

Post-Newtonian reference ellipsoid for relativistic geodesySergei Kopeikin^{*,†}*Department of Physics & Astronomy, 322 Physics Building, University of Missouri,
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We apply general relativity to construct the post-Newtonian background manifold that serves as a reference spacetime in relativistic geodesy for conducting a relativistic calculation of the geoid's undulation and the deflection of the plumb line from the vertical. We chose an axisymmetric ellipsoidal body made up of a perfect homogeneous fluid uniformly rotating around a fixed axis, as a source generating the reference geometry of the background manifold through Einstein's equations. We then reformulate and extend hydrodynamic calculations of rotating fluids done by a number of previous researchers for astrophysical applications to the realm of relativistic geodesy to set up algebraic equations defining the shape of the post-Newtonian reference ellipsoid. To complete this task, we explicitly perform all integrals characterizing gravitational field potentials inside the fluid body and represent them in terms of the elementary functions depending on the eccentricity of the ellipsoid. We fully explore the coordinate (gauge) freedom of the equations describing the post-Newtonian ellipsoid and demonstrate that the fractional deviation of the post-Newtonian level surface from the Maclaurin ellipsoid can be made much smaller than the previously anticipated estimate based on the astrophysical application of the coordinate gauge advocated by Bardeen and Chandrasekhar. We also derive the gauge-invariant relations of the post-Newtonian mass and the constant angular velocity of the rotating fluid with the parameters characterizing the shape of the post-Newtonian ellipsoid including its eccentricity, a semiminor axis, and a semimajor axis. We formulate the post-Newtonian theorems of Pizzetti and Clairaut that are used in geodesy to connect the geometric parameters of the reference ellipsoid to the physically measurable force of gravity at the pole and equator of the ellipsoid. Finally, we expand the post-Newtonian geodetic equations describing the post-Newtonian ellipsoid to the Taylor series with respect to the eccentricity of the ellipsoid and discuss their practical applications for geodetic constants and relations adopted in fundamental astronomy.

DOI: [10.1103/PhysRevD.93.044069](https://doi.org/10.1103/PhysRevD.93.044069)**I. INTRODUCTION**

Accurate definition, determination, and realization of celestial and terrestrial reference frames in the Solar System is essential for a deeper understanding of the underlying principles and concepts of fundamental gravitational physics, astronomy, and geophysics as well as for practical purposes of precise satellite and aircraft navigation, positioning, and mapping. It was suggested long ago to separate

the conceptual meaning of a reference frame and a reference system [1]. The former is understood as a theoretical construction, including mathematical models and standards for its implementation. The latter is its practical realization through observations and materialization of coordinates of a set of reference benchmarks, e.g., a set of fundamental stars—for the International Celestial Reference Frame (ICRF)—or a set of fiducial geodetic stations—for the International Terrestrial Reference Frame (ITRF). Continuous monitoring and maintenance of the reference frames and a self-consistent set of geodetic and astronomical constants associated with them is rendered by the International Earth Rotation Service (IERS) and the International Union of Geodesy and Geophysics (IUGG).

Nowadays, four main geodetic techniques are used to compute accurate terrestrial coordinates and velocities of stations—global positioning system (GPS), very long-baseline interferometry (VLBI), satellite laser ranging

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(SLR), and Doppler orbitography and radiopositioning integrated by satellite (DORIS), for the realizations of the ITRF referred to different epochs. The observations are so accurate that geodesists have to model and to include in the data processing the secular Earth's crust changes to reach self-consistency between various ITRF realizations which are available at http://itrf.ensg.ign.fr/ITRF_solutions/index.php. The higher frequencies of the station displacements (mainly due to geophysical phenomena) can be accessed with the formulations present in Ch. 7 of the IERS conventions [2] (see also <http://62.161.69.131/iers/convupdt/convupdt.html>). Continuity between the ITRF realizations has been ensured when adopting conventions for ICRF and ITRF definitions [2][Ch. 4]. It is recognized that to maintain the up-to-date ITRF realization as accurately as possible the development of the most precise theoretical models and parametric relationships is of paramount importance.

Currently, SLR and GPS allow us to determine the transformation parameters between coordinates and velocities of the collocation points of the ITRF realizations with the precision of ~ 1 mm and ~ 1 mm/yr, respectively [[2], Table 4.1]. On the other hand, the dimensional analysis applied to estimate the relativistic effects in geodesy predicts that the post-Newtonian contribution to the coordinates of the ITRF points on the Earth's surface (as compared with the Newtonian theory of gravity) is expected to be of the order of the Earth's gravitational radius that is about 9 mm [3,4] or, maybe, less [5]. This post-Newtonian geodetic effect emerges as an irremovable long-wave deformation of the three-dimensional coordinate grid of the ITRF which might be potentially measurable with the currently available geodetic techniques and hence deserves to be taken into account when building the ITRF realization of a next generation.

ITRF solutions are specified by the Cartesian equatorial coordinates $x^i = \{x, y, z\}$ of the reference geodetic stations. For the purposes of geodesy and gravimetry, the Cartesian coordinates are often converted to geographical coordinates h, θ, λ (h : height, θ : colatitude, λ : longitude) referring to an international reference ellipsoid which is a solution found by Maclaurin [6] for the figure of a fluid body with a homogeneous mass density that slowly rotates around a fixed z axis with a constant angular velocity ω . Because the post-Newtonian effects deforms the shape of the reference ellipsoid [7,8] and modifies the basic equations of classical geodesy, they must be properly calculated to ensure the adequacy of the geodetic coordinate transformations at the millimeter level of accuracy. To evaluate more precisely the post-Newtonian effects in the shape of the Earth's reference ellipsoid and the geodetic equations, we have decided to conduct a more precise mathematical study of equations of relativistic geodesy which is given in the present paper.

