves that it corresponds to dual forms of the same interacting theory. Recently, a new idea concerning the construction of the so-called Noether vector, the concept of which can be directly analyzed from an initial master action [22]. We will discuss this issue here in the future.

Recently, the soldering formalism was used to investigate the self-dual theories with spin $s \geq 2$ and opposite helicities. In [23,24] the authors have demonstrated that the linearized Fierz-Pauli action, which describes a doublet of massive spin-2 particles can be obtained via the soldering procedure of two second order self-dual models of opposite helicities. Besides, one can recover the new massive gravity [25,26] (also at the linearized level) by soldering two self-dual models of opposite helicities of either third or fourth order in derivatives.

Usually the noncommutativity of spacetime coordinates can be implemented by using the Weyl operators or, for the sake of practical applications, through the way of normal functions with a suitable definition of star products [27]. Generally the noncommutativity of spacetime may be encoded through ordinary products in the noncommutative (NC) $\star$ algebra of Weyl operators. Or equivalently through the deformation of the product of the commutative $\star$ algebra of functions to a NC star product. For instance, in the canonical NC spacetime the star product is simply the Moyal product [28], while on the κ-deformed Minkowski spacetime the star product requires a more complicated expression [29].

To treat the κ-deformed Minkowski spacetime in a very similar way to the usual Minkowski spacetime, the authors in [30,31] have proposed a quite different approach to the implementation of noncommutativity. To this aim a well-defined proper time from the κ-deformed Minkowski spacetime has been defined in such a way that it corresponds to the standard basis. Therefore we encode enough information of noncommutativity of the κ-Minkowski spacetime to a commutative spacetime in this new parameter, and then we set up a NC extension of the Minkowski spacetime. This extended Minkowski spacetime is as commutative as the Minkowski spacetime, but it contains noncommutativity already. Therefore, one can somehow investigate the NC field theories defined on the κ-deformed Minkowski spacetime by following the way of the ordinary (commutative) field theories on the NC extension of the Minkowski spacetime, and thus depict the noncommutativity within the framework of this commutative spacetime. With this simplified treatment of the noncommutativity of the κ-Minkowski spacetime, we unveil the fuzziness in the temporal dimension and build NC chiral boson models in [30].

The organization of the issues through this paper obeys the following sequence: in Sec. II, we have written a review of the κ-Minkowski noncommutativity, and in Sec. III, we have provided a review of the essentials of the soldering formalism. In Sec. IV, we have analyzed the soldering of the NC chiral Schwinger model (CSM), and in Sec. V, we have presented the NC version of the generalized CSM. In Sec. VI, we have discussed the gauge invariant CSM. The (anti-)self-dual model in $D = 2 + 1$ was analyzed in Sec. VII. As usual, the conclusions and perspectives are described in the last section.

II. THE NC EXTENSION OF MINKOWSKI SPACETIME

We have talked so far about the noncommutativity that appears naturally through the Moyal-Weyl product which introduces the NC parameter that deforms the Heisenberg algebra. However, there are other generalizations of noncommutativity beyond the Moyal space [32]. The κ-deformed space is one of theses formulations that will be analyzed here. The κ-deformed space, which is the simplest case of a deformation of the Poincaré group, is described by the coordinates satisfying Lie algebra type commutation relations [33–38], which are Lie algebra type

$$[\hat{\mathcal{X}}^\mu, \hat{\mathcal{X}}^\nu] = i(a^\mu \hat{\mathcal{X}}^\nu - a^\nu \hat{\mathcal{X}}^\mu), \tag{1}$$

where κ is a mass dimensional parameter. We can classify this κ-deformed noncommutativity as timelike if $a^\mu a_\mu < 0$, spacelike if $a^\mu a_\mu > 0$, as well as a light-cone κ-commutation relation if $a^\mu a_\mu = 0$ [39].

The unitarity property of quantum theories with Lie algebra type noncommutativity was studied in [40]. In [41] the authors discussed the violation of the basics concerning causality and unitarity of NC theories. They have claimed that it occurs at energies higher than the inverse scale of the NC parameter. The obvious conclusion would be that, for a NC theory, an upper bound on possible values of energy would free the theory from these problems concerning the violation of the basic physical concepts. In [42], dealing with a Lie-algebraic theory, it was demonstrated that the spacetime quantization leads to an energy spectrum confined in an interval $E \in [0, \pi/\lambda]$, where λ is the NC parameter. This interval for the energy shows an ultraviolet behavior of the field theory on a specific NC Lie-algebraic space. Namely, even planar diagrams in this specific case have been shown to be convergent, which is an opposite behavior occurring in the flat NC Minkowski space.

Back to Eq. (1), in this work we have that a_μ $(\mu = 0, 1, 2, \ldots, n-1)$ are real and dimensionful constants parametrizing the deformation of the Minkowski space [43]. As is well known, in (1), the κ space is defined by the values $a_i = 0$ $i = 1, 2, \ldots, n-1$; $a_0 = a = 1/\kappa$. With these values, the commutation relations concerning the coordinates of κ space (Lie-algebraic type) are given by

$$[\hat{\mathcal{X}}^i, \hat{\mathcal{X}}^j] = 0; \qquad [\hat{\mathcal{X}}^0, \hat{\mathcal{X}}^j] = \frac{i}{\kappa}\hat{\mathcal{X}}^j; \qquad i, j = 1, 2, 3, \tag{2}$$

and using the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$, we can define $\hat{\mathcal{X}}^\mu = \eta_{\mu\nu}\hat{\mathcal{X}}_\nu$. Realizations of the NC coordinates in terms of the commuting coordinates $\hat{\mathcal{X}}_\mu$ and the correspondent derivative operators ∂_μ can be reviewed in [43].

In other words, we can say that the spacetime noncommutativity can be differentiated following the Hopf-algebraic classification by the three kinds, namely, the canonical, Lie-algebraic, and quadratic noncommutativity. The κ-deformed Minkowski spacetime is a specific case of the Lie-algebraic type [30]. The interest in this specific algebraic formulation has grown recently thanks to its connection to the basics of the doubly special relativity [44].

Following the ideas in [35], the concept of quantum deformations applied to the $D = 4$ Poincaré algebra and $D = 4$ Poincaré group (see [35] and references within) leads us to the modification of relativistic symmetries but, at the same time, not changing the $D = 3$ $E(3)$ subalgebra. The deformation parameter κ depicts the underlying mass in the theory. The limit $\kappa \to \infty$ is relative to the undeformed case.

Hence, the so-called noncommutativity is closely connected to a NC spacetime like the κ-deformed Minkowski spacetime. It has no connection to a commutative spacetime, such as the Minkowski spacetime. Consequently, the NC field theories constructed in κ-deformed Minkowski spacetime can be discussed by the way of the ordinary field theories in the NC formulation (extension) of the Minkowski spacetime. The noncommutativity may be described using the framework of this commutative spacetime.

