Quantization of charged fields in the presence of critical potential steps

S. P. Gavrilov[1,3,*] and D. M. Gitman[1,2,4,†]

[1]*Department of Physics, Tomsk State University, 634050 Tomsk, Russia*
[2]*P.N. Lebedev Physical Institute, 53 Leninskiy prospect, 119991 Moscow, Russia*
[3]*Department of General and Experimental Physics, Herzen State Pedagogical University of Russia,*
Moyka embankment 48, 191186 St. Petersburg, Russia
[4]*Institute of Physics, University of São Paulo, CP 66318, CEP 05315-970 São Paulo, SP, Brazil*

QED with strong external backgrounds that can create particles from the vacuum is well developed for the so-called *t*-electric potential steps, which are time-dependent external electric fields that are switched on and off at some time instants. However, there exist many physically interesting situations where external backgrounds do not switch off at the time infinity. E.g., these are time-independent nonuniform electric fields that are concentrated in restricted space areas. The latter backgrounds represent a kind of spatial *x*-electric potential steps for charged particles. They can also create particles from the vacuum, the Klein paradox being closely related to this process. Approaches elaborated for treating quantum effects in the *t*-electric potential steps are not directly applicable to the *x*-electric potential steps and their generalization for *x*-electric potential steps was not sufficiently developed. We believe that the present work represents a consistent solution of the latter problem. We have considered a canonical quantization of the Dirac and scalar fields with *x*-electric potential step and have found in- and out-creation and annihilation operators that allow one to have particle interpretation of the physical system under consideration. To identify in- and out-operators we have performed a detailed mathematical and physical analysis of solutions of the relativistic wave equations with an *x*-electric potential step with subsequent QFT analysis of correctness of such an identification. We elaborated a nonperturbative (in the external field) technique that allows one to calculate all characteristics of zero-order processes, such, for example, scattering, reflection, and electron-positron pair creation, without radiation corrections, and also to calculate Feynman diagrams that describe all characteristics of processes with interaction between the in-, out-particles and photons. These diagrams have formally the usual form, but contain special propagators. Expressions for these propagators in terms of in- and out-solutions are presented. We apply the elaborated approach to two popular exactly solvable cases of *x*-electric potential steps, namely, to the Sauter potential and to the Klein step.

I. INTRODUCTION

The effect of particle creation by strong electromagnetic and gravitational fields has been attracting attention already for a long time. The effect has a pure quantum nature and was first considered in the framework of the relativistic quantum mechanics with understanding that all the questions can be answered only in the framework of quantum field theory (QFT). QFT with external background is to a certain extent an appropriate model for such calculations. In the framework of such a model, the particle creation is closely related to a violation of the vacuum stability with time. Backgrounds (external fields) that violate the vacuum stability are to be electriclike fields that are able to produce nonzero work when interacting with charged particles. Depending on the structure of such backgrounds, different approaches for calculating the effect were proposed and realized. Initially, the effect of particle creation was

considered for time-dependent external electric fields that are switched on and off at the initial and the final time instants, respectively. In what follows, we call such kind of external fields the *t*-electric potential steps. Scattering, particle creation from the vacuum and particle annihilation by the *t*-electric potential steps were considered in the framework of the relativistic quantum mechanics, see Refs. [1–4]; a more complete list of relevant publications can be found in [4]. A general formulation of quantum electrodynamics (QED) and QFT with *t*-electric potential steps was developed in Refs. [5]. However, there exist many physically interesting situations where external backgrounds formally are not switched off at the time infinity, the corresponding backgrounds formally being not *t*-electric potential steps. As an example, we may point out time-independent nonuniform electric fields that are concentrated in restricted space areas. The latter fields represent a kind of spatial or, as we call them conditionally, *x*-electric potential steps for charged particles. The *x*-electric potential steps can also create particles from the vacuum, the Klein paradox being closely related to this

*gavrilovsergeyp@yahoo.com
†gitman@if.usp.br

process [6–8]. We recall that Klein considered the reflection and transmission of relativistic electrons incident on a sufficiently high rectangular potential step (the Klein step) and he had found that there exists a range of energy where the transmission coefficient is negative and the reflection coefficient is greater than one. There would apparently be more reflected fermions than incoming. This is called the Klein paradox. One can find a broader interpretation of what should be called the Klein paradox, e.g., see [9–11]. These authors propose to speak about the Klein paradox when one encounters a special behavior of stationary solutions both for fermions and bosons, unusual for the nonrelativistic quantum mechanics. For example, if the electron kinetic energy belongs to the so-called Klein zone, which is situated just below the range of obvious total reflection, the stationary solutions penetrate through the step, the sign of the kinetic energy being, however, reversed. Just after the original Klein's paper the problem was studied by Sauter, who considered both the Klein step [7] and a more realistic smoothed potential step, $-\alpha E \tanh(x/\alpha)$, which is called the Sauter potential [8]. To avoid confusion, the Klein paradox should be distinguished from the Klein tunneling through the square barrier, e.g., see [12] and references therein. This tunneling without an exponential suppression occurs when an electron is incident on a high barrier, even when it is not high enough to create particles. Approaches elaborated for treating quantum effects in the t-electric potential steps are not directly applicable to the x-electric potential steps. Some heuristic calculations of the particle creation by x-electric potential steps in the framework of the relativistic quantum mechanics were presented by Nikishov in Refs. [2,13] and later developed by Hansen and Ravndal in Ref. [14]. One should also mention the Damour work [15], that contributed significantly in applying semiclassical methods for treating strong field problems in astrophysics. In fact, this work presents a first step to bridge the gap between approaches to quantum effects in potential steps developed within relativistic quantum mechanics and QFT. Using the Damour's approach, mean numbers of pairs created by a strong uniform electric field confined between two capacitor plates separated by a finite distance was calculated in Ref. [16]. A detailed historical review can be found in Refs. [12,14]. Nikishov had tested his way of calculation using the special case of a constant and uniform electric field, which is possible both for the t-electric potential steps and the x-electric potential steps, see [2,13,17]). At that time, however, no justification for such calculations from the QFT point of view was known.

Thus, we face a situation when a material that is commonly treated as a part of an introductory discussion to the relativistic quantum mechanics, see, e.g., [18], is not studied completely and has not an unique interpretation in the research literature. For example, Nikishov [17] has pointed out an inconsistency in the interpretation given by

Hansen and Ravndal [14]. In spite of the recognized achievements of this author in this area, there was no response to his observation. For the first time numerical simulations on space-time resolved data for the Klein paradox in a three-particle problem were reported in [10]. It was shown how electron-positron pairs are created and found that the results contradict to conclusions made in several works where the incoming electron was noted to "knock out" electrons from the step [3] or to "stimulate" [19] pair production. A clear reduction was shown instead of the suggested enhancement of the pair-production rate at those time instances, when the incoming electron wave packet overlaps spatially with the potential step. Recently, quantum simulations for the evolution of the Dirac spinor in the presence of linear potential with trapped ions were interpreted again as an electron transition to the negative energy branch, see Ref. [11].

Although the attempts to formulate a consistent QFT with potential steps as a background field [2,13–15,17] have not been completed, a general understanding was achieved that the Klein paradox is absent from a future consistent QFT. In the recent years the main attention of researchers was drawn to improving calculation technics, as well as to special calculations and applications. Though our present work is devoted to the formulation of such consistent QFT and contains some applications only in order to illustrate the general construction, we overview briefly, for completeness, some recent results obtained after the review [12] had appeared. We restrict this brief review only to works, where the particle creation effect by potential steps is considered, leaving the works devoted to quantum relativistic motion in potential barriers or well potentials, where the problem of bound states is principal, aside. The main motivation for applications is due to a close connection between particle creation by strong electrostatic potentials, in particular, by the steps, and the Unruh effect, the phenomenon of particle emission from black holes and cosmological horizons, e.g., see reviews [4,20,21]. Recent progress in laser physics allows one to hope that particle creation effect will be experimentally observed in laboratory conditions in the near future (see Refs. [22] for a review). In achieving extreme field strengths, the inhomogeneity of the realistic field becomes important. There appears an interest to study the effect in realistic inhomogeneous fields, in the main by using semiclassical and numerical methods. The recent experimental demonstration of the dynamic Casimir effect, and a soon expected verification of an analog of the Hawking radiation by using superconducting circuit devices, associates these two academic problems with possible applications to condensed matter physics [23]. In the recent years, in what concerns applications to condensed matter physics, particle creation by external fields became an observable effect in physics of graphene and similar nanostructures, say, in topological insulators and Weyl semimetals, this area being currently

under intense development [24–27]. This is explained by the fact that although the physics that gives rise to the massless Dirac fermions in each of the above-mentioned materials is different, the low-energy properties are governed by the same Dirac kinematics. The gap between the upper and lower branches in the corresponding Dirac particle spectra is very small, so that the particle creation effect turns out to be dominant (under certain conditions) as a response to the external electriclike field action on such materials. In particular, the particle creation effect is crucial for understanding the conductivity of graphene, especially in the so-called nonlinear regime. Electron-hole pair creation (which is an analog of the electron-positron pair creation from the vacuum) was recently observed in graphene by its indirect influence on the graphene conductivity [28]. The conductivity of graphene modified by the particle creation was calculated in the framework of QED with t-electric potential steps in [29]. It should be noted that the proof of the masslessness of charge carriers in graphene was obtained in studying the Klein tunneling through special potential barriers there. Soon after its theoretical prediction [30], the Klein tunneling in graphene, where the role of potential steps is played by p-n junctions, was observed by several experimental groups [31] (For a colloquium-style introduction to this subject see [32] and recent review [25]). Numerical modeling of the Klein tunneling through potential barriers in n-p-n and n-n' junctions are in good agreement with theoretical predictions [33]. Note that a sharp n-n' junction can be described by the step potential. However, electron's kinetic energies considered in that work do not allow one to speak about particle creation effect. Possible experimental configurations for testing the pair creation by a linear step of finite length were proposed in [34]. Observation of the Klein paradox in the context of so-called quantum quenches, which can nowadays be performed in experiments with ultracold atoms, is proposed in [35]. In achieving extreme field strengths, the inhomogeneity of the field becomes important. A number of works have been done to estimate the pair production rate for electric fields, whose direction is fixed and whose magnitude varies in one spatial dimension [36,37]. These approaches are semiclassical, essentially WKB. A Monte Carlo world line loop method has been developed and applied to the vacuum pair production problem in Ref. [38]. The world line instanton method, in which the Monte Carlo sum is effectively dominated by a single instanton loop, was extended to one-dimensional inhomogeneous fields in Refs. [39]. The agreement between these results is excellent. The world line instanton technique was extended in Ref. [40] to compute the vacuum pair production rate for spatially inhomogeneous electric background fields, with the spatial inhomogeneity being genuinely two or three dimensional. A mathematical analysis of pair production rate by the Sauter potential was presented in Refs. [41,42]. Loop

corrections to two-point correlation functions in the case of the time-independent electric field given by a linear potential step were studied in Ref. [43].

In the present article, we consider a canonical quantization of the Dirac and scalar (Klein-Gordon) fields in the presence of the x-electric potential step as a background [quantization of the action (2.7)] in terms of adequate in- and out-particles and develop a calculation technique of different quantum processes, such as scattering, reflection, and electron-positron pair creation. At the first stage of the quantization, a Dirac Heisenberg operator $\hat{\Psi}(X)$ that satisfies equal-time anticommutation relations and the Dirac equation, as well, is assigned to the Dirac field $\psi(X)$, see Sec. IV. However, the complete program of quantization includes also the second stage. At that stage we have to construct a Hilbert space of state vectors, where anticommutation relations are realized, and construct operators of all physical quantities of the system under consideration. For the free Dirac field this second stage of the quantization program is well known, see e.g., [44]. The result is formulated in terms of a Fock space constructed on the base of creation and annihilation operators of free Dirac particles. For the Dirac field interacting with any external background, the first stage of the canonical quantization of the action (2.7) gives the same above-mentioned result. But the second stage of the program has no universal solution suitable for any background. Each specific background, or a class of backgrounds, has to be analyzed separately. Here it is desirable to find in- and out-creation and annihilation operators that allow one to have particle interpretation of the quantized Dirac field, as it was done in the case of t-electric potential steps in [5]. In the case under consideration, the corresponding construction is possible, but it is more complicated (probably, not any background allows such interpretation of the quantized Dirac field). Another problem is related to the time-independence of the background under consideration. Whereas when considering t-electric potential steps that vanish at the time infinity one can naturally introduce certain in- and out-creation and annihilation operators starting first with the Schrödinger representation and then passing to the Heisenberg picture, this way of action is not possible in time-independent backgrounds. Now one must quantize in the Heisenberg picture from the very beginning. Then a new problem appears: how to identify in- and out-operators. When doing this, we believe that the time independent x-electric potential steps, as well as any constant electromagnetic field, is an idealization. In fact, any external field was switched on at a remote time instant t_{in}, then it was acting during a very large period of time T, and finally it was switched off at another remote time instant $t_{\text{out}} = t_{\text{in}} + T$. We also believe that if T is large enough one can ignore effects of switching the external field on and off. Besides, to identify in- and out-operators it is important to perform a detailed mathematical and physical analysis of solutions of

the Dirac and Klein-Gordon equations with an x-electric potential step. Such an analysis is presented below in Secs. II and III, where we introduce special in- and out-solutions of the Dirac and Klein-Gordon equations that are used in the quantization program. It is here that we give their physical interpretation to be confirmed further after the quantization has been fulfilled in Secs. V, VI, and VII. In the latter sections we present a calculation of different quantum processes such as scattering, reflection, and electron-positron pair creation. In Sec. VIII we discuss quantum processes in complete QED and construct effective particle propagators in terms of the in- and out-solutions introduced. In Sec. IX the developed theory is applied to exactly solvable cases of x-electric potential steps, namely, to the Sauter potential, and to the Klein step. Finally, we present a consistent QFT treatment of processes where a naive one-particle consideration may lead to the Klein paradox. In Appendix A we consider some peculiarities of the quantization of the scalar field in the Klein zone. In Appendix B we have studied the orthogonality relations on the hyperplane $t = \text{const}$ for the different ranges of quantum numbers. In Appendix C we have studied one-particle mean values of the charge, the kinetic energy, the number of particles, the current, and the energy flux through the surfaces $x = x_L$ and $x = x_R$. In Appendix D we show that stable electron wave packets in the Klein zone can exist only in the left asymptotic region, whereas stable positron wave packets can exist only in the right asymptotic region. The main contributions to the particle creation processes in a slowly alternating Sauter field are found in Appendix E.

II. DIRAC EQUATION WITH x-ELECTRIC POTENTIAL STEP AS AN EXTERNAL BACKGROUND

First, we describe a typical x-electric potential step, for which our construction is elaborated.

Potentials of an external electromagnetic field $A^\mu(X)$ in $d = D + 1$ dimensional Minkowski space-time parametrized by coordinates X,

$$X = (X^\mu, \mu = 0, 1, ..., D) = (t, \mathbf{r}),$$
$$X^0 = t, \qquad \mathbf{r} = (X^1, ..., X^D) \qquad (2.1)$$

that correspond to an x-electric potential step are chosen to be

$$A^\mu(X) = (A^0(x), A^j = 0, j = 1, 2, ..., D), \quad x = X^1, \quad (2.2)$$

so that the magnetic field $\mathbf{B}$ is zero and the electric field $\mathbf{E}$ reads

$$\mathbf{E}(X) = \mathbf{E}(x) = (E_x(x), 0, ..., 0),$$
$$E_x(x) = -A'_0(x) = E(x). \qquad (2.3)$$

The electric field (2.3) is directed along the axis $x^1 = x$, it is inhomogeneous in the x-direction, and does not depend on time t [$\mathbf{E}(x)$ is a constant field]. The main property of any x-electric potential step is

$$A_0(x) \xrightarrow{x \to \pm\infty} A_0(\pm\infty), \qquad E(x) \xrightarrow{|x| \to \infty} 0, \qquad (2.4)$$

where $A_0(\pm\infty)$ are some constant quantities, which means that the electric field under consideration is switched off at spatial infinity. In addition, we suppose[1] that the first derivative of the scalar potential $A_0(x)$ does not change its sign for any $x \in \mathbb{R}$. For definiteness sake, we suppose that

$$A'_0(x) \le 0 \Rightarrow \begin{cases} E(x) = -A'_0(x) \ge 0 \\ A_0(-\infty) > A_0(+\infty) \end{cases}, \qquad (2.5)$$

and that there exist points x_L and x_R ($x_R > x_L$) such that for $x \in S_L = (-\infty, x_L]$ and for $x \in S_R = [x_R, \infty))$ the electric field is already switched off,

$$A_0(x)|_{x \in S_L} = A_0(-\infty), \qquad E(x)|_{x \in S_L} = 0,$$
$$A_0(x)|_{x \in S_R} = A_0(+\infty), \qquad E(x)|_{x \in S_R} = 0, \qquad (2.6)$$

whereas the electric field is not zero in the region $S_{\text{int}} = (x_L, x_R)$. It accelerates positrons along the axis x in positive direction, and electrons in the negative direction.

As an example of the x-electric potential step, we refer to the Sauter potential [8] given by Eq. (9.1).

Both the scalar potential $A_0(x)$ and the corresponding electric field $E(x)$ are shown on the same figure Fig. 1.

Then in the $d = D + 1$ dimensional (dim.) Minkowski space-time, we consider a classical Dirac field $\psi(X)$ interacting with an external electromagnetic field $A^\mu(X)$ (external background) representing an x-electric potential step. The action of this system has the form[2]

$$S = \int \bar{\psi}(X)[\gamma^\mu P_\mu - m]\psi(X)dX,$$
$$P_\mu = i\partial_\mu - qA_\mu(X), \qquad (2.7)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and $dX = dt d\mathbf{r}$.

The classical Dirac field describes particles of a mass m and with a charge q. The corresponding quantum theory is already charge symmetric, and contains particles of the

---

[1]That means that we do not consider external electric fields that can create bound states for charged particles.
[2]Here we are using the natural system of units $\hbar = c = 1$.
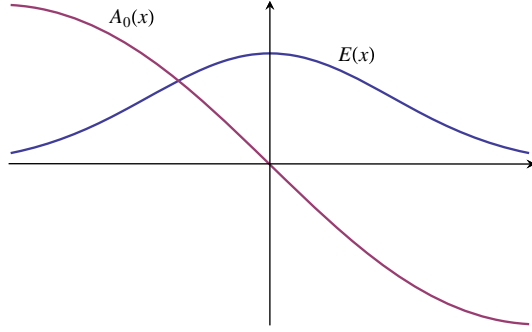
FIG. 1. x-electric potential step.

both signs of charges $\pm|q|$. We suppose that[3] $q = -e$, which means that we consider electrons as basic particles and positrons as their antiparticles.

In d dimensions, the Dirac field $\psi(X)$ is a column with $2^{[d/2]}$ components (we call it just a spinor in what follows), and γ^μ are $2^{[d/2]} \times 2^{[d/2]}$ gamma-matrices, see, e.g., [45],

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \qquad \eta^{\mu\nu} = \mathrm{diag}(\underbrace{1, -1, ..., -1}_{d}),$$

$$\mu, \nu = 0, 1, ..., D.$$

The Dirac field $\psi(X)$ satisfies the Dirac equation $(\gamma^\mu P_\mu - m)\psi = 0$, which, when written in the Hamiltonian form, reads (we have taken into account that $\mathbf{A} = 0$ for x-electric potential steps),

$$i\partial_0 \psi(X) = \hat{H}\psi(X),$$
$$\hat{H} = \gamma^0(-i\gamma^j \partial_j + m) + U(x), \qquad (2.8)$$

where $\hat{H}$ is the one-particle Dirac Hamiltonian and

$$U(x) = qA_0(x) = -eA_0(x) \qquad (2.9)$$

is the potential energy of the electron in the x-electric potential step. It should be noted that $\hat{H}$ does not depend on the time t in the background under consideration.

III. STATIONARY STATES WITH GIVEN LEFT AND RIGHT ASYMPTOTICS

A. General

Let us consider stationary solutions of Dirac equation (2.8), having the following form

$$\psi(X) = \exp(-ip_0 t + i\mathbf{p}_\perp \mathbf{r}_\perp)\psi_{p_0,\mathbf{p}_\perp}(x), \quad X = (t, x, \mathbf{r}_\perp),$$
$$\psi_{p_0,\mathbf{p}_\perp}(x) = \{\gamma^0[p_0 - U(x)] - \gamma^1 \hat{p}_x - \boldsymbol{\gamma}_\perp \mathbf{p}_\perp + m\}\phi_{p_0,\mathbf{p}_\perp}(x),$$
$$\mathbf{r}_\perp = (X^2, ..., X^D), \quad \mathbf{p}_\perp = (p^2, ..., p^D),$$
$$\boldsymbol{\gamma}_\perp = (\gamma^2, ..., \gamma^D), \quad \hat{p}_x = -i\partial_x, \qquad (3.1)$$

---

[3] $e > 0$ is the magnitude of the electron charge.

where $\psi_{p_0,\mathbf{p}_\perp}(x)$ and $\phi_{p_0,\mathbf{p}_\perp}(x)$ are spinors that depend on x alone. In fact these are stationary states with the energy p_0 and with definite momenta $\mathbf{p}_\perp$ in the perpendicular to the axis x directions. Substituting (3.1) into Dirac equation (2.8) (i.e., partially squaring the Dirac equation), we obtain a second-order differential equation for the spinor $\phi_{p_0,\mathbf{p}_\perp}(x)$,

$$\{\hat{p}_x^2 - i\gamma^0\gamma^1 U'(x) - [p_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2\}\phi_{p_0,\mathbf{p}_\perp}(x) = 0.$$
$$(3.2)$$

We can separate spinning variables by the substitution $\phi_{p_0,\mathbf{p}_\perp}(x) = \varphi(x)v$, where ν is a constant spinor that is an eigenvector for the operator $\gamma^0\gamma^1$, and $\varphi(x)$ are some scalar functions. The latter operator has two eigenvalues $\chi = \pm 1$, such that $\gamma^0\gamma^1 v_\chi = \chi v_\chi$. In addition, in $d > 3$, equation (3.2) allows one to subject the constant spinors v_χ to some supplementary conditions that, for example, can be chosen in the form

$$i\gamma^{2s}\gamma^{2s+1}v_{\chi,\sigma} = \sigma_s v_{\chi,\sigma} \quad \text{for even } d,$$
$$i\gamma^{2s+1}\gamma^{2s+2}v_{\chi,\sigma} = \sigma_s v_{\chi,\sigma} \quad \text{for odd } d,$$
$$\sigma = (\sigma_s = \pm 1, s = 1, 2, ..., [d/2] - 1)), \qquad (3.3)$$

where the quantum numbers σ_s describe spinning degrees of freedom (in $(1 + 1)$ and $(2 + 1)$ dimensions ($d = 2, 3$) there are no spinning degrees of freedom described by the quantum numbers σ). In what follows we choose the spinors $v_{\chi,\sigma}$ orthonormalized, $v_{\chi,\sigma}^\dagger v_{\chi',\sigma} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'}$. Taking all the said into account, we use the following substitution in Eq. (3.2):

$$\phi_{p_0,\mathbf{p}_\perp}(x) = \varphi_n^{(\chi)}(x)v_{\chi,\sigma}, n = (p_0, \mathbf{p}_\perp, \sigma). \qquad (3.4)$$

Then scalar functions $\varphi_n^{(\chi)}(x)$ have to obey the second order differential equation

$$\{\hat{p}_x^2 - i\chi U'(x) - [p_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2\}\varphi_n^{(\chi)}(x) = 0.$$
$$(3.5)$$

Thus, we are going to deal with solutions (3.1) of the form

$$\psi_n^{(\chi)}(X) = \exp(-ip_0 t + i\mathbf{p}_\perp \mathbf{r}_\perp)\varrho_n^{(\chi)}(x), \qquad \hat{H}\psi_n^{(\chi)} = p_0\psi_n^{(\chi)},$$
$$\varrho_n^{(\chi)}(x) = \{\gamma^0[p_0 - U(x)] - \gamma^1 \hat{p}_x - \boldsymbol{\gamma}_\perp \mathbf{p}_\perp + m\}\varphi_n^{(\chi)}(x)v_{\chi,\sigma}.$$
$$(3.6)$$

One can easily verify (this is a well-known property related to the specific structure of the projection operator in the brackets $\{...\}$) that solutions $\psi_n^{(\chi)}(X)$ and $\psi_n^{(\chi')}(X)$ (3.6), which only differ by values of χ are linearly dependent. Because of this, it suffices to work with

solutions corresponding to one of the possible two values of χ. That is why, we sometimes omit the subscript χ in the solutions, in such cases it is supposed that the spin quantum number χ is fixed in a certain way. Due to the same reason, there exists, in fact, only $J_{(d)} = 2^{[d/2]-1}$ different spin states (labeled by the quantum numbers σ) for a given set p_0, $\mathbf{p}_\perp$. Special examples of solutions (3.6) are given in Sec. IX.

A formal transition to the case of scalar field can be done by setting $\chi = 0$ in Eq. (3.5), $\varphi_n(x) = \varphi_n^{(0)}(x)$. Then one can find a complete set of solutions of the Klein-Gordon equation in the following form

$$\psi_n(X) = \exp\left(-ip_0 t + i\mathbf{p}_\perp \mathbf{r}_\perp\right)\varphi_n(x),$$
$$n = (p_0, \mathbf{p}_\perp). \tag{3.7}$$

In what follows, we are going to use solutions (3.6) and (3.7) with special left and right asymptotics. Let us describe these solutions. In such solutions the functions $\varphi_n(x)$ (with the omitted index χ) are denoted as ${}_\zeta\varphi_n(x)$ or ${}^\zeta\varphi_n(x)$, respectively, and satisfy the following asymptotic conditions [we recall that $A_0'(x) = 0$ and the scalar potential $A_0(x)$ takes constant values in the regions S_L and S_R, see Eqs. (2.6).]

$${}_\zeta\varphi_n(x) = \varphi_{n,\zeta}^L(x), \qquad x \in S_L = (-\infty, x_L],$$
$$\{\hat{p}_x^2 - [\pi_0(L)]^2 + \pi_\perp^2\}\varphi_n^L(x) = 0; \tag{3.8}$$

$${}^\zeta\varphi_n(x) = \varphi_{n,\zeta}^R(x), \qquad x \in S_R = [x_R, \infty),$$
$$\{\hat{p}_x^2 - [\pi_0(R)]^2 + \pi_\perp^2\}\varphi_n^R(x) = 0. \tag{3.9}$$

Here we have introduced the quantities $\pi_0(L/R)$,

$$\pi_0(R) = p_0 - U_R, \quad \pi_0(L) = p_0 - U_L, \quad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2},$$
$$U_L = U(-\infty), \quad U_R = U(+\infty), \tag{3.10}$$

Since p_0 is the total energy of a particle, we interpret $\pi_0(R)$ and $\pi_0(L)$ as its asymptotic kinetic energies in the regions S_R and S_L respectively. We call the quantity $\pi_\perp$ the total transversal energy or, for simplicity, the transversal energy (in spite of the fact that it includes the rest energy as well). The introduced kinetic energies satisfy the relation

$$\pi_0(L) = \pi_0(R) + \mathbb{U} > \pi_0(R), \tag{3.11}$$

where $\mathbb{U}$ is the magnitude of the electric step,

$$\mathbb{U} = U_R - U_L > 0. \tag{3.12}$$

At the same time one can see that in the asymptotic regions S_L and S_R solutions of the Dirac equation $\psi_n(X)$ are eigenfunctions of the operator $\hat{H} - U(x)$ with the eigenvalues $\pi_0(L)$ and $\pi_0(R)$, respectively. Thus, it is natural to call this operator the kinetic energy operator $\hat{H}^{\text{kin}}$,

$$\hat{H}^{\text{kin}} = \hat{H} - U(x),$$
$$\hat{H}^{\text{kin}}\psi_n(X)|_{x\to\pm\infty} = \pi_0(R/L)\psi_n(X)|_{x\to\pm\infty}. \tag{3.13}$$

One-particle kinetic energy operator $\hat{H}^{\text{kin}}$ of the scalar field [this field satisfies the Klein-Gordon equation written in the Hamiltonian form (4.2)] has the same asymptotic properties.

Equation (3.8) has a complete set of solutions (right asymptotics) in the form of plane waves,

$$\varphi_{n,\zeta}^R(x) = {}^\zeta\mathcal{N} \exp\left(ip^R x\right), \tag{3.14}$$

with real momenta p^R along the axis x,

$$p^R = \zeta\sqrt{[\pi_0(R)]^2 - \pi_\perp^2}, \qquad \zeta = \text{sgn}(p^R) = \pm. \tag{3.15}$$

The corresponding solutions (3.6) of the Dirac equation are denoted as ${}^\zeta\psi_n(X)$. They satisfy the condition

$$\hat{p}_x{}^\zeta\psi_n(X) = p^{R\zeta}\psi_n(X), \qquad x \to +\infty, \tag{3.16}$$

i.e., these are states with definite momenta p^R as $x \to +\infty$.

Nontrivial solutions ${}^\zeta\psi_n(X)$ exist only for quantum numbers n that obey the following relation

$$[\pi_0(R)]^2 > \pi_\perp^2 \Leftrightarrow \begin{cases} \pi_0(R) > \pi_\perp \\ \pi_0(R) < -\pi_\perp \end{cases}. \tag{3.17}$$

Equation (3.9) has a complete set of solutions (left asymptotics) $\varphi_n^L(x)$ in the form of plane waves,

$$\varphi_{n,\zeta}^L(x) = {}_\zeta\mathcal{N} \exp\left(ip^L x\right), \tag{3.18}$$

with real momenta p^L along the axis x,

$$p^L = \zeta\sqrt{[\pi_0(L)]^2 - \pi_\perp^2}, \qquad \zeta = \text{sgn}(p^L) = \pm. \tag{3.19}$$

The corresponding solutions (3.6) and (3.7) are denoted as ${}_\zeta\psi_n(X)$. They satisfy the condition

$$\hat{p}_x{}_\zeta\psi_n(X) = p^L{}_\zeta\psi_n(X), \qquad x \to -\infty, \tag{3.20}$$

i.e., these are states with definite momenta p^L as $x \to -\infty$.

The normalization factors $\mathcal{N}$ are determined in the next section.

There exists a useful relation between absolute values of the momenta p^R and p^L,

$$|p^{\mathrm{L}}| = \sqrt{|p^{\mathrm{R}}|^2 + 2\eta_{\mathrm{R}}\mathbb{U}\sqrt{|p^{\mathrm{R}}|^2 + \pi_\perp^2} + \mathbb{U}^2},$$

$$|p^{\mathrm{R}}| = \sqrt{|p^{\mathrm{L}}|^2 - 2\eta_{\mathrm{L}}\mathbb{U}\sqrt{|p^{\mathrm{L}}|^2 + \pi_\perp^2} + \mathbb{U}^2}, \qquad (3.21)$$

where $\eta_{\mathrm{L}} = \mathrm{sgn}\pi_0(\mathrm{L})$, $\eta_{\mathrm{R}} = \mathrm{sgn}\pi_0(\mathrm{R})$.

Nontrivial solutions $_\zeta\psi_n(X)$ exist only for quantum numbers n that obey the following relation

$$[\pi_0(\mathrm{L})]^2 > \pi_\perp^2 \Leftrightarrow \begin{cases} \pi_0(\mathrm{L}) > \pi_\perp \\ \pi_0(\mathrm{L}) < -\pi_\perp \end{cases}. \qquad (3.22)$$

In what follows we distinguish two types of electric steps, noncritical and critical, by their magnitudes as follows:

$$\mathbb{U} = \begin{cases} \mathbb{U} < \mathbb{U}_c = 2m, & \text{noncritical steps} \\ \mathbb{U} > \mathbb{U}_c, & \text{critical steps} \end{cases}. \qquad (3.23)$$

B. Ranges of quantum numbers

There exist some ranges of quantum numbers n where solutions $\varphi_n^{\mathrm{L/R}}(x)$ have similar forms. These ranges and the corresponding solutions are described below.

In the case of the critical steps, which is of the main interest for us in the present article, there exist five ranges Ω_k, $k = 1, \ldots, 5$. We denote the corresponding quantum numbers by n_k, so that $n_k \in \Omega_k$, see Fig. 2.

It should be noted that the curve plotted on Fig. 2 is the potential energy $U(x)$ (2.9) of an electron in the x-electric step, such that electrons are accelerated to the left and positrons to the right by the electric field (2.3).

In the case of critical steps $\mathbb{U} > \mathbb{U}_c$, the range Ω_3 does exist for the quantum numbers $p_\perp$ restricted by the inequality,

$$2\pi_\perp \leq \mathbb{U}. \qquad (3.24)$$

Note that the range Ω_3 often referred to as the Klein zone. In the case of noncritical steps $\mathbb{U} < \mathbb{U}_c$, there exist only four ranges, the range Ω_3 is absent.

The manifold of all the quantum numbers n is denoted by Ω, so that $\Omega = \Omega_1 \cup \cdots \cup \Omega_5$.

Below, we describe all the ranges in detail. In this connection, it should be noted that the ranges Ω_4 and Ω_5 are similar to the ranges Ω_2 and Ω_1 under the change of electrons by positrons.

1. Ranges Ω_1 and Ω_5

The ranges Ω_1 and Ω_5 exist for any $\mathbb{U}$ and includes the quantum numbers n_1 and n_5, respectively, that obey the inequalities
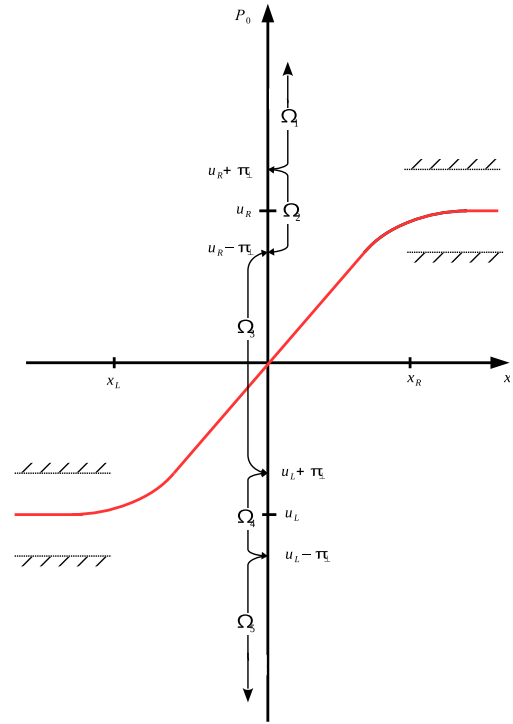

FIG. 2. Potential energy $U(x)$ of electrons in an x-electric step.

$$p_0 \geq U_{\mathrm{R}} + \pi_\perp \Leftrightarrow \pi_0(\mathrm{R}) \geq \pi_\perp \quad \text{if } n \in \Omega_1,$$

$$p_0 \leq U_{\mathrm{L}} - \pi_\perp \Leftrightarrow -\pi_0(\mathrm{L}) \geq \pi_\perp \quad \text{if } n \in \Omega_5 \qquad (3.25)$$

for a given $\pi_\perp$, see Fig. 2. Note that the inequalities (3.25) imply the inequalities (3.22) for n_1 and (3.17) for n_5. Then it follows from Eqs. (3.25) that in the ranges Ω_1 and Ω_5 there exist solutions $^\zeta\psi_n(X)$ and $_\zeta\psi_n(X)$. $^\zeta\psi_n(X)$ can be interpreted as either a wave function of electron for n_1 or a wave function of positron for n_5 with momenta p^{R} along the axis x, given by Eq. (3.15), whereas $_\zeta\psi_n(X)$ can be interpreted as either a wave function of electron for n_1 or a wave function of positron for n_5 with momenta p^{L} along the axis x, given by Eq. (3.19). The further analysis of these solutions in the framework of QED, given in the Sec. IV confirms this interpretation.