Certainly, we are touching upon the topic which has been already discussed in the literature by research teams from

the USA [7,8,11–18], Lithuania [19–21], USSR [22–25], and, most recently, by theorists from the University of Jena in Germany [26–30]. We note that the previous papers focused primarily on studying the astrophysical aspects of the problem like the instability of the equilibrium rotating configurations and the points of bifurcations, finding exact solutions of Einstein's equations for axially symmetric spacetimes, the emission of gravitational waves, etc. Our treatment concerns different aspects and is focused on the post-Newtonian effects in physical geodesy. More specifically, we extend the research on the figures of equilibrium into the realm of relativistic geodesy and pay attention mostly to the possible geodetic applications for an adequate numerical processing of the high-precise data obtained by various geodetic techniques that include but are not limited to SLR, lunar laser ranging (LLR), VLBI, DORIS and global navigation satellite system (GNSS) [31,32]. This vitally important branch of the theory of equilibrium of rotating bodies was not covered by the above-referenced astrophysical works.

An additional stimulating factor for pursuing more advanced research on relativistic geodesy and the Earth figure of equilibrium is related to the recent breakthrough in manufacturing quantum clocks [33], ultraprecise time scale dissemination over the globe [34], and geophysical applications of the clocks [35,36]. Clocks at rest in a gravitational potential tick slower than clocks outside of it. On Earth, this translates to a relative frequency change of 10^{-16} per meter of height difference [37]. Comparing the frequency of a probe clock with a reference clock provides a direct measure of the gravity potential difference between the two clocks. This novel technique has been dubbed *chronometric levelling*. It is envisioned as one of the most promising applications of the relativistic geodesy in the near future [10,38,39]. Optical frequency standards have recently reached a stability of 2.2×10^{-16} at 1 s and demonstrated an overall fractional frequency uncertainty of 2.1×10^{-18} [40], which enables their use for relativistic geodesy at an absolute level of 1 cm.

The chronometric levelling directly measures the equipotential surface of a gravity field (geoid) without conducting a complicated gravimetric survey and solving the differential equations for anomalous gravity potentials [39]. Combining the data of the chronometric levelling with those of the conventional geodetic techniques will allow us to determine the normal heights of reference points with an unprecedented accuracy [35]. An adequate physical interpretation of this type of measurements is inconceivable without an accompanying development of the corresponding mathematical algorithms accounting for the major relativistic effects in geodesy.

Basic theoretical concepts of relativistic geodesy have been discussed in a number of textbooks, most notably [9,41–43] and review papers [5,10,39]. Nonetheless, the theoretical problem of the determination of the reference

level surface in relativistic geodesy has not yet been discussed in the scientific literature with full mathematical rigour. The objective of the present paper is to give its comprehensive post-Newtonian solution. To this end, Sec. II explains briefly the principles of the post-Newtonian approximations and describes the post-Newtonian metric tensor. Section III discusses the post-Newtonian ellipsoid which generalizes the Maclaurin ellipsoid and is the surface of the fourth order. Section IV introduces the reader to the concept of the post-Newtonian gauge freedom and shows how this freedom can be used to simplify the mathematical description of the post-Newtonian (PN) ellipsoid. Sections V and VI calculate, respectively, the Newtonian and post-Newtonian gravitational potentials inside the rotating PN ellipsoid. Section VII gives the post-Newtonian definitions of the conserved mass and angular momentum of the rotating PN ellipsoid. Section VIII is devoted to the derivation of the post-Newtonian equations defining the geometric structure of the reference level surface and its kinematic relation to the angular velocity of rotation ω . Sections IX and X provide the reader with the relativistic generalization of the Pizzetti and Clairaut theorem of classical geodesy [44,45] which connects parameters of the reference ellipsoid with the measured value of the force of gravity. Finally, Sec. XI gives truncated versions of the relativistic formulas which can be used in practical applications of relativistic geodesy. Appendix A contains details of the mathematical calculation of integrals.

We adopt the following notations:

- (i) The Greek indices α, β, \dots run from 0 to 3.
- (ii) The Roman indices i, j, \dots run from 1 to 3.
- (iii) Repeated Greek indices mean Einstein's summation from 0 to 3.
- (iv) Repeated Roman indices mean Einstein's summation from 1 to 3.
- (v) The unit matrix (also known as the Kroneker symbol) is denoted by $\delta_{ij} = \delta^{ij}$,
- (vi) The fully antisymmetric symbol Levi-Civita is denoted as $\varepsilon_{ijk} = \varepsilon^{ijk}$ with $\varepsilon_{123} = +1$.
- (vii) The bold letters $\mathbf{a} = (a^1, a^2, a^3) \equiv (a^i)$, $\mathbf{b} = (b^1, b^2, b^3) \equiv (b^i)$, and so on denote spatial three-dimensional vectors.
- (viii) A dot between two spatial vectors, for example, $\mathbf{a} \cdot \mathbf{b} = a^1 b^1 + a^2 b^2 + a^3 b^3 = \delta_{ij} a^i b^j$, means the Euclidean dot product.
- (ix) The cross between two vectors, for example, $(\mathbf{a} \times \mathbf{b})^i \equiv \varepsilon^{ijk} a^j b^k$, means the Euclidean cross product.
- (x) We use a shorthand notation for partial derivatives $\partial_\alpha = \partial/\partial x^\alpha$.
- (xi) $g_{\alpha\beta}$ is the spacetime metric.
- (xii) The Greek indices are raised and lowered with the metric $g_{\alpha\beta}$.
- (xiii) The Minkowski (flat) space-time metric $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ is used to raise and lower indices of the gravitational metric perturbation, $h_{\alpha\beta}$.