Let us describe from now on the relations that will be used in the paper concerning the κ algebra. The commutative spacetime is characterized by the canonical Heisenberg commutation relations

$$[\hat{\mathcal{X}}^\mu, \hat{\mathcal{X}}^\nu] = 0, \qquad [\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}_\nu] = i\delta^\mu_\nu, \qquad [\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu] = 0, \tag{3}$$

where μ, $\nu = 0, 1, 2, 3$. To introduce the κ-deformed Minkowski spacetime we have, following the algebraic details of [30,35], the relations that can be chosen are

$$\hat{x}^0 = \hat{\mathcal{X}}^0 - \frac{1}{k}[\hat{\mathcal{X}}^i, \hat{\mathcal{P}}_j]_+,$$
$$\hat{x}^i = \hat{\mathcal{X}}^i + A\eta^{ij}\hat{\mathcal{P}}_j \exp\left(\frac{2}{k}\hat{\mathcal{P}}_0\right), \tag{4}$$

where $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]_+ \equiv \frac{1}{2}(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 + \hat{\mathcal{O}}_2\hat{\mathcal{O}}_1)$, $\eta^{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$, $i, j = 1, 2, 3$, and A is an arbitrary constant [30,35]. The NC parameter κ has mass dimension, and it is real and positive. The Casimir operator related to the κ-deformed Poincaré algebra is

$$\hat{\mathcal{C}}_1 = \left(2k \sinh\frac{\hat{p}_0}{2k}\right)^2 - \hat{p}_i^2, \tag{5}$$

and the momentum operators connected with the commutative spacetime and the κ-Minkowski spacetime can be written as

$$\hat{p}_0 = 2k\sinh^{-1}\frac{\hat{\mathcal{P}}_0}{2k}, \qquad \hat{p}_i = \hat{\mathcal{P}}_i. \tag{6}$$

With these last results we can construct our NC phase space $(\hat{x}^\mu, \hat{p}_\nu)$,

$$[\hat{x}^0, \hat{x}^j] = \frac{i}{k}\hat{x}^j, \qquad [\hat{x}^i, \hat{x}^j] = 0,$$
$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \qquad [\hat{x}^i, \hat{p}_j] = i\delta^i_j, \tag{7}$$

$$[\hat{x}^0, \hat{p}_0] = i\left(\cosh\frac{\hat{p}_0}{2k}\right)^{-1}, \qquad [\hat{x}^0, \hat{p}_i] = -\frac{i}{k}\hat{p}_i,$$
$$[\hat{x}^i, \hat{p}_0] = 0, \tag{8}$$

which satisfies the Jacobi identity. It is easy to see that when $k \to \infty$ we recover the commutative phase space in Eq. (3).

The Casimir operator described above in Eq. (5) can now be written in a standard way,

$$\hat{\mathcal{C}}_1 = \hat{\mathcal{P}}_0^2 - \hat{\mathcal{P}}_i^2, \tag{9}$$

where it is easy to see that this selection coincides with the ones in Eq. (3). In the case that $\hat{p}_\mu$ has standard forms like

$$\hat{p}_0 = -i\frac{\partial}{\partial t} \quad \text{and} \quad \hat{p}_i = -i\frac{\partial}{\partial x^i}, \tag{10}$$

the operator $\hat{\mathcal{P}}_0$ then reads

$$\hat{\mathcal{P}}_0 = -2ik\left(\sin\frac{1}{2k}\frac{\partial}{\partial t}\right). \tag{11}$$

In [30] the author has introduced a proper time τ through the operator

$$\hat{\mathcal{P}}_0 \equiv -i\frac{\partial}{\partial \tau}, \tag{12}$$

and using Eqs. (11) and (12) we have that

$$2k\left(\sin\frac{1}{2k}\frac{d}{dt}\right)\tau = 1, \tag{13}$$

where the solution is

$$\tau = t + \sum_{n=0}^{+\infty} c_{-n}\exp(-2kn\pi t), \tag{14}$$

where $n \geq 0$, $n \in \mathbb{N}$. The coefficients c_{-n} are arbitrary real constants. This property implies a kind of temporal fuzziness coherent concerning the κ-Minkowski spacetime. Notice that as $k \to \infty$, the proper time turns back to the ordinary time variable.

To construct a NC extension of Minkowski spacetime (τ, x^i) (where the NC feature is inside the proper time), let us define a twisted t coordinate, such that the metric is

$$g_{00} = \dot{\tau}^2 = \left[1 - 2k\pi\sum_{n=0}^{+\infty} nc_{-n}\exp(-2kn\pi t)\right]^2,$$
$$g_{11} = g_{22} = g_{33} = -1. \tag{15}$$

So, we can use Eq. (15) to construct NC models in the commutative framework. Namely, we can construct a Lagrangian theory for the NC model in the extended framework of the Minkowski spacetime.

III. THE CANONICAL SOLDERING FORMALISM

The basic idea about the soldering procedure is to raise a global Noether symmetry of the self- and anti-self-dual constituents into a local one, but for an effective composite system, with dual components and an interference term. The objective in [45] is to systematize the procedure like an algorithm and consequently to define the soldered action. The physics considerations will be taken based on the resulting action. For example, in [6], one of us has obtained a mass generation method, which was the final result of the soldering process.

An iterative Noether procedure was adopted to lift the global symmetries into local ones. Therefore, we will assume that the symmetries in question are being described by the local actions $S_\pm(\phi^\eta_\pm)$, invariant under a global multiparametric transformation

$$\delta\phi^\eta_\pm = \alpha^\eta, \tag{16}$$

where η represents the tensorial character of the basic fields in the dual actions $S_\pm$ and, for notational simplicity, it will be dropped out from now on. Here the $\pm$ subscript refers to the opposite/complementary aspects of two models at hand. For instance, ϕ_+ may refer to a left chiral field and ϕ_- to a field with right chirality. As it is well known, we can write

$$\delta S_\pm = J^\pm \partial_\pm \alpha, \tag{17}$$

where $J^\pm$ are the Noether currents.