We believe that each pair of solutions $_\zeta\psi_n(X)$, $\zeta = \pm$, forms a complete set for any given $n \in \Omega_1 \cup \Omega_5$, the same is true for each pair of solutions $^\zeta\psi_n(X)$. This means that any given solution $^\zeta\psi_n(X)$ can be decomposed in terms of two solutions $_-\psi_n(X)$ and $_+\psi_n(X)$, whereas any given solution $_\zeta\psi_n(X)$ can be decomposed in terms of two solutions $^+\psi_n(X)$ and $^-\psi_n(X)$ in the ranges Ω_1 and Ω_5.

2. Ranges Ω_2 and Ω_4

The ranges Ω_2 and Ω_4 exist for any $\mathbb{U}$ and include the quantum numbers n_2 that obey the inequalities

$$U_R - \pi_\perp < p_0 < U_R + \pi_\perp, \qquad \pi_0(L) > \pi_\perp \quad \text{if } 2\pi_\perp \leq \mathbb{U},$$
$$U_L + \pi_\perp < p_0 < U_R + \pi_\perp \quad \text{if } 2\pi_\perp > \mathbb{U}, \qquad (3.26)$$

and the quantum numbers n_4 that obey the inequalities

$$U_L - \pi_\perp < p_0 < U_L + \pi_\perp, \qquad \pi_0(R) < -\pi_\perp \quad \text{if } 2\pi_\perp \leq U,$$
$$U_L - \pi_\perp < p_0 < U_R - \pi_\perp \quad \text{if } 2\pi_\perp > U. \qquad (3.27)$$

As a consequence of Eqs. (3.26) and (3.27) there exist solutions $_\zeta\psi_{n_2}(X)$ with definite left asymptotics and $^\zeta\psi_{n_4}(X)$ with definite right asymptotics. $_\zeta\psi_{n_2}(X)$ and $^\zeta\psi_{n_4}(X)$ can be interpreted as a wave function of electron and positron, respectively. Nontrivial solutions $^\zeta\psi_{n_2}(X)$ and $_\zeta\psi_{n_4}(X)$ do not exist, since Eq. (3.26) contradicts Eq. (3.17) and Eq. (3.27) contradicts Eq. (3.22).

The fact that any solution with quantum numbers n_2 has zero right asymptotic and any solution with quantum numbers n_4 has zero left asymptotic imposes restrictions on the form of the solutions $_\zeta\psi_{n_2}(X)$ and $^\zeta\psi_{n_4}(X)$. In particular, they cannot be independent for different ζ. For example, because the set $_\zeta\psi_{n_2}(X)$ is complete, any solution $\psi_{n_2}(X)$ can be represented as

$$\psi_{n_2}(X) = {}_+\psi_{n_2}(X)c_+ + {}_-\psi_{n_2}(X)c_-. \qquad (3.28)$$

In the region S_R the superposition $\psi_{n_2}(X)$ has zero asymptotics and therefore the corresponding Dirac current in x-direction is zero. Using this consideration one can easily find that $|c_+| = |c_-|$, such that Eq. (3.28) can be written as

$$\psi_{n_2}(X) = {}_+\psi_{n_2}(X)e^{+i\theta_{n_2}} + {}_-\psi_{n_2}(X)e^{-i\theta_{n_2}}. \qquad (3.29)$$

Following by the same logic, one sees that in the region S_L the only allowed superposition of $^+\psi_{n_4}(X)$ and $^-\psi_{n_4}(X)$ has zero asymptotics and therefore any solution $\psi_{n_2}(X)$ can be written as

$$\psi_{n_4}(X) = {}^+\psi_{n_4}(X)e^{+i\theta_{n_4}} + {}^-\psi_{n_4}(X)e^{-i\theta_{n_4}}. \qquad (3.30)$$

In fact, $\psi_{n_2}(X)$ are wave functions that describe an unbounded motion in $x \to -\infty$ direction while $\psi_{n_4}(X)$ are wave functions that describe an unbounded motion toward $x = +\infty$. Equations (3.29) and (3.30) provide asymptotic forms for these wave functions. These forms are sums of two waves traveling in opposite directions, with equal in magnitude currents, which means that we deal with standing waves.

3. Range Ω_3

In the Ω_3-range (in the Klein zone) the quantum numbers $\mathbf{p}_\perp$ are restricted by the inequality (3.24) and for any of

such $\pi_\perp$ quantum numbers p_0 obey the double inequality, see Fig. 2,

$$U_L + \pi_\perp \leq p_0 \leq U_R - \pi_\perp. \qquad (3.31)$$

Inequality (3.31) implies also that $\pi_0(L) \geq \pi_\perp$ and $\pi_0(R) \leq -\pi_\perp$. It follows from this inequality that there exist solutions $^\zeta\psi_{n_3}(X)$ [see condition (3.17)]. On the other hand, since the inequality (3.31) implies $\pi_0(L) > \pi_\perp$ and (3.22), there exist solutions $_\zeta\psi_{n_3}(X)$ as well. Thus, in the range Ω_3 there exist the following sets of solutions

$$\{_\zeta\psi_{n_3}(X)\}, \{^\zeta\psi_{n_3}(X)\}, \qquad \zeta = \pm. \qquad (3.32)$$

However, the one-particle interpretation of these solutions based on energy spectrum in a similar way as has been done in the ranges Ω_1 and Ω_5 becomes inconsistent. Indeed, it is enough to see the following contradiction: from the point of view of the left asymptotic area, only electron states are possible in the range Ω_3, whereas from the point of view of the right asymptotic area, only positron states are possible in this range. Detailed consideration in the framework of QED shows (see Sec. VII) that in a certain sense the solutions $^\zeta\psi_{n_3}(X)$ describe electrons, whereas the solutions $_\zeta\psi_{n_3}(X)$ describe positrons.

C. Orthogonality, normalization and completeness

In this subsection we study orthonormalization, mutual decompositions and completeness of the solutions introduced above. To this end it is convenient to use two types of inner products in the Hilbert space of Dirac spinors. One of them is defined on the hyperplane $x = $ const, and the other on the hyperplane $t = $ const.

1. Using inner product on x-constant hyperplane

Let us start with the inner product on the hyperplane $x = $ const. For any two Dirac spinors $\psi(X)$ and $\psi'(X)$ it has the form

$$(\psi, \psi')_x = \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X)dtd\mathbf{r}_\perp. \qquad (3.33)$$

Due to the t-independence of the external field that corresponds to the x-electric potential steps, we can provide x-independence of the inner product (3.33) for two solutions of the Dirac equation by imposing periodic boundary conditions in the variables t and X^j, $j = 2, \ldots, D$. Thus, we consider our theory in a large space-time box that has a spatial volume $V_\perp = \prod_{j=2}^{D} K_j$ and the time dimension T, where all K_j and T are macroscopically large. It is supposed that all the solutions $\psi(X)$ are periodic under transitions from one box to another. Then the integration in (3.33) over the transverse coordinates is fulfilled from

$-K_j/2$ to $+K_j/2$, and over the time t from $-T/2$ to $+T/2$. Under these suppositions, the inner product (3.33) does not depend on x. The limits $K_j \to \infty$ and $T \to \infty$ are assumed in final expressions.

It should be noted that usually QFT deals with physical quantities that are presented by volume integrals on the hyperplane $t = \text{const}$. However, if we wish to extract results of the one-particle scattering theory from a classical field theory, all the constituent quantities, such as reflection and transmission coefficients etc., have to be represented with the help of the inner product (3.33) on the hyperplane $x = \text{const}$. Indeed, in such a theory we observe, for example, electric current, energy flux, or other types of currents flowing through surfaces $x = \text{const}$ situated in asymptotic regions. In addition, we suppose that all the measurements are performed during a macroscopic time (say, the time T) when the external field can be considered as constant. Then the currents under consideration can be represented by integral of the form $T^{-1} \int_T dt \ldots$.

We note that for $\psi' = \psi$ the inner product (3.33) divided by T represents the current of the Dirac field $\psi(X)$ across the hyperplane $x = \text{const}$. For nondecaying wave functions this current differs from zero.

For two different solutions of the form (3.6) the integral in the right-hand side of Eq. (3.33) can be easily calculated to be

$$(\psi_n, \psi'_{n'})_x = V_\perp T \delta_{n,n'} \mathcal{I}, \qquad \mathcal{I} = \varrho_n^\dagger(x) \gamma^0 \gamma^1 \varrho'_n(x). \quad (3.34)$$

Using the structure of spinors v_σ, one can represent the current density $\mathcal{I}$ as follows

$$\mathcal{I} = \varphi_n^*(x)(i\overleftarrow{\partial}_x - i\overrightarrow{\partial}_x)[p_0 - U(x) + \chi i \partial_x]\varphi'_n(x). \quad (3.35)$$

Let us consider solutions $\{\zeta\psi_n(X)\}$ and $\{^\zeta\psi_n(X)\}$ with left and right plane-wave asymptotics, respectively. They are orthogonal for different n and can be subjected to the following orthonormality conditions

$$(\zeta\psi_n, \zeta'\psi_{n'})_x = \zeta \eta_L \delta_{\zeta,\zeta'} \delta_{n,n'}, \qquad \eta_L = \text{sgn}\pi_0(L),$$
$$n \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_5,$$
$$(^\zeta\psi_n, {}^{\zeta'}\psi_{n'})_x = \zeta \eta_R \delta_{\zeta,\zeta'} \delta_{n,n'}, \qquad \eta_R = \text{sgn}\pi_0(R),$$
$$n \in \Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5. \quad (3.36)$$

In justification of (3.36) one has to take into account that for these states the relations

$$|\pi_0(L)| > |p^L|, \qquad |\pi_0(R)| > |p^R|$$

hold. This fact explains why the sign of the inner products, which coincides with the sign of $\mathcal{I}$ is due to the sign of the quantity $\pi_0(L/R)$.

Then the normalization factors [with respect to the inner product (3.33)] in solutions of the Dirac equation

that have plane-wave asymptotics (3.14) and (3.18) have the form

$$_\zeta\mathcal{N} = {_\zeta C}Y, \qquad ^\zeta\mathcal{N} = {^\zeta C}Y, \qquad Y = (V_\perp T)^{-1/2},$$
$$_\zeta C = [2|p^L||\pi_0(L) - \chi p^L|]^{-1/2},$$
$$^\zeta C = [2|p^R||\pi_0(R) - \chi p^R|]^{-1/2}. \quad (3.37)$$

In the $K_j \to \infty$ and $T \to \infty$ limit one has to replace $\delta_{n,n'}$ in normalization conditions (3.36) by $\delta_{\sigma,\sigma'}\delta(p_0 - p'_0) \times \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)$ and to set $Y = (2\pi)^{-(d-1)/2}$ in Eqs. (3.37).

As was assumed, see subsection III B, each pair of solutions $_\zeta\psi_n(X)$ and $^\zeta\psi_n(X)$ with given quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ is complete in the space of solutions with each n. Due to (3.36) the corresponding mutual decompositions of such solutions have the form

$$\eta_L {}^\zeta\psi_n(X) = {}_+\psi_n(X)g(_+|^\zeta) - {}_-\psi_n(X)g(_-|^\zeta),$$
$$\eta_R {}_\zeta\psi_n(X) = {}^+\psi_n(X)g(^+|_\zeta) - {}^-\psi_n(X)g(^-|_\zeta), \quad (3.38)$$

where decomposition coefficients g are given by the relations:

$$(_\zeta\psi_n, {}^{\zeta'}\psi_{n'})_x = \delta_{n,n'}g(_\zeta|^{\zeta'}), \qquad g(^{\zeta'}|_\zeta) = g(_\zeta|^{\zeta'})^*,$$
$$n \in \Omega_1 \cup \Omega_3 \cup \Omega_5. \quad (3.39)$$

Substituting (3.38) into orthonormality conditions (3.36), we derive the following unitary relations for the decomposition coefficients[4]:

$$g(^{\zeta'}|_+)g(_+|^\zeta) - g(^{\zeta'}|_-)g(_-|^\zeta) = \zeta\eta_L\eta_R\delta_{\zeta,\zeta'},$$
$$g(_{\zeta'}|^+)g(^+|_\zeta) - g(_{\zeta'}|^-)g(^-|_\zeta) = \zeta\eta_L\eta_R\delta_{\zeta,\zeta'}. \quad (3.40)$$

In particular, these relations imply that

$$|g(_-|^+)|^2 = |g(_+|^-)|^2, \qquad |g(_+|^+)|^2 = |g(_-|^-)|^2,$$
$$\frac{g(_+|^-)}{g(_-|^-)} = \frac{g(^+|_-)}{g(^+|_+)}. \quad (3.41)$$

One can see that all the coefficients g can be expressed via only two of them, e.g., via $g(_+|^+)$ and $g(_+|^-)$. However, even the latter coefficients are not completely independent, they are related as follows:

$$|g(_+|^-)|^2 - |g(_+|^+)|^2 = -\eta_L\eta_R. \quad (3.42)$$

Nevertheless, in what follows, we will use both coefficients $g(_+|^-)$ and $g(_+|^+)$ in our consideration. This maintains a certain symmetry in writing formulas and, moreover, allows one to generalize the consideration to the cases

---

[4]Similar relations are known in the t-electric potential steps, see [5].

when the inner potential may depend on several space coordinates and to the case when solutions $\{_\zeta\psi_n(X)\}$ and $\{^\zeta\psi_m(X)\}$ are characterized by different sets of quantum numbers, i.e., the sets n and m do not coincide.

For any two solutions $\psi(X)$ and $\psi'(X)$ of the Klein-Gordon equation, the inner product on the hyperplane $x = \text{const}$ has the form

$$(\psi, \psi')_x = \int \psi^*(X)(i\overleftrightarrow{\partial}_x - i\overrightarrow{\partial}_x)\psi'(X)dtd\mathbf{r}_\perp. \quad (3.43)$$

Orthonormality conditions for solutions of the Klein-Gordon equation can be presented in form (3.36) where $\eta_L = \eta_R = 1$. The normalization factors with respect to inner product (3.43) are

$$_\zeta\mathcal{N} = {}_\zeta CY, \qquad ^\zeta\mathcal{N} = {}^\zeta CY,$$
$$_\zeta C = |2p^L|^{-1/2}, \qquad ^\zeta C = |2p^R|^{-1/2}, \quad (3.44)$$

where the factor Y is given by (3.37). In the Klein-Gordon case all the relations presented above and that include coefficients g hold true with the setting $\eta_L = \eta_R = 1$.

2. Using inner product on t-constant hyperplane

We recall that usually the inner product between two solutions $\psi(X)$ and $\psi'(X)$ of the Dirac equation is defined on t-const. hyperplane as follows:

$$(\psi, \psi') = \int \psi^\dagger(X)\psi'(X)d\mathbf{r}. \quad (3.45)$$

Such an inner product does not depend on the choice of the hyperplane (does not depend on t) if the solutions obey certain boundary conditions that allow one to integrate by parts in (3.45) neglecting boundary terms. Since physical states are wave packets that vanish on the remote boundaries, the inner product (3.45) is time-independent for such states. However, considering solutions that are generalized states, which do not vanish at the spatial infinity, one should take some additional steps to keep the inner product (3.45) time independent. Sometimes to do this it is enough to impose periodic boundary conditions (in all spatial directions) on Dirac wave functions and on the corresponding external field. However, in the case under consideration, the external field $A_0(x)$ of x-electric potential steps with different asymptotics at $x \to \pm\infty$ cannot be subjected to any periodic boundary conditions in x-direction without changing its physical meaning. That is why, to provide the time independence of the inner product, one has to extend the definition of the inner product. Under the assumption that solutions with quantum numbers n form a complete set of function in the corresponding Hilbert space at each time instant t, it is enough to make such an extension for a pair

$\psi_n(X)$ and $\psi'_{n'}(X)$ with all possible n and n'. Such an inner product is described below.

Let $\psi_n(X)$ and $\psi'_{n'}(X)$ be solutions of the Dirac equation that were described in the previous subsections III A and III B. They allow one to impose periodic conditions in the coordinates X^j, $j = 2, \ldots, D$ with periodicity K_j (of course it implies quantization of the corresponding transverse momenta). Then the inner product for the pair $\psi_n(X)$ and $\psi'_{n'}(X)$ is defined as follows:

$$(\psi_n, \psi'_{n'}) = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(L)}}^{K^{(R)}} \psi_n^\dagger(X)\psi'_{n'}(X)dx, \quad V_\perp = \prod_{j=2}^{D} K_j,$$
$$(3.46)$$

and the limits $K^{(L/R)} \to \infty$ are assumed in final expressions. As it is demonstrated in Appendix B, the so-defined inner product is time-independent. In what follows the improper integral over x in the right-hand side of Eq. (3.46) is reduced to its special principal value to provide a certain additional property important for us.

Finally, we obtain the following orthonormality relations, see details in Appendix B,

$$({}_\zeta\psi_n, {}_\zeta\psi_{n'}) = ({}^\zeta\psi_n, {}^\zeta\psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, \quad n \in \Omega_1 \cup \Omega_3 \cup \Omega_5,$$
$$(\psi_n, \psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, \quad n, n' \in \Omega_2 \cup \Omega_4;$$
$$\mathcal{M}_n = 2\frac{K^{(R)}}{T}\left|\frac{\pi_0(R)}{p^R}\right||g(_+|^+)|^2 + O(1), \quad n \in \Omega_1 \cup \Omega_5,$$
$$\mathcal{M}_n = 2\frac{K^{(R)}}{T}\left|\frac{\pi_0(R)}{p^R}\right||g(_+|^-)|^2 + O(1), \quad n \in \Omega_3,$$
$$(3.47)$$

where $\mathcal{M}_n$ for $n \in \Omega_2 \cup \Omega_4$ are given by Eqs. (B6). It follows from Eqs. (3.47) that densities (in the x-direction) of the wave functions $_\zeta\psi_{n_i}$, $i = 1, 3, 5$ are dominating in the region S_R, whereas the densities of the wave functions $^\zeta\psi_{n_i}$, $i = 1, 3, 5$ are dominating in the region S_L.

In the limit $K^{(L/R)} \to \infty$, with the account of condition (B15), we obtain solutions normalized to the δ-function,

$$({}_\zeta\psi_n, {}_\zeta\psi_{n'}) = ({}^\zeta\psi_n, {}^\zeta\psi_{n'}) = \delta_{\sigma,\sigma'}\delta(p_0 - p'_0)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\mathcal{M}_n,$$
$$n, n' \in \Omega_1 \cup \Omega_3 \cup \Omega_5,$$
$$(\psi_n, \psi_{n'}) = \delta_{\sigma,\sigma'}\delta(p_0 - p'_0)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp), \quad n, n' \in \Omega_2 \cup \Omega_4;$$
$$\mathcal{M}_n = |g(_+|^+)|^2, \quad n \in \Omega_1 \cup \Omega_5;$$
$$\mathcal{M}_n = |g(_+|^-)|^2, \quad n \in \Omega_3. \quad (3.48)$$

We recall that the inner product between two solutions $\psi(X)$ and $\psi'(X)$ of the Klein-Gordon equation is defined on t-const. hyperplane as follows as a charge,

$$(\psi, \psi') = \int \Phi^\dagger(X) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi(X) d\mathbf{r},$$

$$\Phi(X) = \begin{pmatrix} [i\partial_t - U(x)]\psi(X) \\ \psi(X) \end{pmatrix}. \tag{3.49}$$

It is proportional to the field charge if $\psi = \psi'$. It is assumed that the improper integral over x in the right-hand side of Eq. (3.49) is treated as its special principal value, in the same spirit as in the Dirac case. One can verify that with this inner product the following orthonormality relations hold

$$(\psi_n, \psi'_{n'}) = 0, \quad n \neq n', \forall \, n, n'; \quad (\psi_{n_2}, \psi_{n'_2}) = \delta_{n_2, n'_2} \mathcal{M}_{n_2}, \quad (\psi_{n_4}, \psi_{n'_4}) = -\delta_{n_4, n'_4} \mathcal{M}_{n_4};$$

$$({}_\zeta\psi_n, {}^{-\zeta}\psi_n) = 0, \quad ({}^\zeta\psi_n, {}^\zeta\psi_{n'}) = ({}_\zeta\psi_n, {}_\zeta\psi_{n'}) = \mathrm{sgn}\pi_0(L)\delta_{n,n'}\mathcal{M}_n, \quad n \in \Omega_1 \cup \Omega_5;$$

$$({}_\zeta\psi_n, {}^\zeta\psi_n) = 0, \quad ({}^\zeta\psi_n, {}^\zeta\psi_{n'}) = -({}_\zeta\psi_n, {}_\zeta\psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, \quad n \in \Omega_3, \tag{3.50}$$

where $\mathcal{M}_n$ are given by Eqs. (3.47). Unlike the Dirac case, there are $\pm$ signs in the right-hand side of Eq. (3.50). Then it is natural to suppose that the solutions ${}_\zeta\psi_{n_3}$, ψ_{n_4}, ${}_\zeta\psi_{n_5}$, and ${}^\zeta\psi_{n_5}$ describe positron states.

Thus, both for the Dirac and the Klein-Gordon equations, for each set of quantum numbers n, there exist one or two complete sets of solutions

(a) For $\forall \, n \in \Omega_1 \cup \Omega_5$ we have two $(\zeta = \pm)$ sets: $\{{}_\zeta\psi_n(X), {}^{-\zeta}\psi_n(X)\}$.

(b) For $\forall \, n \in \Omega_3$ we have two $(\zeta = \pm)$ sets: $\{{}_\zeta\psi_n(X), {}^\zeta\psi_n(X)\}$.

(c) For $\forall \, n \in \Omega_2 \cup \Omega_4$ we have the set $\{\psi_n(X)\}$.

We believe that all these solutions allow one to construct two complete (at any time instant t) systems both in the Hilbert space of Dirac spinors and in the Hilbert space of scalar fields. In the Dirac case, this assumption is equivalent to the existence of the propagation function $G(X, X')$ in the space of solutions, which satisfies the boundary condition

$$G(X, X')|_{t=t'} = \delta(\mathbf{r} - \mathbf{r}'), \tag{3.51}$$

and has the following form

$$G(X, X') = \sum_{i=1}^{5} G_i(X, X'),$$

$$G_i(X, X') = \sum_{n_i} \mathcal{M}_{n_i}^{-1} \psi_{n_i}(X) \psi_{n_i}^\dagger(X'), \quad i = 2, 4;$$

$$G_i(X, X') = \sum_{n_i} \mathcal{M}_{n_i}^{-1} [{}_+\psi_{n_i}(X){}_+\psi_{n_i}^\dagger(X') + {}^-\psi_{n_i}(X){}^-\psi_{n_i}^\dagger(X')]$$

$$= \sum_{n_i} \mathcal{M}_{n_i}^{-1} [{}_-\psi_{n_i}(X){}_-\psi_{n_i}^\dagger(X') + {}^+\psi_{n_i}(X){}^+\psi_{n_i}^\dagger(X')], \quad i = 1, 5;$$

$$G_3(X, X') = \sum_{n_3} \mathcal{M}_{n_3}^{-1} [{}_+\psi_{n_3}(X){}_+\psi_{n_3}^\dagger(X') + {}^+\psi_{n_3}(X){}^+\psi_{n_3}^\dagger(X')]$$

$$= \sum_{n_3} \mathcal{M}_{n_3}^{-1} [{}_-\psi_{n_3}(X){}_-\psi_{n_3}^\dagger(X') + {}^-\psi_{n_3}(X){}^-\psi_{n_3}^\dagger(X')]. \tag{3.52}$$

The propagation function $G(X, X')$ in the space of scalar fields satisfies the boundary condition

$$G(X, X')|_{t=t'} = 0, \quad [i\partial_t - U(x)]G(X, X')|_{t=t'} = \delta(\mathbf{r} - \mathbf{r}'), \tag{3.53}$$

and can be presented as

$$G(X, X') = \sum_{i=1}^{3} G_i(X, X') - \sum_{i=4}^{5} G_i(X, X'), \tag{3.54}$$

where $G_i(X, X')$, $i = 1, 2, 4, 5$, have the form given by Eq. (3.52) with clear understanding that $\psi_{n_i}(X)$ are scalar fields. The only $G_3(X, X')$ component has distinct form

$$G_3(X, X') = \sum_{n_3} \mathcal{M}_{n_3}^{-1} [{}_+\psi_{n_3}(X){}_+\psi_{n_3}^*(X')$$

$$+ {}^+\psi_{n_3}(X){}^+\psi_{n_3}^*(X')]$$

$$= \sum_{n_3} \mathcal{M}_{n_3}^{-1} [{}_-\psi_{n_3}(X){}_-\psi_{n_3}^*(X')$$

$$+ {}^-\psi_{n_3}(X){}^-\psi_{n_3}^*(X')]. \tag{3.55}$$

It should be stressed that there are two equivalent representations for each propagation function $G_i(X, X')$, $i = 1, 3, 5$. In addition, our further construction is based on the assumption that Dirac spinors (a) and (b) are divided in the in-and out-solutions as follows:

in-solutions: $_+\psi_{n_1}, {}^-\psi_{n_1}; \quad _-\psi_{n_5}, {}^+\psi_{n_5}; \quad _-\psi_{n_3}, {}^-\psi_{n_3},$

out-solutions: $_-\psi_{n_1}, {}^+\psi_{n_1}; \quad _+\psi_{n_5}, {}^-\psi_{n_5}; \quad _+\psi_{n_3}, {}^+\psi_{n_3},$

$$\text{(3.56)}$$

while for scalar fields, the classification in the range Ω_3 differs from the one given by Eq. (3.56) due to positions of the left superscripts and subscripts $\pm$,

in-solutions: $_+\psi_{n_1}, {}^-\psi_{n_1}; \quad _-\psi_{n_5}, {}^+\psi_{n_5}; \quad _+\psi_{n_3}, {}^+\psi_{n_3},$

out-solutions: $_-\psi_{n_1}, {}^+\psi_{n_1}; \quad _+\psi_{n_5}, {}^-\psi_{n_5}; \quad _-\psi_{n_3}, {}^-\psi_{n_3}.$

$$\text{(3.57)}$$

There exist difficulties in interpreting the states $_\zeta\psi_{n_3}(X)$ and ${}^\zeta\psi_{n_3}(X)$ in the framework of the one-particle theory. There existed different point of view on such an interpretation, see [2,17] and [14]. A consistent interpretation can be obtained in the framework of QED and is presented in Sec. VII.

IV. QUANTIZATION IN TERMS OF PARTICLES

In this and in the following sections, we treat the Dirac and the scalar fields in terms of adequate in- and out-particles. On the base of results of Sec. III, we decompose quantum field operators in complete sets of the corresponding solutions introducing some annihilation and creation operators. Then using the well-known equal time (anti) commutation relations for the quantum fields we establish (anti)commutation relations for the introduced operators. The problem of an identification of these operators as in and out annihilation and creation operators is considered in Secs. V–VII.

The Dirac Heisenberg operator $\hat{\Psi}(X)$ is assigned to the Dirac field $\psi(X)$. This operator satisfies the Dirac equation (2.8) and the anticommutation relations

$$[\hat{\Psi}(X), \hat{\Psi}(X')]_+|_{t=t'} = 0,$$
$$[\hat{\Psi}(X), \hat{\Psi}^\dagger(X')]_+|_{t=t'} = \delta(\mathbf{r} - \mathbf{r}'), \quad \text{(4.1)}$$

e.g., see [44,46].

The Klein-Gordon Heisenberg operator $\hat{\Phi}(X)$ is assigned to the scalar field $\psi(X)$. In terms of the canonical pair

$$\hat{\Phi}(X) = \begin{pmatrix} i\hat{\Pi}^\dagger(X) \\ \hat{\Psi}(X) \end{pmatrix},$$

it satisfies the Klein-Gordon equation,

$$[i\partial_t - U(x)]\hat{\Phi}(X) = \hat{H}^{\text{kin}}\hat{\Phi}(X),$$
$$\hat{H}^{\text{kin}} = \begin{pmatrix} 0 & -(\partial_j)^2 + m^2 \\ 1 & 0 \end{pmatrix}, \quad \text{(4.2)}$$

and the commutation relations

$$[\hat{\Phi}(X), \hat{\Phi}(X')]_-|_{t=t'} = 0,$$
$$[\hat{\Phi}(X), \hat{\Phi}^\dagger(X')]_-|_{t=t'} = \delta(\mathbf{r} - \mathbf{r}')\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{(4.3)}$$

e.g., see [44,46]. Here $\hat{H}^{\text{kin}}$ is the one-particle kinetic energy operator $\hat{H}^{\text{kin}}$. It follows from Eq. (4.2) that

$$i\hat{\Pi}^\dagger(X) = [i\partial_t - U(x)]\hat{\Psi}(X).$$

The further quantization of the scalar field can be done in the same manner as the quantization for Dirac field. That is why, in what follows up to Sec. VIII, we consider only the quantization of Dirac field in detail, adding some comments about the scalar field when necessary, see Appendix A for some peculiarities of the quantization in the range Ω_3.

A. Introducing creation and annihilation operators

We consider the most interesting case $\mathbb{U} > 2m$ of the critical steps. In this case we decompose the Heisenberg operator of Dirac field $\hat{\Psi}(X)$ in two sets of solutions $\{_\zeta\psi_n(X)\}$ and $\{{}^\zeta\psi_n(X)\}$ of the Dirac equation (2.8) complete on the hyperplane $t = \text{const}$. Operator-valued coefficients in such decompositions do not depend on coordinates because both $\hat{\Psi}(X)$ and the complete sets satisfy the same Dirac equation (2.8). Our division of the quantum numbers n in five ranges, implies the representation for $\hat{\Psi}(X)$ as a sum of five operators $\hat{\Psi}_i(X)$, $i = 1, 2, 3, 4, 5$,

$$\hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X). \quad \text{(4.4)}$$

For each of three operators $\hat{\Psi}_i(X), i = 1, 3, 5$, there exist two possible decompositions according to the existence of two different complete sets of solutions with the same quantum numbers n in the ranges Ω_1, Ω_3, and Ω_5, see completeness relation (3.52). Thus, we have:

$$\hat{\Psi}_1(X) = \sum_{n_1} \mathcal{M}_{n_1}^{-1/2} [_+a_{n_1}(\text{in})_+\psi_{n_1}(X) + {}^-a_{n_1}(\text{in})^-\psi_{n_1}(X)]$$

$$= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2} [^+a_{n_1}(\text{out})^+\psi_{n_1}(X) + {}_-a_{n_1}(\text{out})_-\psi_{n_1}(X)],$$

$$\hat{\Psi}_3(X) = \sum_{n_3} \mathcal{M}_{n_3}^{-1/2} [^-a_{n_3}(\text{in})^-\psi_{n_3}(X) + {}_-b_{n_3}^\dagger(\text{in})_-\psi_{n_3}(X)]$$

$$= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2} [^+a_{n_3}(\text{out})^+\psi_{n_3}(X) + {}_+b_{n_3}^\dagger(\text{out})_+\psi_{n_3}(X)],$$

$$\hat{\Psi}_5(X) = \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} [^+b_{n_5}^\dagger(\text{in})^+\psi_{n_5}(X) + {}_-b_{n_5}^\dagger(\text{in})_-\psi_{n_5}(X)]$$

$$= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} [_+b_{n_5}^\dagger(\text{out})_+\psi_{n_5}(X) + {}^-b_{n_5}^\dagger(\text{out})^-\psi_{n_5}(X)]. \tag{4.5}$$

There may exist only one complete set of solutions with the same quantum numbers n_2 and n_4. Therefore, we have only one possible decomposition for each of the two operators $\hat{\Psi}_i(X)$, $i = 2, 4$,

$$\hat{\Psi}_2(X) = \sum_{n_2} \mathcal{M}_{n_2}^{-1/2} a_{n_2} \psi_{n_2}(X),$$

$$\hat{\Psi}_4(X) = \sum_{n_4} \mathcal{M}_{n_4}^{-1/2} b_{n_4}^\dagger \psi_{n_4}(X). \tag{4.6}$$

We interpret all a and b as annihilation and all $a^\dagger$ and $b^\dagger$ as creation operators; all a and $a^\dagger$ are interpreted as describing electrons and all b and $b^\dagger$ as describing positrons; all the operators labeled by the argument in are interpreted as in-operators, whereas all the operators labeled by the argument out as out-operators.

In this connection, we reiterate that the time-independence of the external field under consideration is an idealization. In fact, it is supposed that the external field was switched on at a time instant t_{in}, then it was acting as a constant field during a large time T, and finally it was switched off at a time instant $t_{\text{out}} = t_{\text{in}} + T$, and that one can ignore effects of its switching on and off. Then we suppose that in the Schrödinger picture one can introduce creation and annihilation operators of particles at the initial and final time instants. In the Heisenberg representation, these operators when being developed to zero time instant are called in-operators and out-operators. It is the well-known procedure in QFT with t-electric potential steps. Technical realization of this construction was presented in Refs. [5]. In QED with constant fields, in particular, with the x-electric potential steps, we quantize directly in the Heisenberg representation. And here we encounter the problem of identification of in-operators and out-operators. Its final solution is presented in Secs. V, VI, and VII.

Taking into account the orthonormalization relations (3.47) and the completeness relations (3.52), we find that the anticommutation relations (4.1) for the Heisenberg

operator (4.4) yield the following anticommutation rules for the introduced creation and annihilation in- or out-operators (here we do not discuss commutation relations between sets of in and out-operators):

All creation (annihilation) operators with different quantum numbers n anticommute between themselves; all the operators from different ranges Ω_i anticommute between themselves; all anticommutators in each range Ω_i, $i = 1, 2, 3, 4, 5$ have the form:

$$[_+a_{n_1}(\text{in}), {}_+a_{n'_1}^\dagger(\text{in})]_+ = [^-a_{n_1}(\text{in}), {}^-a_{n'_1}^\dagger(\text{in})]_+$$

$$= [^+a_{n_1}(\text{out}), {}^+a_{n'_1}^\dagger(\text{out})]_+$$

$$= [_-a_{n_1}(\text{out}), {}_-a_{n'_1}^\dagger(\text{out})]_+ = \delta_{n_1,n'_1};$$

$$[a_{n_2}, a_{n'_2}^\dagger]_+ = \delta_{n_2,n'_2}; \qquad [b_{n_4}, b_{n'_4}^\dagger]_+ = \delta_{n_4,n'_4};$$

$$[^-a_{n_3}(\text{in}), {}^-a_{n'_3}^\dagger(\text{in})]_+ = [^+a_{n_3}(\text{out}), {}^+a_{n'_3}^\dagger(\text{out})]_+$$

$$= [_-b_{n_3}(\text{in}), {}_-b_{n'_3}^\dagger(\text{in})]_+$$

$$= [_+b_{n_3}(\text{out}), {}_+b_{n'_3}^\dagger(\text{out})]_+ = \delta_{n_3,n'_3};$$

$$[^+b_{n_5}(\text{in}), {}^+b_{n'_5}^\dagger(\text{in})]_+ = [_-b_{n_5}(\text{in}), {}_-b_{n'_5}^\dagger(\text{in})]_+$$

$$= [_+b_{n_5}(\text{out}), {}_+b_{n'_5}^\dagger(\text{out})]_+$$

$$= [^-b_{n_5}(\text{out}), {}^-b_{n'_5}^\dagger(\text{out})]_+ = \delta_{n_5,n'_5}. \tag{4.7}$$

Some preliminary remarks about the division of creation and annihilation operators into in- and out-type are in order. Usually, when quantizing a field theory with a time-dependent external background, we work in the Schrödinger picture, where we have to define initial and final asymptotic states. Even if the external field is switched off at the time infinity, its potentials may be different from zero there. Thus, the Schrödinger initial and final asymptotic states are different as it occurs in QED with t-electric

potential steps. Then the Schrödinger initial and final asymptotic states give rise to in- and out-states (the corresponding operators) in the Heisenberg picture. In the case under consideration, where we formally deal with time-independent backgrounds, and quantize directly in the Heisenberg picture, there appears the problem of identifying in- and out-operators. In the case of x-electric potential steps, we will be guided by the following physical considerations: All the in-particles (created by the in-creation operators from the vacuum) are moving from the asymptotic region S_L or S_R to the step, whereas all the out-particles (created by the out-creation operators from the vacuum) are moving from the step to the asymptotic region S_L or S_R.