- (xiv) G is the universal gravitational constant.
 - (xv) c is the speed of light in vacuum.
 - (xvi) ω is a constant rotational velocity of the Earth.
 - (xvii) ρ is a constant density of the reference ellipsoid.
 - (xviii) a is a semimajor axis of the Maclaurin ellipsoid.
 - (xix) b is a semiminor axis of the Maclaurin ellipsoid.
 - (xx) $\kappa \equiv \pi G \rho a^2 / c^2$ is a dimensional parameter characterizing the strength of the gravitational field.
 - (xxi) R_\oplus is the mean (volumetric) radius of the Earth, $R_\oplus \approx 6.3710 \times 10^8$ cm.
 - (xxii) a_\oplus is the equatorial radius of the Earth reference ellipsoid, $a_\oplus \approx 6.3781 \times 10^8$ cm.
 - (xxiii) b_\oplus is the polar radius of the Earth reference ellipsoid, $b_\oplus \approx 6.3568 \times 10^8$ cm.
- Other notations are explained in the text as they appear.

II. POST-NEWTONIAN METRIC

Einstein's field equations represent a system of ten nonlinear differential equations in partial derivatives for the metric tensor, $g_{\alpha\beta}$, and we have to find their solutions for the case of an isolated rotating fluid. Because the equations are difficult to solve exactly due to their non-linearity, we apply the post-Newtonian approximations (PNAs) for their solution [7].

The PNAs are applied in case of slowly moving matter having a weak gravitational field. This is exactly the situation in the Solar System which makes PNAs highly appropriate for constructing a relativistic theory of reference frame reference frames [46] and relativistic celestial mechanics celestial mechanics!relativistic in the Solar System [9,41,47].

The PNAs are based on the assumption that a Taylor expansion of the metric tensor can be done in inverse powers of the fundamental speed c that is equal to the speed of light in vacuum. Exact mathematical formulation of a set of basic axioms required for doing the post-Newtonian expansion was given by Rendall [48]. Practically, it requires having several small parameters characterizing the source of gravity. They are $\varepsilon_i \sim v_i/c$, $\varepsilon_e \sim v_e/c$ and $\eta_i \sim U_i/c^2$, $\eta_e \sim U_e/c^2$, where v_i is a characteristic velocity of the motion of matter inside a body, v_e is a characteristic velocity of the relative motion of the bodies with respect to each other, U_i is the internal gravitational potential of each body, and U_e is the external gravitational potential between the bodies. If one denotes a characteristic radius of a body as L and a characteristic distance between the bodies as R , the internal and external gravitational potentials will be $U_i \approx GM/L$ and $U_e \approx GM/R$, where M is a characteristic mass of the body. Because of the virial theorem of the Newtonian gravity [7], the small parameters are not independent. Specifically, one has $\varepsilon_i^2 \sim \eta_i$ and $\varepsilon_e^2 \sim \eta_e$. Hence, parameters ε_i and ε_e are sufficient in doing post-Newtonian approximations. Because within the Solar System these parameters do not significantly differ from each other, they will be not distinguished when doing the

post-Newtonian iterations. In particular, notation $\varepsilon \equiv 1/c$ is used to mark the powers of the fundamental speed c in the post-Newtonian terms. This parameter is considered as a primary expansion parameter of the PNA scheme because all the other parameters are proportional to it, for example, $\varepsilon_i = \varepsilon v_i$, $\eta_i = \varepsilon^2 U_i$, etc.

We work in the framework of general relativity and adopt the harmonic coordinates $x^\alpha = (x^0, x^i)$, where $x^0 = ct$ and t is the coordinate time. The class of the harmonic coordinates is defined by imposing the de Donder gauge condition on the metric tensor [49,50],

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}) = 0. \quad (1)$$

Our choice of the harmonic coordinates is not of the principal value. Calculations could be performed in arbitrary coordinates. Nonetheless, the choice of the harmonic coordinates is dictated by their long-term use in relativistic celestial mechanics, astrometry, and geodesy [4,9,47,51]. Furthermore, all relativistic algorithms of the data processing of high-precise astronomical and geodetic observations are written down in harmonic coordinates and are recommended to use by the International Astronomical Union (IAU) resolutions [2,46]