Now, under local transformations these actions will not remain invariant, and Noether counterterms become necessary to reestablish the invariance, along with appropriate auxiliary fields $B^{(N)}$, the so-called soldering fields which have no dynamics where the N superscript is referring to the level of the iteration. This makes a wider range of gauge-fixing conditions available. In this way, the N action can be written as

$$S_\pm(\phi_\pm)^{(0)} \to S_\pm(\phi_\pm)^{(N)} = S_\pm(\phi_\pm)^{(N-1)} - B^{(N)}J^{(N)}_\pm. \tag{18}$$

Here $J_{\pm}^{(N)}$ are the N-iteration Noether currents. For the self- and anti-self-dual systems we have in mind that this iterative gauging procedure is (intentionally) constructed not to produce invariant actions for any finite number of steps. However, if after N repetitions, the noninvariant piece ends up being only dependent on the gauging parameters, but not on the original fields, there will exist the possibility of mutual cancellation if both gauged versions of self- and anti-self-dual systems are put together. Then, suppose that after N repetitions we arrive at the following simultaneous conditions:

$$\begin{aligned} \delta S_{\pm}(\phi_{\pm})^{(N)} &\neq 0, \\ \delta S_{B}(\phi_{\pm}) &= 0, \end{aligned} \tag{19}$$

with S_B being the so-called soldered action

$$S_{B}(\phi_{\pm}) = S_{+}^{(N)}(\phi_{+}) + S_{-}^{(N)}(\phi_{-}) + \text{contact terms}, \tag{20}$$

and the "contact terms" being generally quadratic functions of the soldering fields. Then we can immediately identify the (soldering) interference term as

$$S_{\text{int}} = \text{contact terms} - \sum_{N} B^{(N)} J_{\pm}^{(N)}. \tag{21}$$

Incidentally, these auxiliary fields $B^{(N)}$ may be eliminated, for instance, through their equations of motion, from the resulting effective action, in favor of the physically relevant degrees of freedom. It is important to notice that after the elimination of the soldering fields, the resulting effective action will not depend on either self- or anti-self-dual fields $\phi_{\pm}$ but only in some collective field, say Φ, defined in terms of the original ones in a (Noether) invariant way

$$S_{B}(\phi_{\pm}) \rightarrow S_{\text{eff}}(\Phi). \tag{22}$$

Analyzing in terms of the classical degrees of freedom, it is obvious that we have now a theory with bigger symmetry groups. Once such effective action has been established, and the physical consequences of the soldering are readily obtained by simple inspection.

IV. SOLDERING OF NC BOSONIZED CHIRAL SCHWINGER MODEL

The CSM is a 2D (1 spatial dimension + 1 time dimension) Euclidean quantum electrodynamics for a Dirac fermion. This model exhibits a spontaneous symmetry breaking of the U(1) group due to a chiral condensate from a pool of instantons [46]. The photon in this model becomes a massive particle at low temperatures. This model can be solved exactly, and it is used as a toy model for other complex theories. The bosonization of this theory can be done in several ways that apparently lead to different bosonized models. But these (apparently) inequivalent models are related by some gauge transformations [4]. Here we shall not enter into the details of this equivalence. We will discuss the application of the soldering mechanism in the different forms concerning these chiral models.

The CSM is described by the Lagrangian density

$$\begin{aligned} \mathcal{L}_{ch} &= \dot{\phi}\phi' - (\phi')^2 + 2e\phi'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 \\ &\quad + \frac{1}{2}e^2 a A_\mu A^\mu, \end{aligned} \tag{23}$$

where the last term is the CSM mass term for the gauge field A_μ. In fact, this Lagrangian is the gauged version of the FJ's Lagrangian, $\mathcal{L}_0 = \dot{\phi}\phi' - (\phi')^2$ [2]. On the 2D extended Minkowski spacetime (τ, x) the Lagrangian (23) takes the following action form:

$$\begin{aligned} \hat{\mathcal{S}} = \int d\tau dx \bigg[&\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial x} - \left(\frac{\partial\phi}{\partial x}\right)^2 + 2e\frac{\partial\phi}{\partial x}(A_0 - A_1) \\ &- \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 a\eta^{\mu\nu}A_\mu A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\bigg], \end{aligned} \tag{24}$$

where $\eta^{\mu\nu} = \text{diag}(1, -1)$ is the flat metric of the extended Minkowski spacetime (τ, x) and a is a real parameter $(a > 1)$.

By the coordinate transformation (14) we can rewrite the above action in terms of (t, x) with explicit noncommutativity,

$$\begin{aligned} \hat{\mathcal{S}} = \int dt dx \sqrt{-g} \bigg[&\frac{1}{\dot{\tau}}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} - \left(\frac{\partial\phi}{\partial x}\right)^2 + 2e\frac{\partial\phi}{\partial x}(A_0 - A_1) \\ &- \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 a\eta^{\mu\nu}A_\mu A_\nu \\ &+ \frac{1}{2}\left(\frac{1}{\dot{\tau}}\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x}\right)^2\bigg], \end{aligned} \tag{25}$$

where $\sqrt{-g}$ is the Jacobian of the transformation and also the nontrivial measure of the κ-deformed Minkowski spacetime. Note that always $\sqrt{-g} = |\dot{\tau}|$ but here we only focus on the case $\dot{\tau} > 0$.

Until now we have considered only the left chiral Schwinger model, but the bosonization process gives us both the left and right chiral bosons which depend on the "chiral constraint" that we have imposed on it. The corresponding Lagrangians for these chiral models in the extended Minkowski spacetime are given by

$$\begin{aligned} \hat{\mathcal{L}}_{+} &= \dot{\phi}\phi' - \sqrt{-g}(\phi')^2 \\ &\quad + \sqrt{-g}\bigg\{2e\phi'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 \\ &\quad + \frac{1}{2}e^2 a[(A_0)^2 - (A_1)^2]\bigg\} \\ &\quad + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2, \end{aligned} \tag{26}$$

$$\hat{\mathcal{L}}_- = -\dot{\rho}\rho' - \sqrt{-g}(\rho')^2 + \sqrt{-g}\left\{2e\rho'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 b[(A_0)^2 - (A_1)^2]\right\} + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2. \tag{27}$$

Notice that + and − signs are associated with the left and right moving chiral bosons, respectively. These models contain noncommutativity through the proper time τ with the finite NC parameter k. In the limit $k \to +\infty$, $\sqrt{-g} = \dot{\tau} = 1$, these Lagrangians turn back to their ordinary forms on the Minkowski spacetime.

Now we are ready to solder these two chiral Lagrangians. To accomplish the task we calculate the variations of Eqs. (26) and (27) under the following local variations:

$$\delta\phi = \eta(t, x) = \delta\rho. \tag{28}$$

In fact, we are imposing this local symmetry into these models in order to obtain a gauge invariant Lagrangian. Under this variation we have, after some algebra, that

$$\delta(\hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_-) = (J_+ + J_-)\delta B_1, \tag{29}$$

where

$$J_+ = 2\dot{\phi} - 2\sqrt{-g}\phi' + 2e\sqrt{-g}(A_0 - A_1) \tag{30}$$

and

$$J_- = -2\dot{\rho} - 2\sqrt{-g}\rho' + 2e\sqrt{-g}(A_0 - A_1), \tag{31}$$

where B_1 (mentioned in the previous section) and B_2 (which will be necessary) are auxiliary fields where variations can be defined as

$$\delta B_1 = \partial_x \eta \quad \text{and} \quad \delta B_2 = \partial_t \eta, \tag{32}$$

and we can see, obviously from Eq. (29), that $\delta(\mathcal{L}_+ + \mathcal{L}_-) \neq 0$.