Below, after analyzing properties of one-particle states created by the introduced operators, we confirm the consistence of their division into in- and out-types.

B. Physical quantities

1. Classical physical quantities

Energy $\mathcal{H}$ of the classical Dirac field has the form

$$\mathcal{H} = \int \psi^\dagger(X)\hat{H}\psi(X)d\mathbf{r}, \qquad (4.8)$$

where one-particle Hamiltonian $\hat{H}$ is given by Eq. (2.8). The energy (4.8) is a gauge dependent quantity. Substituting the one-particle kinetic energy operator $\hat{H}^{kin}$, given by Eq. (3.13), for $\hat{H}$ in the right-hand side (RHS) of Eq. (4.8) we obtain a gauge invariant quantity $\mathcal{H}^{kin}$, which we call the kinetic energy of the classical Dirac field $\psi(X)$,

$$\mathcal{H}^{kin} = \int \psi^\dagger(X)\hat{H}^{kin}\psi(X)d\mathbf{r},$$
$$\hat{H}^{kin} = \hat{H} - U(x). \qquad (4.9)$$

Decomposing the field $\psi(X)$ over the complete set $\psi_n(X)$, and dividing integral (4.9) in three integrals within the regions S_L, S_{int}, and S_R, as was done for the quantity $\mathcal{R}$ in Eq. (B2), we reduce calculating the quantity (4.9) to calculating the following matrix elements

$$\mathcal{H}_{nl}^{kin} = \mathcal{H}_{nl}^L + \mathcal{H}_{nl}^{int} + \mathcal{H}_{nl}^R,$$
$$\mathcal{H}_{nl}^L = \int_{-K^{(L)}}^{x_L} h_{nl}dx, \qquad \mathcal{H}_{nl}^{int} = \int_{x_L}^{x_R} h_{nl}dx,$$
$$\mathcal{H}_{nl}^R = \int_{x_R}^{K^{(R)}} h_{nl}dx,$$
$$h_{nl} = \int \psi_n^\dagger(X)\hat{H}^{kin}\psi_l(X)d\mathbf{r}_\perp. \qquad (4.10)$$

The matrix elements $\mathcal{H}_{nl}^{int}$ are finite for any n and l, and the terms $\mathcal{H}_{nl}^L$ and $\mathcal{H}_{nl}^R$ dominate in the limit $K^{(L/R)} \to \infty$.

In the asymptotic regions S_L and S_R solutions $\psi_n(X)$ with any n are eigenfunctions of the operator $\hat{H}^{kin}$ with the eigenvalues $\pi_0(L)$ and $\pi_0(R)$, respectively. That is why only diagonal matrix elements $\mathcal{H}_{nl}^L$ and $\mathcal{H}_{nl}^R$ differ from zero in the limit $K^{(L/R)} \to \infty$. Thus, the stationary states introduced in Sec. III A diagonalize the introduced kinetic energy (4.9). This is an important necessary condition in the quantization procedure which provides in what follows an interpretation in terms of particles.

The kinetic energy of a stationary state $\psi_n(X)$ reads

$$\mathcal{E}_n = \mathcal{M}_n^{-1} \int \psi_n^\dagger(X)[p_0 - U(x)]\psi_n(X)d\mathbf{r}. \qquad (4.11)$$

The kinetic energy of a wave packet $\psi(X)$ composed of stationary states $\psi_n(X)$ is a sum of the partial energies (4.11).

One can easily see that stationary states with quantum numbers n from the regions Ω_2 and Ω_4 have the following kinetic energies

$$\mathcal{E}_{n_2} = \pi_0(L), \qquad \mathcal{E}_{n_4} = \pi_0(R). \qquad (4.12)$$

Let us consider stationary states $\{_\zeta\psi_n(X)\}$ and $\{^\zeta\psi_n(X)\}$ with quantum numbers n from the ranges Ω_1, Ω_3, and Ω_5. Their kinetic energies are denoted as $_\zeta\mathcal{E}_n$ and $^\zeta\mathcal{E}_n$, respectively. In the same manner, which was used in finding the orthonormality relations of the corresponding solutions, and retaining only the leading terms in the limit $K^{(L/R)} \to \infty$, we obtain

$$_\zeta\mathcal{E}_n = V_\perp\mathcal{M}_n^{-1}(_\zeta E_n^L + _\zeta E_n^R), \qquad ^\zeta\mathcal{E}_n = V_\perp\mathcal{M}_n^{-1}(^\zeta E_n^L + ^\zeta E_n^R),$$
$$_\zeta E_n^L = \pi_0(L)_\zeta\mathcal{R}_L, \qquad _\zeta E_n^R = \pi_0(R)_\zeta\mathcal{R}_R,$$
$$^\zeta E_n^L = \pi_0(L)^\zeta\mathcal{R}_L, \qquad ^\zeta E_n^R = \pi_0(R)^\zeta\mathcal{R}_R, \qquad (4.13)$$

where $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, and the quantities $^\zeta\mathcal{R}_{L/R}$ and $_\zeta\mathcal{R}_{L/R}$ are given by Eqs. (B5) and (B7); in fact, they depend also on the index n. Then, using Eqs. (B5), (B7), (3.40), and Eq. (B15), we find

$$_\zeta\mathcal{E}_n = \pi_0(R) + \frac{\mathbb{U}}{2}|g(_+|^+)|^{-2},$$
$$^\zeta\mathcal{E}_n = \pi_0(L) - \frac{\mathbb{U}}{2}|g(_+|^+)|^{-2}, \quad n \in \Omega_1 \cup \Omega_5, \quad (4.14)$$

$$_\zeta\mathcal{E}_n = \pi_0(R) + \frac{\mathbb{U}}{2}|g(_+|^-)|^{-2},$$
$$^\zeta\mathcal{E}_n = \pi_0(L) - \frac{\mathbb{U}}{2}|g(_+|^-)|^{-2}, \qquad n \in \Omega_3. \qquad (4.15)$$

The energies $\mathcal{E}_{n_i}, i = 1, 2$ are positive, and the corresponding solutions in Eq. (4.11) are electron states, whereas all the energies $\mathcal{E}_{n_i}, i = 4, 5$ are negative, and

the corresponding solutions in Eq. (4.11) are positron states,

$$\mathcal{E}_n > 0, \quad \forall\, n \in \Omega_1 \cup \Omega_2; \qquad \mathcal{E}_n < 0, \quad \forall\, n \in \Omega_4 \cup \Omega_5. \tag{4.16}$$

To estimate signs of energies (4.15), we note that they contain two terms with opposite signs. It follows from Eq. (3.42) that $|g(_+|^-)|^{-2} \le 1$. Then Eqs. (4.15) imply

$$^{\zeta}\mathcal{E}_{n_3} - {}_{\zeta}\mathcal{E}_{n_3} = \mathbb{U}(1 - |g(_+|^-)|^{-2}) \ge 0. \tag{4.17}$$

It has to be stressed that the latter condition is enough to provide the existing of a vacuum in the further quantum field theory.

Note that energies $\mathcal{E}_{n_i}$, $i = 1, 2, 4, 5$ for the stationary states of bosons have the same form as for fermions in (4.12) and (4.14). However, the corresponding relations for bosons

$$^{\zeta}\mathcal{E}_n = \pi_0(\mathrm{R}) - \frac{\mathbb{U}}{2}|g(_+|^-)|^{-2},$$

$$_{\zeta}\mathcal{E}_n = \pi_0(\mathrm{L}) + \frac{\mathbb{U}}{2}|g(_+|^-)|^{-2}, \quad n \in \Omega_3. \tag{4.18}$$

differ from their fermionic versions (4.15). Equations (4.18) imply the inequalities

$$^{\zeta}\mathcal{E}_{n_3} > 0, \qquad {}_{\zeta}\mathcal{E}_{n_3} < 0, \tag{4.19}$$

which provide the positiveness of any boson excitations over the vacuum.

2. Quantum physical quantities

After the second quantization, physical quantities of the classical Dirac field turn out to be operators. In what follows, we are going to consider some of them. The first one is kinetic energy operator $\hat{\mathbb{H}}^{\mathrm{kin}}$, which, just from the beginning, we write in a renormalized form,

$$\hat{\mathbb{H}}^{\mathrm{kin}} = \int \hat{\Psi}^{\dagger}(X)\hat{H}^{\mathrm{kin}}\hat{\Psi}(X)d\mathbf{r} - \mathbb{H}_0, \tag{4.20}$$

where the constant (in general, infinite) term $\mathbb{H}_0$ corresponds to the energy of vacuum fluctuations. Its explicit expression will be given below.

Inserting decompositions (4.4), (4.5), and (4.6) in the right-hand side of Eq. (4.20), we obtain a representation of the kinetic energy in terms of the introduced creation and annihilation operators,

$$\hat{\mathbb{H}}^{\mathrm{kin}} = \sum_{i=1}^{5}\sum_{n_i} \hat{\mathbb{H}}_{n_i},$$

$$\mathbb{H}_0 = \sum_{n_3}{}_{+}\mathcal{E}_{n_3} + \sum_{n_4}\pi_0(\mathrm{R}) + \sum_{n_5}({}_{+}\mathcal{E}_{n_5} + {}^{-}\mathcal{E}_{n_5}),$$

$$\hat{\mathbb{H}}_{n_1} = {}_{+}\mathcal{E}_{n_1}{}_{+}a_{n_1}^{\dagger}(\mathrm{in})_{+}a_{n_1}(\mathrm{in}) + {}^{-}\mathcal{E}_{n_1}{}^{-}a_{n_1}^{\dagger}(\mathrm{in})^{-}a_{n_1}(\mathrm{in})$$
$$= {}_{-}\mathcal{E}_{n_1}{}_{-}a_{n_1}^{\dagger}(\mathrm{out})_{-}a_{n_1}(\mathrm{out}) + {}^{+}\mathcal{E}_{n_1}{}^{+}a_{n_1}^{\dagger}(\mathrm{out})^{+}a_{n_1}(\mathrm{out}),$$

$$\hat{\mathbb{H}}_{n_2} = \pi_0(\mathrm{L})a_{n_2}^{\dagger}a_{n_2}, \qquad \hat{\mathbb{H}}_{n_4} = -\pi_0(\mathrm{R})b_{n_4}^{\dagger}b_{n_4},$$

$$\hat{\mathbb{H}}_{n_3} = {}^{+}\mathcal{E}_{n_3}{}^{+}a_{n_3}^{\dagger}(\mathrm{out})^{+}a_{n_3}(\mathrm{out}) - {}_{+}\mathcal{E}_{n_3}{}_{+}b_{n_3}^{\dagger}(\mathrm{out})_{+}b_{n_3}(\mathrm{out})$$
$$= {}^{-}\mathcal{E}_{n_3}{}^{-}a_{n_3}^{\dagger}(\mathrm{in})^{-}a_{n_3}(\mathrm{in}) - {}_{-}\mathcal{E}_{n_3}{}_{-}b_{n_3}^{\dagger}(\mathrm{in})_{-}b_{n_3}(\mathrm{in}),$$

$$\hat{\mathbb{H}}_{n_5} = -{}_{+}\mathcal{E}_{n_5}{}_{+}b_{n_5}^{\dagger}(\mathrm{out})_{+}b_{n_5}(\mathrm{out}) - {}^{-}\mathcal{E}_{n_5}{}^{-}b_{n_5}^{\dagger}(\mathrm{out})^{-}b_{n_5}(\mathrm{out})$$
$$= -{}_{-}\mathcal{E}_{n_5}{}_{-}b_{n_5}^{\dagger}(\mathrm{in})_{-}b_{n_5}(\mathrm{in}) - {}^{+}\mathcal{E}_{n_5}{}^{+}b_{n_5}^{\dagger}(\mathrm{in})^{+}b_{n_5}(\mathrm{in}). \tag{4.21}$$

We stress that the existence of two different representations for physical observables in the ranges Ω_1, Ω_3, and Ω_5 corresponds to the existence of two different complete sets of solutions in these ranges. According to Eqs. (4.14) and (4.15), we have

$$_{+}\mathcal{E}_{n_3} = {}_{-}\mathcal{E}_{n_3}, \qquad {}_{-}\mathcal{E}_{n_5} + {}^{+}\mathcal{E}_{n_5} = {}_{+}\mathcal{E}_{n_5} + {}^{-}\mathcal{E}_{n_5},$$

that is why the constant H_0 has the same value for any possible choice of $\hat{\mathbb{H}}_{n_i}$, $i = 1, 3, 5$ in representation (4.21).

The formal expression of the charge operator $\hat{Q}$ is

$$\hat{Q} = \frac{q}{2}\int [\hat{\Psi}^{\dagger}(X), \hat{\Psi}(X)]_{-}d\mathbf{r}. \tag{4.22}$$

Its decomposition in the creation and annihilation operators introduced reads

$$\hat{Q} = \sum_{i=1}^{5}\sum_{n_i} \hat{Q}_{n_i},$$

$$\hat{Q}_{n_1} = -e[{}_{+}a_{n_1}^{\dagger}(\mathrm{in})_{+}a_{n_1}(\mathrm{in}) + {}^{-}a_{n_1}^{\dagger}(\mathrm{in})^{-}a_{n_1}(\mathrm{in})]$$
$$= -e[{}_{-}a_{n_1}^{\dagger}(\mathrm{out})_{-}a_{n_1}(\mathrm{out}) + {}^{+}a_{n_1}^{\dagger}(\mathrm{out})^{+}a_{n_1}(\mathrm{out})],$$

$$\hat{Q}_{n_2} = -ea_{n_2}^{\dagger}a_{n_2}, \qquad \hat{Q}_{n_4} = eb_{n_4}^{\dagger}b_{n_4},$$

$$\hat{Q}_{n_3} = -e[{}^{+}a_{n_3}^{\dagger}(\mathrm{out})^{+}a_{n_3}(\mathrm{out}) - {}_{+}b_{n_3}^{\dagger}(\mathrm{out})_{+}b_{n_3}(\mathrm{out})]$$
$$= -e[{}^{-}a_{n_3}^{\dagger}(\mathrm{in})^{-}a_{n_3}(\mathrm{in}) - {}_{-}b_{n_3}^{\dagger}(\mathrm{in})_{-}b_{n_3}(\mathrm{in})],$$

$$\hat{Q}_{n_5} = e[{}_{+}b_{n_5}^{\dagger}(\mathrm{out})_{+}b_{n_5}(\mathrm{out}) + {}^{-}b_{n_5}^{\dagger}(\mathrm{out})^{-}b_{n_5}(\mathrm{out})]$$
$$= e[{}_{-}b_{n_5}^{\dagger}(\mathrm{in})_{-}b_{n_5}(\mathrm{in}) + {}^{+}b_{n_5}^{\dagger}(\mathrm{in})^{+}b_{n_5}(\mathrm{in})]. \tag{4.23}$$

We also will consider the energy flux of the Dirac field through the surface $x = \mathrm{const}$. Its QFT operator has the form

$$\hat{F}(x) = \frac{1}{T} \int \hat{T}^{10} dt d\mathbf{r}_\perp, \qquad (4.24)$$

where $\hat{T}^{10}$ is the component of the operator of the energy momentum tensor $\hat{T}^{\mu\nu}$, and the integral over $d\mathbf{r}_\perp$ is defined in Eq. (3.33). For our purposes, it is enough to work with the canonical energy momentum tensor

$$T_{\mu\nu} = \frac{1}{2} \{ \bar{\psi}(x) \gamma_\mu P_\nu \psi(x) + [P_\nu^* \bar{\psi}(x)] \gamma_\mu \psi(x) \}.$$

Then the QFT operator $\hat{F}(x)$ reads

$$\hat{F}(x) = \frac{1}{T} \int \hat{\Psi}^\dagger(X) \gamma^0 \gamma^1 \hat{H}^{\mathrm{kin}} \hat{\Psi}(X) dt d\mathbf{r}_\perp. \qquad (4.25)$$

Another physical quantity useful for the further analysis is the electric current of the Dirac field through the surface $x = \mathrm{const}$. The corresponding QFT operator has the form

$$\hat{J} = -\frac{e}{T} \int \hat{\Psi}^\dagger(X) \gamma^0 \gamma^1 \hat{\Psi}(X) dt d\mathbf{r}_\perp. \qquad (4.26)$$

Inserting decompositions (4.4), (4.5), and (4.6) in the right-hand part of the quantities (4.25) and (4.26), we can obtain their representations in terms of the introduced creation and annihilation operators. Then we can see that the right-hand part of (4.25) is diagonal with respect to the quantum numbers n for $x \in S_L$ and for $x \in S_R$, whereas the right-hand part of (4.26) does not depend on x and is diagonal with respect to the quantum numbers n for any x.

C. Partial and total vacuum states

Let us define two vacuum vectors $|0, \mathrm{in}\rangle$ and $|0, \mathrm{out}\rangle$, one of which is the zero-vector for all in-annihilation operators and the other is zero-vector for all out-annihilation operators. Besides, the both vacua are zero-vectors for the annihilation operators a_{n_2} and b_{n_4}, which is consistent with the anticommutation relations (4.7). Thus, we have

$$_+a_{n_1}(\mathrm{in})|0, \mathrm{in}\rangle = {}^-a_{n_1}(\mathrm{in})|0, \mathrm{in}\rangle = 0,$$
$$_-b_{n_5}(\mathrm{in})|0, \mathrm{in}\rangle = {}^+b_{n_5}(\mathrm{in})|0, \mathrm{in}\rangle = 0,$$
$$^-a_{n_3}(\mathrm{in})|0, \mathrm{in}\rangle = {}_-b_{n_3}(\mathrm{in})|0, \mathrm{in}\rangle = 0,$$
$$a_{n_2}|0, \mathrm{in}\rangle = b_{n_4}|0, \mathrm{in}\rangle = 0, \qquad (4.27)$$

and

$$_-a_{n_1}(\mathrm{out})|0, \mathrm{out}\rangle = {}^+a_{n_1}(\mathrm{out})|0, \mathrm{out}\rangle = 0,$$
$$_+b_{n_5}(\mathrm{out})|0, \mathrm{out}\rangle = {}^-b_{n_5}(\mathrm{out})|0, \mathrm{out}\rangle = 0,$$
$$_+b_{n_3}(\mathrm{out})|0, \mathrm{out}\rangle = {}^+a_{n_3}(\mathrm{out}|0, \mathrm{out}\rangle = 0,$$
$$a_{n_2}|0, \mathrm{out}\rangle = b_{n_4}|0, \mathrm{out}\rangle = 0. \qquad (4.28)$$

Then we postulate that the state space of the system under consideration is the Fock space constructed, say, with the help of the vacuum $|0, \mathrm{in}\rangle$ and the corresponding creation operators. One can verify (see Sec. V) that this Fock space is unitary equivalent to the other Fock space constructed with the help of the vacuum $|0, \mathrm{out}\rangle$ and the corresponding creation operators.

In this case, one can see that vacuum mean values of the operator $\hat{H}^{\mathrm{kin}}$ (4.21) and of the charge operator $\hat{Q}$ are zero in the both Fock spaces,

$$\langle 0, \mathrm{in}|\hat{H}^{\mathrm{kin}}|0, \mathrm{in}\rangle = \langle 0, \mathrm{out}|\hat{H}^{\mathrm{kin}}|0, \mathrm{out}\rangle$$
$$= \langle 0, \mathrm{in}|\hat{Q}|0, \mathrm{in}\rangle = \langle 0, \mathrm{out}|\hat{Q}|0, \mathrm{out}\rangle = 0.$$

Thus, according to Eqs. (4.14), (4.16), and (4.17), these are uncharged states with minimal kinetic energy and the operator $\hat{H}^{\mathrm{kin}}$ is positively-defined in the ranges Ω_i, $i = 1, 2, 4, 5$. One can verify that the introduced vacua have minimum energy with respect to any uncharged quasistationary states in the range Ω_3. Indeed, in our construction it is assumed that electrons and positrons with quantum numbers in this range being in one of corresponding asymptotic regions occupy quasistationary states, i.e., they are described by wave packets that maintain their forms sufficiently long in these regions. Only such electron and positron wave packets have a physical meaning. As we demonstrate in Secs. VII C and D in the range Ω_3, any electron wave packets that really have a physical meaning can be localized only in the asymptotic region S_L (as in the range Ω_2), whereas any positron wave packets that really have a physical meaning can be localized only in the asymptotic regions S_R (as in the range Ω_4). Kinetic energies of these wave packets are formed for electrons by contributions of $\pi_0(\mathrm{L})$, for positrons by contribution $|\pi_0(\mathrm{R})|$ and are, therefore, always positive.

Because any annihilation operators with quantum numbers n_i corresponding to different i anticommute between themselves, we can represent the introduced vacua as tensor products of the corresponding vacua in the five ranges,

$$|0, \mathrm{in}\rangle = \prod_{i=1}^5 {}^\otimes |0, \mathrm{in}\rangle^{(i)},$$

$$|0, \mathrm{out}\rangle = \prod_{i=1}^5 {}^\otimes |0, \mathrm{out}\rangle^{(i)}, \qquad (4.29)$$

where the partial vacua $|0,\text{in}\rangle^{(i)}$ and $|0,\text{out}\rangle^{(i)}$ obey relations (4.27) and (4.28) for any n_i and ζ.

It follows from relations (4.27) and (4.28) that

$$|0,\text{in}\rangle^{(i)} = |0,\text{out}\rangle^{(i)}, \quad i = 2,4. \qquad (4.30)$$

Let us rewrite relations (3.38) for solutions with quantum numbers n_1 as follows

$$\begin{aligned}
_{+}\psi_n(X) &= g(_{+}|^{+})^{-1}[\eta_{\text{L}}{}^{+}\psi_n(X) + {}_{-}\psi_n(X)g(_{-}|^{+})], \\
_{-}\psi_n(X) &= g(_{-}|^{-})^{-1}[-\eta_{\text{L}}{}^{-}\psi_n(X) + {}_{+}\psi_n(X)g(_{+}|^{-})]; \\
{}^{+}\psi_n(X) &= g(^{+}|_{+})^{-1}[\eta_{\text{R}+}\psi_n(X) + {}^{-}\psi_n(X)g(^{-}|_{+})], \\
{}^{-}\psi_n(X) &= g(^{-}|_{-})^{-1}[-\eta_{\text{R}-}\psi_n(X) + {}^{+}\psi_n(X)g(^{+}|_{-})], \quad (4.31)
\end{aligned}$$

where pairs of solutions in the RHS of Eqs. (4.31) are orthogonal due to relations (3.47) at any fixed t. We recall that the relation

$$|g(_{+}|^{+})|^2 = |g(_{+}|^{-})|^2 + 1 \qquad (4.32)$$

holds as a consequence of Eqs. (3.42). Using these relations and two possible representations (4.5) for the operator $\hat{\Psi}_1(X)$, one can find direct and inverse canonical transformations between the initial and final pairs of annihilation operators of electrons:

$$\begin{aligned}
{}^{+}a_n(\text{out}) &= \eta_{\text{R}}g(_{+}|^{+})^{-1}{}_{+}a_n(\text{in}) + g(^{-}|_{-})^{-1}g(^{+}|_{-})^{-}a_n(\text{in}), \\
{}_{-}a_n(\text{out}) &= g(_{+}|^{+})^{-1}g(_{-}|^{+})_{+}a_n(\text{in}) - \eta_{\text{L}}g(^{-}|_{-})^{-1}{}^{-}a_n(\text{in}); \\
{}_{+}a_n(\text{in}) &= g(_{-}|^{-})^{-1}g(_{+}|^{-})_{-}a_n(\text{out}) + \eta_{\text{L}}g(^{+}|_{+})^{-1+}a_n(\text{out}), \\
{}^{-}a_n(\text{in}) &= -\eta_{\text{R}}g(_{-}|^{-})^{-1}{}_{-}a_n(\text{out}) \\
&\quad + g(^{+}|_{+})^{-1}g(^{-}|_{+})^{+}a_n(\text{out}). \qquad (4.33)
\end{aligned}$$

Canonical transformations between the initial and final pairs of creation operators of electrons can be derived from relations (4.33).

In the same manner, using relations (3.38) for solutions with quantum numbers $n \in \Omega_5$, and two possible representations for the operator $\hat{\Psi}_5(X)$ given by (4.5), one can find canonical transformations between the initial and final pairs of positron creation operators. They can be obtained from relations (4.33) by the substitution $_{+}a_{n_1}(\text{in}) \to {}_{+}b^{\dagger}_{n_5}(\text{out})$, $^{-}a_{n_1}(\text{in}) \to {}^{-}b^{\dagger}_{n_5}(\text{out})$, $^{+}a_{n_1}(\text{out}) \to {}^{+}b^{\dagger}_{n_5}(\text{in})$, and $_{-}a_{n_1}(\text{out}) \to {}_{-}b^{\dagger}_{n_5}(\text{in})$.

Since the canonical transformations (4.33) do not mix creation and annihilation operators, we can choose

$$|0,\text{in}\rangle^{(i)} = |0,\text{out}\rangle^{(i)}, \quad i = 1,5. \qquad (4.34)$$

Together with their adjoint relations Eqs. (4.33) define an unitary transformation V_{Ω_1} between in- and out-operators in the range Ω_1,

$$\begin{aligned}
\{_{+}a^{\dagger}(\text{in}), {}^{-}a^{\dagger}(\text{in}), {}_{+}a(\text{in}), {}^{-}a(\text{in}), \} \\
= V_{\Omega_1}\{_{+}a^{\dagger}(\text{out}), {}^{-}a^{\dagger}(\text{out}), {}^{+}a(\text{out}), {}_{-}a(\text{out})\}V^{\dagger}_{\Omega_1}.
\end{aligned}$$

The unitary operator V_{Ω_1} has the form

$$\begin{aligned}
V_{\Omega_1} &= \exp\left\{\sum_{n\in\Omega_1}{}^{+}a^{\dagger}_n(\text{out})g(_{-}|^{+})^{-1}{}_{-}a_n(\text{out})\right\} \\
&\times \exp\left\{-\sum_{n\in\Omega_1}{}_{-}a^{\dagger}_n(\text{out})\ln[g(^{-}|_{-})g(^{-}|_{+})^{-1}]_{-}a_n(\text{out})\right\} \\
&\times \exp\left\{\sum_{n\in\Omega_1}{}^{+}a^{\dagger}_n(\text{out})\ln[g(^{+}|_{+})g(^{-}|_{+})^{-1}]^{+}a_n(\text{out})\right\} \\
&\times \exp\left\{-\sum_{n\in\Omega_1}{}_{-}a^{\dagger}_n(\text{out})g(^{-}|_{+})^{-1+}a_n(\text{out})\right\}. \qquad (4.35)
\end{aligned}$$

Similar results take place in the range Ω_5:

$$\begin{aligned}
\{_{-}b^{\dagger}(\text{in}), {}^{+}b^{\dagger}(\text{in}), {}_{-}b(\text{in}), {}^{+}b(\text{in})\} \\
= V_{\Omega_5}\{_{-}b^{\dagger}(\text{out}), {}^{+}b^{\dagger}(\text{out}), {}^{-}b(\text{out}), {}_{+}b(\text{out})\}V^{\dagger}_{\Omega_5},
\end{aligned}$$

$$\begin{aligned}
V_{\Omega_5} &= \exp\left\{-\sum_{n\in\Omega_5}{}_{+}b^{\dagger}(\text{out})g(_{+}|^{-})^{-1-}b(\text{out})\right\} \\
&\times \exp\left\{-\sum_{n\in\Omega_5}{}^{-}b^{\dagger}(\text{out})\ln[g(^{-}|_{-})g(^{+}|_{-})^{-1}]^{-}b(\text{out})\right\} \\
&\times \exp\left\{\sum_{n\in\Omega_5}{}_{+}b^{\dagger}(\text{out})\ln[g(^{+}|_{+})g(^{+}|_{-})^{-1}]_{+}b(\text{out})\right\} \\
&\times \exp\left\{\sum_{n\in\Omega_5}{}^{-}b^{\dagger}(\text{out})g(^{+}|_{-})^{-1}{}_{+}b(\text{out})\right\}. \qquad (4.36)
\end{aligned}$$

We recall, that the similar result (4.30) takes place in the regions Ω_i, $i = 2, 4$. Both relations (4.30) and (4.34) mean that the partial vacua $|0,\text{in}\rangle^{(i)}$, $i = 1, 2, 4, 5$ are stable under the action of the external field. In what follows, we denote the tensor product of these partial vacua by $|0\rangle$,

$$|0\rangle = \prod_{i=1,2,4,5}{}^{\otimes}|0,\text{in}\rangle^{(i)} = \prod_{i=1,2,4,5}{}^{\otimes}|0,\text{out}\rangle^{(i)}. \qquad (4.37)$$

Below, we are going to consider one-particle states $a^{\dagger}_n(\text{in/out})|0,\text{in/out}\rangle$ and $b^{\dagger}_n(\text{in/out})|0,\text{in/out}\rangle$ created by the introduced creation operators from the vacuum.

The physical meaning of these states will be discussed separately for each range of the quantum numbers n.

V. ONE-PARTICLE STATES IN THE RANGES Ω_1 AND Ω_5

A. General

We believe that according to the structure of the Dirac energy spectra in the asymptotic regions S_{L} and S_{R}, there

exist only one-electron states in the range Ω_1, whereas in the range Ω_5 there exist only one-positron states.

We recall that condition $(_\zeta\psi_n, {}^{-\zeta}\psi_n) = 0$ (B14) for $n \in \Omega_1 \cup \Omega_5$ means that the sets of solution s $\{_\zeta\psi_n(X)\}$ and $\{^{-\zeta}\psi_n(X)\}$ with opposite ζ represent independent physical states.

We associate the independent pair $\{_+\psi_{n_1}(X), {}^-\psi_{n_1}(X)\}$ with electron in-solutions and the independent pair $\{_-\psi_{n_1}(X)\}$ and $\{^+\psi_{n_1}(X)\}$ with electron out-solutions. Correspondingly, in the range Ω_1, one-electron states are

$$_+a^\dagger_{n_1}(\text{in})|0\rangle, \qquad {}^-a^\dagger_{n_1}(\text{in})|0\rangle,$$
$$_-a^\dagger_{n_1}(\text{out})|0\rangle, \qquad {}^+a^\dagger_{n_1}(\text{out})|0\rangle. \qquad (5.1)$$

We associate the independent pair $\{_-\psi_{n_5}(X)\}$ and $\{^+\psi_{n_5}(X)\}$ with positron in-solutions and the independent pair $\{_+\psi_{n_5}(X)\}$ and $\{^-\psi_{n_5}(X)\}$ with positron out-solutions. Correspondingly, in the range Ω_5, one-positron states are

$$_+b^\dagger_{n_5}(\text{out})|0\rangle, \qquad {}^-b^\dagger_{n_5}(\text{out})|0\rangle,$$
$$_-b^\dagger_{n_5}(\text{in})|0\rangle, \qquad {}^+b^\dagger_{n_5}(\text{in})|0\rangle. \qquad (5.2)$$

B. Interpretation of states in Ω_1 and Ω_5

To give an interpretation of states in Ω_1 and Ω_5 we have studied one-particle mean values of the charge, the kinetic energy, the number of particles, the current, and the energy flux through the surfaces $x = x_L$ and $x = x_R$. The corresponding calculations are placed in the Appendix C 1. Based on results of such calculations, we can finally conclude:

(1) The states (5.1) are states with the charge $-e$. The states (5.2) are states with the charge $+e$.
(2) All these states have positive energies and therefore they can be treated as physical particle states, namely states (5.1) represent electrons, whereas states (5.2) represent positrons. Their currents (C7) and (C8) also confirm this interpretation.
(3) Mean energy fluxes (C10) and (C11) of states under consideration through the surfaces $x = x_L$ and $x = x_R$ together with expressions for their currents allow us to believe that:
  (a) electrons $_+a^\dagger_{n_1}(\text{in})|0\rangle$ and $^+a^\dagger_{n_1}(\text{out})|0\rangle$ are moving to the right,
  (b) electrons $_-a^\dagger_{n_1}(\text{out})|0\rangle$ and $^-a^\dagger_{n_1}(\text{in})|0\rangle$ are moving to the left;
  (c) positrons $_-b^\dagger_{n_5}(\text{in})|0\rangle^-b^\dagger_{n_5}(\text{out})|0\rangle$ are moving to the right,
  (d) positrons $_+b^\dagger_{n_5}(\text{out})|0\rangle$ and $^+b^\dagger_{n_5}(\text{in})|0\rangle$ are moving to the left.

Thus, we see that the asymptotic longitudinal physical momenta of electrons are $p^L_{ph} = p^L$ and $p^R_{ph} = p^R$, whereas for positrons they are $p^L_{ph} = -p^L$ and $p^R_{ph} = -p^R$.

(4) We classify electron states $_+a^\dagger_{n_1}(\text{in})|0\rangle$ and $^-a^\dagger_{n_1}(\text{in})|0\rangle$ as in- states, because they are moving to the step from the asymptotic regions S_L and S_R, respectively, with definite asymptotic behavior there. We classify electron states $_-a^\dagger_{n_1}(\text{out})|0\rangle$ and $^+a^\dagger_{n_1}(\text{out})|0\rangle$ as out- states because they are moving from the step to the asymptotic regions S_L and S_R, respectively, having there definite asymptotics.

We classify positrons states $_-b^\dagger_{n_5}(\text{in})|0\rangle$ and $^+b^\dagger_{n_5}(\text{in})|0\rangle$ as in- states because they are moving to the step from the asymptotic regions S_L and S_R, respectively, having their definite asymptotics. We classify positron states $_+b^\dagger_{n_5}(\text{out})|0\rangle$ and $^-b^\dagger_{n_5}(\text{out})|0\rangle$ as out- states because they are moving from the step to the asymptotic regions S_L and S_R, respectively, having their definite asymptotics.

In Fig. 3 we show in- and out-electron states in the range Ω_1 and in- and out-positron states in the range Ω_5. Here electrons are drawn by circles with the sign minus inside and positrons with the sign plus inside. The associated arrows show the energy flux directions given by Eqs. (C10) and (C11). Thus, these arrows show the directions of motion.