Einstein equations for the metric tensor are a complicated nonlinear system of differential equations in partial derivatives. Because the gravitational field of the Solar System is weak and the motion of matter is slow, we can focus on the first post-Newtonian approximation of general relativity. Furthermore, we assume that Earth rotates uniformly with angular velocity ω around a fixed axis, which we identify with the z axis. We shall also neglect tides and consider Earth as an isolated body. Under these assumptions, the spacetime becomes stationary with the post-Newtonian metric having the form [9]

$$g_{00} = -1 + \frac{2V}{c^2} + \frac{2}{c^4}(\Phi - V^2) + \mathcal{O}(c^{-6}), \quad (2)$$

$$g_{0i} = -\frac{4V^i}{c^3} + \mathcal{O}(c^{-5}), \quad (3)$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2V}{c^2} \right) + \mathcal{O}(c^{-4}), \quad (4)$$

where the gravitational potentials entering the metric satisfy the Poisson equations,

$$\Delta V = -4\pi G\rho, \quad (5)$$

$$\Delta V^i = -4\pi G\rho v^i, \quad (6)$$

$$\Delta\Phi = -4\pi G\rho \left(2v^2 + 2V + \Pi + \frac{3p}{\varepsilon} \right), \quad (7)$$

with p and v^i being the pressure and velocity of matter, respectively, and Π being the internal energy of matter per

unit mass. We emphasize that ρ is the local mass density of baryons per a unit of the invariant (three-dimensional) volume element $dV = \sqrt{-g}u^0 d^3x$, where u^0 is the time component of the 4-velocity of matter. The local mass density, ρ , relates in the post-Newtonian approximation to the invariant mass density $\rho^* = \sqrt{-g}u^0\rho$. In the first post-Newtonian approximation, this equation reads

$$\rho^* = \rho + \frac{\rho}{c^2} \left(\frac{1}{2} v^2 + 3V \right). \quad (8)$$

We assume that the matter consists of a perfect fluid. Then, the internal energy, Π , is related to pressure, p , and the local density, ρ , by the thermodynamic equation

$$d\Pi + p d\left(\frac{1}{\rho}\right) = 0, \quad (9)$$

and the equation of state, $p = p(\rho)$. In the present paper, we consider the case of a body consisting of a fluid with a constant mass density $\rho = \text{const}$. Equation (9) states then that inside such a fluid the internal energy, Π , is also constant.

In the stationary spacetime, the mass density ρ^* obeys the *exact* equation of continuity

$$\partial_i(\rho^* v^i) = 0. \quad (10)$$

The velocity of rigidly rotating fluid is

$$v^i = \varepsilon^{ijk} \omega^j x^k, \quad (11)$$

where ω^i is the constant angular velocity. Replacing the velocity v^i in Eq. (10) with Eq. (11), and differentiating, reveals that

$$v^i \partial_i \rho = 0, \quad (12)$$

which means that the velocity of the fluid is tangent to the surfaces of constant density ρ .

Modelling the real Earth as a rotating fluid ball of constant density is, of course, unrealistic from a geophysical point of view as it is inconsistent with the seismological data [52] and with IERS data on the Earth's rotation, which clearly indicates the presence of several layers of different densities. Nonetheless, when one considers the geodetic problem of the terrestrial reference frame, the realistic model of the Earth's interior leads to enormous practical difficulties in establishing the geoid's surface. Indeed, calculation of the geoid requires a well-defined reference level surface occupied by rotating fluid and approximating the geoid with the height's deviations as minimal as possible. If one takes a realistic mass distribution, the reference level surface cannot be an ellipsoid of rotation [53]. Furthermore, the gravity field of such a figure of equilibrium (the so-called *normal gravity field* [54]) will be described by too-complicated mathematical equations

which are not suitable for practical applications. A compromise is to take the reference level of the rotating fluid body as an ellipsoid, which allows us to derive the normal gravity field in a concise meaningful form. However, such a reference ellipsoid of rotation can be maintained only by a homogeneous density distribution [53]. Fortunately, the maximum deviation between the level surfaces of the realistic density distribution and the surfaces of equal density are of the order of $e^2 \approx 1/149$ only, and the differences in stress at the model remain considerably smaller than in the real Earth [43,55]. This explains why the classic geodesy operates with the reference Maclaurin ellipsoid of constant density as a reference surface.

The same reasons are applied for the justification of using the constant density ρ to build the reference level surface in relativistic geodesy. The constant density allows us to solve the post-Newtonian equations exactly so that we can write down precise mathematical equations to describe the gravitational field of the post-Newtonian rotating fluid configuration. However, as we shall see in the next section, the rotating fluid of constant density cannot be an ellipsoid of revolution in the post-Newtonian approximation but a surface of the fourth order. In the post-Newtonian approximation, it is possible to build the rotating ellipsoid of revolution which is a surface of the second order, only under the assumption that the density of the fluid has an inhomogeneous ellipsoidal distribution of mass [56]. The deviations from the inhomogeneity are of the post-Newtonian order of magnitude, $\delta\rho/\rho \approx GM_\oplus/c^2 R_\oplus \approx 7 \times 10^{-10}$, and are practically unmeasurable in local experiments. Nonetheless, the relativistic effects in the normal gravity field produced by such a density inhomogeneity over the global scale might be noticed in precise measurements of the gravity field conducted with the next generation of gravimeters [57,58] and/or gravity gradiometers [59–61]. Thus, we again have to compromise between two models of the Earth’s interior—a nonhomogeneous density distribution of matter inside a reference level ellipsoid or a homogeneous density with the small, post-Newtonian deviations of the reference level surface from the precise ellipsoid of revolution. It is not the goal of the present paper to decide between the two cases. Instead, we focus on the solution of the problem of the rotating fluid having a homogeneous distribution of density as the most tractable mathematical case extrapolating the Newtonian case of the Maclaurin ellipsoid to relativistic geodesy. The case of the inhomogeneous density distribution and comparison with the homogeneous density model is considered in [56].