So, following the method, we must add a counterterm to both original Lagrangians (26) and (27) to cover the above extra terms. Hence,

$$\hat{\mathcal{L}}_{+1} = \hat{\mathcal{L}}_+ - J_+ B_1, \tag{33}$$

$$\hat{\mathcal{L}}_{-1} = \hat{\mathcal{L}}_- - J_- B_1. \tag{34}$$

Now let us check the variation of the above Lagrangians,

$$\delta\hat{\mathcal{L}}_{+1} = -(\delta J_+)B_1 = -(2\dot{\eta} - 2\sqrt{-g}\eta')B_1 = -2B_1(\delta B_2) + 2\sqrt{-g}B_1(\delta B_1), \tag{35}$$

$$\delta\hat{\mathcal{L}}_{-1} = -(\delta J_-)B_1 = (2\dot{\eta} - 2\sqrt{-g}\eta')B_1 = 2B_1(\delta B_2) + 2\sqrt{-g}B_1(\delta B_1). \tag{36}$$

As we can see, it is not zero, but the extra terms are independent of the original fields. Therefore, the iteration will finish in this second step by adding another counterterm.

Finally we can solder these two Lagrangians in order to construct an invariant one,

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- - (J_+ + J_-)B_1 - 2\sqrt{-g}(B_1)^2, \tag{37}$$

where the B_2 field was eliminated algebraically. On the other hand, we can eliminate the auxiliary field B_1 by its equation of motion

$$\frac{\delta\mathcal{W}}{\delta B_1} = 0 \Rightarrow -(J_+ + J_-) - 4\sqrt{-g}B_1 = 0 \Rightarrow B_1 = \frac{-1}{4\sqrt{-g}}(J_+ + J_-). \tag{38}$$

By substituting Eq. (38) into $\mathcal{W}$ we find

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- + \frac{1}{8\sqrt{-g}}(J_+ + J_-)^2. \tag{39}$$

Here we define a new field, the soldering field $\Psi = \phi - \rho$. By this definition we can rewrite $\mathcal{W}$ in a compact and nice form,

$$\hat{\mathcal{W}} = -\frac{\sqrt{-g}}{2}\Psi'^2 + \frac{1}{2\sqrt{-g}}\dot{\Psi}^2 + 2e\dot{\Psi}(A_0 - A_1) + 2\xi, \tag{40}$$

where ξ is given by

$$\xi = \sqrt{-g}\left\{\frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{4}e^2(a + b)[(A_0)^2 - (A_1)^2]\right\} + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2. \tag{41}$$

We can see that the final result, action (40), is not a "chiral" theory anymore. It also has a symmetry group that is bigger than both initial models. Hence, we have soldered both chiral models, and as a consequence we have gained an additional term in the final Lagrangian that did not exist initially. One of the peculiar consequences of this action is that the electromagnetic field interacts just with the temporal derivative of the soldered field. This peculiarity has its origin in the noncovariant initial Jackiw-Floreanini

Lagrangian. In fact, one can decompose the above action into two distinct ones using the dual projection approach [15]. The result is a self-dual and a free massive scalar field.

This mechanism, in some sense, is analogous to adding a mass term into the Dirac action. Without this mass term, the Dirac equation describes two chiral electrons, and by adding the mass, we have merged these two chiral electrons to obtain the real electron.

V. THE SOLDERING OF THE GENERALIZED BOSONIZED CSM

Bassetto *et al.* [47] have constructed the generalized chiral Schwinger model (GCSM), i.e., a vector and axial-vector theory characterized by a parameter that interpolates between pure vector and chiral Schwinger models. This 2D model is given by the action

$$\hat{\mathcal{S}} = \int \mathrm{d}t\mathrm{d}x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + eA_\mu(\epsilon^{\mu\nu} - r\eta^{\mu\nu})\partial_\nu\phi + \frac{1}{2}e^2aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{42}$$

The quantity r is a real interpolating parameter between the vector $(r=0)$ and the chiral Schwinger models $(r=\pm1)$. This action can be rewritten in the extended Minkowski spacetime

$$\hat{\mathcal{L}} = \frac{1}{2\sqrt{-g}}\dot{\phi}^2 - \frac{\sqrt{-g}}{2}\phi'^2 - k_1\dot{\phi} + k_2\phi' + \xi, \tag{43}$$

where

$$k_1 = e(rA_0 + A_1), \tag{44}$$

$$k_2 = e\sqrt{-g}(A_0 + rA_1), \tag{45}$$

$$\xi = \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A'_0)^2 + \frac{1}{2}e^2a\sqrt{-g}[(A_0)^2 - (A_1)^2]. \tag{46}$$

By defining the value of the parameter r in two extreme points ±1 we obtain two chiral Lagrangians

$$\hat{\mathcal{L}}_+ = \frac{1}{2\sqrt{-g}}\dot{\phi}^2 - \frac{\sqrt{-g}}{2}\phi'^2 - e(A_0 + A_1)\dot{\phi} + e\sqrt{-g}(A_0 + A_1)\phi' + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A'_0)^2 + \frac{1}{2}ae^2\sqrt{-g}[(A_0)^2 - (A_1)^2], \tag{47}$$

$$\hat{\mathcal{L}}_- = \frac{1}{2\sqrt{-g}}\dot{\rho}^2 - \frac{\sqrt{-g}}{2}\rho'^2 - e(-A_0 + A_1)\dot{\rho} + e\sqrt{-g}(A_0 - A_1)\rho' + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A'_0)^2 + \frac{1}{2}be^2\sqrt{-g}[(A_0)^2 - (A_1)^2], \tag{48}$$

where a and b are the Jackiw-Rajaraman coefficients for each chirality, respectively. Here, through the iterative Noether embedding procedure, we will transform both Lagrangians (47) and (48) into two embedded Lagrangians that are invariant under transformations $\delta\phi = \eta(x)$ and $\delta\rho = \eta(x)$. After that, we will be able to solder these new Lagrangians in order to obtain an invariant one that describes a fermionic system. By varying the Lagrangians with respect to the variables $\partial_t\Phi$ and $\partial_x\Phi$ $[\Phi = (\phi, \rho)]$, we obtain the following Noether currents:

$$J_{1+} = \frac{1}{\sqrt{-g}}\dot{\phi} - e(A_0 + A_1), \tag{49}$$

$$J_{2+} = -\sqrt{-g}[\phi' - e(A_0 + A_1)], \tag{50}$$

$$J_{1-} = \frac{1}{\sqrt{-g}}\dot{\rho} + e(A_0 - A_1), \tag{51}$$

$$J_{2-} = -\sqrt{-g}[\rho' - e(A_0 - A_1)]. \tag{52}$$

After two iterations and by adding the counterterms to the original Lagrangians, we can find that

$$\hat{\mathcal{L}}_+^{(2)} = \mathcal{L}_+ - J_{1+}B_1 - J_{2+}B_2 + \frac{1}{2\sqrt{-g}}(B_1)^2 - \frac{\sqrt{-g}}{2}(B_2)^2 + \xi_+, \tag{53}$$

$$\hat{\mathcal{L}}_-^{(2)} = \mathcal{L}_- - J_{1-}B_1 - J_{2-}B_2 + \frac{1}{2\sqrt{-g}}(B_1)^2 - \frac{\sqrt{-g}}{2}(B_2)^2 + \xi_-, \tag{54}$$

where $\xi_\pm$ are the nondynamical terms of $\mathcal{L}_\pm$. The embedding process ends after these two steps and these Lagrangians are invariant under the desired transformation $\delta\phi = \eta(x)\delta\rho$. Now we can solder them by adding up two Lagrangians, Eqs. (53) and (54),

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+^{(2)} + \hat{\mathcal{L}}_-^{(2)} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- - (J_{1+} + J_{1-})B_1 - (J_{2+} + J_{2-})B_2 + \frac{1}{\sqrt{-g}}(B_1)^2 - \sqrt{-g}(B_2)^2. \tag{55}$$

To express this Lagrangian just in terms of the original fields, we can eliminate B_1 and B_2 through their equations of motions, which read

$$B_1 = \frac{\sqrt{-g}}{2}(J_{1+} + J_{1-}), \tag{56}$$

$$B_2 = -\frac{1}{2\sqrt{-g}}(J_{2+} + J_{2-}). \tag{57}$$

After substituting these results into $\mathcal{W}$, defining a new field $\Psi = \phi - \rho$, and fixing the Jackiw-Rajaraman coefficients $a = b = 1$, for simplicity, we can write that

$$\hat{\mathcal{W}} = \frac{1}{4\sqrt{-g}}\dot{\Psi}^2 - \frac{\sqrt{-g}}{4}\Psi'^2 - eA_0\dot{\Psi} + eA_1\sqrt{-g}\Psi' + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2 + e^2\sqrt{-g}[(A_0)^2 - (A_1)^2]. \tag{58}$$

This Lagrangian describes a 2D fermionic system and has a larger symmetry group than the initial Lagrangians (47) and (48). As the previous case, the soldering process included an extra noton term in the original Lagrangians to fuse the chiral states. This nondynamical term can acquire dynamics upon quantization [45].

VI. THE SOLDERING OF THE GAUGE INVARIANT GENERALIZED BOSONIZED CSM

In [48], the authors have introduced the Wess-Zumino (WZ) term for the GCSM and constructed its gauge invariant formulation by adding the WZ term into the Lagrangian of the model. This gauge invariant model is described by

$$\hat{\mathcal{S}} = \int dt dx \Big\{ \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + eA^\mu(\epsilon_{\mu\nu} - r\eta_{\mu\nu})\partial^\nu\phi + \frac{1}{2}e^2 aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(a - r^2)(\partial_\mu\theta)(\partial^\mu\theta) + eA^\mu[r\epsilon_{\mu\nu} + (a - r^2)\eta_{\mu\nu}]\partial^\nu\theta \Big\}, \tag{59}$$

where $\theta(x)$ is the WZ field. The Lagrangians of left/right moving bosons are given by defining the parameter r at its two opposite points ± 1

$$\begin{aligned}\hat{\mathcal{L}}_+ &= \frac{1}{2\sqrt{-g}}(\dot{\phi})^2 - \frac{\sqrt{-g}}{2}(\phi')^2 - b_1\dot{\phi} + b_1\sqrt{-g}\phi' + \frac{b_2}{\sqrt{-g}}(\dot{\theta})^2 - b_2\sqrt{-g}(\theta')^2 + b_3\dot{\theta} + b_4\theta' + \xi_+,\\ \hat{\mathcal{L}}_- &= \frac{1}{2\sqrt{-g}}(\dot{\rho})^2 - \frac{\sqrt{-g}}{2}(\rho')^2 - b_5\dot{\rho} + b_5\sqrt{-g}\rho' + \frac{b_2'}{\sqrt{-g}}(\dot{\eta})^2 - b_2'\sqrt{-g}(\eta')^2 + b_6\dot{\eta} + b_7\eta' + \xi_-,\end{aligned} \tag{60}$$

where η is also another WZ field and

$$\begin{aligned}&b_1 \equiv e(A_0 + A_1), \quad b_2 \equiv \frac{a-1}{2}, \quad b_2' \equiv \frac{b-1}{2},\\ &b_3 \equiv e[A_0(a-1) - A_1], \quad b_4 \equiv e\sqrt{-g}[A_0 - A_1(a-1)],\\ &b_5 \equiv e(A_0 - A_1), \quad b_6 \equiv e[A_0(b-1) + A_1],\\ &b_7 \equiv e\sqrt{-g}[-A_0 - A_1(b-1)],\\ &\xi_{(\pm)} \equiv \frac{1}{2\sqrt{-g}}(\dot{A}_1)^2 - \frac{\sqrt{-g}}{2}\left((A_0')^2 + \frac{\sqrt{-g}}{2}e^2\binom{a}{b}\right) \times [(A_0)^2 - (A_1)^2] - \dot{A}_1 A_0'.\end{aligned} \tag{61}$$

The goal here is to gauge these Lagrangians under the following transformations:

$$\begin{aligned}\delta\phi &= \delta\rho = \alpha(x),\\ \delta\theta &= \delta\eta = \beta(x).\end{aligned} \tag{62}$$

The Noether currents under these transformations are

$$\begin{aligned}&J_{1+} = \frac{1}{\sqrt{-g}}\dot{\phi} - b_1, \quad J_{1-} = \frac{1}{\sqrt{-g}}\dot{\rho} - b_5,\\ &J_{2+} = -\sqrt{-g}\phi' + b_1\sqrt{-g}, \quad J_{2-} = -\sqrt{-g}\rho' + b_5\sqrt{-g},\\ &J_{3+} = \frac{2b_2}{\sqrt{-g}}\dot{\theta} + b_3, \quad J_{3-} = \frac{2b_2'}{\sqrt{-g}}\dot{\eta} + b_6,\\ &J_{4+} = -2b_2\sqrt{-g}\theta' + b_4, \quad J_{4-} = -2b_2'\sqrt{-g}\eta' + b_7.\end{aligned} \tag{63}$$