To justify completely our interpretation of the in- and out-states, we first recall that it is impossible to refer (even in the nonrelativistic quantum mechanics) to a direction of motion of plane waves, which has no physical meaning, since they are not localized. The scattering problem is formulated for particles that are represented by wave packets localized in some space areas. What do we demand
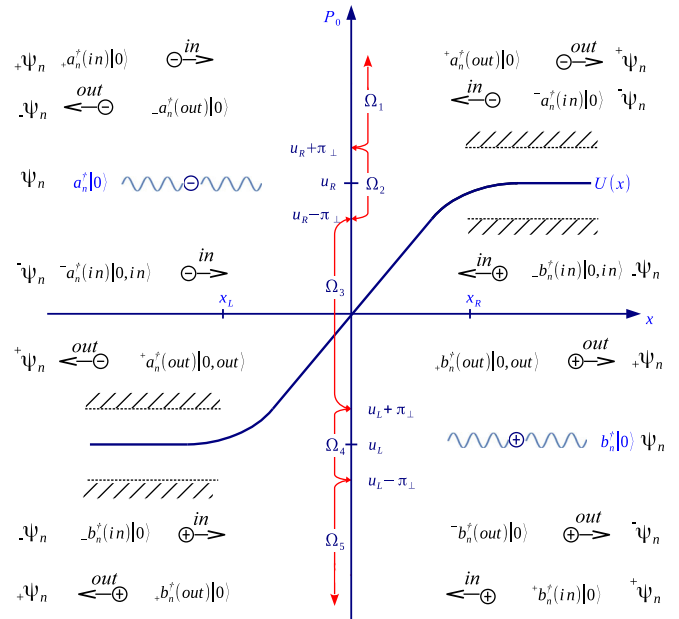

FIG. 3. in and out-particles near an x-potential step.

from such localized packets? First of all, the localization areas have to belong to one of the asymptotic regions S_L or S_R. Each of the localized wave packets must be composed of states with asymptotic physical momenta p_{ph}^L or p_{ph}^R, respectively, that have the same directions and belong to one and the same range Ω_i. We call these packets quasilocalized, because of not very rigid requirements for their localization. Such wave packets are moving in the same direction as their constituent waves. In the scattering problem under consideration, we consider four types of wave packets in each range Ω_1 and Ω_5, two of them being quasilocalized in the asymptotic region S_L and two of them in the asymptotic region S_R. All the packets quasilocalized in S_L are formed of solutions $_\zeta\psi_n(X)$, whereas all the packets quasilocalized in S_R are formed of solutions $^\zeta\psi_n(X)$. Indeed, in the asymptotic region S_L and S_R solutions $_\zeta\psi_n(X)$ and $^\zeta\psi_n(X)$, respectively, are reduced to waves with definite asymptotic physical momenta p_{ph}^L and p_{ph}^R, respectively. That is why directions of motion of the wave packets in these regions are well defined.

The electron wave packets in the range Ω_1, composed of solutions $_+\psi_n(X)$ or $^-\psi_n(X)$, are moving to the region S_{int} (in the QM scattering theory they are called incoming waves), whereas the electron wave packets composed of solutions $_-\psi_n(X)$ or $^+\psi_n(X)$ are moving away from the region S_{int} (in the QM scattering theory they are called outgoing waves). Thus, we believe that the first type of the wave packets describe in-electron states with asymptotic behavior formed before they meet the external field, and the second type of wave packets describe out-electrons that have asymptotic behavior observed after they have left the region where the external field is present. That is the reason for our definitions of in- and out- creation and annihilation operators with quantum numbers n_1 in the decomposition of the quantized Dirac field (4.5). It is not difficult to give similar interpretation for positron wave packets in the range Ω_5.

According to these definitions, we introduce absolute $\tilde{R}$ and relative R amplitudes of an electron reflection, and absolute $\tilde{T}$ and relative T amplitudes of an electron transmission in the range Ω_1 as

$$
\begin{aligned}
\tilde{R}_{+,n_1} &= c_v R_{+,n_1}, &\quad R_{+,n_1} &= \langle 0|_- a_{n_1}(\text{out})_+ a_{n_1}^\dagger(\text{in})|0\rangle, \\
\tilde{T}_{+,n_1} &= c_v T_{+,n_1}, &\quad T_{+,n_1} &= \langle 0|^+ a_{n_1}(\text{out})_+ a_{n_1}^\dagger(\text{in})|0\rangle, \\
\tilde{R}_{-,n_1} &= c_v R_{-,n_1}, &\quad R_{-,n_1} &= \langle 0|^+ a_{n_1}(\text{out})^- a_{n_1}^\dagger(\text{in})|0\rangle, \\
\tilde{T}_{-,n_1} &= c_v T_{-,n_1}, &\quad T_{-,n_1} &= \langle 0|_- a_{n_1}(\text{out})^- a_{n_1}^\dagger(\text{in})|0\rangle,
\end{aligned}
\tag{5.3}
$$

and similar quantities for a positron in the range Ω_5 as

$$
\begin{aligned}
\tilde{R}_{+,n_5} &= c_v R_{+,n_5}, &\quad R_{+,n_5} &= \langle 0|^- b_{n_5}(\text{out})^+ b_{n_5}^\dagger(\text{in})|0\rangle, \\
\tilde{T}_{+,n_5} &= c_v T_{+,n_5}, &\quad T_{+,n_5} &= \langle 0|_+ b_{n_5}(\text{out})^+ b_{n_5}^\dagger(\text{in})|0\rangle, \\
\tilde{R}_{-,n_5} &= c_v R_{-,n_5}, &\quad R_{-,n_5} &= \langle 0|_+ b_{n_5}(\text{out})_- b_{n_5}^\dagger(\text{in})|0\rangle, \\
\tilde{T}_{-,n_5} &= c_v T_{-,n_5}, &\quad T_{-,n_5} &= \langle 0|^- b_{n_1}(\text{out})_- b_{n_5}^\dagger(\text{in})|0\rangle,
\end{aligned}
\tag{5.4}
$$

where R and T are the corresponding relative amplitudes, and $c_v = \langle 0, \text{out}|0, \text{in}\rangle$, see Sec. VII D.

Using canonical transformations (4.33) one can calculate the relative electron amplitudes,

$$
\begin{aligned}
R_{+,n} &= g(_+|^+)^{-1} g(_-|^+), &\quad T_{+,n} &= \eta_L g(_+|^+)^{-1}, \\
R_{-,n} &= g(^-|_-)^{-1} g(^+|_-), &\quad T_{-,n} &= -\eta_R g(^-|_-)^{-1}.
\end{aligned}
\tag{5.5}
$$

Similar expressions can be obtained for the corresponding positron amplitudes. They differ from Eqs. (5.5) only by phases.

As it follows from Eqs. (3.41) and (3.42) the corresponding probabilities satisfy the following relations

$$
\begin{gathered}
|R_{+,n}|^2 = |R_{-,n}|^2, \qquad |T_{+,n}|^2 = |T_{-,n}|^2, \\
|R_{\zeta,n}|^2 + |T_{\zeta,n}|^2 = 1, \qquad n \in \Omega_1, \qquad \Omega_5.
\end{gathered}
\tag{5.6}
$$

Equation (5.6) is just the condition of the probability conservation, written in terms of relative probabilities of reflection and transmission, under the condition that in all other states with quantum numbers $m \neq n$ partial vacua remain vacua.

Now we see that according to Eqs. (C4) the relative probabilities coincide with the corresponding mean values,

$$
\begin{aligned}
N_{\zeta,n}^{(a)}(n, -\zeta) &= |R_{\zeta,n}|^2, \\
N_{\zeta,n}^{(a)}(n, \zeta) &= |T_{\zeta,n}|^2, \qquad n \in \Omega_1, \Omega_5.
\end{aligned}
\tag{5.7}
$$

This nontrivial result may be interpreted as QFT justification of rules of time-independent potential scattering theory, see Ref. [47], in the ranges Ω_1 and Ω_5. To clarify this point of view, let us consider one specific process in the range Ω_1, namely, the evolution of the in-state $_+a_{n_1}^\dagger(\text{in})|0\rangle$. From the point of view of causal evolution this state can be reflected, i.e., to pass to the out-state $_-a_{n_1}^\dagger(\text{out})|0\rangle$ with the probability $|R_{+,n}|^2$ and can be transmitted, i.e., to pass to the out-state $^+a_{n_1}^\dagger(\text{out})|0\rangle$ with the probability $|T_{+,n}|^2$. Let us try to apply the potential scattering theory to the same problem, using our QFT picture. Then, we have to calculate two mean currents in our in-state, one J_R related to the out-particles $_-a_{n_1}^\dagger(\text{out})|0\rangle$ and another one J_T related to the out-particles $^+a_{n_1}^\dagger(\text{out})|0\rangle$. Both currents are proportional to the mean numbers of the corresponding out-particles in our

in-state and can be represented by these numbers in the example under consideration. Then

$$
\begin{aligned}
J_R &= \langle 0|_+ a_{n_1}(\text{in})[_- a^\dagger_{n_1}(\text{out})_- a_{n_1}(\text{out})]_+ a^\dagger_{n_1}(\text{in})|0\rangle \\
&= |g(_+|^+)|^{-2}|g(_-|^+)|^2 = |R_{+,n}|^2, \\
J_T &= \langle 0|_+ a_{n_1}(\text{in})[^+ a^\dagger_{n_1}(\text{out})^+ a_{n_1}(\text{out})]_+ a^\dagger_{n_1}(\text{in})|0\rangle \\
&= |g(_+|^+)|^{-2} = |T_{+,n}|^2.
\end{aligned}
$$

Thus, we see that in the range Ω_1 realization of rules of the potential scattering theory in the framework of QFT allows one to obtain the correct result $J_R + J_T = 1$.

VI. ONE-PARTICLE STATES IN THE RANGES Ω_2 AND Ω_4

In the range Ω_2 there exist only one-electron states $a^\dagger_{n_2}|0\rangle$, whereas in the range Ω_4 there exist only one-positron states $b^\dagger_{n_4}|0\rangle$,

$$
a^\dagger_{n_2}|0\rangle, \qquad b^\dagger_{n_4}|0\rangle. \tag{6.1}
$$

Below, we study their interpretations and properties.

Using Eqs. (3.36) we see that the renormalized QFT currents, given by the operator $\hat{\mathbb{J}}$ (C6) is zero in the states under consideration,

$$
J_{n_2} = \langle 0|a_{n_2}\hat{\mathbb{J}}a^\dagger_{n_2}|0\rangle = J_{n_4} = \langle 0|b_{n_4}\hat{\mathbb{J}}b^\dagger_{n_4}|0\rangle = 0. \tag{6.2}
$$

We interpret the QFT states (6.1) as standing waves (stationary waves) that present a result of interference between two waves traveling in opposite directions, see Eqs. (3.29) and (3.30). In Fig. 3, we show these standing waves in the ranges Ω_2 and Ω_4. Here electron standing waves are drawn as circles with the minus inside and positrons with the plus inside.

It should be stressed that the case of the ranges $\Omega_{2,4}$ can be considered as a degenerate one with respect to the case of the ranges $\Omega_{1,5}$. This case could formally by extracted from relation (5.5) by considering the limit $T_{n,+} = \eta_L g(_+|^+)^{-1} \to 0$, which implies that $|g(_+|^+)|^2 = |T_n|^{-2} \to \infty$ for the potential step under consideration. Then it follows from the relation $|R_n|^2 + |T_n|^2 = 1$ that $|R_n|^2 \to 1$, which corresponds to the almost total reflection, when the currents of incoming and outgoing waves with a given n almost cancel each other. According to the definition of the charge operator (4.22), the space distributions of the charge are given by the densities $|^\zeta\psi_n(X)|^2$ and $|_\zeta\psi_n(X)|^2$. In turn this means that linear charge density in the asymptotic regions S_L and S_R are given by the quantities $^\zeta\mathcal{R}_{L/R}/K^{(R)}$ and $_\zeta\mathcal{R}_{L/R}/K^{(R)}$ which were obtained in (B5) and (B7). We note that $|g(_+|^-)|^2 \simeq |g(_+|^+)|^2$ in the case $|g(_+|^+)|^2 \gg 1$ such that distributions for the charge density in the ranges $\Omega_{1,5}$ become similar to the distributions in the ranges $\Omega_{2,4}$.

It is natural to expect that such a case is realized when quantum numbers n_1 are close to the upper bound of the range Ω_2, and the quantum numbers n_5 are close to the lower bound of the range Ω_4.

Finally, in the ranges $\Omega_{2,4}$ only the total reflection takes place. This process can be well described in the framework of one-particle quantum mechanics, see [48]. Its rigorous description in the framework of QFT as a time-dependent process leads to the same results confirming heuristic one-particle quantum-mechanical interpretation.

One can, in principle, define in- and out-states related to these opposite waves. It should be noted that, on the one hand, this is not a trivial task, and, on the other hand, this problem is the scope of our main goal, which is the consideration of particle creation processes. The latter processes are specific for the Ω_3 range, as it will be clear in what follows. However, we represent below a brief discussion of in- and out-states in the $\Omega_{2,4}$ ranges.

We believe that physical state vectors that correspond to localized in- and out-electrons or positrons are some wave packets composed of formal solutions introduced in the ranges Ω_2 and Ω_4. In the region S_R the constituent waves $\psi_{n_2}(X)$ have zero asymptotic values which implies that any wave packet describing an electron is quasilocalized in the region S_L. In the region S_L the constituent waves $\psi_{n_4}(X)$ have zero asymptotic values, which implies that any wave packet describing a positron state is quasilocalized in the region S_R.

VII. ONE-PARTICLE STATES IN THE RANGE Ω_3

A. General

First we recall (see Sec. III B 3) that according to the structure of the Dirac energy spectra in the asymptotic regions S_L and S_R, there exist two sets $\{_\zeta\psi_{n_3}(X)\}$ and $\{^\zeta\psi_{n_3}(X)\}$ of solutions in the range Ω_3 that obey the orthogonality relations $(_\zeta\psi_{n_3}, ^\zeta\psi_{n_3}) = 0$, see (B16). Thus, in this range we have two pairs $\{_-\psi_{n_3}(X), ^-\psi_{n_3}(X)\}$ and $\{_+\psi_{n_3}(X), ^+\psi_{n_3}(X)\}$ of independent solutions. Each pair forms a complete set of solutions in the range Ω_3.

According to Eqs. (3.38), there exist relations between solutions $\{_\zeta\psi_{n_3}(X)\}$ and $\{^\zeta\psi_{n_3}(X)\}$,

$$
\begin{aligned}
{}_+\psi_n(X) &= g(_+|^-)^{-1}[_-\psi_n(X)g(_-|^-) + {}^-\psi_n(X)], \\
{}_-\psi_n(X) &= g(_-|^+)^{-1}[_+\psi_n(X)g(_+|^+) - {}^+\psi_n(X)], \\
{}^+\psi_n(X) &= g(^+|_-)^{-1}[^-\psi_n(X)g(^-|_-) - {}_-\psi_n(X)], \\
{}^-\psi_n(X) &= g(^-|_+)^{-1}[^+\psi_n(X)g(^+|_+) + {}_+\psi_n(X)].
\end{aligned} \tag{7.1}
$$

As it follows from Eqs. (3.42), in the range under consideration the coefficients g satisfy the following relation

$$|g(_+|^-)|^2 = |g(_+|^+)|^2 + 1, \qquad (7.2)$$

which differs from similar relation (4.32) in the range Ω_1.

We associate the first independent pair $\{_-\psi_{n_3}(X), {}^-\psi_{n_3}(X)\}$ with in-solutions and the second independent pair $\{_+\psi_{n_3}(X), {}^+\psi_{n_3}(X)\}$ with out-solutions. Thus, solutions $\{{}^-\psi_{n_3}(X), {}^+\psi_{n_3}(X)\}$ are associated with in- and out-electron states, whereas solutions $\{_-\psi_{n_3}(X), {}_+\psi_{n_3}(X)\}$ are associated with in- and out-positron states. Correspondingly, in the range Ω_3 there exist four types of one-particle QFT states,

$$
\begin{aligned}
&{}^-a_{n_3}^\dagger(\text{in})|0, \text{in}\rangle, && {}^+a_{n_3}^\dagger(\text{out})|0, \text{out}\rangle, \\
&{}_-b_{n_3}^\dagger(\text{in})|0, \text{in}\rangle, && {}_+b_{n_3}^\dagger(\text{out})|0, \text{out}\rangle.
\end{aligned} \qquad (7.3)
$$

Since no other ranges are considered in this section, the quantum numbers n_3 are sometimes denoted by n for simplicity.

Using both alternative decompositions (4.5) for $\Psi_3(X)$ and relations (7.1), we find the following linear canonical transformations between the introduced in- and out- creation and annihilation operators

$$
\begin{aligned}
{}^+a_n(\text{out}) &= -g(_-|^+)^{-1}\,_-b_n^\dagger(\text{in}) + g(^-|_+)^{-1}g(^+|_+)^-a_n(\text{in}), \\
{}_+b_n^\dagger(\text{out}) &= g(_-|^+)^{-1}g(_+|^+)\,_-b_n^\dagger(\text{in}) + g(^-|_+)^{-1}\,^-a_n(\text{in}), \\
{}_-b_n^\dagger(\text{in}) &= g(_+|^-)^{-1}g(_-|^-)\,_+b_n^\dagger(\text{out}) - g(^+|_-)^{-1}\,^+a_n(\text{out}), \\
{}^-a_n(\text{in}) &= g(_+|^-)^{-1}\,_+b_n^\dagger(\text{out}) \\
&\quad + g(^+|_-)^{-1}g(^-|_-)\,^+a_n(\text{out}).
\end{aligned} \qquad (7.4)
$$

Because these transformations entangle annihilation and creation operators, the partial vacua $|0, \text{in}\rangle^{(3)}$ and $|0, \text{out}\rangle^{(3)}$ are essentially different. That is why, the total vacua $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are different as well, see Eqs. (4.29) and (4.37).

B. Interpretation of states in Ω_3

To give an interpretation of states in Ω_3 we have studied one-particle mean values of the charge, the kinetic energy, the number of particles, the current, and the energy flux through the surfaces $x = x_{\text{L}}$ and $x = x_{\text{R}}$. The corresponding calculations are placed in the Appendix C 2. Based on results of such calculations, we can finally conclude:

(1) The states

$$
{}^-a_{n_3}^\dagger(\text{in})|0, \text{in}\rangle, \qquad {}^+a_{n_3}^\dagger(\text{out})|0, \text{out}\rangle \qquad (7.5)
$$

are states with the charge $-e$. The states

$$
{}_-b_{n_3}^\dagger(\text{in})|0, \text{in}\rangle, \qquad {}_+b_{n_3}^\dagger(\text{out})|0, \text{out}\rangle \qquad (7.6)
$$

are states with the charge $+e$. Equations (C17) confirm these conclusions.

(2) Each of these states belongs to one of the physical wave packets that have positive energies as is demonstrated in Sec. VII C, and, therefore, they can be treated as physical particle states, namely, states (7.5) represent electrons, whereas states (7.6) represent positrons. Their currents (C17) also confirm this interpretation.

(3) Mean energy fluxes (C19) of states (7.3) through the surfaces $x = x_{\text{L}}$ and $x = x_{\text{R}}$ together with expressions (C17) for their currents allow us to conclude that:
　(a) electrons $\quad {}^-a_{n_3}^\dagger(\text{in})|0, \text{in}\rangle \quad$ and $\quad$ positrons $_+b_{n_3}^\dagger(\text{out})|0, \text{out}\rangle$ are moving to the right;
　(b) electrons $\quad {}^+a_{n_3}^\dagger(\text{out})|0, \text{out}\rangle \quad$ and $\quad$ positrons $_-b_{n_3}^\dagger(\text{in})|0, \text{in}\rangle$ are traveling to the left.
　We stress that in contrast to the ranges Ω_1 and Ω_5, in the range Ω_3 signs of the energy fluxes of the electrons are determined by signs of the quantum number p^{R} whereas signs of the energy fluxes of the positrons are determined by signs of the quantum number p^{L}. This correlation follows from the fact that in the region S_{R} the directions of the positron motion coincides with the directions of positron current and in the region S_{L} the directions of the electron motion is opposite to the directions of the electron current.

(4) We classify electron states ${}^-a_{n_3}^\dagger(\text{in})|0, \text{in}\rangle$ as in-states, because they can move to the step only from the asymptotic region S_{L}. The latter statement is based on our belief that the structure of the Dirac spectrum in the region S_{R} forbids electrons to be present in this region.

We classify electron states ${}^+a_{n_3}^\dagger(\text{out})|0, \text{out}\rangle$ as out-states, because they can move from the step only to the asymptotic regions S_{L}. The latter statement is based on our belief that the structure of the Dirac spectrum in the region S_{R} forbids electrons to be present in this region.

We classify positrons states ${}_-b_{n_3}^\dagger(\text{in})|0, \text{in}\rangle$ as in- states, because they can move to the step only from the asymptotic region S_{R}. The latter statement is based on our belief that the structure of the Dirac spectrum in the region S_{L} forbids positrons to be present in this region.

We classify positrons states ${}_+b_{n_3}^\dagger(\text{out})|0, \text{out}\rangle$ as out-states, because they can move to the step only from the asymptotic region S_{R}. The latter statement is based on our belief that the structure of the Dirac spectrum in the region S_{L} forbids positrons to be present in this region.

In Fig. 3 we show in- and out-states in the range Ω_3.

C. Discussion of the localization properties

Taking into account Eqs. (3.41), one can verify that all differential mean numbers (C13) are equal. That is why, we introduce the unique notation N_n^{cr} for all of them,

$$N_n^{\text{cr}} = N_n^b(\text{out}) = N_n^a(\text{out}) = N_n^b(\text{in}) = N_n^a(\text{in}) = |g(_+|^-)|^{-2}.$$
$$(7.7)$$

For the fermions under consideration, it is natural that the quantity N_n^{cr} is always less or equal than one, $N_n^{\text{cr}} \leq 1$. We also note that the quantities $\pi_0(\text{L})$ and $|\pi_0(\text{R})|$ achieve their minimal values on the boundaries of the range Ω_3, namely,

$$\min \pi_0(\text{L}) = \pi_\perp, \qquad \min |\pi_0(\text{R})| = \pi_\perp. \quad (7.8)$$

If N_n^{cr} tends to zero, $N_n^{\text{cr}} \to 0$, then $|g(_+|^-)|^2 \to \infty$ and, at the same time, $|g(_+|^+)|^2 \to \infty$ in accordance to relation (7.2).

The quantity N_n^{cr} can be calculated explicitly in the so-called exactly solvable cases (see Introduction and Sec. IX). However, we can derive some additional useful properties of this quantity in the general case.

Taking into account the fact that no pair creation takes place in the regions Ω_2 and Ω_4, we suppose that pair creation vanishes near the boundaries of the range Ω_3,

$$N_{n_3}^{\text{cr}}|_{n_3 \to \Omega_2} \to 0 \Leftrightarrow N_{n_3}^{\text{cr}}|_{\pi_0(\text{R}) \to -\pi_\perp} \to 0,$$
$$N_{n_3}^{\text{cr}}|_{n_3 \to \Omega_4} \to 0 \Leftrightarrow N_{n_3}^{\text{cr}}|_{\pi_0(\text{L}) \to \pi_\perp} \to 0. \quad (7.9)$$

Note that in this case kinetic energies of electron or positron plane waves given by Eqs. (4.15) tend to their values, ${}^\zeta \mathcal{E}_{n_3} \to \pi_0(\text{L})$ and ${}_\zeta \mathcal{E}_{n_3} \to \pi_0(\text{R})$, near the boundaries of the range Ω_3. Therefore, conditions (7.9) provide that

$$^\zeta \mathcal{E}_{n_3} > 0, \qquad {}_\zeta \mathcal{E}_{n_3} < 0 \quad (7.10)$$

near the boundaries of the range Ω_3 and operator $\hat{H}^{\text{kin}}$ (4.21) is positively-defined in this part of the range Ω_3. We believe that inequality (7.9) takes place for all nonpathological x-electric steps.

As was already mentioned, Eqs. (7.9) imply that $|g(_+|^-)|^2 \simeq |g(_+|^+)|^2 \to \infty$. Then it follows from Eqs. (B5) and (B7) that the electron density $|^\zeta\psi_n(X)|^2$ is concentrated in the region S_L, whereas the positron density $|_\zeta\psi_n(X)|^2$ is concentrated in the region S_R,

$$|^\zeta\psi_{n_3}(X)|^2|_{\pi_0(\text{R}) \to -\pi_\perp} \to 0, \quad x \in S_\text{R},$$
$$|_\zeta\psi_{n_3}(X)|^2|_{\pi_0(\text{L}) \to \pi_\perp} \to 0, \quad x \in S_\text{L}, \quad (7.11)$$

which means that these densities are continuous near the boundaries of the range Ω_3. We see that conditions (7.9) and (7.11) are equivalent. Thus we believe that conditions (7.11) imply (7.9) near the boundaries of the range Ω_3.

For arbitrary $n \in \Omega_3$ some properties of N_n^{cr} can be established first in the case of weak external fields (but still strong enough, $\mathbb{U} - 2\pi_\perp > 0$, to provide the existence of

the Ω_3-range) using a semiclassical approximation. In the latter case a really strong restriction has to be imposed, that N_n^{cr} is exponentially small, $N_n^{\text{cr}} \ll 1$. Then it is natural to expect that for any finite $\mathbb{U} < \infty$ and $m \neq 0$ the inequality $|g(_+|^-)|^{-2} < 2\pi_\perp/\mathbb{U}$ holds, such that inequalities (7.10) hold for arbitrary $n \in \Omega_3$. The condition $N_n^{\text{cr}} \ll 1$ and Eqs. (B5) and (B7) imply that for arbitrary $n \in \Omega_3$ the electron density $|^\zeta\psi_n(X)|^2$ is concentrated in the region S_L, whereas the positron density $|_\zeta\psi_n(X)|^2$ is concentrated in the region S_R,

$$|^\zeta\psi_{n_3}(X)|^2 \to 0, \quad x \in S_\text{R},$$
$$|_\zeta\psi_{n_3}(X)|^2 \to 0, \quad x \in S_\text{L}. \quad (7.12)$$

Thus, the wave functions ${}^\zeta\psi_{n_3}(X)$ and ${}_\zeta\psi_{n_3}(X)$ behave quite similarly to the behavior of the corresponding functions with $n \in \Omega_2$, Ω_4.

In the general case when the quantities N_n^{cr} are not small, it is natural to expect a similar behavior, namely: the region S_R is not available for electrons, a the region S_L is not available for positrons. However, when the quantities N_n^{cr} are not small, the latter property may hold only for the corresponding wave packets, but not for the separate plane waves. That means that Eq. (7.12) may not hold, that is, these plane waves may be different from zero in the whole space. Within our context it is assumed that electrons and positrons in one of corresponding asymptotic regions may occupy quasistationary states, i.e. they should be described by wave packets that pertain their form sufficiently long in one of corresponding asymptotic regions. In other words, only such electron and positron wave packets have a physical meaning in the problem under consideration. This is what we shall keep in mind when discussing the wave packets in what follows. We can demonstrate that in the general case the electron wave packets that really have a physical meaning can be localized only in the asymptotic region S_L, whereas the positron wave packets that really have a physical meaning can be localized only in the asymptotic regions S_R. This is a consequence of a specific structure of plane waves ${}^\zeta\psi_{n_3}(X)$ and ${}_\zeta\psi_{n_3}(X)$ in asymptotic regions S_L and S_R. Indeed, this structure is quite different from the structure of plane waves in the ranges Ω_1 and Ω_5. As was mentioned above, electron states with given quantum numbers n_3 are states with a definite quantum number p^R, whereas positron states with given quantum numbers n_3 are states with a definite quantum number p^L. This fact together with relation (7.1) implies, for example, that a partial wave of an in-electron, $^-\psi_{n_3}(X)$, in the region where the electron can really be observed, i.e., in the region S_L, is always a superposition of two waves with opposite signs of the quantum number p^L, $_+\psi_{n_3}(X)$ and $_-\psi_{n_3}(X)$. In turn, this implies that in contrast to the ranges Ω_1 and Ω_5, the sign of p^L is not related to the sign of the mean energy flux in the region S_L. The same holds true for a partial wave

of an out-electron $^+\psi_{n_3}(X)$. Similarly, one can see that partial waves of both in- positron $_-\psi_{n_3}(X)$, and out-positron $_+\psi_{n_3}(X)$, in the region S_R, are always superpositions of two waves with quantum number p^R of opposite signs and, therefore, signs of these quantum numbers are not connected to the sign of the mean energy flux in the region S_R. However, as it was demonstrated above, these are states with well-defined asymptotic energy flux, and therefore with the corresponding well-defined asymptotic field momentum. One can demonstrate that namely these properties of the constituent plane waves are responsible for the fact that stable electron wave packets can exist only in the region S_L, whereas stable positron wave packets can exist only in the region S_R, see details in the Appendix D.

We stress that in the range Ω_3, within each pair of independent states with the same n, the in- particles and out- particles always move in opposite directions.

Mean values (C14) and (C17) are typical for the total reflection. Indeed, all the mean particle numbers do not change in the course of the interaction with the step field, and the both electron currents are equal in magnitude and have opposite directions, the same holding for the both positron currents. We have demonstrated that electron densities in Ω_3 have a behavior similar to that in the Ω_2 range, vanishing in the S_R region, while positron densities in Ω_3 behave similarly to the Ω_4 range, vanishing in the S_L region.

Since the above-described properties of mean values (C14) and (C17) hold true in the whole range Ω_3, we believe that all the in-states in this range are subjected to the total reflection. Once this is the case, the wave functions of the in-states and of the out-states corresponding to them have to be concentrated in the same regions on the left or on the right of the x-electric step, similar to the behavior of the particles in the ranges Ω_2 and Ω_4, respectively. Note that kinetic energies of these physical states are sums of the positive partial energies $V_\perp M_n^{-1}{}^\zeta E_n^L$ for electrons and $-V_\perp M_n^{-1}{}_\zeta E_n^R$ for positrons, where the quantities $^\zeta E_n^L$ and $_\zeta E_n^R$ are given by Eqs. (4.13).

Some additional arguments in favor of the given interpretation are in order.

In the range Ω_3 no particles exist that could maintain the direction of their motion after the interaction with the external field. This peculiarity allows one to classify these one-particle states as in- or out-states by using mean currents (C17). The electric field under consideration accelerates positrons along the axis x and electrons in the opposite direction, that is why in the range Ω_3 the current of out-electron states coincides with the electric field direction, whereas the current of in-electron states is opposite to the electric field direction. Thus, $^+a_n^\dagger(\text{out})|0,\text{out}\rangle$ and $_+b_{n_3}^\dagger(\text{out})|0,\text{out}\rangle$ are out- states of electrons and positrons, respectively, whereas

$^-a_{n_3}^\dagger(\text{in})|0,\text{in}\rangle$ and $_-b_{n_3}^\dagger(\text{in})|0,\text{in}\rangle$ are in− states of electrons and positrons, respectively. This also implies that $|0,\text{in}\rangle$ is the in− vacuum and $|0,\text{out}\rangle$ is the out− vacuum. Finally, $^-a_{n_3}^\dagger(\text{in})$, $^-a_{n_3}(\text{in})$, $_-b_{n_3}^\dagger(\text{in})$, $_-b_{n_3}(\text{in})$ are creation and annihilation operators of in− electrons and positrons respectively, whereas $^+a_{n_3}^\dagger(\text{out})$, $^+a_{n_3}(\text{out})$, $_+b_{n_3}^\dagger(\text{out})$, $_+b_{n_3}(\text{out})$ are creation and annihilation operators of out-electrons and out-positrons, respectively.

We come to the same conclusion considering the mean values (C13). Thus, we see that the quantities $N_n^a(\text{out})$ and $N_n^b(\text{out})$ are differential mean numbers of out-electrons and out-positrons, respectively, created from the vacuum, since electric currents composed of the corresponding states coincide with the direction of the electric field. In this case we face electron and positron pairs outgoing from the state where no incoming particles were present. The mean numbers of created electrons are equal to mean numbers of created positrons, in full agreement with the charge conservation law.

Besides, it follows from Eq. (C13) that $N_n^b(\text{in})$ and $N_n^a(\text{in})$ are differential mean numbers of electrons and positrons, respectively, annihilated from the initial neutral state of an electron-positron pair. The electric current corresponding to this initial pair is directed opposite to the electric field, that is why after the annihilation its electron and positron components are equally reduced by the quantity $e\mathcal{M}^{-1}N_n^{cr}$.

Thus, we believe that the wave functions $^-\psi_{n_3}(X)$ and $_-\psi_{n_3}(X)$ describe initial or incoming states of an electron and a positron, respectively, whereas the wave functions $^+\psi_{n_3}(X)$ and $_+\psi_{n_3}(X)$ describe final or outgoing states of an electron and a positron, respectively.

This causal identification coincides with the one proposed by Nikishov in the frame of relativistic quantum mechanics in Refs. [2,17]. Note that it differs from another identification in the frame of relativistic quantum mechanics given in Refs. [14] and repeated, for example, in Refs. [3,12]. In the Sec. VII E we discuss this problem in more detail.

D. Reflection and creation of particles in Ω_3

The total number N_n^{cr} of pairs created from the vacuum is the sum over the range Ω_3 of the differential mean numbers N_n^{cr},

$$N = \sum_{n\in\Omega_3} N_n^{cr} = \sum_{n\in\Omega_3} |g(_+|^-)|^{-2}. \qquad (7.13)$$

Here we consider probability amplitudes of some simplest processes in the range Ω_3. First of all, this is the vacuum-to-vacuum transition amplitude which coincides [due to Eq. (4.37)] with the total vacuum-to-vacuum transition amplitude c_v,

$$c_v^{(3)} = {}^{(3)}\langle 0, \text{out}|0, \text{in}\rangle^{(3)} = c_v = \langle 0, \text{out}|0, \text{in}\rangle. \quad (7.14)$$

Among other nonzero amplitudes we have to consider two relative scattering amplitudes of electrons and positrons,

$$w(+|+)_{n'n} = c_v^{-1}\langle 0, \text{out}|{}^+a_{n'}(\text{out})^-a_n^\dagger(\text{in})|0, \text{in}\rangle,$$

$$w(-|-)_{n'n} = c_v^{-1}\langle 0, \text{out}|_+b_{n'}(\text{out})_-b_n^\dagger(\text{in})|0, \text{in}\rangle, \quad (7.15)$$

and two relative amplitudes of a pair creation and a pair annihilation,

$$w(+-|0)_{n'n} = c_v^{-1}\langle 0, \text{out}|{}^+a_{n'}(\text{out})_+b_n(\text{out})|0, \text{in}\rangle,$$

$$w(0|-+)_{nn'} = c_v^{-1}\langle 0, \text{out}|_-b_n^\dagger(\text{in})^-a_{n'}^\dagger(\text{in})|0, \text{in}\rangle. \quad (7.16)$$

As can be derived from relations (7.4), all the amplitudes (7.15) and (7.16) are diagonal in the quantum numbers n and can be expressed in terms of the coefficients $g({}^{\zeta'}|_\zeta)$ as follows:

$$w(+|+)_{n'n} = \delta_{n,n'}w_n(+|+),$$
$$w_n(+|+) = g({}^+|_-)g({}^-|_-)^{-1} = g(_+|^-)g(_+|^+)^{-1},$$
$$w(-|-)_{nn'} = \delta_{n,n'}w_n(-|-),$$
$$w_n(-|-) = g({}^-|_+)g({}^-|_-)^{-1} = g(_-|^+)g(_+|^+)^{-1},$$
$$w(0|-+)_{nn'} = \delta_{n,n'}w_n(0|-+),$$
$$w_n(0|-+) = -g({}^-|_-)^{-1},$$
$$w(+-|0)_{n'n} = \delta_{n,n'}w_n(+-|0),$$
$$w_n(+-|0) = g(_+|^+)^{-1}. \quad (7.17)$$

Recalling the physical meaning of the one-particle states (7.3), we conclude that in the range Ω_3 the total reflection is the only possible form of particle scattering, with $w(+|+)_n$ and $w(-|-)_n$ being relative probability amplitudes of a particle reflection. The relative probability amplitude $w(+-|0)_n$ describes creation of an electron-positron pair of out-particles with given quantum numbers n, and the relative probability amplitude $w(0|-+)_n$ describes annihilation of an electron-positron pair of in-particles, each of them having quantum numbers n.