III. POST-NEWTONIAN REFERENCE ELLIPSOID

In classical geodesy, the reference figure for the calculation of geoid’s undulation is the Maclaurin ellipsoid of a rigidly rotating fluid of a constant density ρ . Maclaurin’s ellipsoid is a surface described by a polynomial of the second order [53],

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad (13)$$

where a and b are the semimajor and semiminor axes of the ellipsoid. This property is because the differential Euler equation defining the equilibrium of gravity and pressure is of the first-order partial differential equation of which the first integral is the Newtonian gravity potential that is a scalar function represented by a polynomial of the second order with respect to the Cartesian spatial coordinates. In what follows, we assume $a > b$ and define the eccentricity of the Maclaurin ellipsoid by a standard formula [43,54],

$$e \equiv \frac{\sqrt{a^2 - b^2}}{a}. \quad (14)$$

We shall demonstrate in the following sections that in the post-Newtonian approximation the gravity potential, W , of the rotating homogeneous fluid is a polynomial of the fourth order as was first noticed by Chandrasekhar [7]. Hence, the level surface of a rigidly rotating fluid is expected to be a surface of the fourth order. We shall assume that the surface remains axisymmetric in the post-Newtonian approximation and call the body with such a surface as a PN ellipsoid [62].

We shall denote all quantities taken on the surface of the PN ellipsoid with a bar to distinguish them from the coordinates outside of the surface. Let the barred coordinates $\bar{x}^i = \{\bar{x}, \bar{y}, \bar{z}\}$ denote a point on the surface of the PN ellipsoid with the axis of symmetry directed along the rotational axis and with the origin located at its post-Newtonian center of mass. The post-Newtonian definitions of mass, the center of mass, and the other multipole moments of an extended astronomical body can be found, for example, in Ref. [9][chapter 4.5.3] and are also given in Sec. VII of the present paper. Let the rotational axis coincide with the direction of the z axis. Then, the most general equation of the PN ellipsoid is

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1 + \kappa F(\mathbf{x}), \quad (15)$$

where $\sigma^2 \equiv x^2 + y^2$, $\kappa \equiv \pi G \rho a^2 / c^2$ is the post-Newtonian parameter which is convenient in the calculations that follow,

$$F(\mathbf{x}) \equiv K_1 \frac{\sigma^2}{a^2} + K_2 \frac{z^2}{b^2} + E_1 \frac{\sigma^4}{a^4} + E_2 \frac{z^4}{b^4} + E_3 \frac{\sigma^2 z^2}{a^2 b^2}, \quad (16)$$

and K_1, K_2, E_i ($i = 1, 2, 3$) are arbitrary numerical coefficients.

Let $x^i = \{x, y, z\}$ be any point inside the PN ellipsoid. We introduce a quadratic polynomial,

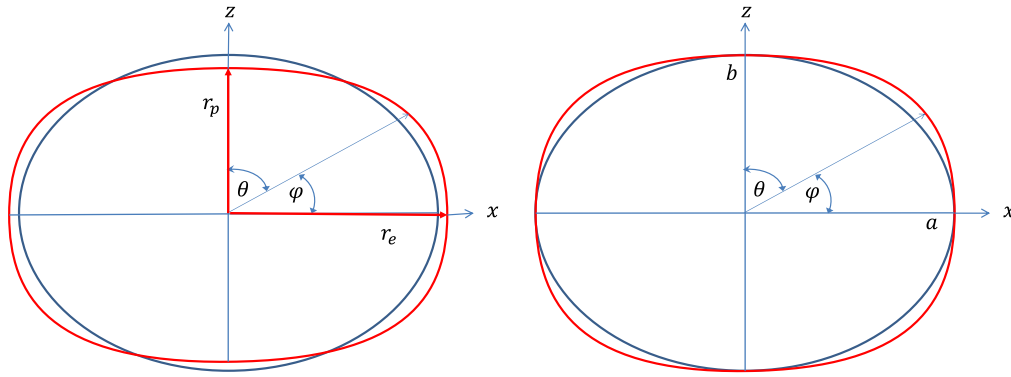


FIG. 1. Meridional cross section of the PN ellipsoid (a red curve in the online version) vs the Maclaurin ellipsoid (a blue curve in the online version). The left panel represents the most general case with arbitrary values of the shape parameters E_1, E_2, E_3 when the equatorial, r_e , and polar, r_p , radii of the PN ellipsoid differ from the semimajor, a , and semiminor, b , axes of the Maclaurin ellipsoid, $r_e \neq a, r_p \neq b$. The right panel shows the case of $E_1 = E_2 = 0$ when the equatorial and polar radii of the PN ellipsoid and the Maclaurin ellipsoid are equal. The angle φ is the geographic latitude ($-90^\circ \leq \varphi \leq 90^\circ$), and the angle θ is a complementary angle (colatitude) used for the calculation of integrals in the Appendix of the present paper ($0 \leq \theta \leq \pi$). In general, when $E_1 \neq E_2 \neq 0$, the maximal radial difference (the “height” difference) between the surfaces of the PN ellipsoid and the Maclaurin ellipsoid depends on the choice of the post-Newtonian coordinates and can amount to a few cm. Carefully operating with the residual gauge freedom of the post-Newtonian theory by choosing $E_1 = E_2 = 0$ allows us to make the difference between the two surfaces much less than 1 mm that is practically unobservable at the present time (for more discussion, see Sec. XI).