The first iteration Lagrangians read

$$\begin{aligned}\hat{\mathcal{L}}_+^{(1)} &= \hat{\mathcal{L}}_+ - J_{1+}B_1 - J_{2+}B_2 - J_{3+}B_3 - J_{4+}B_4,\\ \hat{\mathcal{L}}_-^{(1)} &= \hat{\mathcal{L}}_- - J_{1-}B_1 - J_{2-}B_2 - J_{3-}B_3 - J_{4-}B_4,\end{aligned} \tag{64}$$

where B_1, B_2, B_3, and B_4 are new auxiliary fields that have the following variations:

$$\delta B_1 = \partial_t\alpha, \delta B_2 = \partial_x\alpha, \delta B_3 = \partial_t\beta, \delta B_4 = \partial_x\beta. \tag{65}$$

The variation of the first iterated Lagrangians are given by

$$\delta\hat{\mathcal{L}}_+^{(1)} = -\frac{1}{\sqrt{-g}}(\delta B_1)B_1 + \sqrt{-g}(\delta B_2)B_2 - \frac{2b_2}{\sqrt{-g}}(\delta B_3)B_3 + 2b_2\sqrt{-g}(\delta B_4)B_4, \tag{66}$$

$$\delta\hat{\mathcal{L}}_-^{(1)} = -\frac{1}{\sqrt{-g}}(\delta B_1)B_1 + \sqrt{-g}(\delta B_2)B_2 - \frac{2b_2}{\sqrt{-g}}(\delta B_3)B_3 + 2b_2\sqrt{-g}(\delta B_4)B_4. \tag{67}$$

As we can see, these variations are completely independent of the original fields. Therefore the embedding process has finished here, and, by adding the counterterms associated with these variations, we can obtain our desired invariant Lagrangian. Now we are ready to fuse both Lagrangians in Eqs. (64) by adding them up and by introducing a counterterm,

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- - J_{1+}B_1 - J_{2+}B_2 - J_{3+}B_3 - J_{4+}B_4 - J_{1-}B_1 - J_{2-}B_2 - J_{3-}B_3 - J_{4-}B_4 + \frac{1}{\sqrt{-g}}(B_1)^2 - \sqrt{-g}(B_2)^2 + \frac{2b_2}{\sqrt{-g}}(B_3)^2 - 2b_2\sqrt{-g}(B_4)^2, \tag{68}$$

where we have fixed the Jackiw-Rajaraman coefficients $a = b$ for simplicity. To express the final result only in terms of the original fields, one can eliminate the auxiliary fields by using their equations of motions

$$B_1 = \frac{\sqrt{-g}}{2}(J_{1+} + J_{1-}),$$
$$B_2 = \frac{-1}{2\sqrt{-g}}(J_{2+} + J_{2-}),$$
$$B_3 = \frac{\sqrt{-g}}{4b_2}(J_{3+} + J_{3-}),$$
$$B_4 = \frac{-1}{4b_2\sqrt{-g}}(J_{4+} + J_{4-}). \tag{69}$$

By substituting these results into Eq. (68) and introducing two soldering fields $\Psi = \phi - \rho$ and $\Omega = \theta - \eta$, we obtain an effective action

$$\hat{\mathcal{W}}_{\rm eff} = \frac{1}{4\sqrt{-g}}(\dot{\Psi})^2 - \frac{\sqrt{-g}}{4}(\Psi')^2 - eA_0\dot{\Psi} + e\sqrt{-g}A_1\Psi' + \frac{b_2}{2\sqrt{-g}}(\dot{\Omega})^2 - \frac{b_2\sqrt{-g}}{2}(\Omega')^2 - eA_1\dot{\Omega} + \frac{1}{2}[eA_0 + e\sqrt{-g}(A_0 - 2A_1b_2) + 2eA_1b_2]\Omega' - 2e^2b_2\sqrt{-g}(A_0)^2 + \frac{e^2\sqrt{-g}}{8b_2}(A_0 - 2A_1b_2)^2 - \frac{2e^2A_0}{8b_2}(A_0 - 2A_1b_2) + \frac{e^2}{8\sqrt{-g}b_2}(A_0)^2 + \frac{e^2}{2\sqrt{-g}}A_0A_1 + \frac{e^2b_2}{2\sqrt{-g}}(A_1)^2 - \frac{e^2}{2}A_1(A_0 - 2A_1b_2) + 2\xi, \tag{70}$$

where $\xi = \xi_- + \xi_+$. The initial Lagrangians were invariant under a semilocal gauge group, but this effective Lagrangian is invariant under the local version of the initial gauge group; moreover, it is invariant under the gauge transformations (62).

One can ask about the counterpart of this model in the commutative spacetime. We can find it just by putting $\sqrt{-g} = 1$. It reads

$$\mathcal{W}_{\rm eff} = \frac{1}{4}\partial_\mu\Psi\partial^\mu\Psi + e\epsilon^{\mu\nu}A_\mu\partial_\nu\Psi + (a-1)\partial_\mu\Omega\partial^\mu\Omega - eA_\mu\epsilon^{\mu\nu}\partial_\nu\Omega + \frac{1}{2}e^2aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{71}$$

where $\xi' = \xi|_{\sqrt{-g}=1}$. We have succeeded in the issue concerning the inclusion of the effects of interference between rightons and leftons (right/left moving scalar). Consequently, these components have lost their individuality in favor of a new, gauge invariant, collective field that does not depend on ϕ or ρ separately.

As it can be seen, this Lagrangian is apparently different from the initial ones, and the new fields Ψ and Ω are not chiral anymore. If we fix the Jackiw-Rajaraman coefficients $a = b = 1$, the field Ω becomes nondynamical, and it will just interact with the electromagnetic field. The combination of the massless modes lead us to a massive vectorial mode as a consequence of the chiral interference. The noton field that was defined before propagates neither to the left nor to the right directions.

VII. THE SOLDERING OF NC (ANTI-)SELF-DUAL MODELS IN $D = 2+1$

The Thirring model is an exactly solvable QFT that describes the self-interactions of a Dirac theory in $(2+1)$ dimensions. For the first time, Coleman has discovered an equivalence between this model and the sine-Gordon one, which is a bosonic theory [49].