Unitary relations (3.40) and their consequences (3.41) and (3.42) imply the following connections for the introduced amplitudes w:

$$|w_n(+|+)|^2 = |w_n(-|-)|^2,$$
$$|w_n(+-|0)|^2 = |w_n(0|-+)|^2,$$
$$|w_n(+|+)|^2 - |w_n(+-|0)|^2 = 1,$$
$$\frac{w_n(-|-)^*}{w_n(+|+)} = -\frac{w_n(+-|0)^*}{w_n(0|-+)}. \quad (7.18)$$

Referring to Eqs. (7.17), relations (7.4) can be rewritten as

$$^-a_n(\text{in}) = w_n(+|+)^{-1}[^+a_n(\text{out}) + w_n(+-|0)_+b_n^\dagger(\text{out})],$$

$$_-b_n(\text{in}) = w_n(-|-)^{-1}[_+b_n(\text{out}) - w_n(+-|0)^+a_n^\dagger(\text{out})].$$

$$(7.19)$$

Together with their adjoint relations they determine a unitary transformation V_{Ω_3} between the in- and out-operators,

$$\{^-a^\dagger(\text{in}), {}^-a(\text{in}), {}_-b^\dagger(\text{in}), {}_-b(\text{in})\}$$
$$= V_{\Omega_3}\{^+a^\dagger(\text{out}), {}^+a(\text{out}), {}_+b^\dagger(\text{out}), {}_+b(\text{out})\}V_{\Omega_3}^\dagger.$$

Since Eqs. (7.18) and (7.19) coincide formally with the corresponding equations for t-electric potential steps, the unitary operator V_{Ω_3} can be taken from the works [5,49]. It has the form

$$V_{\Omega_3} = \exp\left\{-\sum_{n\in\Omega_3}{}^+a_n^\dagger(\text{out})w_n(+-|0)_+b_n^\dagger(\text{out})\right\}$$

$$\times \exp\left\{-\sum_{n\in\Omega_3}{}_+b_n(\text{out})\ln w_n(-|-)_+b_n^\dagger(\text{out})\right\}$$

$$\times \exp\left\{\sum_{n\in\Omega_3}{}^+a_n^\dagger(\text{out})\ln w_n(+|+)^+a_n(\text{out})\right\}$$

$$\times \exp\left\{-\sum_{n\in\Omega_3}{}_+b_n(\text{out})w_n(0|-+)^+a_n(\text{out})\right\}.$$

$$(7.20)$$

At the same time, the operator V_{Ω_3} relates the in- and out-vacua, $|0, \text{in}\rangle = V_{\Omega_3}|0, \text{out}\rangle$ and therefore it determines the vacuum-to-vacuum transition amplitude c_v,

$$c_v = \langle 0, \text{out}|V_{\Omega_3}|0, \text{out}\rangle = \prod_n w_n(-|-)^{-1}. \quad (7.21)$$

The probabilities of a particle reflection, a pair creation, and the probability for a vacuum to remain a vacuum can be expressed via differential mean numbers of created pairs N_n^{cr}. By using the relation $|w_n(-|-)|^2 = (1 - N_n^{\text{cr}})^{-1}$, one finds

$$P(+|+)_{n',n} = |\langle 0, \text{out}|{}^+a_{n'}(\text{out})^-a_n^\dagger(\text{in})|0, \text{in}\rangle|^2$$

$$= \delta_{n,n'}\frac{1}{1 - N_n^{\text{cr}}}P_v,$$

$$P(+-|0)_{n',n} = |\langle 0, \text{out}|{}^+a_{n'}(\text{out})_+b_n(\text{out})|0, \text{in}\rangle|^2$$

$$= \delta_{n,n'}\frac{N_n^{\text{cr}}}{1 - N_n^{\text{cr}}}P_v,$$

$$P_v = |c_v|^2 = \prod_n p_v^n, \quad p_v^n = (1 - N_n^{\text{cr}}). \quad (7.22)$$

The probabilities for a positron scattering $P(-|-)_{n,n'}$ and a pair annihilation $P(0|-+)_{n,n'}$ coincide with the expressions $P(+|+)$ and $P(+-|0)$, respectively.

We finish this section with some important remarks. Note that p_v^n given by Eq. (7.22) is the probability that the partial vacuum state with given n remains a vacuum. One can see that in the framework of developed QFT quantization the total reflection of a particle off the x-electric potential step is described by the conditional probability of reflection of a particle with given quantum numbers n, under the condition that all other partial vacua remain vacua. Such a probability has the form $|w_n(+|+)|^2 p_v^n$. Consequently, $|w_n(+|+)|^2 p_v^n = 1$. However, the probability $P(+|+)_{n,n}$ is not equal to one as it follows from Eq. (7.22). If all $N_n^{\mathrm{cr}} \ll 1$ then

$$1 - P_v \approx N = \sum_{n \in \Omega_3} N_n^{\mathrm{cr}}. \qquad (7.23)$$

The vacuum instability is not essential if $N_n^{\mathrm{cr}} \to 0$. Then $P_v \to 1$, $P(+|+)_{n,n} \to 1$ and $P(+-|0)_{n,n} \to N_n^{\mathrm{cr}}$.

It should be noted that semiclassical approximations can be used namely under the latter condition. We recall that, by using the proper-time method, Schwinger calculated the one-loop effective Lagrangian L in electric field and assumed that the probability P_v that no actual pair-creation has occurred in the history of the field during the time T in the volume V can be presented as $P_v = \exp\{-VT2\mathrm{Im}L\}$ [50] (for a subsequent development, see the review [51]). Schwinger interpreted $2\mathrm{Im}L$ as the probability, per time unit, and per volume unit, of creating a pair by a constant electric field. This interpretation remains approximately valid as long as the WKB calculation is applicable, that is, $VT2\mathrm{Im}L \ll 1$. Then the total probability of pair-creation reads as $1 - P_v \approx VT2\mathrm{Im}L$. To calculate the differential probabilities of pair-creation with quantum numbers m (for instance, momentum and spin polarization), one can represent the probability P_v as an infinite product:

$$P_v = \prod_m e^{-2\mathrm{Im}S_m}, \qquad (7.24)$$

where a certain discretization scheme is used, so that the effective action $S = VTL$ is written as $S = \sum_m S_m$. The above-said is possible only if m are selected as integrals of motion. Then, $e^{-2\mathrm{Im}S_m}$ is the vacuum-persistence probability in a cell of the space of quantum numbers m. Using the WKB approximation in the case $2\mathrm{Im}S_m \ll 1$, one obtains for the probability of a single pair-production with quantum numbers m and for the corresponding mean values N_m^{cr} of created pairs the following relation:

$$N_m^{\mathrm{cr}} \approx P(+-|0)_{m,m} \approx 2\mathrm{Im}S_m. \qquad (7.25)$$

E. Some comments on in- and out- states in the Klein zone

It should be noted that in the works [2,13] of Nikishov he gave a consistent (and correct to our mind) resolution of the Klein's paradox based on his original approach, which is a combination of elements of second quantized theory and relativistic quantum mechanics. In particular, treating the Klein step, he interpreted solutions $^{\zeta}\psi_n(X)$ and $_{\zeta}\psi_n(X)$ in the way that correspond to our choice of in- and out- states. After Nikishov's works there appear a work of Hansen and Ravndal [14] where they tried to use second quantized quantum field theory to describe quantum effects near x-electric potential steps. However, their interpretation of the solutions $^{\zeta}\psi_n(X)$ and $_{\zeta}\psi_n(X)$ was different from Nikishov's one. In terms of proposed by us quantization they interpreted our in-states as out-states and our out-states as in-states. Nikishov in his work [17] has pointed out that this is a wrong interpretation. Our detailed analysis confirms his opinion. Nevertheless the interpretation of Hansen and Ravndal was repeated in the textbook [3], in the review [12], and in some other publications. To our mind Hansen and Ravndal came to their interpretation treating processed in the Klein zone by a misleading analogy with the one-particle scattering, which really takes place in the ranges Ω_1 and Ω_5. That is why they believed that in the range Ω_3 signs of the quantum numbers p^{L} and p^{R} define signs of particle asymptotic longitudinal kinetic momenta in the regions S_{L} and S_{R}, respectively. If so, then one has to treat solutions $_{\zeta}\psi_n$ as describing one-electron states and solutions $^{\zeta}\psi_n$ as describing one-positron states, such that signs of the corresponding quantum numbers p^{L} and p^{R} label incoming and outgoing particles. However, our analysis of states in the range Ω_3, given in Sec. VII, does not confirm such a correlation between signs of the asymptotical physical longitudinal energy flux and quantum numbers p^{L} and p^{R}. We have demonstrated that the wave functions $^-\psi_{n_3}(X)$ and $_-\psi_{n_3}(X)$ describe incoming states of an electron and a positron, respectively, whereas the wave functions $^+\psi_{n_3}(X)$ and $_+\psi_{n_3}(X)$ describe outgoing states of an electron and a positron, respectively. That is why we believe that the particle-antiparticle and causal identification of wave functions $^{\zeta}\psi_n(X)$ and $_{\zeta}\psi_n(X)$ given in Ref. [14] is erroneous. Accepting this identification, one makes a mistake in defining what are in- and out-vacua, and calculates, e.g., the quantity $N_n^b(\mathrm{in})$ (C13) instead of the mean number of created pairs $N_n^a(\mathrm{out})$ with all the ensuing consequences. The correct causal interpretation is extremely important in all problems where one has to use the causal (in-out) propagator. Probably, the wrong interpretation did not attract for a long time an attention since in majority works devoted to the pair creation due to x-electric potential steps, the only mean numbers $N_n^b(\mathrm{in})$ [instead of $N_n^a(\mathrm{out})$] and functions of these numbers were calculated. However, in all

the considered cases the quantities $N_n^b(\text{in})$ and $N_n^a(\text{out})$ coincide numerically.

VIII. COMPLETE QED AND MASSIVE PARTICLE PROPAGATORS

Finally, it should be noted that quantization of the Dirac field in the presence of an x-electric potential steps developed in the present article serves as the base for constructing the Furry picture in the complete QED, which includes, apart from the matter (Dirac) field, the electromagnetic field as well. Formally, this Furry picture can be formulated in full analogy with the case of the t-electric potential steps considered in detail in Refs. [5]. Such a theory allows one to consider all quantum processes with charged particles moving in external background and interacting with photons. Processes of zero order with respect to the radiative interaction in such a theory do not include the interaction with the photons, their treatment was already presented above in Secs. IV–VII.

In complete QED with an external background any possible process is described by the following matrix element

$$\langle 0, \text{out}|a(\text{out})\cdots b(\text{out})\cdots c\cdots S(\hat{\Psi},\hat{\Psi}^\dagger,\hat{A}_\mu)\cdots$$
$$c^\dagger\cdots b^\dagger(\text{in})\cdots a^\dagger(\text{in})|0,\text{in}\rangle,$$

where c and $c^\dagger$ are creation and annihilation operators of photons and $S(\hat{\Psi},\hat{\Psi}^\dagger,\hat{A}_\mu)$ is the S-matrix in the interaction representation.

One sees specific terms influenced by vacuum instability when the initial and final states of charged particles belong to the range Ω_3. In this range, every operator F can be expressed exclusively in terms of in-annihilating operators and out-creation operators, using relations (7.19), and then divided into two parts,

$$F = F^{(-)} + F^{(+)}, \qquad F^{(-)}|0,\text{in}\rangle = 0, \qquad 0 = \langle 0,\text{out}|F^{(+)},$$

Then one can introduce a generalized normal form $\mathcal{N}_{\text{out-in}}(\ldots)$ in which all $F^{(-)}$ are placed to the right from all $F^{(+)}$. Wick's theorem holds with generalized chronological couplings,

$$\underbrace{FG} = FG - \mathcal{N}_{\text{out-in}}(FG) = \langle 0,\text{out}|FG|0,\text{in}\rangle c_v^{-1},$$

$$\overbrace{F(X)G(Y)} = \hat{T}F(X)G(Y) - \mathcal{N}_{\text{out-in}}[F(X)G(Y)]$$
$$= \langle 0,\text{out}|\hat{T}F(X)G(Y)|0,\text{in}\rangle c_v^{-1},$$

where $\hat{T}$ denotes the chronological ordering operation, see [5].

To reduce any functional of field operators to the generalized normal form in the range Ω_3, we have to represent the operator $\hat{\Psi}_3(X)$ (4.5) with the help of Eqs. (7.1) and (7.17) in the following form

$$\hat{\Psi}_3(X) = \sum_{n_3}\mathcal{M}_{n_3}^{-1/2}[\,^-a_{n_3}(\text{in})w_{n_3}(+|+)^+\psi_{n_3}(X)$$
$$+\,_+b_{n_3}^\dagger(\text{out})w_{n_3}(-|-)_-\psi_{n_3}(X)],$$
$$\hat{\Psi}_3^\dagger(X) = \sum_{n_3}\mathcal{M}_{n_3}^{-1/2}[\,^+a_{n_3}^\dagger(\text{out})w_{n_3}(+|+)^-\psi_{n_3}^\dagger(X)$$
$$+\,_-b_{n_3}(\text{in})w_{n_3}(-|-)_+\psi_{n_3}^\dagger(X)]. \qquad (8.1)$$

Processes of higher orders are described by the Feynman diagrams with two kinds of charged particle propagators in the external field under consideration, namely, the so-called in-out propagator $S^c(X,X')$, which is just the causal Feynman propagator, and the so-called in-in propagator $S_{\text{in}}^c(X,X')$,

$$S^c(X,X') = i\langle 0,\text{out}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^\dagger(X')\gamma^0|0,\text{in}\rangle c_v^{-1},$$
$$S_{\text{in}}^c(X,X') = i\langle 0,\text{in}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^\dagger(X')\gamma^0|0,\text{in}\rangle, \qquad (8.2)$$

where $\hat{T}$ in Eqs. (8.2) denotes the chronological ordering operation.

Using Eqs. (4.4), (4.5), (4.6), and anticommutation relations (4.7), we find for the in-in propagator:

$$S_{\text{in}}^c(X,X') = \theta(t-t')S_{\text{in}}^-(X,X') - \theta(t'-t)S_{\text{in}}^+(X,X'),$$

$$S_{\text{in}}^-(X,X') = i\sum_{j=1}^{2}G_j(X,X')\gamma^0 + \tilde{S}_{\text{in}}^-(X,X'),$$

$$S_{\text{in}}^+(X,X') = i\sum_{j=4}^{5}G_j(X,X')\gamma^0 + \tilde{S}_{\text{in}}^+(X,X'),$$

$$\tilde{S}_{\text{in}}^-(X,X') = i\sum_{n_3}\mathcal{M}_{n_3}^{-1}\,^-\psi_{n_3}(X)^-\bar{\psi}_{n_3}(X'),$$

$$\tilde{S}_{\text{in}}^+(X,X') = i\sum_{n_3}\mathcal{M}_{n_3}^{-1}\,_-\psi_{n_3}(X)_-\bar{\psi}_{n_3}(X'), \qquad (8.3)$$

where the functions $G_j(X,X')$ are given by Eq. (3.52).

Calculation of the in-out propagator can be done in a similar manner. Then, taking into account Eq. (7.19), we obtain

$$S^c(X,X') = \theta(t-t')S^-(x,x') - \theta(t'-t)S^+(x,x'),$$

$$S^-(X,X') = i\sum_{j=1}^{2}G_j(X,X')\gamma^0 + \tilde{S}^-(X,X'),$$

$$S^+(X,X') = i\sum_{j=4}^{5}G_j(X,X')\gamma^0 + \tilde{S}^+(X,X'),$$

$$\tilde{S}^-(X,X') = i\sum_{n_3}\mathcal{M}_{n_3}^{-1}[\,^+\psi_{n_3}(X)w_{n_3}(+|+)^-\bar{\psi}_{n_3}(X')],$$

$$\tilde{S}^+(X,X') = i\sum_{n_3}\mathcal{M}_{n_3}^{-1}[\,_-\psi_{n_3}(X)w_{n_3}(-|-)_+\bar{\psi}_{n_3}(X')], \qquad (8.4)$$

Using relations (7.1), we can represent the difference between both propagators as follows

$$S^p(X, X') = S^c_{in}(X, X') - S^c(X, X')$$
$$= -i \sum_{n_3} \mathcal{M}^{-1}_{n_3} [_-\psi_{n_3}(X) w_{n_3}(0| - +)^- \bar{\psi}_{n_3}(X')].$$

(8.5)

It is formed in the range Ω_3 only and vanishes if there is no pair creation.

IX. SAUTER POTENTIAL

A. Scattering, reflection, and pair creation on the Sauter potential

Let us consider an x-electric potential step in the form of the Sauter potential [8]. In this case

$$A_0(x) = -\alpha E \tanh(x/\alpha), \quad \alpha > 0,$$
$$E(x) = E \cosh^{-2}(x/\alpha),$$
$$U(x) = -eA_0(x) = eE\alpha \tanh(x/\alpha), \quad (9.1)$$

see Fig. 1, and the asymptotic quantities introduced in Sec. II are

$$U_R = -U_L = U(+\infty) = eE\alpha, \qquad \mathbb{U} = 2eE\alpha,$$
$$\pi_0(L) = p_0 + eE\alpha, \qquad \pi_0(R) = p_0 - eE\alpha.$$

Solutions (3.6) of Dirac equation (2.8) with special asymptotic behavior at $x \to \pm\infty$ are expressed via the corresponding solutions of Eq. (3.5). The latter have the form

$$_\zeta\varphi_n(x) = {}_\zeta\mathcal{N} \exp(i\zeta|p^L|x)[1 + e^{2x/\alpha}]^{-i(\zeta|p^L|+|p^R|)\alpha/2} {}_\zeta u(x),$$
$$_+u(x) = F(a, b; c; \xi), \quad _-u(x) = F(a+1-c, b+1-c; 2-c; \xi);$$
$$^\zeta\varphi_n(x) = {}^\zeta\mathcal{N} \exp(i|p^L|x)[1 + e^{2x/\alpha}]^{i(\zeta|p^R|-|p^L|)\alpha/2\zeta} u(x),$$
$$^+u(x) = F(c-a, c-b; c+1-a-b; 1-\xi),$$
$$^-u(x) = F(a, b; a+b+1-c; 1-\xi),$$
$$a = \frac{i\alpha}{2}(|p^L| + |p^R|) + \frac{1}{2} + \left(\frac{1}{4} - (eE\alpha^2)^2 + i\chi eE\alpha^2\right)^{1/2},$$
$$b = \frac{i\alpha}{2}(|p^L| + |p^R|) + \frac{1}{2} - \left(\frac{1}{4} - (eE\alpha^2)^2 + i\chi eE\alpha^2\right)^{1/2},$$
$$c = 1 + i\alpha|p^L|, \qquad \xi = \frac{1}{2}\left(1 + \tanh\frac{x}{\alpha}\right),$$

(9.2)

where $F(a, b; c; \xi)$ is the hypergeometric series of variable ξ with the normalization $F(a, b; c; 0) = 1$, [52]. As was already mentioned in Sec. II, the quantity χ can be chosen to be either $\chi = +1$ or $\chi = -1$, and $_\zeta\mathcal{N}$ and $^\zeta\mathcal{N}$ are normalization factors given by Eq. (3.37).

A formal transition to the Bose case can be done by setting $\chi = 0$ in Eqs. (9.2). In this case $n = (p_0, \mathbf{p}_\perp)$, and $_\zeta\mathcal{N}$ and $^\zeta\mathcal{N}$ are normalization factors given by Eq. (3.44).

For fermions, using Kummer's relations and Eq. (3.39), one can find coefficients $g(_+|^-)^*$ to be

$$g(_+|^-)^* = -\eta_R \frac{_+C\Gamma(c)\Gamma(c-a-b)}{^-C\Gamma(c-a)\Gamma(c-b)}, \qquad (9.3)$$

where $_+C$ and ^-C are constants given by Eq. (3.37). Then

$$|g(_+|^-)|^{-2} = \frac{\sinh(\pi\alpha|p^L|)\sinh(\pi\alpha|p^R|)}{|\sinh\{\pi\alpha[eE\alpha + \frac{1}{2}(|p^L| - |p^R|)]\}\sinh\{\pi\alpha[eE\alpha - \frac{1}{2}(|p^L| - |p^R|)]\}|}. \qquad (9.4)$$

In a similar manner we obtain coefficients $g(_+|^-)^*$ for bosons,

$$g(_+|^-)^* = -\frac{_+C\Gamma(c)\Gamma(c-a-b)}{^-C\Gamma(c-a)\Gamma(c-b)}, \qquad (9.5)$$

where $_+C$ and ^-C are given by Eqs. (3.44) and parameters a, b, and c are given by Eq. (9.2) at $\chi = 0$. Then

$$|g(_+|^-)|^{-2} = \frac{\sinh(\pi\alpha|p^L|)\sinh(\pi\alpha|p^R|)}{\cosh^2\left[\pi\sqrt{(eE\alpha^2)^2 - \frac{1}{4}}\right] + \sinh^2\left[\frac{\pi\alpha}{2}(|p^L| - |p^R|)\right]}. \qquad (9.6)$$

Relations (9.4) and (9.6) for $|g(_+|^-)|^{-2}$ hold in the ranges Ω_1, Ω_5, and Ω_3. However, their interpretations in

the range Ω_3 and in the ranges Ω_1, Ω_5 are completely different.

Using Eqs. (5.5) and (5.6), we find reflection $|R_{\zeta,n}|^2$ and transmission $|T_{\zeta,n}|^2$ probabilities for electrons in the range Ω_1 and for the positrons in the range Ω_5, that formally have the same form in terms of the quantities $|g(_+|^-)|$,

$$|R_{\zeta,n}|^2 = 1 - |T_{\zeta,n}|^2,$$
$$|T_{\zeta,n}|^2 = |g(_+|^+)|^{-2} = [1 + |g(_+|^-)|^2]^{-1}. \quad (9.7)$$

However, the latter quantities are given by Eq. (9.4) for fermions and by Eq. (9.6) for bosons. As was already noted in Sec. V B, Eqs. (9.7) imply that $|T_{\zeta,n}|^2 \leq 1$.

In the range Ω_3, the quantity N_n^{cr} has the form $N_n^{cr} = |g(_+|^-)|^{-2}$ (7.7), where $|g(_+|^-)|^{-2}$ are given by Eq. (9.4) for fermions and by Eq. (9.6) for bosons. As was already demonstrated, in the general case, for fermions one has $N_n^{cr} \leq 1$.

According to our interpretation presented in Sec. VII D, it is known that in the range Ω_3, similar to the ranges Ω_2 and Ω_4, the in-electron or the in-positron are subjected to the total reflection such that the corresponding transmission probabilities vanish. In this case, $w_n(+|+)$ and $w_n(-|-)$ represent relative probability amplitudes of the reflection. In spite of the fact that the transmission probabilities vanish, the relative probabilities of the reflection are not equal to unit due to the vacuum instability. The quantities $w_n(+-|0)$ and $w_n(0|-+)$ are relative probability amplitudes of an electron-positron pair creation and annihilation, respectively.

Using Eqs. (7.17) and (A13), we can rewrite these relative probabilities and the probability for a vacuum to remain a vacuum as follows:

$$|w_n(+-|0)|^2 = |g(_+|^+)|^{-2} = [|g(_+|^-)|^2 - \kappa]^{-1}$$
$$= N_n^{cr}(1 - \kappa N_n^{cr})^{-1},$$
$$[p_v^n]^{-\kappa} = |w_n(-|-)|^2 = |g(_+|^-)|^2 |g(_+|^+)|^{-2}$$
$$= (1 - \kappa N_n^{cr})^{-1},$$
$$P_v = |c_v|^2 = \prod_{n \in \Omega_3} p_v^n = \prod_{n \in \Omega_3} (1 - \kappa N_n^{cr})^\kappa,$$
$$\kappa = \begin{cases} +1, & \text{for fermions} \\ -1, & \text{for bosons} \end{cases}. \quad (9.8)$$

For the first time these formulas were obtained by Nikishov in the framework of one-particle relativistic quantum mechanics in Refs. [2,13].

Equations (9.4) and (9.6) allow one to verify that for any $\pi_\perp \neq 0$ one of the following limits holds true:

$$|g(_+|^-)|^{-2} \sim |\alpha p^R| \to 0,$$
$$|g(_+|^-)|^{-2} \sim |\alpha p^L| \to 0. \quad (9.9)$$

These limits imply the following properties of the coefficients $|g(_+|^-)|$ in the case of the Sauter step:

(a) $|g(_+|^-)|^{-2} \to 0$ in the range Ω_1 if n tends to the boundary with the range Ω_2 ($|p^R| \to 0$)

(b) $|g(_+|^-)|^{-2} \to 0$ in the range Ω_5 if n tends to the boundary with the range Ω_4 ($|p^L| \to 0$)

(c) $|g(_+|^-)|^{-2} \to 0$ in the range Ω_3 if n tends to the boundary with the range Ω_2 ($|p^R| \to 0$)

(d) $|g(_+|^-)|^{-2} \to 0$ in the range Ω_3 if n tends to the boundary with the range Ω_4 ($|p^L| \to 0$)

Namely these properties are essential for supporting the interpretation proposed by us in Secs. VI and VII C.

B. Integral quantities

Usually Sauter potential is used for imitating a slowly alternating electric field or a small-gradient field. To this end the parameter α is taken to be sufficiently large. Let us consider just this case, supposing that

$$eE\alpha^2 \gg 1. \quad (9.10)$$

Let us consider the total number N^{cr} of pairs created from the vacuum by Sauter potential with a large parameter α. This quantity can be calculated using Eq. (7.13) with differential numbers N_n^{cr} given by Eq. (E3) in the Appendix E. In the case under consideration these numbers are the same for fermions and bosons and do not depend on the spin polarization parameters σ_s. Thus, for fermions, the probabilities and mean numbers summed over all σ_s are $J_{(d)} = 2^{\lfloor \frac{d}{2} \rfloor - 1}$ times greater than the corresponding differential quantities. To get the total number N^{cr} of fermion pairs created in all possible states one has to sum over the spin projections and then over the momenta $\mathbf{p}_\perp$ and energy p_0. The latter sum can be easily transformed into an integral as follows

$$N^{cr} = \sum_{\mathbf{p}_\perp, p_0 \in \Omega_3} \sum_\sigma N_n^{cr}$$
$$= \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \int_{\Omega_3} dp_0 d\mathbf{p}_\perp N_n^{cr}, \quad (9.11)$$

where $V_\perp$ is the spatial volume of the $(d-1)$- dimensional hypersurface orthogonal to the electric field direction and T is the time duration of the electric field. The total number of boson pairs created in all possible states follows from Eq. (9.11) at $J_{(d)} = 1$.

To calculate the integral in the right-hand side of Eq. (9.11), we can find a subrange $D \subset \Omega_3$ where this integral is collected. It is demonstrated in Appendix E that the quantity N_n^{cr} is almost zero in some areas near the boundary of the range Ω_3. Such areas are characterized by the conditions

$$\pi\alpha|p^R| < 1 \quad \text{or} \quad \pi\alpha|p^L| < 1.$$

For areas that are closer to the center of the range Ω_3, where either $1 \lesssim \pi\alpha|p^R| \lesssim \pi k m \alpha$ or $1 \lesssim \pi\alpha|p^L| \lesssim \pi k m \alpha$, the quantity N_n^{cr} satisfies Eqs. (E12). Therefore it is almost zero if

$$k \ll \frac{\pi m \alpha}{2}. \tag{9.12}$$

We assume that inequality (9.12) holds true together with Eq. (E9), found in the Appendix E. Therefore, the main contribution to integral (9.11) is due to the subrange $D \subset \Omega_3$ that is defined by Eqs. (E7) and (E14) (see Appendix E). In this subrange the functions N_n^{cr} can be approximated by Eq. (E15), and the integral (9.11) can be represented as

$$N^{cr} \approx \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \int_{\alpha\pi_\perp < K_\perp} d\mathbf{p}_\perp I_{p_\perp},$$

$$I_{p_\perp} = 2 \int_0^{eE\alpha - K/\alpha} dp_0 e^{-\pi\tau}, \tag{9.13}$$

where τ is given by Eq. (E15). By using a variable s, defined as $\tau = \lambda(s^2 + 1)$, one can represent the quantity $I_{p_\perp}$ as follows

$$I_{p_\perp} = 2 \int_0^{s_{max}} e^{-\pi\lambda(s^2+1)} f(p_0(s)) ds, \tag{9.14}$$

where the number s_{max} is defined by the relation $\tau_{max} = \lambda(s_{max}^2 + 1)$ and τ_{max} is given by Eq. (E17). Note that the expansion of τ in powers of p_0 has the form

$$\tau = \lambda + \frac{\lambda}{(eE\alpha)^2} p_0^2 + \cdots \tag{9.15}$$

The leading contribution to integral (9.14) is formed at $s \to 0$, or equivalently as $p_0/eE\alpha \to 0$. Using expansions

$$p_0(s) = eE\alpha s(1 + c_2 s^2 + c_4 s^4 + \cdots)$$
$$\Rightarrow f(p_0(s)) = eE\alpha(1 + 3c_2 s^2 + \cdots),$$

where finite coefficients $c_2, c_4, \ldots$, can be found from Eq. (9.15), we obtain the following asymptotic expressions for the quantity $I_{p_\perp}$,

$$I_{p_\perp} \approx 2eE\alpha \int_0^{s_{max}} e^{-\pi\lambda(s^2+1)} ds$$

$$\approx 2eE\alpha \int_0^\infty e^{-\pi\lambda(s^2+1)} ds. \tag{9.16}$$

Substituting it into integral (9.13) and neglecting exponentially small contribution from the integration over $\pi_\perp > K_\perp/\alpha$, we find

$$N^{cr} \approx V_\perp T n^{cr},$$

$$n^{cr} = \frac{J_{(d)} 2eE\alpha}{(2\pi)^{d-1}} \int_0^\infty ds \int d\mathbf{p}_\perp e^{-\pi\lambda(s^2+1)}. \tag{9.17}$$

It should be noted that the density n^{cr} (per the $d-1$ space volume) of pairs created by t-electric potential step given by the Sauter-type vector potential $A_1(x^0) = \alpha E \tanh(t/\alpha)$ with large $\alpha(eE\alpha^2 \gg \max(1, m^2/eE))$ is given by the same integral (9.17) as was demonstrated for the first time in our work [53].

Finally, performing the integration over $\mathbf{p}_\perp$, we obtain

$$n^{cr} = \frac{J_{(d)}\alpha\delta}{(2\pi)^{d-1}} (eE)^{\frac{d}{2}} \exp\left\{-\pi\frac{m^2}{eE}\right\}. \tag{9.18}$$

Here

$$\delta = \int_0^\infty dt\, t^{-1/2} (t+1)^{-(d-2)/2} \exp\left(-t\pi\frac{m^2}{eE}\right)$$
$$= \sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; \pi\frac{m^2}{eE}\right),$$

where $\Psi(a, b; x)$ is the confluent hypergeometric function [52], and $J_{(d)} = 1$ for bosons.

The vacuum-to-vacuum transition probability P_v reads

$$P_v = \exp(-\mu N^{cr}),$$

$$\mu = \sum_{j=0}^\infty \frac{(-1)^{(1-\kappa)j/2} \epsilon_{j+1}}{(j+1)^{d/2}} \exp\left(-j\pi\frac{m^2}{eE}\right),$$

$$\epsilon_j = \delta^{-1}\sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; j\pi\frac{m^2}{eE}\right). \tag{9.19}$$

If $eE/m^2 \ll 1$, one can use the asymptotic expression for the Ψ-function [52],

$$\Psi(1/2, -(d-2)/2; j\pi m^2/eE)$$
$$= (eE/j\pi m^2)^{1/2} + O([eE/m^2]^{3/2}).$$

Then $\delta \approx \sqrt{eE/m^2}$, $\epsilon_j \approx j^{-\frac{1}{2}}$ and $\mu \approx 1$.

In $d = 4$, the formula (9.19) reproduces a result obtained in Ref. [41] for bosons, and a result obtained in Ref. [42] for fermions.

C. The Klein step

The Klein paradox was discovered by Klein [6] who calculated, using the Dirac equation, reflection and transmission probabilities of charged particles incident on a sufficiently high rectangular potential step (Klein step) of the form

$$qA_0(x) = \begin{cases} U_L, x < 0 \\ U_R, x > 0 \end{cases}, \qquad (9.20)$$

where U_R and U_L are constants. According to calculations of Klein and other authors, for certain energies and sufficient high magnitude $U = U_R - U_L$ of the Klein step, there are more reflected fermions than incident. This is what many articles and books call the Klein paradox. Let us study quantum processes near the Klein step, applying our approach to the Sauter potential with α sufficiently small, $\alpha \to 0$.

The Sauter potential with constant asymptotic potentials, $U_R = -U_L = \mathbb{U}/2 = eE\alpha$ and with small α,

$$\mathbb{U}\alpha \ll 1, \qquad (9.21)$$

imitates the Klein step (9.20) sufficiently well, and coincides with the latter as $\alpha \to 0$. Thus, the Sauter potential can be considered as the regularization of the Klein step.

In the ranges Ω_1 and Ω_5 the energy $|p_0|$ is not restricted from the above, that is why in what follows we consider only the subranges, where

$$\max \{\alpha|p^L|, \alpha|p^R|\} \ll 1. \qquad (9.22)$$

Then in the leading-term approximation in α it follows from Eqs. (9.4) and (9.6) that

$$|g(_+|^-)|^{-2} \approx \frac{4k}{(1-k)^2},$$

$$k = \begin{cases} k_f = k_b \frac{\pi_0(L) + \pi_\perp}{\pi_0(R) + \pi_\perp}, & \text{for fermions} \\ k_b = \frac{|p^R|}{|p^L|}, & \text{for bosons} \end{cases}, \qquad (9.23)$$

where k is called the kinematic factor.

Note that in the ranges Ω_1 and Ω_5 we have that both k_b and k_f are positive and do not achieve the unit values, $k_b \neq 1$, $k_f \neq 1$.

It should be noted that the quantity $|g(_+|^-)|^{-2}$ was calculated in Refs. [6,7,12,14] only at $p_\perp = 0$. Equation (9.23) contains these results as a particular case.