$$C(\mathbf{x}) \equiv \frac{\sigma^2}{a^2} + \frac{z^2}{b^2} - 1, \quad (17)$$

where $\sigma^2 \equiv x^2 + y^2$. This polynomial vanishes on the boundary surface of the Maclaurin ellipsoid (13). However, in the post-Newtonian approximation, we have on the boundary of the PN ellipsoid (15) the following condition:

$$C(\bar{\mathbf{x}}) = \kappa F(\bar{\mathbf{x}}). \quad (18)$$

In terms of the polynomial $C(\mathbf{x})$, function $F(\mathbf{x})$ in the right-hand side of (15) can be formally recast to

$$F(\mathbf{x}) = K_1 + E_1 - (K_1 - K_2 + 2E_1 - E_3) \frac{z^2}{b^2} + (E_1 + E_2 - E_3) \frac{z^4}{b^4} + \mathfrak{R}(\mathbf{x}), \quad (19)$$

where the remainder

$$\mathfrak{R}(\mathbf{x}) \equiv \left[K_1 + E_1 + E_1 \frac{\sigma^2}{a^2} + (E_3 - E_1) \frac{z^2}{b^2} \right] C(\mathbf{x}). \quad (20)$$

The remainder $\mathfrak{R}(\mathbf{x})$ can be discarded on the boundary of the PN ellipsoid because $\mathfrak{R}(\bar{\mathbf{x}}) = 0$. This property indicates a specific freedom in the definition of the surface of the PN ellipsoid. Namely, Eq. (15) is defined up to a class of equivalence *modulo* function $C(\mathbf{x})$ given in Eq. (17) that vanishes on the surface of the Maclaurin ellipsoid, $C(\bar{\mathbf{x}}) = 0$. It means that the surface of the PN ellipsoid defined by Eq. (15) is always determined in the

post-Newtonian terms only up to a function $(\alpha_1 \frac{\sigma^2}{a^2} + \alpha_2 \frac{z^2}{b^2}) C(\mathbf{x})$ where α_1, α_2 are arbitrary constant parameters. By making a specific choice of the parameters α_1, α_2 , we can eliminate any two of the five coefficients K_1, K_2, E_1, E_2, E_3 entering function $F(\mathbf{x})$ in Eq. (15). It is convenient to choose $K_1 = K_2 = 0$ that simplifies the equations which follow. The choice $K_1 = K_2 = 0$ is equivalent to rescaling the semimajor and semiminor axes, a and b , in Eq. (15), respectively.

Each cross section of the PN ellipsoid being orthogonal to the rotational axis represents a circle. The equatorial cross section has an equatorial radius, $\bar{\sigma} = r_e$, being determined from Eq. (15) by the condition $\bar{z} = 0$. It yields

$$r_e = a \left(1 + \frac{1}{2} \kappa E_1 \right). \quad (21)$$

The meridional cross section of the PN ellipsoid is no longer an ellipse (as it was in case of the Maclaurin ellipsoid) but a curve of the fourth order. Nonetheless, we can define the polar radius, $\bar{z} = r_p$, of the PN ellipsoid by the condition, $\bar{\sigma} = 0$. Equation (15) yields

$$r_p = b \left(1 + \frac{1}{2} \kappa E_2 \right). \quad (22)$$

In terms of the parameters r_e and r_p , Eq. (15) of the PN ellipsoid takes on the following form:

$$\frac{\sigma^2}{r_e^2} + \frac{z^2}{r_p^2} = 1 - \kappa(E_1 + E_2 - E_3) \left(\frac{z^2}{r_p^2} - \frac{z^4}{r_p^4} \right). \quad (23)$$

This reveals that only a single combination, $E_1 + E_2 - E_3$, of the parameters explicitly appears in the description of the shape of the PN ellipsoid in the harmonic coordinates, while the other two parameters, E_1 and E_2 , can be absorbed (like the coefficients K_1 and K_2 above) in its equatorial and polar radii. The combination $E_1 + E_2 - E_3$ is determined by the physical equation of the equilibrium of the rotating fluid as explained in Sec. VIII. Parameters E_1 and E_2 are not limited by physics and can be chosen arbitrary within the accuracy allowed by the post-Newtonian approximation. We discuss their possible choices in Secs. VIII and XI.

We characterize the ‘‘oblateness’’ of the PN ellipsoid (23) by the post-Newtonian ‘‘eccentricity’’

$$\epsilon \equiv \frac{\sqrt{r_e^2 - r_p^2}}{r_e}. \quad (24)$$

It differs from the eccentricity (14) of the Maclaurin ellipsoid by the relativistic correction

$$\epsilon = e - \kappa \frac{1 - e^2}{2e} (E_2 - E_1). \quad (25)$$

If we choose either $E_2 = E_1$ or $E_1 = E_2 = 0$, the two eccentricities will coincide. The possible configurations of the PN ellipsoid vs Maclaurin’s ellipsoid are visualized in Fig. 1.