In $D = 1+1$, the starting point is to consider two distinct fermionic theories with opposite chiralities. The analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures,

$$\mathcal{L}_+ = \bar{\psi}(i\partial + m)\psi - \frac{\lambda^2}{2}(\bar{\psi}\gamma_\mu\psi)^2,$$
$$\mathcal{L}_- = \bar{\xi}(i\partial - m')\xi - \frac{\lambda^2}{2}(\bar{\xi}\gamma_\mu\xi)^2, \tag{72}$$

where the bosonized Lagrangians are, respectively,

$$\mathcal{L}_+ = \frac{1}{2M}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda + \frac{1}{2}f_\mu f^\mu,$$
$$\mathcal{L}_- = -\frac{1}{2M}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda + \frac{1}{2}g_\mu g^\mu, \tag{73}$$

where f_μ and g_μ are the distinct bosonic vector fields. The current bosonization formulas in both cases are given by

$$j_\mu^+ = \bar{\psi}\gamma_\mu\psi = \frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho,$$
$$j_\mu^- = \bar{\xi}\gamma_\mu\xi = -\frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu g^\rho. \tag{74}$$

These models are known as the self- and anti-self-dual models [50–52].

On the extended Minkowski spacetime (τ, x) the Lagrangian (73) takes the following action form:

$$\hat{S}_\pm = \int d\tau d^2x\left[\frac{1}{2}h^\mu h_\mu \pm \frac{1}{2M}\left(\epsilon_{\mu 0\lambda}h^\mu\frac{\partial h^\lambda}{\partial\tau} + \epsilon_{\mu i\lambda}h^\mu\partial^i h^\lambda\right)\right], \tag{75}$$

where $h^\mu = f^\mu, g^\mu$.

After making the coordinate transformation, we obtain the action written in terms of the coordinates (t, x),

$$\hat{S}_\pm = \int dt d^2x\sqrt{-g}\left[\frac{1}{2}h^\mu h_\mu \pm \frac{1}{2M}\epsilon_{\mu i\lambda}h^\mu\partial^i h^\lambda\right]$$
$$\pm\frac{1}{2M}\epsilon_{\mu 0\lambda}h^\mu\frac{\partial h^\lambda}{\partial t}. \tag{76}$$

By taking a hint from the two-dimensional case, let us consider the gauging of the following symmetry:

$$\delta f_\mu = \delta g_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho\alpha^\sigma. \tag{77}$$

Under these transformations the bosonized Lagrangians change as

$$\delta\hat{S}_\pm = \int dt d^2x\left[\sqrt{-g}\left\{\epsilon^{\mu\rho\sigma}h_\mu \pm \frac{1}{M}\epsilon_{\mu i\lambda}\epsilon^{\mu\rho\sigma}\partial^i h^\lambda\right\}\right.$$
$$\left.\pm\frac{1}{M}\epsilon_{\mu 0\lambda}\epsilon^{\mu\rho\sigma}\partial^0 h^\lambda\right]\partial_\rho\alpha_\sigma. \tag{78}$$

We can identify the Noether currents

$$J_\pm^{\rho\sigma}(h_\mu) = \sqrt{-g}\left\{\epsilon^{\mu\rho\sigma}h_\mu \pm \frac{1}{M}\epsilon_{\mu i\lambda}\epsilon^{\mu\rho\sigma}\partial^i h^\lambda\right\}$$
$$\pm\frac{1}{M}\epsilon_{\mu 0\lambda}\epsilon^{\mu\rho\sigma}\partial^0 h^\lambda. \tag{79}$$

As a comment about the form of the gauge transformation in Eq. (77) we can say that the simpler form, such as the one we have assumed in the 2D case, will not be suitable and the variations cannot be combined to give a single structure like (79). Now we will introduce the auxiliary field coupled to the antisymmetric currents. In the two-dimensional case, this field was a vector. In the three dimensional case, as a natural generalization, we adopt an antisymmetric second rank Kalb-Ramond tensor field $B_{\rho\sigma}$ where its transformation is given by

$$\delta B_{\rho\sigma} = \partial_\rho\alpha_\sigma - \partial_\sigma\alpha_\rho. \tag{80}$$

It is worthwhile to mention that in the canonical NC approach, one must include the variation of the current associated with the NC field/parameter concerning the transformation of the auxiliary tensor field in order to obtain an effective Lagrangian after the soldering procedure [53].

To eliminate the nonvanishing change (78), we add a counterterm into the original Lagrangian. So, the first iterated Lagrangian is

$$\mathcal{L}_\pm^{(1)} = \mathcal{L}_\pm - \frac{1}{2}J_\pm^{\rho\sigma}(h_\mu)B_{\rho\sigma}, \tag{81}$$

which transforms as

$$\delta\mathcal{L}_\pm^{(1)} = -\frac{1}{2}\delta J_\pm^{\rho\sigma}B_{\rho\sigma}. \tag{82}$$

The variation of the currents coupled to the auxiliary field is given by

$$\delta J_\pm^{\rho\sigma}B_{\rho\sigma} = \sqrt{-g}\left[\delta B^{\rho\sigma}B_{\rho\sigma} \mp \frac{1}{M}\epsilon^{\lambda\gamma\theta}(\partial^i\partial_\gamma\alpha_\theta)B_{i\lambda}\right]$$
$$\mp\frac{2}{M}\epsilon^{\lambda\gamma\theta}(\partial^0\partial_\gamma\alpha_\theta)B_{0\lambda}. \tag{83}$$

As we can see, the above Lagrangians also are not invariant under the transformation (77); hence we must go further and add another counterterm. As a key point in the soldering formalism, the invariance of one Lagrangian alone is not the task. We are looking for a combination of both Lagrangians that is gauge invariant. To this aim, the second iteration Lagrangians are defined by

$$\mathcal{L}_\pm^{(2)} = \mathcal{L}_\pm^{(1)} + \frac{\sqrt{-g}}{4}B^{\rho\sigma}B_{\rho\sigma}. \tag{84}$$

By this definition, a straightforward algebra shows that the following combination is invariant under transformations (77) and (80). So,

$$\mathcal{L}_S = \mathcal{L}_+^{(2)} + \mathcal{L}_-^{(2)}$$
$$= \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{2}B^{\rho\sigma}(J_{\rho\sigma}^+(f) + J_{\rho\sigma}^-(g)) + \frac{\sqrt{-g}}{2}B^{\rho\sigma}B_{\rho\sigma}. \tag{85}$$

The gauging procedure of the symmetry is finished now. But the final result would be more interesting if we write the above Lagrangian in terms of the original fields. By using the equation of motion for $B_{\rho\sigma}$ we can eliminate this auxiliary field,

$$B_{\rho\sigma} = \frac{1}{2\sqrt{-g}}(J^{+}_{\rho\sigma}(f) + J^{-}_{\rho\sigma}(g)). \tag{86}$$