In the ranges Ω_1 and Ω_5, coefficients g satisfy the same relations for bosons and fermions,

$$|g(_+|^+)|^2 = |g(_+|^-)|^2 + 1. \qquad (9.24)$$

Therefore, reflection and transmission probabilities derived from Eqs. (9.23) have the same forms, common for bosons and fermions

$$|T_{\zeta,n}|^2 = |g(_+|^+)|^{-2} = \frac{4k}{(1+k)^2},$$

$$|R_{\zeta,n}|^2 = |g(_+|^-)|^2|g(_+|^+)|^{-2} = \frac{(1-k)^2}{(1+k)^2}. \qquad (9.25)$$

To compare our exact results with results of the non-relativistic consideration obtained in any textbook for one dimensional quantum motion, we set $p_\perp = 0$, then $\pi_\perp = m$, $\pi_0(L) = p_0 = m + E$, and $\pi_0(R) = p_0 - \mathbb{U} = m + E - \mathbb{U}$. In this case

$$k_f = \mu k_b, \qquad \mu = \frac{\pi_0(L) + m}{\pi_0(R) + m} = [1 - \mathbb{U}/(E + 2m)]^{-1}.$$

For sufficiently small steps $\mathbb{U} \ll E + 2m$, we have $\mu \approx 1 + \mathbb{U}/(E + 2m)$. In the nonrelativistic limit, when $E \ll m$, we obtain

$$k_b = k_f = k^{NR} = \sqrt{\frac{E - \mathbb{U}}{E}},$$

which can be identified with nonrelativistic results, e.g., see [48]. Relativistic corrections have different forms for bosons and fermions,

$$k_b \approx k^{NR}\left(1 - \frac{\mathbb{U}}{4m}\right), \qquad k_f \approx k^{NR}\left(1 + \frac{\mathbb{U}}{4m}\right).$$

Let us consider the range Ω_3. Here quantum numbers $\mathbf{p}_\perp$ are restricted by the inequality $2\pi_\perp \leq \mathbb{U}$ and for any of such $\pi_\perp$ quantum numbers p_0 obey the strong inequality (3.31), see Fig. 3. In this range the quantity $|g(_+|^-)|^{-2}$ represents the differential mean numbers of electron-positron pairs created from the vacuum, $N_n^{cr} = |g(_+|^-)|^{-2}$. In this range for any given $\pi_\perp$ the absolute values of $|p^R|$ and $|p^L|$ are restricted from above, see (D6). Therefore, condition (9.21) implies Eq. (9.22). Then it follows from Eq. (9.4) that for fermions in the leading approximation the following result holds true

$$|g(_+|^-)|^{-2} \approx \frac{4|p^L||p^R|}{\mathbb{U}^2 - (|p^L| - |p^R|)^2} = \frac{4|k_f|}{(1 + |k_f|)^2}. \qquad (9.26)$$

Note that expression (9.26) differs from expression (9.23) only by the sign of the kinematic factor k_f. This factor is positive in Ω_1 and Ω_5, and it is negative in Ω_3. In the range Ω_3, the difference $|p^L| - |p^R|$ may be zero at $p_0 = 0$, which corresponds to $k_f = -(\mathbb{U} + 2\pi_\perp)/(\mathbb{U} - 2\pi_\perp)$. Namely in this case the quantity $|g(_+|^-)|^{-2}$ has a maximum at a given $\pi_\perp$,

$$\max |g(_+|^-)|^{-2} = 1 - (2\pi_\perp/\mathbb{U})^2. \quad (9.27)$$

As it follows from Eq. (9.6) for bosons in the range Ω_3 and in the leading approximation, the quantity $|g(_+|^-)|^{-2}$ reads

$$|g(_+|^-)|^{-2} \approx \frac{4|p^{\mathrm{L}}||p^{\mathrm{R}}|}{\alpha^2 \mathbb{U}^4/2 + (|p^{\mathrm{L}}| - |p^{\mathrm{R}}|)^2}. \quad (9.28)$$

It has a maximum at $|p^{\mathrm{L}}| - |p^{\mathrm{R}}| = 0$,

$$\max |g(_+|^-)|^{-2} = \frac{2}{(\alpha \mathbb{U})^2}\left[1 - \left(\frac{2\pi_\perp}{\mathbb{U}}\right)^2\right]. \quad (9.29)$$

However, in contrast with the Fermi case the limit $\alpha \to 0$ in (9.28) is possible only when the difference $|p^{\mathrm{L}}| - |p^{\mathrm{R}}|$ is not very small, namely when

$$\alpha^2 \mathbb{U}^4/2 \ll (|p^{\mathrm{L}}| - |p^{\mathrm{R}}|)^2.$$

Only under the latter condition one can neglect an α-depending term in Eq. (9.28) to obtain

$$|g(_+|^-)|^{-2} \approx \frac{4k_b}{(1 - k_b)^2}, \quad (9.30)$$

which coincides with Eq. (9.23).

The same results for $|g(_+|^-)|^{-2}$ in the forms (9.26) and (9.30) at $p_\perp = 0$ were obtained in Refs. [6,7,12,14].

In the range Ω_3, relation (9.24) still holds for bosons, whereas for fermions we have

$$|g(_+|^+)|^2 = |g(_+|^-)|^2 - 1. \quad (9.31)$$

Relative probability amplitudes of the reflection and of electron-positron pair creation follows from Eqs. (9.8), (9.26), and (9.30) to be

$$|w_n(+ - |0)|^2 = |g(_+|^+)|^{-2} = \frac{4|k|}{(1 + k)^2},$$

$$|w_n(-|-)|^2 = |g(_+|^-)|^2|g(_+|^+)|^{-2} = \frac{(1 - k)^2}{(1 + k)^2}. \quad (9.32)$$

We see that expressions (9.32) for $|w_n(+ - |0)|^2$ and $|w_n(-|-)|^2$ are quite similar to the forms of transmission and reflection probabilities given by Eqs. (9.25), respectively. However, in case of fermions, the range of values of these functions is quite different because $k_f < 0$ in Eq. (9.32). This is natural, since the interpretation of these quantities in the range Ω_3 differ essentially from their interpretation in the ranges Ω_1 and Ω_5. This formal similarity was the reason for the systematic misunderstanding in treating quantum processes in the Klein zone.

D. Klein paradox

We remind that the Klein paradox dating back to the works of Klein [6] and Sauter [7,8] (see Sommerfeld [54], as well) is that when considering scattering of relativistic electrons on a high step potential in the context of the Dirac equation one comes to a strange result that there is more reflected electrons than incoming. One of the initial resolutions of the paradox was reduced to the impossibility that there is no possibility of establishing the appropriate high-step potential. However, later Hund studied the paradox in connection with pair production [55] and it seems that Feynman was the first to point out that the paradox should disappear in a field-theoretical treatment [56]. A detailed historical review can be found in Refs. [12,14]. The absence of the Klein paradox in the framework of appropriate field theoretical interpretation of solutions of the Dirac and Klein-Gordon equations was first demonstrated by Nikishov in Refs. [2,13]. In these works Nikishov used a reformulation of the Green theorem to demonstrate an analogy between the propagation in time t and in the space coordinate x and thus to identify solutions of the Dirac equation that describe electrons and positrons. Nikishov had tested his way of calculation using the special case of a constant and uniform electric field. In this case, explicit solutions of the Dirac and Klein-Gordon equations can be found in the constant electric field, which can be described either by a vector potential with only one nonzero component $A_1(t) = Et$, or by a scalar potential $A_0(x) = -Ex$ alone, only, see details in [17]. The first case can be treated as a degenerated t-electric potential step and the second case can be treated as a degenerated x-electric potential step. Comparison of exact solutions in these cases allowed that author to confirm his interpretation for x-potential steps referring to the well developed Feynman interpretation of the t-electric potential step. His calculations give a clear qualitative explanation of the physics involved in Klein scattering. However complete consideration of the scattering on arbitrary x-electric potential steps in the frame work of a consistent QFT consideration was not given.

Applying our general approach to particular cases that were studied by Nikishov, we obtain the same results. In particular, his point of view that the scattering theory that works within the ranges Ω_1 and Ω_5 cannot be applied to the range Ω_3 has a clear support in our general approach.

In the ranges Ω_1 and Ω_5 transmission and reflection probabilities for bosons and fermions are expressed via the coefficients g as follows

$$|T_{\zeta,n}|^2 = |g(_+|^+)|^{-2},$$
$$|R_{\zeta,n}|^2 = |g(_+|^-)|^2|g(_+|^+)|^{-2}. \quad (9.33)$$

As follows from unitarity relations the sum of these probabilities satisfies the relation of probability conservation,

$$|R_{\zeta,n}|^2 + |T_{\zeta,n}|^2 = 1. \qquad (9.34)$$

In particular, for the Klein step we obtain Eq. (9.25)

$$|g(_+|^+)|^{-2} = \frac{4k}{(1+k)^2}, \qquad |g(_+|^-)|^2|g(_+|^+)|^{-2} = \frac{(1-k)^2}{(1+k)^2},$$

$$k = \begin{cases} k_f, & \text{Fermi case} \\ k_b, & \text{Bose case} \end{cases}. \qquad (9.35)$$

In the range Ω_3, we obtain for the Klein step[5]

$$|g(_+|^+)|^{-2} = \frac{4|k|}{(1+k)^2},$$

$$|g(_+|^-)|^2|g(_+|^+)|^{-2} = \frac{(1-k)^2}{(1+k)^2}, \qquad (9.36)$$

where for fermions one has $k = k_f < 0$, see Eq. (9.26) and Eq. (9.30).

If by analogy with the ranges Ω_1 and Ω_5 we believe that $|g(_+|^+)|^{-2}$ and $|g(_+|^-)|^2|g(_+|^+)|^{-2}$ are transmission and reflection probabilities, respectively, then we have to accept that there will apparently be more fermions reflected than coming in. This is the situation first considered by Klein. Besides, in this case there will apparently be more fermions transmitted than coming in and the following relation will hold true

$$|g(_+|^-)|^2|g(_+|^+)|^{-2} - |g(_+|^+)|^{-2} = 1, \qquad (9.37)$$

which does not imply Eq. (9.34). These contradictions do not exist in the framework of our approach with correct interpretation of the quantities (9.36). Indeed, as follows from Eq. (7.17) the quantity $|g(_+|^+)|^{-2}$ is the relative probability of the electron-positron pair creation,

$$|g(_+|^+)|^{-2} = |w_n(+ - |0)|^2, \qquad (9.38)$$

and $|g(_+|^-)|^2|g(_+|^+)|^{-2}$ is the relative probability of the electron (positron) reflection,

$$|g(_+|^-)|^2|g(_+|^+)|^{-2} = |w_n(- |-)|^2 = |w_n(+ |+)|^2, \qquad (9.39)$$

in the range Ω_3. Besides, the x-electric potential step creates pairs in the region Ω_3, the differential numbers of such pairs being $N_n^{cr} = |g(_+|^-)|^{-2}$ (7.7). In this situation the many-particle nature of the problem is essential, and relations of probability conservation are quite different in the range Ω_3 and in the ranges Ω_1 and Ω_5. Unitarity of canonical transformation between the in- and out- creation and annihilation operators was proved in the general case in

Sec. VII D. Here it is enough to mention that for fermions the quantity

$$p_v^n = |w_n(+|+)|^{-2} = (1 - N_n^{cr}) \qquad (9.40)$$

is the probability that the partial vacuum state with a given n remains a vacuum. Due to the Pauli principle, if an initial state is vacuum, there are only two possibilities in a cell of the space with given quantum number n, namely, this partial vacuum remains a vacuum, or with the probability $p_v^n|w_n(+ - |0)|^2$ a pair with the quantum number n will be created. Then Eq. (9.37) is just the condition of probability conservation,

$$p_v^n + p_v^n|w_n(+ - |0)|^2 = 1. \qquad (9.41)$$

It is obvious that many particle consideration is necessary for correct interpretation of relation (9.37). The same is true when we consider one-electron initial state with a given n. Here again due to the Pauli principle, creation of a pair of fermions with the same quantum number is impossible. Then the total reflection of the initial particle on the x-electric potential step is described by the conditional probability of reflection under the condition that in all other cells of the space with quantum numbers $m \neq n$ partial vacua remain vacua. Such a probability has the form $|w_n(+|+)|^2 p_v^n$. Consequently,

$$|w_n(+|+)|^2 p_v^n = 1. \qquad (9.42)$$

This result is consistent with calculations of mean occupation numbers given in Sec. C 2.

For bosons, in the range Ω_3, the correct interpretation of quantities $|g(_+|^+)|^{-2}$, $|g(_+|^-)|^2|g(_+|^+)|^{-2}$ and $|g(_+|^-)|^{-2}$ coincides with the one for fermions, they are relative probabilities of a pair creation and electron/positron reflection respectively. However, for bosons an additional relation holds,

$$|g(_+|^-)|^2|g(_+|^+)|^{-2} + |g(_+|^+)|^{-2} = 1. \qquad (9.43)$$

It formally can be understood as relation (9.34). However, here equation (9.43) is a relation for relative probabilities, and does not admit an interpretation in the framework of one-particle theory. Due to the vacuum instability the probability conservation has to be considered in the framework of many-particle theory, taking into account that in the bosons case in a given state n any number of pairs can be created. The unitarity of canonical transformation between the in- and out- operators was proven in Appendix A. Here it should be noted that for boson the probability that the partial vacuum state with given n remains a vacuum is

$$p_v^n = |w_n(+|+)|^2 = (1 + N_n^{cr})^{-1} \qquad (9.44)$$

---

[5]In Sec. IX C we have demonstrated that for bosons Eq. (9.30) holds true only in a part of Ω_3, where the difference $|p^L| - |p^R|$ is not very small.

The conditional probability of a pair creation with a given quantum numbers n, under the condition that all other partial vacua with the quantum numbers $m \neq n$ remain the vacua is the sum of probabilities of creation for any number l of pairs

$$P(\text{pairs}|0)_n = p_v^n \left[\sum_{l=1}^{\infty} |w_n(+-|0)|^{2l} \right]. \quad (9.45)$$

In this case Eq. (9.43) is the probability conservation law in the form of a sum of probabilities of all possible events in a cell of the space of quantum numbers n:

$$P(\text{pairs}|0)_n + p_v^n = 1. \quad (9.46)$$

X. SUMMARY

When quantizing charged fields (those of Dirac and Klein-Gordon) in the presence of x-electric potential steps, we succeeded to describe quantum theory of the systems under consideration in terms of adequate in- and out-particles. These particles represent positive-energy excitations above the corresponding in- and out-vacua and have all natural physical properties inherent to such particles in various examples of known QFT models. In fact, the idea of introducing such particles is an advancement of the well-known Furry picture in QED with external magnetic field [57] and of the generalized Furry picture in QED with t-electric potential steps [5]. For the class of external electromagnetic field, which we identify as the x-electric potential steps, we define special solutions of the relativistic Dirac and Klein-Gordon wave equations that expand the corresponding Heisenberg field operators in adequate in- and out-creation and annihilation operators related to the in- and out-particles. Solutions, which we have used in the implementation of the quantization program, were chosen as stationary solutions with special asymptotic behavior at the remote left and remote right sides of the potential step (of course, this choice is not unique). These solutions, by their asymptotic behavior, are labeled also by a set of quantum numbers n that include the total energy p_0, transverse momenta $\mathbf{p}_\perp$, and the spin polarization (the latter in the case of the Dirac field). In the most general case of critical steps with the potential difference $\mathbb{U} > 2m$, there exist five ranges of quantum numbers n, where these solutions have similar forms, and physical processes with the corresponding in- and out-particles have similar interpretation. A detailed consideration of various physical processes with these particles had confirmed justified their definitions and had demonstrated that:

(a) In the first range $p_0 \geq U_R + \pi_\perp$ ($\pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}$) there exist only in- and out-electrons, whereas in the fifth range $p_0 \leq U_L - \pi_\perp$ there exist only in- and out-positrons. In these ranges electrons and positrons are subjected to the scattering and the reflection only. No particle creation in these ranges is possible.

(b) In the second range $U_R - \pi_\perp < p_0 < U_R + \pi_\perp$ if $2\pi_\perp \leq \mathbb{U}$, similar to the first range, there exist only electrons that are subjected to the total reflection. In the fourth range $U_L - \pi_\perp < p_0 < U_L + \pi_\perp$ there exist only positrons that are also subjected to the total reflection.

(c) In the third range $U_L + \pi_\perp \leq p_0 \leq U_R - \pi_\perp$, which exists only for critical steps and for transversal momenta that satisfy the inequality $2\pi_\perp \leq \mathbb{U}$, there exist in- and out-electrons that can be situated only to the left of the step, and in- and out-positrons that can be situated only to the right of the step. In the third range, all the partial vacua are unstable, processes of pair creation are possible. The pairs consist of out-electrons and out-positrons that appear on the left and on the right of the step and move there to the left and to the right, respectively. At the same time, the in-electrons that move to the step from the left are subjected to the total reflection. After being reflected they move to the left of the step already as out-electrons. Similarly, the in-positrons that move to the step from the right are subjected to the total reflection. After being reflected they move to the right of the step already as out-positrons.

We elaborated a technique that allows one to calculate all the above described processes (zero-order processes) and also to calculate Feynman diagrams that describe all the processes of interaction between the introduced in- and out-particles and photons. These diagrams have formally the usual form, but contain special propagators. Constructions of these propagators in terms of introduced in- and out-solutions are presented. It should be noted that calculations in terms of these Feynman diagrams (as well as calculations of zero-order processes) are nonperturbative, in such calculations interaction with external field of x-electric potential steps are taken into account exactly. Another interesting feature is worth noting: when considering reflection and transmission of in-particles in the first and fifth ranges the formalism of QFT allows one to calculate both the probability amplitudes of transitions between in- and out-states and the mean currents of out-particles in the in-states, testing in such a way the rules of one-particle time-independent potential scattering theory and its applicability.

Finally, the developed theory is applied to exactly solvable cases of x-electric potential steps, namely, to the Sauter potential, and to the Klein step. We present a consistent QFT treatment of processes, where a naive one-particle consideration might lead to the Klein paradox. From this point of view we comment various approaches known in the literature that use pure one-particle consideration or its partial combination with elements of QFT.

ACKNOWLEDGMENTS

APPENDIX A: SOME DETAILS OF SCALAR FIELD QUANTIZATION

The quantization of scalar field in terms of adequate in- and out-particles can be given along the same lines as the quantization for Dirac field. Having results of Sec. III, a quantum scalar field $\hat{\Psi}(X)$ can be written as the sum $\hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X)$ of five operators, each one $\hat{\Psi}_i(X)$ in the range Ω_i. The operators $\hat{\Psi}_i(X)$, $i = 1, 2, 4, 5$. have similar to the fermionic case [see Eqs. (4.5) and (4.6)] decompositions in terms of the creation and annihilation operators and only the operator $\hat{\Psi}_3(X)$,

$$\hat{\Psi}_3(X) = \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}[^+a_{n_3}(\text{in})^+\psi_{n_3}(X) + {}_+b_{n_3}^\dagger(\text{in})_+\psi_{n_3}(X)]$$
$$= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}[^-a_{n_3}(\text{out})^-\psi_{n_3}(X)$$
$$+ {}_-b_{n_3}^\dagger(\text{out})_-\psi_{n_3}(X)], \qquad (A1)$$

is distinct from the corresponding form in Eqs. (4.5) due to the $\pm$ signs in the right-hand side of Eq. (3.50). In what follows, we consider some peculiarities of the quantization of the scalar field in the range Ω_3.

Taking into account relations (3.50) and Eqs. (3.53)–(3.55), one can see that commutation relations (4.3) imply the commutation rules for the introduced creation and annihilation in- and out-operators: all creation (annihilation) operators with different quantum numbers n commute between themselves; all the operators from different ranges Ω_i commute between themselves, and satisfy the canonical commutation relations. One defines two vacuum vectors $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$, using introduced annihilation operators. In the ranges $\Omega_i, i = 1, 2, 4, 5$ the corresponding equations have exactly the same form as in (4.27) and (4.28), whereas in the range Ω_3 they are different since here the in- and out-operators are different,

$$^+a_{n_3}(\text{in})|0, \text{in}\rangle = {}_+b_{n_3}(\text{in})|0, \text{in}\rangle = 0,$$
$$_-b_{n_3}(\text{out})|0, \text{out}\rangle = {}^-a_{n_3}(\text{out})|0, \text{out}\rangle = 0. \quad (A2)$$

Partial and total vacuum states are related by Eqs. (4.29), (4.30), (4.34), and (4.37). The introduced vacua have zero energy and electric charge and all the excitations above the vacuum have positive energies.

Canonical transformations between the in- and out-operators in the ranges Ω_1 and Ω_5 are similar to the fermionic case. In particular, amplitudes of electron reflection and transmission in the range Ω_1 have the same form (5.5) as in the fermionic case. The form of amplitudes of positron reflection and transmission in the range Ω_5 differs from Eqs. (5.5) only by phases.

Justification of the presented choice of in- and out-operators can be done similarly to the fermionic case. In the range Ω_3 there appear some peculiarities, since here we have different positions of $\pm$ superscripts and subscripts in comparing to the fermionic case, nevertheless the interpretation of sets $\{^\zeta\psi_n(X)\}$ as electron states and sets $\{_\zeta\psi_n(X)\}$ as positron states is the same. The canonical transformations between the in- and out-operators are

$$^-a_n(\text{out}) = g(_+|^-)^{-1}_+b_n^\dagger(\text{in}) + g(^+|_-)^{-1}g(^-|_-)^+a_n(\text{in}),$$
$$_-b_n^\dagger(\text{out}) = g(_+|^-)^{-1}g(_-|^-)_+b_n^\dagger(\text{in}) + g(^+|_-)^{-1+}a_n(\text{in}),$$
$$_+b_n^\dagger(\text{in}) = g(_-|^+)^{-1}g(_+|^+)_-b_n^\dagger(\text{out}) - g(^-|_+)^{-1-}a_n(\text{out}),$$
$$^+a_n(\text{in}) = -g(_-|^+)^{-1}_-b_n^\dagger(\text{out})$$
$$+ g(^-|_+)^{-1}g(^+|_+)^-a_n(\text{out}), \qquad n \in \Omega_3.$$
$$(A3)$$

Differential mean numbers of out-particles created from the in-vacuum are

$$N_{n_3}^a(\text{out}) = \langle 0, \text{in}|^-a_{n_3}^\dagger(\text{out})^-a_{n_3}(\text{out})|0, \text{in}\rangle = |g(_+|^-)|^{-2},$$
$$N_{n_3}^b(\text{out}) = \langle 0, \text{in}|_-b_{n_3}^\dagger(\text{out})_-b_{n_3}(\text{out})|0, \text{in}\rangle = |g(_-|^+)|^{-2}.$$
$$(A4)$$

It can be shown that they are equal and define differential mean number $N_{n_3}^{\text{cr}}$ of created pairs similarly to the fermionic case given by Eq. (7.7). In contrast to the case of fermions, the quantity N_n^{cr} is unbounded from above due to relation (3.42) with formal setting $\eta_L = \eta_R = 1$ for bosons. If N_n^{cr} tends to zero, $N_n^{\text{cr}} \to 0$, then $|g(_+|^-)|^2 \to \infty$ and, at the same time, $|g(_+|^+)|^2 \to \infty$ similar to the fermionic case. The total number N of pairs created from the vacuum is defined similarly to the fermionic case given by Eq. (7.13).

One can see that due to relation (3.42) with $\eta_L = \eta_R = 1$ the differential mean numbers of out-particles in one-particle in-states are

$$\langle 0, \text{in}|^+a_n(\text{in})^-a_n^\dagger(\text{out})^-a_n(\text{out})^+a_n^\dagger(\text{in})|0, \text{in}\rangle = 1 + 2N_n^{\text{cr}},$$
$$\langle 0, \text{in}|^+a_n(\text{in})_-b_n^\dagger(\text{out})_-b_n(\text{out})^+a_n^\dagger(\text{in})|0, \text{in}\rangle = 2N_n^{\text{cr}},$$
$$\langle 0, \text{in}|_+b_n(\text{in})_-b_n^\dagger(\text{out})_-b_n(\text{out})_+b_n^\dagger(\text{in})|0, \text{in}\rangle = 1 + 2N_n^{\text{cr}},$$
$$\langle 0, \text{in}|_+b_n(\text{in})^-a_n^\dagger(\text{out})^-a_n(\text{out})_+b_n^\dagger(\text{in})|0, \text{in}\rangle = 2N_n^{\text{cr}},$$
$$n \in \Omega_3,$$
$$(A5)$$

and see that Eqs. (A5) are quite different from Eqs. (C14) obtained for fermions. This a consequence of the absence of the Pauli principle. In this case $2N_n^{cr}$ is the differential mean number of scalar electrons (positrons) created by the external field. We see that the presence of a particle at the initial state increases the mean number of created bosons. It is known effect for bosons, e.g., see [49].

In the range Ω_3, we consider relative scattering amplitudes of scalar electrons and positrons,

$$w(+|+)_{n'n} = c_v^{-1} \langle 0, \text{out}|^- a_{n'}(\text{out})^+ a_n^\dagger(\text{in})|0, \text{in}\rangle,$$
$$w(-|-)_{n'n} = c_v^{-1} \langle 0, \text{out}|_- b_{n'}(\text{out})_+ b_n^\dagger(\text{in})|0, \text{in}\rangle, \quad (A6)$$

and relative amplitudes of a pair creation and a pair annihilation,

$$w(+-|0)_{n'n} = c_v^{-1} \langle 0, \text{out}|^- a_{n'}(\text{out})_- b_n(\text{out})|0, \text{in}\rangle,$$
$$w(0|-+)_{nn'} = c_v^{-1} \langle 0, \text{out}|_+ b_n^\dagger(\text{in})^+ a_{n'}^\dagger(\text{in})|0, \text{in}\rangle, \quad (A7)$$

where c_v is the vacuum-to-vacuum transition amplitude for bosons

$$c_v^{(3)} = {}^{(3)}\langle 0, \text{out}|0, \text{in}\rangle^{(3)} = c_v = \langle 0, \text{out}|0, \text{in}\rangle. \quad (A8)$$

As follows from relations (A3), all the amplitudes (A6) and (A7) are diagonal in the quantum numbers n and can be expressed in terms of the coefficients $g(^{\zeta'}|_\zeta)$ as follows:

$$w(+|+)_{n'n} = \delta_{n,n'} w_n(+|+),$$
$$w_n(+|+) = g(^-|_+)g(^+|_+)^{-1} = g(_-|^+)g(_-|^-)^{-1},$$
$$w(-|-)_{nn'} = \delta_{n,n'} w_n(-|-),$$
$$w_n(-|-) = g(^+|_-)g(^+|_+)^{-1} = g(_+|^-)g(_-|^-)^{-1},$$
$$w(0|-+)_{nn'} = \delta_{n,n'} w_n(0|-+),$$
$$w_n(0|-+) = -g(^+|_+)^{-1},$$
$$w(+-|0)_{n'n} = \delta_{n,n'} w_n(+-|0),$$
$$w_n(+-|0) = g(_-|^-)^{-1}, n \in \Omega_3. \quad (A9)$$

In the range Ω_3, similar to the case of fermions, the total reflection is the only possible form of particle scattering, with $w(+|+)_n$ and $w(-|-)_n$ being relative probability amplitudes of a particle reflection.

Unitary relations (3.40) and their consequences (3.41) and (3.42) with setting $\eta_L = \eta_R = 1$ for bosons imply the following connections for the introduced amplitudes w:

$$|w_n(+|+)|^2 = |w_n(-|-)|^2,$$
$$|w_n(+-|0)|^2 = |w_n(0|-+)|^2,$$
$$|w_n(+|+)|^2 + |w_n(+-|0)|^2 = 1,$$
$$\frac{w_n(-|-)^*}{w_n(+|+)} = -\frac{w_n(+-|0)^*}{w_n(0|-+)}. \quad (A10)$$

Using Eqs. (A9), two lower lines in relations (A3) can be rewritten as

$$^+ a_n(\text{in}) = w_n(+|+)^{-1}[^- a_n(\text{out}) - w_n(+-|0)_- b_n^\dagger(\text{out})],$$
$$_+ b_n(\text{in}) = w_n(-|-)^{-1}[_- b_n(\text{out}) - w_n(+-|0)^- a_n^\dagger(\text{out})]. \quad (A11)$$

Together with their adjoint relations they define an unitary transformation V_{Ω_3} between the in- and out-operators,

$$\{^+ a^\dagger(\text{in}), {}^+ a(\text{in}), {}_+ b^\dagger(\text{in}), {}_+ b(\text{in})\}$$
$$= V_{\Omega_3}\{^- a^\dagger(\text{out}), {}^- a(\text{out}), {}_- b^\dagger(\text{out}), {}_- b(\text{out})\} V_{\Omega_3}^\dagger.$$

Since Eqs. (A10) and (A11) formally coincide with the corresponding equations for the general case of time-dependent external field, the unitary operator V_{Ω_3} can be taken, for example, from [49] or book [5]. The operator V_{Ω_3} relates the in- and out-vacua, $|0, \text{in}\rangle = V_{\Omega_3}|0, \text{out}\rangle$ and determines the vacuum-to-vacuum transition amplitude,

$$c_v = \langle 0, \text{out}|V_{\Omega_3}|0, \text{out}\rangle = \prod_n w_n(-|-). \quad (A12)$$

The probabilities of a particle reflection, a pair creation, and the probability for a vacuum to remain a vacuum can be expressed via differential mean numbers of created pairs N_n^{cr}. By using the relation $|w_n(-|-)|^2 = (1 + N_n^{cr})^{-1}$, one finds

$$P(+|+)_{n,n'} = |\langle 0, \text{out}|^+ a_n(\text{out})^- a_{n'}^\dagger(\text{in})|0, \text{in}\rangle|^2$$
$$= \delta_{n,n'}(1 + N_n^{cr})^{-1} P_v,$$
$$P(+-|0)_{n,n'} = |\langle 0, \text{out}|^+ a_n(\text{out})_+ b_{n'}(\text{out})|0, \text{in}\rangle|^2$$
$$= \delta_{n,n'} N_n^{cr}(1 + N_n^{cr})^{-1} P_v,$$
$$P_v = |c_v|^2 = \prod_{n \in \Omega_3} p_v^n, \quad p_v^n = (1 + N_n^{cr})^{-1}. \quad (A13)$$

The probabilities for a positron scattering $P(-|-)$ and a pair annihilation $P(0|-+)$ coincide with the expressions $P(+|+)$ and $P(+-|0)$, respectively.

Note that p_v^n given by Eq. (A13) is the probability that the partial vacuum state with given n remains a vacuum. If all $N_n^{cr} \ll 1$ then in the leading approximation the relation $1 - P_v \approx N$ is the same with the case of fermions. The

vacuum instability is not essential if $N_n^{\mathrm{cr}} \to 0$. Then like for fermions $P_v \to 1$, $P(+|+)_{n,n} \to 1$ and $P(+-|0)_{n,n} \to N_n^{\mathrm{cr}}$.

Processes of higher orders are described by the Feynman diagrams with two kinds of charged scalar particle propagators in the external field under consideration, namely, the so-called in-out propagator $\Delta^c(X, X')$, which is just the causal Feynman propagator, and the so-called in-in propagator $\Delta_{\mathrm{in}}^c(X, X')$,

$$\Delta^c(X, X') = i\langle 0, \mathrm{out}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^\dagger(X')|0, \mathrm{in}\rangle c_v^{-1},$$
$$\Delta_{\mathrm{in}}^c(X, X') = i\langle 0, \mathrm{in}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^\dagger(X')|0, \mathrm{in}\rangle. \tag{A14}$$

We find the in-in propagator as

$$\Delta_{\mathrm{in}}^c(X, X') = \theta(t - t')\Delta_{\mathrm{in}}^-(X, X') - \theta(t' - t)\Delta_{\mathrm{in}}^+(X, X'),$$

$$\Delta_{\mathrm{in}}^-(X, X') = i\sum_{j=1}^{2} G_j(X, X') + \tilde{\Delta}_{\mathrm{in}}^-(X, X'),$$

$$\Delta_{\mathrm{in}}^+(X, X') = -i\sum_{j=4}^{5} G_j(X, X') + \tilde{\Delta}_{\mathrm{in}}^+(X, X'),$$

$$\tilde{\Delta}_{\mathrm{in}}^-(X, X') = i\sum_{n_3} \mathcal{M}_{n_3}^{-1+}\psi_{n_3}(X)^+\psi_{n_3}^*(X'),$$

$$\tilde{\Delta}_{\mathrm{in}}^+(X, X') = -i\sum_{n_3} \mathcal{M}_{n_3}^{-1}{}_+\psi_{n_3}(X)_+\psi_{n_3}^*(X'), \tag{A15}$$

where the functions $G_j(X, X')$ are given by Eq. (3.52), and obtain for the in-out propagator that

$$\Delta^c(X, X') = \theta(t - t')\Delta^c(X, X') - \theta(t' - t)\Delta^+(X, X'),$$

$$\Delta^-(X, X') = i\sum_{j=1}^{2} G_j(X, X') + \tilde{\Delta}^-(X, X'),$$

$$\Delta^+(X, X') = -i\sum_{j=4}^{5} G_j(X, X') + \tilde{\Delta}^+(X, X'),$$

$$\tilde{\Delta}^-(X, X') = i\sum_{n_3} \mathcal{M}_{n_3}^{-1}[^-\psi_{n_3}(X)w_{n_3}(+|+)^+\psi_{n_3}^*(X')],$$

$$\tilde{\Delta}^+(X, X') = -i\sum_{n_3} \mathcal{M}_{n_3}^{-1}[_+\psi_{n_3}(X)w_{n_3}(-|-)_-\psi_{n_3}^*(X')]. \tag{A16}$$

Using relations (3.38) with $\eta_{\mathrm{L}} = \eta_{\mathrm{R}} = 1$ for bosons, we can represent the difference between both propagators as follows

$$\Delta^p(X, X') = \Delta_{\mathrm{in}}^c(X, X') - \Delta^c(X, X')$$
$$= -i\sum_{n_3} \mathcal{M}_{n_3}^{-1}[_+\psi_{n_3}(X)w_{n_3}(0|-+)^+\psi_{n_3}^*(X')]. \tag{A17}$$

It is formed in the range Ω_3 only and vanishes if there is no pair creation.

APPENDIX B: ORTHOGONALITY AND NORMALIZATION ON t-CONSTANT HYPERPLANE

Integrating in (3.46) over the coordinates $\mathbf{r}_\perp$ and using the structure of constant spinors v_σ that enter the states $\psi_n(X)$ and $\psi'_{n'}(X)$, we obtain:

$$(\psi_n, \psi'_{n'}) = \delta_{\sigma,\sigma'}\delta_{\mathbf{p}_\perp, \mathbf{p}'_\perp} V_\perp \mathcal{R}, \qquad \mathcal{R} = \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}} \Theta dx,$$

$$\Theta = e^{i(p_0 - p'_0)t}\varphi_n^*(x)[p_0 + p'_0 - 2U(x)]$$
$$\times [p'_0 - U(x) + \chi i\partial_x]\varphi'_{n'}(x). \tag{B1}$$

Then we represent the integral $\mathcal{R}$ as follows

$$\mathcal{R} = \int_{-K^{(\mathrm{L})}}^{x_{\mathrm{L}}} \Theta dx + \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \Theta dx + \int_{x_{\mathrm{R}}}^{K^{(\mathrm{R})}} \Theta dx. \tag{B2}$$

Due to our suppositions about the structure of the scalar potential $A_0(x)$ only the second terms in the right-hand side of Eq. (B2) depends on the external field. At the same time, the smoothness of the scalar potential allows us to believe that this integral is finite. The first and the third terms are calculated as integrals over the areas where the electric field is zero but the scalar potentials, as being constant, differ from zero. Thus, functions $\varphi(x)$ and $\varphi'(x)$ entering the quantity Θ (B1) are different for the left and the right areas even for equal quantum numbers n and n'.