IV. POST-NEWTONIAN GAUGE FREEDOM

The theoretical formalism for the calculation of the post-Newtonian level surface can be worked out in arbitrary coordinates. For mathematical and historic reasons, the most convenient are harmonic coordinates which are also used by the IAU [46] and IERS [2] astro-geodetic data processing centers. The harmonic coordinates are selected by the de Donder condition (1), but it does not pick up a single coordinate system because of the property known as a residual gauge freedom [63]. It means that the gauge condition (1) selects an infinite set of harmonic coordinates interrelating by coordinate transformations which do not violate the gauge condition (1). The field equations (5)–(7) and their solutions are form invariant with respect to the residual gauge transformations. The residual gauge freedom can be further limited by imposing additional constraints on the metric tensor [9].

The residual gauge freedom of the harmonic coordinates is described by a post-Newtonian coordinate transformation,

$$x'^\alpha = x^\alpha + \kappa \xi^\alpha(x), \quad (26)$$

where functions, ξ^α , obey the Laplace equation [63],

$$\Delta \xi^\alpha = 0. \quad (27)$$

We discuss the case of the solution of the post-Newtonian equations (5)–(7) inside the rotating fluid. Therefore, the solution of the Laplace equation (27) must be convergent at the origin of the coordinate system. It is well known that such a solution is given in terms of the harmonic polynomials which are selected by the condition that the shape of the PN ellipsoid given by Eq. (15) remains the polynomial of the fourth order with the rotational symmetry about the z axis. This condition reduces the functions ξ^α in Eq. (26) to the harmonic polynomials of the third order having the form

$$\xi^1 = hx + \frac{px}{a^2}(\sigma^2 - 4z^2), \quad (28a)$$

$$\xi^2 = hy + \frac{py}{a^2}(\sigma^2 - 4z^2), \quad (28b)$$

$$\xi^3 = kz + \frac{qz}{b^2}(3\sigma^2 - 2z^2), \quad (28c)$$

where h , k , p , and q are arbitrary constant parameters. It can be checked by direct inspection that the polynomials (28a)–(28c) satisfy the Laplace equation (27). We have chosen $\xi^0 = 0$ because we consider a stationary spacetime, which means that all functions are time independent. We emphasize that the transformation (28a)–(28c) does not preserve the element of the coordinate volume d^3x in the most general case. The coordinate volume would be preserved if $\partial_\alpha \xi^\alpha = 0$, which implies $k = -2h$ and $q = -4b^2 p/3a^2$. This constraint on the gauge transformation was originally employed by Chandrasekhar [11], who treated the volume-preserved transformations as the Lagrange displacements of the Maclaurin ellipsoid. Later on, Chandrasekhar abandoned this constraint [14] to adjust his theory to physical criteria for comparison of the post-Newtonian and Newtonian configurations of rotating fluids proposed by Bardeen [8] (see Sec. VIII for further details).

Coordinate transformation (26) with ξ^i taken from (28a)–(28c) does not violate the harmonic gauge condition (1), but it changes Eqs. (15) and (16) to

$$\frac{\sigma'^2}{a^2} + \frac{z'^2}{b^2} = 1 + \kappa F'(x'), \quad (29)$$

$$F'(x') \equiv K_1' \frac{\sigma'^2}{a^2} + K_2' \frac{z'^2}{b^2} + E_1' \frac{\sigma'^4}{a^4} + E_2' \frac{z'^4}{b^4} + E_3' \frac{\sigma'^2 z'^2}{a^2 b^2}, \quad (30)$$

where $\sigma'^2 \equiv x'^2 + y'^2$ and the primed coefficients are

$$K_1' = K_1 + 2h, \quad (31a)$$

$$K_2' = K_2 + 2k, \quad (31b)$$

$$E_1' = E_1 + 2p, \quad (31c)$$

$$E_2' = E_2 - 4q, \quad (31d)$$

$$E_3' = E_3 - 8p \frac{b^2}{a^2} + 6q \frac{a^2}{b^2}. \quad (31e)$$

Transformation equations (31a)–(31e) make it evident that four out of the five coefficients K_1, K_2, E_1, E_2, E_3 are algebraically independent. Moreover, there is one-to-one mapping between four parameters: $K_1 \leftrightarrow h, K_2 \leftrightarrow k, E_1 \leftrightarrow p,$ and $E_2 \leftrightarrow q$. It means that the choice of the coordinate parameters, $h, k, p,$ and $q,$ is actually equivalent to selecting the coefficients K_1, K_2, E_1, E_2 in the original equation (15) of the PN ellipsoid and fixing the residual gauge freedom of the harmonic coordinates. Because the geodetic data in classic geodesy are referred to the surface of the Maclaurin ellipsoid, it would be practically useful to find such a post-Newtonian gauge in which the differences between the surfaces of the Maclaurin and PN ellipsoid were minimized. It would allow us to avoid unnecessary complications in adjusting the results of classic geodesy to the realm of the general theory of relativity. Nonetheless, the question about what post-Newtonian gauge is the most convenient for geodesy remains open for the time being. We shall explore some possible options to fix the residual gauge in subsequent sections to see how the post-Newtonian physical equations defining the level surface, mass, angular momentum, etc., depend on the choice of the gauge in Secs. VIII and XI.

We have already fixed $K_1 = K_2 = 0$. It complies with the transformations (31a), (31b) indicating that picking up the gauge parameters h, k can always eliminate the coefficients K_1, K_2 . We shall fix the coefficients E_1, E_2 later on, after solving the field equations and determining the gravitational potentials. The coefficient E_3 linearly depends on the choice of the parameters p and q and is a truly gauge-dependent parameter. Its value is fixed (after choosing the parameters E_1 and E_2) by physics of the rotating fluid in the gravitational field leading to Eq. (134) of the equipotential level surface.