Including this solution in (85) the final soldered Lagrangian is expressed only in terms of the original fields,

$$\mathcal{L}_S = \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{8\sqrt{-g}}(J^{+}_{\rho\sigma}(f) + J^{-}_{\rho\sigma}(g))(J^{+\rho\sigma}(f) + J^{-\rho\sigma}(g)). \tag{87}$$

The crucial point of the soldering formalism becomes clear now. By using the explicit structures for the currents, the above Lagrangian is no longer a function of f_μ and g_μ separately, but it is a function of the combination

$$A_\mu = \frac{1}{\sqrt{2}M}(f_\mu - g_\mu). \tag{88}$$

By this field redefinition we can obtain the final effective action as

$$\mathcal{L}_S = \frac{M^2\sqrt{-g}}{2}A^\mu A_\mu + \partial_i A_0 \partial^0 A^i - \frac{1}{2\sqrt{-g}}\partial_0 A_i \partial^0 A^i - \frac{\sqrt{-g}}{2}(\partial^i A^0 \partial_i A_0 + \partial_i A_j \partial^i A^j - \partial_j A_i \partial^i A^j). \tag{89}$$

In the usual commutative Minkowski spacetime we obtain the Proca theory by soldering two (anti-)self-dual theories [52]. As a generalization, we can claim that the Lagrangian (89) is the NC version of the Abelian Proca theory in the κ-deformed $(2+1)$D Minkowski spacetime. To check that our result is correct we can directly obtain this Lagrangian by applying the coordinate transformation $(\tau, x) \to (t, x)$ in Proca theory. The Abelian Proca model on the extended Minkowski spacetime (τ, x) is

$$\begin{aligned}\hat{S} &= \int d\tau d^2x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}A^\mu A_\mu\right] \\ &= \int d\tau d^2x \left\{-\frac{1}{2}\left[\frac{\partial A^i}{\partial \tau}\left(\frac{\partial A_i}{\partial \tau} - \frac{\partial A_0}{\partial x^i}\right) + \frac{\partial A^0}{\partial x_i}\left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial \tau}\right) + \frac{\partial A^j}{\partial x_i}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)\right] + \frac{M^2}{2}A^\mu A_\mu\right\},\end{aligned} \tag{90}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. By coordinate transformation (14) we can rewrite the above actions in terms of (t, x) with explicit noncommutativity,

$$\hat{\mathcal{S}} = \int dtd^2x\sqrt{-g}\left\{-\frac{1}{2}\left[\frac{1}{\sqrt{-g}}\frac{\partial A^i}{\partial t}\left(\frac{1}{\sqrt{-g}}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i}\right) + \frac{\partial A^0}{\partial x_i}\left(\frac{\partial A_0}{\partial x^i} - \frac{1}{\sqrt{-g}}\frac{\partial A_i}{\partial t}\right) + \frac{\partial A^j}{\partial x_i}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)\right] + \frac{M^2}{2}A^\mu A_\mu\right\}. \tag{91}$$

Here we have assumed that $\dot{\tau} = \sqrt{-g} > 0$. After some straightforward manipulation we find that

$$\hat{\mathcal{S}} = \frac{1}{2}\int dtd^2x\Big(2\partial^0 A^i \partial_i A_0 + \sqrt{-g}\partial^i A^j \partial_j A_i - \sqrt{-g}\partial^i A^j \partial_i A_j - \sqrt{-g}\partial^i A^0 \partial_i A_0 - \frac{1}{\sqrt{-g}}\partial^0 A^i \partial_0 A_i + M^2\sqrt{-g}A^\mu A_\mu\Big), \tag{92}$$

and, as we expected, this action is equal to the model described by the Lagrangian (89).

Notice that, in this NC version, besides the modification of the field dynamics in this new spacetime, the mass term has also changed, and it is not equal to the usual Minkowski spacetime so the particle associated with this field must have a different mass term in this spacetime.

It is noteworthy that the transformations (77) are not the unique ones that lead to this result. We can also use the transformation

$$\delta f_\mu = -\delta g_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho \alpha^\sigma. \tag{93}$$

By assuming the above transformation and defining the final soldered field

$$A_\mu = \frac{1}{\sqrt{2}M}(f_\mu - g_\mu), \tag{94}$$

we can arrive at the same Lagrangian as in (89). This result led the authors of [21] to the idea of generalizing the soldering formalism. As it was mentioned before, the basic idea of soldering is that, adding two independent dual Lagrangians, it does not give us new information and, to construct a gauge invariant model, we have to fuse two Lagrangians via the Noether procedure. This idea was successfully applied to different models in various dimensions such as the chiral Schwinger model with opposite chiralities.

Some years after proposing this idea, it was shown that the usual sum of opposite chiral bosonic models is, in fact, gauge invariant, and it corresponds to a composite model, where the component models are the vector and axial Schwinger models [21]. As a consequence, we can reinterpret the soldering formalism as a kind of degree of freedom reduction mechanism.

In the case at hand, the two transformations (77) and (93) result in the same effective action, but in a general case, we may obtain two apparently different actions. For example, if we add an interaction term to the Lagrangians (73), the final result will be different. This property is the subject of the generalized soldering formalism [21]. Now this question may arise whether these two actions are describing two distinct phenomena. However, by calculating the generating functional of these two Lagrangians we have the same result. This shows that we are dealing with the same physics but described by different Lagrangians.

VIII. CONCLUSIONS AND PERSPECTIVES

The idea concerning the construction of a bosonized version of some fermionic models in order to study the properties of the target model through a theoretically easier version, the bosonized one, has dwelled in the theoretical physicist's mind during the 1980s and 1990s. In two spacetime dimensions, the concept of chirality together with bosonization was discussed after the influence of the chiral boson version in string theories.

In this way, Stone provided a method in which the objective is to put together in the same multiplet two chiral versions of a bosonized model in such a way that an effective final model was obtained. The target is to analyze physically the properties of the last resulting one. Another result obtained in the soldering technique is to discuss the fact that the final action is connected to the first ones through duality properties.

There was a relevant production of papers considering several models but none of them have considered NC models, which in fact is our objective here. We have analyzed the κ-Minkowski noncommutativity where some variations of the CSM were soldered and the soldered (final) action obtained was discussed in the aftermath.

However, as a perspective, we can provide a constraint analysis via Dirac and symplectic formalisms, in order to compare the before-and-after soldering. The comparison can also be made together with the one concerning the commutative models; namely, we can see clearly what is new in the NC parameter introduction. Another path is to investigate soldering in the light of the canonical noncommutativity, where the NC parameter is constant.

The conversion of NC second-class constraints into first-class ones concerning the soldered actions can reveal interesting properties involving the gauge invariance of NC models. These ideas are, as a matter of fact, ongoing research that will be published elsewhere.

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