First, we evaluate the quantity $\mathcal{R}$ for coinciding quantum numbers n and n', and then we calculate the norm squared (ψ_n, ψ_n) of the introduced solutions for any n. In this case Eq. (B2) reads:

$$\mathcal{R}|_{n=n'} = \mathcal{R}_{\mathrm{L}} + \mathcal{R}_{\mathrm{int}} + \mathcal{R}_{\mathrm{R}},$$
$$\mathcal{R}_{\mathrm{L}} = \int_{-K^{(\mathrm{L})}}^{x_{\mathrm{L}}} \Theta_{\mathrm{L}} dx, \qquad \mathcal{R}_{\mathrm{int}} = \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \Theta|_{n=n'} dx < \infty,$$
$$\mathcal{R}_{\mathrm{R}} = \int_{x_{\mathrm{R}}}^{K^{(\mathrm{R})}} \Theta_{\mathrm{R}} dx,$$
$$\Theta_{\mathrm{L/R}} = \varphi_n^*(x)2\pi_0(\mathrm{L/R})[\pi_0(\mathrm{L/R}) + \chi i\partial_x]\varphi'_n(x). \tag{B3}$$

If we consider only solutions from the sets $\{_\zeta\psi_n(X)\}$ and $\{^\zeta\psi_n(X)\}$, then the first line in Eqs. (B3) looks different in the ranges $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, $n \in \Omega_2$, and $n \in \Omega_4$. Namely,

$$\mathcal{R}|_{n=n'} = \mathcal{R}_{\mathrm{L}} + \mathcal{R}_{\mathrm{R}} + O(1), \qquad n \in \Omega_1 \cup \Omega_3 \cup \Omega_5,$$
$$\mathcal{R}|_{n=n'} = \mathcal{R}_{\mathrm{L}} + O(1), \qquad n \in \Omega_2,$$
$$\mathcal{R}|_{n=n'} = \mathcal{R}_{\mathrm{R}} + O(1), \qquad n \in \Omega_4, \tag{B4}$$

where the designation $O(1)$ is used here and in what follows for the terms that satisfy the relation

$$\lim_{K^{(L/R)} \to \infty} \frac{O(1)}{K^{(L/R)}} = 0.$$

Consider the quantities $\mathcal{R}_{L/R}$ (B3) defined by the functions $_\zeta \varphi_n(x)$ and $^\zeta \varphi_n(x)$. In this case we attribute the corresponding index ζ to these quantities as follows: $\mathcal{R}_{L/R} \to {_\zeta}\mathcal{R}_{L/R}$ or $\mathcal{R}_{L/R} \to {^\zeta}\mathcal{R}_{L/R}$. Using Eqs. (3.14), (3.18), and (3.37), we obtain

$$_\zeta\mathcal{R}_L = Y^2 K^{(L)} \left| \frac{\pi_0(L)}{p^L} \right| + O(1),$$

$$^\zeta\mathcal{R}_R = Y^2 K^{(R)} \left| \frac{\pi_0(R)}{p^R} \right| + O(1). \tag{B5}$$

This result allows one to find the square norm of the states with $n \in \Omega_2 \cup \Omega_4$,

$$(\psi_n, \psi_n) = \mathcal{M}_n, \qquad n \in \Omega_2 \cup \Omega_4;$$

$$\mathcal{M}_{n_2} = 2\frac{K^{(L)}}{T} \left| \frac{\pi_0(L)}{p^L} \right| + O(1),$$

$$\mathcal{M}_{n_4} = 2\frac{K^{(R)}}{T} \left| \frac{\pi_0(R)}{p^R} \right| + O(1). \tag{B6}$$

To calculate the quantities $_\zeta\mathcal{R}_R$ and $^\zeta\mathcal{R}_L$ that correspond to functions $\varphi_n(x)$ with $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ we have to use relations between the functions $_\zeta\varphi_n(x)$ and $^\zeta\varphi_n(x)$. It follows from Eq. (B1) that the matrix elements $(\psi_n, \psi'_{n'})$ are diagonal in quantum numbers σ. Using this fact, one can easily see that relations (3.38) remain valid under changing the functions $_\zeta\psi_{n_i}(X)$ and $^{\zeta'}\psi_{n_i}(X)$ to $_\zeta\varphi_{n_i}(x)$ and $^{\zeta'}\varphi_{n_i}(x)$. Using these relations, and taking into account Eqs. (3.14), (3.18), and (3.37), we find

$$_\zeta\mathcal{R}_R = Y^2 K^{(R)} \left| \frac{\pi_0(R)}{p^R} \right| [|g(_\zeta|^+)|^2 + |g(_\zeta|^-)|^2] + O(1),$$

$$^\zeta\mathcal{R}_L = Y^2 K^{(L)} \left| \frac{\pi_0(L)}{p^L} \right| [|g(_+|^\zeta)|^2 + |g(_-|^\zeta)|^2] + O(1).$$
$$\tag{B7}$$

These results allow us to find square norms of states with $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$. They are

$$({_\zeta}\psi_n, {_\zeta}\psi_n) = {_\zeta}\mathcal{R}_L + {_\zeta}\mathcal{R}_R,$$
$$({^\zeta}\psi_n, {^\zeta}\psi_n) = {^\zeta}\mathcal{R}_L + {^\zeta}\mathcal{R}_R. \tag{B8}$$

Note that these square norms are of the order of the large numbers $K^{(L)}$ and/or $K^{(R)}$.

In the case $n \neq n'$, we already know that $(\psi_n, \psi'_{n'}) \sim \delta_{\sigma,\sigma'}\delta_{\mathbf{p}_\perp, \mathbf{p}'_\perp}$. Thus, it is enough to study the quantity $\mathcal{R}|_{\sigma=\sigma',\mathbf{p}_\perp=\mathbf{p}'_\perp, p_0 \neq p'_0}$ in order to make up a conclusion about

the complete inner product (B1) for $n \neq n'$. Let us consider solutions $\psi_n(X)$ and $\psi'_{n'}(X)$ with a given asymptotic behavior and for $\sigma = \sigma'$, $\mathbf{p}_\perp = \mathbf{p}'_\perp$, $p_0 \neq p'_0$. In this case $p_0 \neq p'_0$ implies $p^L \neq p^{L\prime}$ and/or $p^R \neq p^{R\prime}$. That is why the quantities Θ are oscillating functions of x in the both regions S_L and S_R, and the modulus of the quantity $\mathcal{R}|_{\sigma=\sigma',\mathbf{p}_\perp=\mathbf{p}'_\perp, p_0 \neq p'_0}$ is finite. Then for any $n, n' \in \Omega$ we have

$$(\psi_n, \psi'_{n'}) = O(1), \qquad n \neq n'. \tag{B9}$$

One can easily verify that the relation

$$(\psi_n, \hat{H}\psi'_{n'}) - (\hat{H}\psi_n, \psi'_{n'}) = O(1) \tag{B10}$$

holds true for any stationary states. Thus, the Hamiltonian $\hat{H}$ is Hermitian as $K^{(L/R)} \to \infty$.

In what follows, such matrix elements always appear divided by terms proportional $K^{(L/R)}$ such that they can be neglected in the limits $K^{(L/R)} \to \infty$. Thus, we further assume that all the wave functions (described in subsections III A and III B) having different quantum numbers n are orthogonal with respect to the introduced inner product on the hyperplane $t = \text{const}$,

$$(\psi_n, \psi'_{n'}) = 0, \qquad n \neq n'. \tag{B11}$$

Then the complete orthonormality relations (3.47) for the ranges Ω_2 and Ω_4 follow from Eqs. (B6) and (B11).

Orthonormality relations for solutions with quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ are considered below, they are more complicated since such solutions have an additional quantum number ζ.

There always exist two independent solutions with quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$. In spite of the fact that these solutions are obtained in the constant external field, we believe that they represent asymptotic forms of some unknown solutions of the Dirac equation with the external field that is switched on and off at $t \to \pm\infty$ and that effects of the switching on and off are negligible. We believe that there exist orthogonal pairs of solutions describing independent particle states at the initial and final time instants, and [since the inner product (3.46) does not depend on t in the limits $K^{(L/R)} \to \infty$] that such solutions remain orthogonal at arbitrary time instant. Below, we are going to find out which solutions under consideration form such orthogonal pairs.

Let us consider the inner products $({_\zeta}\psi_n, {^{\zeta'}}\psi_n)$, $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$. They are written [see (B1) and (B3)] in terms of the quantities $\mathcal{R}_{L/R}(_\zeta|^{\zeta'})$ as follows

$$\mathcal{R}_{L}(_\zeta|^{\zeta'}) = \int_{-K^{(L)}}^{x_L} \Theta_{L}(_\zeta|^{\zeta'}) dx,$$

$$\mathcal{R}_{R}(_\zeta|^{\zeta'}) = \int_{x_R}^{K^{(R)}} \Theta_{R}(_\zeta|^{\zeta'}) dx,$$

$$\Theta_{L/R}(_\zeta|^{\zeta'}) = {}_\zeta\varphi_n^*(x) 2\pi_0(L/R)[\pi_0(L/R) + \chi i\partial_x]^{\zeta'}\varphi_n(x),$$

$$n \in \Omega_1 \cup \Omega_3 \cup \Omega_5. \qquad (B12)$$

As was mentioned before, relations (3.38) remain valid if the functions ${}_\zeta\psi_{n_i}(X)$ and ${}^{\zeta'}\psi_{n_i}(X)$ are changed to the functions ${}_\zeta\varphi_{n_i}(x)$ and ${}^{\zeta'}\varphi_{n_i}(x)$. Using this fact, we express the functions ${}^\zeta\varphi_{n_i}(x)$ in terms of ${}_\zeta\varphi_{n_i}(x)$ in $\mathcal{R}_{L}(_\zeta|^{\zeta'})$, and the functions ${}_\zeta\varphi_{n_i}(x)$ in terms of ${}^\zeta\varphi_{n_i}(x)$ in $\mathcal{R}_{R}(_\zeta|^{\zeta'})$. Then taking into account Eqs. (3.37), we obtain

$$\mathcal{R}_{L}(_\zeta|^{\zeta'}) = \zeta\eta_L Y^2 K^{(L)}\left|\frac{\pi_0(L)}{p^L}\right| g(_\zeta|^{\zeta'}) + O(1),$$

$$\mathcal{R}_{R}(_\zeta|^{\zeta'}) = \zeta'\eta_R Y^2 K^{(R)}\left|\frac{\pi_0(R)}{p^R}\right| g(_\zeta|^{\zeta'}) + O(1). \qquad (B13)$$

Then we consider only the case $n \in \Omega_1 \cup \Omega_5$. For $n \in \Omega_1$, due to the inequalities $\pi_0(L) > \pi_0(R) \geq \pi_\perp$, both sets of solutions describe electrons. For $n \in \Omega_5$, due to the inequalities $\pi_0(R) < \pi_0(L) \leq -\pi_\perp$, both sets of solutions describe positrons. In both cases $\eta_L = \eta_R$. Then it follows from Eqs. (B3) and (B13) that

$$({}_\zeta\psi_n, {}^{-\zeta}\psi_n) = 0, \qquad n \in \Omega_1 \cup \Omega_5, \qquad (B14)$$

if we assume that the quantities $K^{(L/R)}$ satisfy the following relation

$$K^{(L)}\left|\frac{\pi_0(L)}{p^L}\right| - K^{(R)}\left|\frac{\pi_0(R)}{p^R}\right| = O(1), \qquad (B15)$$

that was first proposed by Nikishov in Ref. [17]. Condition (B14) means that for $n \in \Omega_1 \cup \Omega_5$ solutions ${}_\zeta\psi_n(X)$ and ${}^{-\zeta}\psi_n(X)$, represent independent physical states. The currents (3.36) of these independent physical states have opposite directions.

Let us consider the range $n \in \Omega_3$. In this range, the inequalities $\pi_0(L) \geq \pi_\perp$ and $\pi_0(R) \leq -\pi_\perp$ hold true and $\eta_L = -\eta_R = +1$. Then it follows from Eqs. (B3), (B13), and (B15) that

$$({}_\zeta\psi_n, {}^\zeta\psi_n) = 0, \qquad n \in \Omega_3. \qquad (B16)$$

Thus, for $n \in \Omega_3$ solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, represent independent physical states. Formally, the difference between the two cases $n \in \Omega_1 \cup \Omega_5$ and $n \in \Omega_3$ is owing to the difference in signs in the unitarity relations for these

cases. The currents (3.36) of these independent physical states have the same directions.

Finally, using Eqs. (3.40), (B8), and (B15), we obtain the orthonormality relations (3.47) for the ranges. Ω_i, $i = 1, 3, 5$.

APPENDIX C: SOME MEAN VALUES

1. Mean values in Ω_1 and Ω_5

I. All mean values of the QFT charge operator $\hat{Q}$ given by Eq. (4.22) in states (5.1) are $-e$, whereas all mean values of the QFT charge operator $\hat{Q}$ in states (5.2) are $+e$.

II. Using Eqs. (4.21) and (4.16), we can verify that all the kinetic energies of the states under consideration are positive. For the electron states, we obtain:

$$\langle 0|_+a_{n_1}(\text{in})\hat{\mathbb{H}}^{\text{kin}}{}_+a_{n_1}^\dagger(\text{in})|0\rangle = {}_+\mathcal{E}_{n_1} > 0,$$

$$\langle 0|^-a_{n_1}(\text{in})\hat{\mathbb{H}}^{\text{kin}-}a_{n_1}^\dagger(\text{in})|0\rangle = {}^-\mathcal{E}_{n_1} > 0,$$

$$\langle 0|_-a_{n_1}(\text{out})\hat{\mathbb{H}}^{\text{kin}}{}_-a_{n_1}^\dagger(\text{out})|0\rangle = {}_-\mathcal{E}_{n_1} > 0,$$

$$\langle 0|^+a_{n_1}(\text{out})\hat{\mathbb{H}}^{\text{kin}+}a_{n_1}^\dagger(\text{out})|0\rangle = {}^+\mathcal{E}_{n_1} > 0, \qquad (C1)$$

whereas for the positron states, we have

$$\langle 0|_-b_{n_5}(\text{in})\hat{\mathbb{H}}^{\text{kin}}{}_-b_{n_5}^+(\text{in})|0\rangle = -{}_-\mathcal{E}_{n_5} > 0,$$

$$\langle 0|^+b_{n_5}(\text{in})\hat{\mathbb{H}}^{\text{kin}+}b_{n_5}^+(\text{in})|0\rangle = -{}^+\mathcal{E}_{n_5} > 0,$$

$$\langle 0|_+b_{n_5}(\text{out})\hat{\mathbb{H}}^{\text{kin}}{}_+b_{n_5}^+(\text{out})|0\rangle = -{}_+\mathcal{E}_{n_1} > 0,$$

$$\langle 0|^-b_{n_5}(\text{out})\hat{\mathbb{H}}^{\text{kin}-}b_{n_5}^+(\text{out})|0\rangle = -{}^-\mathcal{E}_{n_5} > 0. \qquad (C2)$$

III. Let us calculate differential mean values of out-particles with respect to different in-states (5.1) and (5.2). To this end we have to find the corresponding mean values of the following operators,

$$\hat{N}_{+,n_1}^{(a)} = {}^+a_{n_1}^\dagger(\text{out}){}^+a_{n_1}(\text{out}),$$

$$\hat{N}_{-,n_1}^{(a)} = {}_-a_{n_1}^\dagger(\text{out}){}_-a_{n_1}(\text{out}),$$

$$\hat{N}_{-,n_5}^{(b)} = {}^-b_{n_5}^\dagger(\text{out}){}^-b_{n_5}(\text{out}),$$

$$\hat{N}_{+,n_5}^{(b)} = {}_+b_{n_5}^\dagger(\text{out}){}_+b_{n5}(\text{out}). \qquad (C3)$$

Technically it can be done by using canonical transformations (4.33) between in and out operators, derived in Sec. IV C. Thus, we obtain

$$N^{(a)}_{\zeta,n_1}(0) = \langle 0|\hat{N}^{(a)}_{\zeta,n_1}|0\rangle = 0,$$

$$N^{(b)}_{\zeta,n_5}(0) = \langle 0|\hat{N}^{(b)}_{\zeta,n_5}|0\rangle = 0,$$

$$N^{(a)}_{\zeta,n_1}(n_1,+) = \langle 0|_+a_{n_1}(\text{in})\hat{N}^{(a)}_{\zeta,n_1}{}_+a^\dagger_{n_1}(\text{in})|0\rangle$$
$$= \begin{cases} |g(_+|^+)|^{-2}, & \zeta = + \\ |g(_+|^+)|^{-2}|g(_-|^+)|^2, & \zeta = - \end{cases},$$

$$N^{(a)}_{\zeta,n_1}(n_1,-) = \langle 0|^-a_{n_1}(\text{in})\hat{N}^{(a)}_{\zeta,n_1}{}^-a^\dagger_{n_1}(\text{in})|0\rangle$$
$$= \begin{cases} |g(_+|^+)|^{-2}, & \zeta = - \\ |g(_+|^+)|^{-2}|g(_-|^+)|^2, & \zeta = + \end{cases},$$

$$N^{(b)}_{\zeta,n_5}(n_5,+) = \langle 0|^+b_{n_5}(\text{in})\hat{N}^{(b)}_{\zeta,n_5}{}^+b^\dagger_{n_5}(\text{in})|0\rangle$$
$$= \begin{cases} |g(_+|^+)|^{-2}, & \zeta = + \\ |g(_+|^+)|^{-2}|g(_-|^+)|^2, & \zeta = - \end{cases},$$

$$N^{(b)}_{\zeta,n_5}(n_5,-) = \langle 0|_-b_{n_5}(\text{in})\hat{N}^{(b)}_{\zeta,n_5}{}_-b^\dagger_{n_5}(\text{in})|0\rangle$$
$$= \begin{cases} |g(_+|^+)|^{-2}, & \zeta = - \\ |g(_+|^+)|^{-2}|g(_-|^+)|^2, & \zeta = + \end{cases}, \quad \text{(C4)}$$

see (3.39) for the definition of the coefficients g.

Then it follows from relations (3.41) and (3.42) that

$$N^{(a)}_{+,n_1}(n_1,+) + N^{(a)}_{-,n_1}(n_1,+)$$
$$= N^{(a)}_{+,n_1}(n_1,-) + N^{(a)}_{-,n_1}(n_1,-) = 1;$$
$$N^{(b)}_{+,n_5}(n_5,+) + N^{(b)}_{-,n_5}(n_5,+)$$
$$= N^{(b)}_{+,n_5}(n_5,-) + N^{(b)}_{-,n_5}(n_5,-) = 1. \quad \text{(C5)}$$

Thus, the number of electrons with quantum numbers n_1 and positrons with quantum numbers n_5 are conserved in the course of scattering off the x-electric potential step.

III. Using the electric current operator $\hat{J}$ (4.26), we construct the corresponding renormalized operator $\hat{\mathbb{J}}$. Its mean value in the vacuum state is zero,

$$\hat{\mathbb{J}} = \hat{J} - \langle 0|\hat{J}|0\rangle, \qquad \langle 0|\hat{\mathbb{J}}|0\rangle = 0. \quad \text{(C6)}$$

Then, using orthonormality condition (3.36), we calculate currents created by one-electron states in the range Ω_1:

$$\langle 0|_+a_{n_1}(\text{in})\hat{\mathbb{J}}_+a^\dagger_{n_1}(\text{in})|0\rangle = -e(\mathcal{M}_{n_1}T)^{-1} < 0,$$
$$\langle 0|^-a_{n_1}(\text{in})\hat{\mathbb{J}}^-a^\dagger_{n_1}(\text{in})|0\rangle = e(\mathcal{M}_{n_1}T)^{-1} > 0,$$
$$\langle 0|_-a_{n_1}(\text{out})\hat{\mathbb{J}}_-a^\dagger_{n_1}(\text{out})|0\rangle = e(\mathcal{M}_{n_1}T)^{-1} > 0,$$
$$\langle 0|^+a_{n_1}(\text{out})\hat{\mathbb{J}}^+a^\dagger_{n_1}(\text{out})|0\rangle = -e(\mathcal{M}_{n_1}T)^{-1} < 0, \quad \text{(C7)}$$

and currents created by one-positron states in the range Ω_5:

$$\langle 0|_-b_{n_5}(\text{in})\hat{\mathbb{J}}_-b^\dagger_{n_5}(\text{in})|0\rangle = e(\mathcal{M}_{n_5}T)^{-1} > 0,$$
$$\langle 0|^+b_{n_5}(\text{in})\hat{\mathbb{J}}^+b^\dagger_{n_5}(\text{in})|0\rangle = -e(\mathcal{M}_{n_5}T)^{-1} < 0,$$
$$\langle 0|_+b_{n_5}(\text{out})\hat{\mathbb{J}}_+b^\dagger_{n_5}(\text{out})|0\rangle = -e(\mathcal{M}_{n_5}T)^{-1} < 0,$$
$$\langle 0|^-b_{n_5}(\text{out})\hat{\mathbb{J}}^-b^\dagger_{n_5}(\text{out})|0\rangle = e(\mathcal{M}_{n_5}T)^{-1} > 0, \quad \text{(C8)}$$

The quantities $\mathcal{M}_n$ are given by Eqs. (3.48), and the combination $(\mathcal{M}_nT)^{-1}$ is the modulus of the probability flux of a one-particle state through the hyperplane $x = \text{const}$. One can see that signs of currents (C7) and (C8) are always opposite to the signs of the asymptotic values p^R and p^L, respectively. Thus, the one-particle quantum mechanical interpretation of quantum numbers p^R and p^L as momenta holds true in the ranges Ω_1 and Ω_5.

IV. Using energy flux operator $\hat{F}(x)$ (4.25), we construct the corresponding renormalized operator $\hat{\mathbb{F}}(x)$. Its mean value in the vacuum state is zero,

$$\hat{\mathbb{F}}(x) = \hat{F}(x) - \langle 0|\hat{F}(x)|0\rangle, \qquad \langle 0|\hat{\mathbb{F}}(x)|0\rangle = 0. \quad \text{(C9)}$$

Then, with the help of this operator, we calculate mean energy fluxes created by one-particle states (5.1) and (5.2) through the surfaces $x = x_L$ and $x = x_R$. Using orthonormality condition (3.36), we obtain for electrons in the range Ω_1:

$$\mathbb{F}_{n_1,+}(\text{in}) = \langle 0|_+a_{n_1}(\text{in})\hat{\mathbb{F}}(x_L)_+a^\dagger_{n_1}(\text{in})|0\rangle$$
$$= (\mathcal{M}_{n_1}T)^{-1}\pi_0(L) > 0,$$
$$\mathbb{F}_{n_1,+}(\text{out}) = \langle 0|^+a_{n_1}(\text{out})\hat{\mathbb{F}}(x_R)^+a^\dagger_{n_1}(\text{out})|0\rangle$$
$$= (\mathcal{M}T)^{-1}\pi_0(R) > 0,$$
$$\mathbb{F}_{n_1,-}(\text{in}) = \langle 0|^-a_{n_1}(\text{in})\hat{\mathbb{F}}(x_R)^-a^\dagger_{n_1}(\text{in})|0\rangle$$
$$= -(\mathcal{M}T)^{-1}\pi_0(R) < 0,$$
$$\mathbb{F}_{n_1,-}(\text{out}) = \langle 0|_-a_{n_1}(\text{out})\hat{\mathbb{F}}(x_L)_-a^\dagger_{n_1}(\text{out})|0\rangle$$
$$= -(\mathcal{M}T)^{-1}\pi_0(L) < 0, \quad \text{(C10)}$$

and for positrons in the range Ω_5:

$$\mathbb{F}_{n_5,-}(\text{in}) = \langle 0|_-b_{n_5}(\text{in})\hat{\mathbb{F}}(x_L)_-b^\dagger_{n_5}(\text{in})|0\rangle$$
$$= (\mathcal{M}T)^{-1}|\pi_0(L)| > 0,$$
$$\mathbb{F}_{n_5,-}(\text{out}) = \langle 0|^-b_{n_5}(\text{out})\hat{\mathbb{F}}(x_R)^-b^\dagger_{n_5}(\text{out})|0\rangle$$
$$= (\mathcal{M}T)^{-1}|\pi_0(R)| > 0,$$
$$\mathbb{F}_{n_5,+}(\text{in}) = \langle 0|^+b_{n_5}(\text{in})\hat{\mathbb{F}}(x_R)^+b^\dagger_{n_5}(\text{in})|0\rangle$$
$$= -(\mathcal{M}T)^{-1}|\pi_0(R)| < 0,$$
$$\mathbb{F}_{n_5,+}(\text{out}) = \langle 0|_+b_{n_5}(\text{out})\hat{\mathbb{F}}(x_L)_+b^\dagger_{n_5}(\text{out})|0\rangle$$
$$= -(\mathcal{M}T)^{-1}|\pi_0(L)| < 0. \quad \text{(C11)}$$

We believe that the direction of the energy flux indicates the direction of motion of the corresponding particle, which is shown on Fig. 3 by the corresponding arrows.

This fact allows us to find longitudinal momenta of the waves in the asymptotic regions S_L and S_R by integrating the corresponding energy fluxes over x. Such QFT quantities differ from the quantum numbers p^L and p^R that are asymptotic longitudinal momenta of the corresponding unit one-particle flux, for simplicity, we call them QM longitudinal momenta of particles.

For example, QFT longitudinal momenta of electron states are

$$P_{n_1,+}(\text{in}) = \mathbb{F}_{n_1,+}(\text{in})K^{(L)} = \frac{1}{2}|p^L||g(_+|^+)|^{-2},$$

$$P_{n_1,+}(\text{out}) = \mathbb{F}_{n_1,+}(\text{out})K^{(R)} = \frac{1}{2}|p^R||g(_+|^+)|^{-2},$$

$$P_{n_1,-}(\text{out}) = \mathbb{F}_{n_1,-}(\text{out})K^{(L)} = -\frac{1}{2}|p^L||g(_+|^+)|^{-2},$$

$$P_{n_1,-}(\text{in}) = \mathbb{F}_{n_1,-}(\text{in})K^{(R)} = -\frac{1}{2}|p^R||g(_+|^+)|^{-2}, \quad \text{(C12)}$$

see Sec. III C. In the same manner, using Eq. (C11), we can find longitudinal momenta of positron states.

These relations show that quantum numbers $|\pi_0(\text{L/R})|$ can be interpreted as kinetic energies of the unit particle flux, both for electrons and positrons in the asymptotic regions S_L and S_R, respectively. Then it follows from Eqs. (C12) that $|p^L|$ and $|p^R|$ are modulus of asymptotic longitudinal momenta of the unit of the corresponding one-particle fluxes. One can also see that the sign of the mean values $\mathbb{F}_{n,\zeta}(\text{in/out})$ in electron states is ζ and in positron states is $-\zeta$. The directions of the energy fluxes represent the directions of the motion of both electrons and positrons. Thus, we see that the asymptotic longitudinal physical momenta of electrons are $p_{ph}^L = p^L$ and $p_{ph}^R = p^R$, whereas for positrons they are $p_{ph}^L = -p^L$ and $p_{ph}^R = -p^R$. This matches with the standard interpretation of quantum numbers of solutions of the Dirac equation. One can also see that the electric current of electrons is opposite to the direction of their energy flux (and to their asymptotic longitudinal physical momenta), whereas the electric current of positrons coincides with the direction of their energy flux (and with the asymptotic longitudinal physical momenta).

2. Mean values in Ω_3

I. By using relations (7.4), we find differential mean numbers of out-particles in the vacuum $|0, \text{in}\rangle$, and differential mean numbers of in-particles in the out-vacuum $|0, \text{out}\rangle$,

$$N_n^a(\text{out}) = \langle 0, \text{in}|^+a_n^\dagger(\text{out})^+a_n(\text{out})|0, \text{in}\rangle = |g(_-|^+)|^{-2},$$

$$N_n^b(\text{out}) = \langle 0, \text{in}|_+b_n^\dagger(\text{out})_+b_n(\text{out})|0, \text{in}\rangle = |g(_+|^-)|^{-2},$$

$$N_n^a(\text{in}) = \langle 0, \text{out}|^-a_n^\dagger(\text{in})^-a_n(\text{in})|0, \text{out}\rangle = |g(_+|^-)|^{-2},$$

$$N_n^b(\text{in}) = \langle 0, \text{out}|_-b_n^\dagger(\text{in})_-b_n(\text{in})|0, \text{out}\rangle = |g(_-|^+)|^{-2}.$$

$$\text{(C13)}$$

II. By using relations (3.42) we find differential mean numbers of out-particles in one-particle in-states,

$$\langle 0, \text{in}|^-a_n(\text{in})^+a_n^\dagger(\text{out})^+a_n(\text{out})^-a_n^\dagger(\text{in})|0, \text{in}\rangle = 1,$$

$$\langle 0, \text{in}|^-a_n(\text{in})_+b_n^\dagger(\text{out})_+b_n(\text{out})^-a_n^\dagger(\text{in})|0, \text{in}\rangle = 0,$$

$$\langle 0, \text{in}|_-b_n(\text{in})_+b_n^\dagger(\text{out})_+b_n(\text{out})_-b_n^\dagger(\text{in})|0, \text{in}\rangle = 1,$$

$$\langle 0, \text{in}|_-b_n(\text{in})^+a_n^\dagger(\text{out})^+a_n(\text{out})_-b_n^\dagger(\text{in})|0, \text{in}\rangle = 0. \quad \text{(C14)}$$

Thus, the one-electron in-state contains only one out-electron and does not contain any out-positron, whereas the one-positron in-state contains only one out-positron and does not contain any out-electron, which is a consequence of the Pauli principle.

III. Using the operator $\hat{\mathbb{H}}^{\text{kin}}$ (4.21), we calculate kinetic energies of all particles in the range Ω_3,

$$\langle 0, \text{in}|^-a_n(\text{in})\hat{\mathbb{H}}^{\text{kin}\,-}a_n^\dagger(\text{in})|0, \text{in}\rangle = {}^-\mathcal{E}_{n_3},$$

$$\langle 0, \text{in}|_-b_n(\text{in})\hat{\mathbb{H}}^{\text{kin}}{}_-b_n^\dagger(\text{in})|0, \text{in}\rangle = -_-\mathcal{E}_{n_3},$$

$$\langle 0, \text{out}|^+a_n(\text{out})\hat{\mathbb{H}}^{\text{kin}\,+}a_n^\dagger(\text{out})|0, \text{out}\rangle = {}^+\mathcal{E}_{n_3},$$

$$\langle 0, \text{out}|_+b_n(\text{out})\hat{\mathbb{H}}^{\text{kin}}{}_+b_n^\dagger(\text{out})|0, \text{out}\rangle = -_+\mathcal{E}_{n_3}. \quad \text{(C15)}$$

One can verify that, with the account taken of (4.17), the combinations $({}^\zeta\mathcal{E}_{n_3} - _\zeta\mathcal{E}_{n_3})$ are positive (as we demonstrate in Sec. VII C, kinetic energies of physical wave packets of electrons and positrons are also positive).

IV. Let us introduce renormalized (with respect to the corresponding vacua) in- and out-operators of the electric current flowing through the surface $x = \text{const}$,

$$\hat{\mathbb{J}}(\text{in}) = \hat{J} - \langle 0, \text{in}|\hat{J}|0, \text{in}\rangle,$$

$$\hat{\mathbb{J}}(\text{out}) = \hat{J} - \langle 0, \text{out}|\hat{J}|0, \text{out}\rangle, \quad \text{(C16)}$$

where the operator $\hat{J}$ is given by Eq. (4.26). With the help of Eq. (3.36) we find differential mean values of these operators in all one-particle states (7.3),

$$J_n^a(\text{in}) = \langle 0, \text{in}|^- a_n(\text{in})\hat{\mathbb{J}}(\text{in})^- a_n^\dagger(\text{in})|0, \text{in}\rangle = -e(\mathcal{M}_n T)^{-1},$$

$$J_n^b(\text{in}) = \langle 0, \text{in}|_- b_n(\text{in})\hat{\mathbb{J}}(\text{in})_- b_n^\dagger(\text{in})|0, \text{in}\rangle = -e(\mathcal{M}_n T)^{-1},$$

$$J_n^a(\text{out}) = \langle 0, \text{out}|^+ a_n(\text{out})\hat{\mathbb{J}}(\text{out})^+ a_n^\dagger(\text{out})|0, \text{out}\rangle$$
$$= e(\mathcal{M}_n T)^{-1},$$

$$J_n^b(\text{out}) = \langle 0, \text{out}|_+ b_n(\text{out})\hat{\mathbb{J}}(\text{out})_+ b_n^\dagger(\text{out})|0, \text{out}\rangle$$
$$= e(\mathcal{M}_n T)^{-1}. \tag{C17}$$

One can see that the mean currents $J_n^a(\text{out})$ and $J_n^b(\text{out})$ are positive and have the same direction as the applied external electric field, whereas the mean currents $J_n^a(\text{in})$ and $J_n^b(\text{in})$ are negative and have the opposite direction to the applied external electric field. Both in- and out-electron states (7.3) are states with definite quantum numbers p^R, whereas both in- and out-positron states (7.3) are states with definite quantum numbers p^L. Therefore, signs of both currents $J_n^a(\text{in})$ and $J_n^a(\text{out})$ coincide with the sign of p^R, whereas signs of both currents $J_n^b(\text{in})$ and $J_n^b(\text{out})$ coincide with the sign of p^L.

V. Using energy flux operator $\hat{F}(x)$ (4.25), we construct the corresponding renormalized operators $\hat{\mathbb{F}}(x|\text{in})$ and $\hat{\mathbb{F}}(x|\text{out})$,

$$\hat{\mathbb{F}}(x|\text{in}) = \hat{F}(x) - \langle 0, \text{in}|\hat{F}(x)|0, \text{in}\rangle,$$
$$\hat{\mathbb{F}}(x|\text{out}) = \hat{F}(x) - \langle 0, \text{out}|\hat{\mathbb{F}}(x)|0, \text{out}\rangle. \tag{C18}$$

With the help of these operators, we calculate mean energy fluxes through the surfaces $x = x_L$ and $x = x_R$. Since electron wave packets are localized in the region S_L, and positron wave packets are localized in the region S_R, the mean energy flux of electron partial waves is to be defined through the surface $x = x_L$, and of positron partial waves through the surface $x = x_R$. These mean values are expressed via energy fluxes of the Dirac field through the surfaces $x = x_L$ and $x = x_R$, respectively. The latter fluxes can be calculated using Eqs. (3.36),

$$F_n^a(\text{in}) = \langle 0, \text{in}|^- a_n(\text{in})\hat{\mathbb{F}}(x_L, \text{in})^- a_n^\dagger(\text{in})|0, \text{in}\rangle$$
$$= (\mathcal{M}_n T)^{-1}\pi_0(L),$$

$$F_n^a(\text{out}) = \langle 0, \text{out}|^+ a_n(\text{out})\hat{\mathbb{F}}(x_L, \text{out})^+ a_n^\dagger(\text{out})|0, \text{out}\rangle$$
$$= -(\mathcal{M}_n T)^{-1}\pi_0(L),$$

$$F_n^b(\text{in}) = \langle 0, \text{in}|_- b_n(\text{in})\hat{\mathbb{F}}(x_R, \text{in})_- b_n^\dagger(\text{in})|0, \text{in}\rangle$$
$$= -(\mathcal{M}_n T)^{-1}|\pi_0(R)|,$$

$$F_n^b(\text{out}) = \langle 0, \text{out}|_+ b_n(\text{out})\hat{\mathbb{F}}(x_R, \text{out})_+ b_n^\dagger(\text{out})|0, \text{out}\rangle$$
$$= (\mathcal{M}_n T)^{-1}|\pi_0(R)|. \tag{C19}$$

To find the longitudinal momenta of particles in the asymptotic regions S_L and S_R, we have to integrate these energy fluxes over x. Thus we obtain:

$$P_n^a(\text{in/out}) = F_n^a(\text{in/out})K^{(L)} = \pm\frac{1}{2}|p^L||g(_+|^-)|^{-2},$$

$$P_n^b(\text{in/out}) = F_n^b(\text{in/out})K^{(R)} = \mp\frac{1}{2}|p^R||g(_+|^-)|^{-2}. \tag{C20}$$

We stress that in contrast to the ranges Ω_1 and Ω_5, signs of the quantities related to the electrons in Eqs. (C19) and (C20) are determined by the signs of the quantum number p^R, but not by the signs of the quantum number p^L, whereas signs of the quantities related to the positrons are determined by the signs of the quantum number p^L, but not by the signs of the quantum number p^R,

$$\text{sgn}(P_n^a(\text{in/out})) = -\text{sgn}(p^R),$$
$$\text{sgn}(P_n^b(\text{in/out})) = \text{sgn}(p^L). \tag{C21}$$

Relations (C21) indicate a direct correlation between directions of in- and out- energy fluxes and directions of the corresponding currents, which is

$$\text{sgn}\,p^R = \text{sgn}J_n^b(\text{out}) = \text{sgn}J_n^a(\text{out}),$$
$$\text{sgn}\,p^L = \text{sgn}J_n^b(\text{in}) = \text{sgn}J_n^a(\text{in}). \tag{C22}$$

One can see from these relations that in the range Ω_3, as well as in the ranges Ω_1 and Ω_5, the quantum numbers $|\pi_0(L/R)|$ are kinetic energies per unit of one-particle flux in the corresponding asymptotic regions. Direction of the energy flux indicates the direction of the particle motion. This direction coincides with the direction of positron currents situated in the region S_R and is opposite to the direction of the electron currents situated in the region S_L.

APPENDIX D: TOTAL REFLECTION IN THE KLEIN ZONE

To consider the total reflection in the range Ω_3 in the general case of arbitrary strong electric field when $N_n^{\text{cr}} \to 1$ for some $n \in \Omega_3^{\text{str}} \subset \Omega_3$ it is necessary to associate particles with special wave packets that are stable enough. We consider the Dirac case. The case of scalar field can be given along the same lines.

There are in and out electron and positron states each of them associated with the corresponding wave packet. Electron in-states are composed from the plane waves $^-\psi_{n_3}(X)$, whereas electron out-states are composed from the plane waves $^+\psi_{n_3}(X)$. Positron in-states are composed from the plane waves $_-\psi_{n_3}(X)$, whereas positron out-states are composed from the plane waves $_+\psi_{n_3}(X)$. It is natural to suppose that all these wave packets are localized at some time instants in some areas that have a finite length on the axis x. Let $x = x_F$ be a center of such an area at some time instant. We represent such wave packets for electrons and positrons as follows

$$
{}^\zeta\psi_{x_F}(X) = {}^\zeta N_{x_F} \sum_{n\in\Omega_3} {}^\zeta c_n^{(x_F)} \mathcal{M}_n^{-1/2} {}^\zeta\psi_n(X)
$$

$$
- \text{ in/out}(\zeta = -/+) - \text{electron states},
$$

$$
{}_\zeta\psi_{x_F}(X) = {}_\zeta N_{x_F} \sum_{n\in\Omega_3} {}_\zeta c_n^{(x_F)} \mathcal{M}_n^{-1/2} {}_\zeta\psi_n(X)
$$

$$
- \text{ in/out}(\zeta = -/+) - \text{positron states}, \quad \text{(D1)}
$$

where ${}^\zeta c_n^{(x_F)}$ and ${}_\zeta c_n^{(x_F)}$ are some coefficients and ${}^\zeta N_{x_F}$ and ${}_\zeta N_{x_F}$ are normalization factors,

$$
|{}^\zeta N_{x_F}|^{-2} = \sum_{n\in\Omega_3} |{}^\zeta c_n^{(x_F)}|^2, \qquad |{}_\zeta N_{x_F}|^{-2} = \sum_{n\in\Omega_3} |{}_\zeta c_n^{(x_F)}|^2.
$$

We are interested in the cases when particle wave packets are localized in the asymptotic region S_L far enough from the asymptotic region S_R, which means $x_F < x_L^F < x_L$, or in the asymptotic region S_R far enough from the asymptotic region S_L, which means $x_F > x_R^F > x_R$. We assume that minimal extension Δ_F of the whole wave packet along the axis x, is much less than the distance between the points x_L^F and x_R, or x_R^F and x_L, $\Delta_F \ll x_R^F - x_L, x_R - x_L^F$. Similar to the discussion related to Eqs. (B3) and (B4), one can study square norms of the introduced wave packets $({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})$ and $({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})$. One can separate contributions to these square norms from the asymptotic regions S_L and S_R, as follows

$$
({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F}) = ({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_L + ({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_R + O(1),
$$

$$
({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F}) = ({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_L + ({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_R + O(1),
$$
$$
\text{(D2)}
$$

where

$$
({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_L = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(L)}}^{x_L} {}^\zeta\psi_{x_F}^\dagger(X) {}^\zeta\psi_{x_F}(X) dx,
$$

$$
({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_R = \int_{V_\perp} d\mathbf{r}_\perp \int_{x_R}^{K^{(R)}} {}^\zeta\psi_{x_F}^\dagger(X) {}^\zeta\psi_{x_F}(X) dx;
$$

$$
({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_L = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(L)}}^{x_L} {}_\zeta\psi_{x_F}^\dagger(X) {}_\zeta\psi_{x_F}(X) dx,
$$

$$
({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_R = \int_{V_\perp} d\mathbf{r}_\perp \int_{x_R}^{K^{(R)}} {}_\zeta\psi_{x_F}^\dagger(X) {}_\zeta\psi_{x_F}(X) dx. \quad \text{(D3)}
$$

Let us study integrals (D3) following the procedure described in Appendix B. Taking into account the mutual decompositions of the plane waves (3.38) and using spin and coordinate factorization of Dirac spinors given by Eq. (3.6), one can represent integrals (D3) in the following forms

$$
({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_L = |{}^\zeta N_{x_F}|^2 \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(L)}}^{x_L} |{}^\zeta\varphi_{x_F}^L(X)|^2 dx,
$$

$$
({}^\zeta\psi_{x_F}, {}^\zeta\psi_{x_F})_R = |{}^\zeta N_{x_F}|^2 \int_{V_\perp} d\mathbf{r}_\perp \int_{x_R}^{K^{(R)}} |{}^\zeta\varphi_{x_F}^R(X)|^2 dx;
$$

$$
({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_L = |{}_\zeta N_{x_F}|^2 \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(L)}}^{x_L} |{}_\zeta\varphi_{x_F}^L(X)|^2 dx,
$$

$$
({}_\zeta\psi_{x_F}, {}_\zeta\psi_{x_F})_R = |{}_\zeta N_{x_F}|^2 \int_{V_\perp} d\mathbf{r}_\perp \int_{x_R}^{K^{(R)}} |{}_\zeta\varphi_{x_F}^R(X)|^2 dx, \quad \text{(D4)}
$$

where

$$
{}^\zeta\varphi_{x_F}^R(X) = \sum_{n\in\Omega_3} \frac{{}^\zeta c_n^{(x_F)} \exp\left(-ip_0 t + ip^R x + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|},
$$

$$
{}^\zeta\varphi_{x_F}^L(X) = \sum_{n\in\Omega_3} [g(_+|^\zeta) e^{i|p^L|x} - g(_-|^\zeta) e^{-i|p^L|x}]
$$
$$
\times \frac{{}^\zeta c_n^{(x_F)} \exp\left(-ip_0 t + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|};
$$

$$
{}_\zeta\varphi_{x_F}^R(X) = \sum_{n\in\Omega_3} [g(^+|_\zeta) e^{i|p^R|x} - g(^-|_\zeta) e^{-i|p^R|x}]
$$
$$
\times \frac{{}_\zeta c_n^{(x_F)} \exp\left(-ip_0 t + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|},
$$

$$
{}_\zeta\varphi_{x_F}^L(X) = \sum_{n\in\Omega_3} \frac{{}_\zeta c_n^{(x_F)} \exp\left(-ip_0 t + ip^L x + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|}. \quad \text{(D5)}
$$

Absolute values of the asymptotic momenta $|p^L|$ and $|p^R|$ are determined by the quantum numbers p_0 and $\mathbf{p}_\perp$, see Eqs. (3.15) and (3.19). This fact imposes certain correlations between both quantities. In particular, one can see from Eq. (3.21) that $d|p^L|/d|p^R| < 0$, and at any given $\mathbf{p}_\perp$ these quantities are restricted inside the range Ω_3,

$$
0 \le |p^{R/L}| \le p^{\max}, \qquad p^{\max} = \sqrt{\mathbb{U}(\mathbb{U} - 2\pi_\perp)}. \quad \text{(D6)}
$$

As an example, let us consider an electron wave packet with a given spin polarization σ and transversal momentum $\mathbf{p}_\perp$,

$$
{}^\zeta\psi_{x_F}(X) = \frac{{}^\zeta N_{x_F} T}{2\pi} \int_{p_0\in\Omega_3} \mathcal{M}_n^{-1/2} {}^\zeta c_n^{(x_F)} {}^\zeta\psi_n(X) dp_0, \quad \text{(D7)}
$$

where the integration over p_0 is fulfilled for a given fixed $\mathbf{p}_\perp$ and the corresponding asymptotic scalar functions (D5) are

$$\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X) = \frac{T}{2\pi}\int_{p_0\in\Omega_3}\frac{\zeta c_n^{(x_{\mathrm{F}})}\exp\left(-ip_0 t + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|}$$

$$\times\left[g(_+|^\zeta)e^{i|p^{\mathrm{L}}|x} - g(_-|^\zeta)e^{-i|p^{\mathrm{L}}|x}\right]dp_0,$$

$$\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X) = \frac{T}{2\pi}\int_{p_0\in\Omega_3}\frac{\zeta c_n^{(x_{\mathrm{F}})}\exp\left(-ip_0 t + ip^{\mathrm{R}}x + i\mathbf{p}_\perp\mathbf{r}_\perp\right)}{\sqrt{2}|g(_+|^-)|}dp_0.$$

$$(D8)$$

Using Eqs. (3.10) and (3.15), we express p_0 via p^{R} as

$$p_0 = U_{\mathrm{R}} - \sqrt{(p^{\mathrm{R}})^2 + \pi_\perp^2}.$$

Then we denote the mean value of p^{R} in a wave packet in S_{R} as $\bar{p}^{\mathrm{R}}$, and the mean value of $|p^{\mathrm{L}}|$ of the same packet in S_{L} as $|\bar{p}^{\mathrm{L}}|$. Afterwards we chose coefficients $\zeta c_n^{(x_{\mathrm{F}})}$ as follows

$$\frac{\zeta c_n^{(x_{\mathrm{F}})}}{|g(_+|^-)|}dp_0 = -d|p^{\mathrm{R}}|\frac{1}{\Delta_{\mathrm{F}}}\int_{-\Delta_{\mathrm{F}}/2}^{+\Delta_{\mathrm{F}}/2}d\delta x_{\mathrm{F}}$$

$$\times\frac{\exp\left[-ip_0\zeta T/2 - i\zeta|p^{\mathrm{R}}|(x_{\mathrm{F}}+\delta x_{\mathrm{F}})\right]}{|\bar{g}(_+|^-)|},$$

$$(D9)$$

where the quantity $|\bar{g}(_+|^-)| = |g(_+|^-)||_{p^{\mathrm{R}}=\bar{p}^{\mathrm{R}}}$ does not depend on $|p^{\mathrm{R}}|$. This allows one to represent wave packets (D8) as integrals over $|p^{\mathrm{R}}|$,

$$\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X) = \frac{\zeta D}{\Delta_{\mathrm{F}}}\int_{-\Delta_{\mathrm{F}}/2}^{+\Delta_{\mathrm{F}}/2}d\delta x_{\mathrm{F}}\int_0^{p^{\max}}d|p^{\mathrm{R}}|$$

$$\times\exp\left[i\sqrt{(p^{\mathrm{R}})^2 + \pi_\perp^2}(t+\zeta T/2)\right.$$

$$\left.+ i\zeta|p^{\mathrm{R}}|(x - x_{\mathrm{F}} - \delta x_{\mathrm{F}})\right],$$

$$\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X) = \frac{\zeta D}{\Delta_{\mathrm{F}}}\int_{-\Delta_{\mathrm{F}}/2}^{+\Delta_{\mathrm{F}}/2}d\delta x_{\mathrm{F}}\int_0^{p^{\max}}d|p^{\mathrm{R}}|$$

$$\times\exp\left[i\sqrt{(p^{\mathrm{R}})^2 + \pi_\perp^2}(t+\zeta T/2)\right.$$

$$\left.- i\zeta|p^{\mathrm{R}}|(x_{\mathrm{F}} + \delta x_{\mathrm{F}})\right]$$

$$\times\left[g(_+|^\zeta)e^{i|p^{\mathrm{L}}|x} - g(_-|^\zeta)e^{-i|p^{\mathrm{L}}|x}\right],$$

$$\zeta D = \frac{T\exp\left[-iU_{\mathrm{R}}(t+\zeta T/2) + i\mathbf{p}_\perp\mathbf{r}_\perp\right]}{2\sqrt{2}\pi|\bar{g}(_+|^-)|}. \quad (D10)$$

The case $p^{\max}\to 0$ where $|p^{\max}(x_{\mathrm{R}} - x_{\mathrm{F}} - \delta x_{\mathrm{F}})|\lesssim 1$, takes place for relatively weak fields, or near the border between Ω_3 and Ω_2, and, as was already said above, is characterized by big values of the quantity $|g(_+|^-)| \sim |g(_+|^+)| \to \infty$. In this case, it follows from (D10) that the asymptotic densities $|\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X)|^2$ tend to zero,

i.e., electron wave packets do not penetrate in the asymptotic region S_{R}, whereas the absolute values of the coefficients in front of the incoming and outgoing plane waves in the expression for $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ are equal.

In the case when $p^{\max}$ is not small, i.e., $p^{\max}\Delta_{\mathrm{F}} \gg 1$, we consider first the situation when $x_{\mathrm{F}}\in S_{\mathrm{L}}$, $x_{\mathrm{F}} < x_{\mathrm{L}}^{\mathrm{F}}$, and $x_{\mathrm{F}} + \delta x_{\mathrm{F}} < x_{\mathrm{L}}^{\mathrm{F}}$ for all δx_{F}. In our general setting of the problem we have $-T/2 < t < T/2$ that is why $\zeta(t + \zeta T/2) > 0$ and therefore in the region S_{R} where $x > x_{\mathrm{R}} > 0$ the exponent index in the expression (D10) for $\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X)$ is not zero. Moreover, at any time instant, high-frequency oscillations in the latter expression lead to vanishing the asymptotic densities $|\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X)|^2 \to 0$. It is easy to see that situation is quite different in the asymptotic region S_{L}, where $x < x_{\mathrm{L}}^{\mathrm{F}} < x_{\mathrm{L}} < 0$. Here $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ is a superposition of two types of plane waves with opposite signs of the quantum number p^{L}. That is why there always exists such an area on the axis x where the exponent index in the expression (D10) for $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ is zero. In particular, when $|t + \zeta T/2| \sim 0$, there always exists an x such that $|p^{\mathrm{R}}|(x_{\mathrm{F}} + \delta x_{\mathrm{F}}) - |p^{\mathrm{L}}|x = 0$ for any ζ.

This corresponds to incoming wave packets at $t \to -T/2$ for $\zeta = -$ and outgoing wave packets at $t \to T/2$ for $\zeta = +$. Note that mean currents of these electron and positron wave packets are zero unlike the mean currents of constituent plane waves. To understand such a distinct behavior, it useful to recall that mean currents are defined by the inner product (3.33), which is, in particular, the average value over the period of time from $-T/2$ to $T/2$, where T is the time dimension of a large space-time box. Thus, a certain direction of a wave packet in a given time instant matches with the zero average current of this wave packet.

Let us suppose now that $x_{\mathrm{F}}\in S_{\mathrm{R}}$, $x_{\mathrm{F}} > x_{\mathrm{R}}^{\mathrm{F}}$, and $x_{\mathrm{F}} + \delta x_{\mathrm{F}} > x_{\mathrm{R}}^{\mathrm{F}}$ for all δx_{F}. In such a case there exist a coordinate $x \in S_{\mathrm{R}}$ and $x > x_{\mathrm{R}}^{\mathrm{F}} > x_{\mathrm{R}} > 0$, such that the exponent index in the expression (D10) for $\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X)$ is zero. However, since $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ is a superposition of two types of plane waves with opposite signs of the quantum number p^{L}, in the asymptotic region S_{L} there always exists such an area on the axis x where the exponent index in the expression (D10) for $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ is also zero. This means that such wave packets cannot represent an electron. This result holds true for any electron wave packets. Indeed, in our reasonings we have used only the general structure (D5) of functions $\zeta\varphi^{\mathrm{R}}_{x_{\mathrm{F}}}(X)$ which consist of only one type of plane waves with the same sign of the quantum number p^{R}, whereas the functions $\zeta\varphi^{\mathrm{L}}_{x_{\mathrm{F}}}(X)$ represent superpositions of the two types of plane waves with opposite signs of the quantum number p^{L}. Namely this is the reason why electron packets cannot be localized only in S_{R} and cannot represent stable states describing electrons. We see that in the framework of our consideration there are

no electrons in the region S_R with quantum numbers from the range Ω_3.

It is not difficult to give similar example for positron wave packets and prove that they can be localized only in one asymptotic region, namely in S_R.

Thus, in the range Ω_3 there exists the same localization of electrons as in the range Ω_2 and positron localization as in the range Ω_4. That is why in contrast to the ranges Ω_1 and Ω_5, any initial and final wave packets in the range Ω_3 may come in and go out only to the same asymptotic region, which corresponds to the total reflection both for electrons and positrons.

APPENDIX E: DIFFERENTIAL MEAN NUMBER IN SLOWLY ALTERNATING FIELD

The absolute values of $|p^R|$ and $|p^L|$ are related by Eq. (3.21). In the range Ω_3 these relations imply Eq. (D6) and

$$0 \le ||p^L| - |p^R|| \le p^{\max}. \tag{E1}$$

Therefore, for big α that satisfies Eq. (3.21), we obtain

$$\pi\alpha\left[eE\alpha - \frac{1}{2}||p^L| - |p^R||\right] \gg 1. \tag{E2}$$

As a consequence of (E2) the quantities N_n^{cr} given by Eq. (9.4) for fermions, and by Eq. (9.6) for bosons have approximately the same form

$$N_n^{cr} = |g(_+|^-)|^{-2}$$
$$\approx 4\sinh\left(\pi\alpha|p^L|\right)\sinh\left(\pi\alpha|p^R|\right)\exp\left(-2\pi eE\alpha^2\right). \tag{E3}$$

Then it follows from Eq. (E3) that if the range Ω_3 is small enough

$$eE\alpha - \pi_\perp \to 0 \Rightarrow \pi\alpha p^{\max} \ll 1. \tag{E4}$$

then the quantities N_n^{cr} are exponentially small.

Let us consider the opposite case of big ranges Ω_3 when

$$\pi\alpha p^{\max} \gg 1 \tag{E5}$$

and the quantities N_n^{cr} are not small. Such ranges do exist if

$$eE\alpha \gg m \tag{E6}$$

and

$$\alpha\pi_\perp < K_\perp, \tag{E7}$$

where $K_\perp$ is a given arbitrary number, $m\alpha \ll K_\perp \ll eE\alpha^2$.

In this case, we consider first finite subranges adjoining the range Ω_3 from inside. In such subranges

$$N_n^{cr} \simeq 2\sinh\left(\pi\alpha|p^R|\right)$$
$$\times \exp\left[-\pi\alpha\sqrt{(p^R)^2 + \pi_\perp^2}\right] \quad \text{if } \pi\alpha|p^R| < km\alpha,$$
$$N_n^{cr} \simeq 2\sinh\left(\pi\alpha|p^L|\right)$$
$$\times \exp\left[-\pi\alpha\sqrt{(p^L)^2 + \pi_\perp^2}\right] \quad \text{if } \pi\alpha|p^L| < km\alpha, \tag{E8}$$

where $k \gtrsim 1$ is a given arbitrary number, obeying the inequality

$$km\alpha \ll eE\alpha^2. \tag{E9}$$

Near the borders of the range Ω_3, we have

$$N_n^{cr} \approx 2\sinh\left(\pi\alpha|p^R|\right)e^{-\pi\alpha\pi_\perp} \quad \text{if } \pi\alpha|p^R| < K^0,$$
$$N_n^{cr} \approx 2\sinh\left(\pi\alpha|p^L|\right)e^{-\pi\alpha\pi_\perp} \quad \text{if } \pi\alpha|p^L| < K^0, \tag{E10}$$

where $K^0 < 1$ is an arbitrary number. The numbers N_n^{cr} given by Eq. (E10) are exponentially small, $N_n^{cr} \lesssim 2e^{-\pi m\alpha}$.

For border areas situated more close to the center of the range Ω_3, where $1 \lesssim \pi\alpha|p^{R,L}|$, we have

$$N_n^{cr} \approx \exp\left[-\pi\alpha\left[\sqrt{(p^R)^2 + \pi_\perp^2} - |p^R|\right]\right]$$
$$\text{if } 1 \lesssim \pi\alpha|p^R| \lesssim \pi km\alpha,$$
$$N_n^{cr} \approx \exp\left[-\pi\alpha\left[\sqrt{(p^L)^2 + \pi_\perp^2} - |p^L|\right]\right]$$
$$\text{if } 1 \lesssim \pi\alpha|p^L| \lesssim \pi km\alpha. \tag{E11}$$

The numbers N_n^{cr} are growing as n recedes from the borders of the range Ω_3 and for any fixed $\pi_\perp$ achieves his maximum when $\pi\alpha|p^R| \to \pi km\alpha$, or $\alpha|p^L| \to \pi km\alpha$. In turn, this maximum value grows as $\pi_\perp \to m$. Thus, in the subranges under consideration, we can estimate the quantities N_n^{cr} from above as

$$N_n^{cr} < \exp\left(-\frac{\pi m\alpha}{2k}\right). \tag{E12}$$

This quantity is exponentially small if for any given k the ratio $\frac{\pi m\alpha}{2k} > 1$ is big enough.

The main contribution to the particle creation is due to the inner part of the range Ω_3. This range is defined by the following inequalities

$$\pi\alpha|p^R| > \pi km\alpha, \qquad \pi\alpha|p^R| > \pi km\alpha. \tag{E13}$$

They correspond to the following restrictions on the energy p_0:

$$\alpha|p_0| < eE\alpha^2 - K, \qquad K = \alpha\sqrt{(km)^2 + \pi_\perp^2}. \tag{E14}$$

Inequalities (E7) and (E9) imply that $K \ll eE\alpha^2$. In such a case, we can approximate the numbers (E3) as

$$N_n^{\mathrm{cr}} \approx N_{p_0, \mathbf{p}_\perp}^{\mathrm{as}} = e^{-\pi\tau},$$

$$\tau = \alpha(2eE\alpha - |p^{\mathrm{R}}| - |p^{\mathrm{L}}|). \qquad \text{(E15)}$$

The function τ is minimal at $p_0 = 0$,

$$\min \tau = \tau|_{p_0=0} = \lambda = \frac{\pi_\perp^2}{eE}, \qquad \text{(E16)}$$

it grows monotonically as $|p_0|$ grows, and takes its maximum value

$$\tau_{\max} = \tau|_{|p_0|=eE\alpha - K/\alpha} \approx K - km\alpha + \lambda/4 \qquad \text{(E17)}$$

on the boundary of the Ω_3 range. In the wide range of energies where $\alpha|p_0| \ll eE\alpha^2$, the numbers N_n^{cr} do not depend practically on the parameter α and have the form of the differential number of created particles in an uniform electric field [2,13],

$$N_n^{\mathrm{cr}} \approx e^{-\pi\lambda}. \qquad \text{(E18)}$$

[1] A. I. Nikishov, Pair production by a constant electric field, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969) [Transl. Sov. Phys. JETP **30**, 660 (1970)].

[2] A. I. Nikishov, in Quantum Electrodynamics of Phenomena in Intense Fields, *Proc. P.N. Lebedev Phys. Inst.* (Nauka, Moscow, 1979), Vol. 111, p. 153.

[3] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985).

[4] R. Ruffini, G. Vereshchagin, and S. Xue, Electron-positron pairs in physics and astrophysics: From heavy nuclei to black holes, Phys. Rep. **487**, 1 (2010).

[5] D. M. Gitman, Processes of arbitrary order in quantum electrodynamics with a pair-creating external field, J. Phys. A **10**, 2007 (1977); E. S. Fradkin and D. M. Gitman, Furry picture for quantum electrodynamics with pair-creating external field, Fortschr. Phys. **29**, 381 (1981); E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).

[6] O. Klein, Die Reflexion von Elektronen einem Potentialsprung nach der relativistischen Dynamik von Dirac, Z. Phys. **53**, 157 (1929); Elelrtrodynamik und Wellenmechanik vom Standpunkt des Korrespondenzprinzips, Z. Phys. **41**, 407 (1927).

[7] F. Sauter, Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs, Z. Phys. **69**, 742 (1931).

[8] F. Sauter, Zum "Klenschen Paradoxon", Z. Phys. **73**, 547 (1932).

[9] B. R. Holstein, Klein's paradox, Am. J. Phys. **66**, 507 (1998).

[10] P. Krekora, Q. Su, and R. Grobe, Klein Paradox in Spatial and Temporal Resolution, Phys. Rev. Lett. **92**, 040406 (2004).

[11] R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. García-Ripoll, E. Solano, R. Blatt, and C. F. Roos, Quantum Simulation of the Klein Paradox with Trapped Ions, Phys. Rev. Lett. **106**, 060503 (2011).

[12] N. Dombey and A. Calogeracos, Seventy years of the Klein paradox, Phys. Rep. **315**, 41 (1999); History and Physics of the Klein Paradox, Contemp. Phys. **40**, 313 (1999).

[13] A. I. Nikishov, Barrier scattering in field theory: removal of Klein paradox, Nucl. Phys. **B21**, 346 (1970).

[14] A. Hansen and F. Ravndal, Klein's paradox and its resolution, Phys. Scr. **23**, 1036 (1981).

[15] T. Damour, in *Proceedings of the First Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North-Holland, Amsterdam, 1977), p. 459.

[16] R-Ch. Wang and Ch-Y. Wong, Finite-size effect in the Schwinger particle-production mechanism, Phys. Rev. D **38**, 348 (1988).

[17] A. I. Nikishov, Scattering and pair production by a potential barrier, Yad. Fiz. **67**, 1503 (2004) [Transl. Phys. Atom. Nucl. **67**, 1478 (2004)]; On the Theory of Scalar Pair Production by a Potential Barrier, *Problems of Atomic Science and Technology* (Kharkov, Ukraine, 2001), p. 103.

[18] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964); F. Gross, *Relativistic Quantum Mechanics and Field Theory* (Wiley, New York, 1993); B. R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1992).

[19] H. Nitta, T. Kudo, and H. Minowa, Motion of a wave packet in the Klein paradox, Am. J. Phys. **67**, 966 (1999).

[20] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, The Unruh effect and its applications, Rev. Mod. Phys. **80**, 787 (2008).

[21] P. R. Anderson and E. Mottola, On the instability of global de Sitter space to particle creation, Phys. Rev. D **89**, 104038 (2014).

[22] G. V. Dunne, New strong-field QED effects at ELI: Nonperturbative vacuum pair production, Eur. Phys. J. D **55**, 327 (2009); A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Extremely high-intensity laser interactions with fundamental quantum systems, Rev. Mod. Phys. **84**, 1177 (2012); G. Mourou and T. Tajima, Summary of the IZEST science and aspiration, Eur. Phys. J. Spec. Top. **223**, 979 (2014); G. V. Dunne, Extreme quantum field theory and particle physics with IZEST, Eur. Phys. J. Spec. Top. **223**, 1055 (2014); B. M. Hegelich, G. Mourou, and J. Rafelski, Probing the quantum vacuum with ultra intense laser pulses, Eur. Phys. J. Spec. Top. **223**, 1093 (2014).

[23] P. D. Nation, J. R. Johansson, M. P. Blencowe, and F. Nori, Stimulating uncertainty: Amplifying the quantum vacuum with superconducting circuits, Rev. Mod. Phys. **84,** 1 (2012).

[24] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. **81,** 109 (2009); N. M. R. Peres, The transport properties of graphene: An introduction, Rev. Mod. Phys. **82,** 2673 (2010); M. A. H. Vozmediano, M. I. Katsnelson, and F. Guinea, Gauge fields in graphene, Phys. Rep. **496,** 109 (2010).

[25] D. Das Sarma, S. Adam, E. H. Hwang, and E. Rossi, Electronic transport in two-dimensional graphene, Rev. Mod. Phys. **83,** 407 (2011).

[26] M. Z. Hasan and C. L. Kane, Topological insulators, Rev. Mod. Phys. **82,** 3045 (2010); X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83,** 1057 (2011).

[27] O. Vafek and A. Vishwanath, Dirac fermions in solids— from high Tc cuprates and graphene to topological insulators and Weyl semimetals, Annu. Rev. Condens. Matter Phys. **5,** 83 (2014).

[28] N. Vandecasteele, A. Barreiro, M. Lazzeri, A. Bachtold, and F. Mauri, Current-voltage characteristics of graphene devices: Interplay between Zener-Klein tunneling and defects, Phys. Rev. B **82,** 045416 (2010).

[29] S. P. Gavrilov, D. M. Gitman, and N. Yokomizo, Dirac fermions in strong electric field and quantum transport in graphene, Phys. Rev. D **86,** 125022 (2012).

[30] M. I. Katsnelson, K. S. Novoselov, and A. K. Geim, Chiral tunneling and the Klein paradox in graphene, Nat. Phys. **2,** 620 (2006); V. V. Cheianov and V. I. Fal'ko, Selective transmission of Dirac electrons and ballistic magnetoresistance of n-p junctions in graphene, Phys. Rev. B **74,** 041403(R) (2006).

[31] N. Stander, B. Huard, and D. Goldhaber-Gordon, Evidence for Klein Tunneling in Graphene p-n Junctions, Phys. Rev. Lett. **102,** 026807 (2009); A. F. Young and Ph. Kim, Quantum interference and Klein tunnelling in graphene heterojunctions, Nat. Phys. **5,** 222 (2009).

[32] C. W. J. Beenakker, Andreev reflection and Klein tunneling in graphene, Rev. Mod. Phys. **80,** 1337 (2008).

[33] R. Logemann, K. J. A. Reijnders, T. Tudorovskiy, M. I. Katsnelson, and S. Yuan, Modeling Klein tunneling and caustics of electron waves in graphene, Phys. Rev. B **91,** 045420 (2015).

[34] D. Allor, T. D. Cohen, and D. A. McGady, The Schwinger mechanism and graphene, Phys. Rev. D **78,** 096009 (2008).

[35] L. Bucciantini, S. Sotiriadis, and T. Macrì, Probing Klein tunneling through quantum quenches, J. Phys. A **49,** 025002 (2016).

[36] S. P. Kim and D. N. Page, Schwinger pair production via instantons in strong electric fields, Phys. Rev. D **65,** 105002 (2002); Schwinger pair production in electric and magnetic fields, Phys. Rev. D **73,** 065020 (2006); Improved approximations for fermion pair production in inhomogeneous electric fields, Phys. Rev. D **75,** 045013 (2007).

[37] S. P. Kim, H. K. Lee, and Y. Yoon, Effective action of QED in electric field backgrounds. II. Spatially localized fields, Phys. Rev. D **82,** 025015 (2010).

[38] H. Gies and K. Klingmuller, Pair production in inhomogeneous field, Phys. Rev. D **72,** 065001 (2005).

[39] G. V. Dunne and C. Schubert, Worldline instantons and pair production in inhomogeneous fields, Phys. Rev. D **72,** 105004 (2005); G. V. Dunne, H. Gies, C. Schubert, and Q. Wang, Worldline instantons II: The fluctuation prefactor, Phys. Rev. D **73,** 065028 (2006); D. D. Dietrich and G. V. Dunne, Gutzwiller's trace formula and vacuum pair production, J. Phys. A **40,** F825 (2007); R. Schützhold, H. Gies, and G. Dunne, Dynamically Assisted Schwinger Mechanism, Phys. Rev. Lett. **101,** 130404 (2008).

[40] G. V. Dunne and Q. Wang, Multidimensional worldline instantons, Phys. Rev. D **74,** 065015 (2006).

[41] A. Chervyakov and H. Kleinert, Exact pair production rate for a smooth potential step, Phys. Rev. D **80,** 065010 (2009).

[42] A. Chervyakov and H. Kleinert, On electron-positron pair production by a spatially nonuniform electric field, arXiv:1112.4120.

[43] E. T. Akhmedov and F. K. Popov, A few more comments on secularly growing loop corrections in strong electric fields, J. High Energy Phys. 09 (2015) 085.

[44] S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961).

[45] R. Brauer and H. Weyl, Spinors in *n*-dimensions, Am. J. Math. **57,** 425 (1935).

[46] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin-Heidelberg, 1990).

[47] V. De Alfaro and T. Regge, *Potential Scattering* (Interscience Publishers, New York, 1965).

[48] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, Oxford, 1977).

[49] S. P. Gavrilov, D. M. Gitman, and J. L. Tomazelli, Density matrix of a quantum field in a particle-creating background, Nucl. Phys. **B795,** 645 (2008).

[50] J. Schwinger, On gauge invariance and vacuum polarization, Phys. Rev. **82,** 664 (1951).

[51] G. V. Dunne, Heisenberg-Euler Effective Lagrangians: Basics and extensions, in I. Kogan Memorial Volume, *From Fields to Strings: Circumnavigating Theoretical Physics,* edited by M Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2005).

[52] *Higher Transcendental Functions (Bateman Manuscript Project),* edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vols. 1 and 2.

[53] S. P. Gavrilov and D. M. Gitman, Vacuum instability in external fields, Phys. Rev. D **53,** 7162 (1996).

[54] A. Sommerfeld, *Atombau and Spektrallinien,* Vol. 11 (Friedr. Vieweg & Sohn, Braunschweig, 1960).

[55] F. Hund, Materieerzeugung im anschaulichen und im gequantelten Wellenbild der Materie, Z. Phys. **117,** 1 (1941).

[56] R. P. Feynman, *Quantum Electrodynamics* (W. A. Benjamin, New York, 1961).

[57] W. H. Furry, On bound states and scattering in positron theory, Phys. Rev. **81,** 115 (1951).