Needless to say, physical quantities that make sense must be gauge-invariant quantities. In what follows, we will demonstrate how to build the gauge-invariant expressions for the total mass and angular momentum of the rotating fluid. Building the gauge-invariant expressions for the force of gravity is possible as well but takes us away from the canonical expressions adopted in the Newtonian geodesy. The gauge-invariant expressions for geodetic observables including the force of gravity are given, for example, in Refs. [64,65]. However, the gauge-invariant approach in practical geodesy has little, if any, application. Observables are gauge-invariant quantities, but they are taken at different epochs and places and must be interconnected. The interconnection of the observables is done with the help of

the propagation equations mapping the observables to the fixed coordinate systems that are employed in fundamental astronomy and geodesy solely as the intermediate book-keeper in order to compare the observables at one epoch to observables measured at another epoch. The process of the data processing maps one gauge-invariant quantity to another through the intermediate coordinate chart. This gauge-dependent chart is called the reference ellipsoid, stellar fundamental catalog, ITRF, etc., and they cannot be made gauge independent because they are essentially realizations of the fundamental coordinate systems which are chosen and fixed by conventions adopted at general assemblies of the IAU, IUGG, etc. Therefore, not all our results can be presented in the gauge-invariant form because this manuscript is about the fundamental coordinate systems in relativistic geodesy and their comparison.

V. NEWTONIAN POTENTIAL V

Newtonian gravitational potential V satisfies the inhomogeneous Poisson equation

$$\Delta V(\mathbf{x}) = -4\pi G\rho(\mathbf{x}), \quad (32)$$

inside the mass. Its particular solution is given by

$$V(\mathbf{x}) = \int_{\mathcal{V}} \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x} - \mathbf{x}'|}, \quad (33)$$

where \mathcal{V} is the coordinate volume occupied by the matter distribution. Inside the mass and under the assumption of the constant mass density ρ , the integral (33) can be calculated by making use of the spherical coordinates θ, λ on a unit sphere. The procedure is as follows [53].

Let us consider a point $x^i = \{x, y, z\}$ inside the PN ellipsoid (15). It is connected to a point \bar{x}^i on the surface of the ellipsoid by a vector $R^i = \bar{x}^i - x^i$ where $R^i = R\ell^i$, $R = \sqrt{\delta_{ij}R^iR^j}$, and the unit vector, $\ell^i \equiv \{\sin\theta \cos\lambda, \sin\theta \sin\lambda, \cos\theta\}$. In terms of these quantities, we have

$$\bar{x}^i = x^i + \ell^i R. \quad (34)$$

Substituting Eq. (34) into Eq. (15) yields a quadratic equation,

$$AR^2 + 2BR + C = \kappa F(\mathbf{x} + \ell R), \quad (35)$$

where $\mathbf{x} \equiv \{x^i\}$, $\mathbf{l} = \{\ell^i\}$, and

$$\begin{aligned} A &\equiv \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}, \\ B &\equiv \frac{\sin\theta(x \cos\lambda + y \sin\lambda)}{a^2} + \frac{z \cos\theta}{b^2}, \\ C &\equiv \frac{\sigma^2}{a^2} + \frac{z^2}{b^2} - 1. \end{aligned} \quad (36)$$

We solve Eq. (35) iteratively by making use of $R = \hat{R} + c^{-2}\Delta R$, where $\hat{R} = (\hat{R}_+, \hat{R}_-)$ corresponds to two algebraically independent solutions of the quadratic equation (35) with the right side being nil and ΔR being yet unknown. After omitting terms of the order of $\mathcal{O}(\kappa^2) \simeq \mathcal{O}(c^{-4})$, we have two roots,

$$R_{\pm} = -\frac{B}{A} \pm \frac{\sqrt{B^2 - AC + \kappa A F_{\pm}}}{A}, \quad (37)$$

where

$$\begin{aligned} F_{\pm} \equiv & E_1 + (E_3 - 2E_1) \left(\frac{z + \cos \theta \hat{R}_{\pm}}{b} \right)^2 \\ & + (E_1 + E_2 - E_3) \left(\frac{z + \cos \theta \hat{R}_{\pm}}{b} \right)^4 \\ & + \mathfrak{R}(\mathbf{x} + \ell R_{\pm}). \end{aligned} \quad (38)$$

We replace variable \mathbf{x}' in Eq. (33) with $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and use the spherical coordinates to perform the integration with respect to the radial coordinate $r = |\mathbf{x} - \mathbf{x}'|$. After integrating, the integral (33) takes on the form [53]

$$V = \frac{1}{4} G\rho \oint_{S^2} (R_+^2 + R_-^2) d\Omega, \quad (39)$$

where R_+ and R_- are defined in Eq. (37). After making use of Eq. (37) and expanding the integrand in Eq. (39) with respect to the post-Newtonian parameter κ , the Newtonian potential takes on the form

$$\begin{aligned} V = & \frac{1}{2} G\rho \oint_{S^2} \left\{ \frac{2B^2 - AC}{A^2} \right. \\ & \left. + \frac{\kappa}{2A} \left[F_+ + F_- - \frac{B}{\sqrt{B^2 - AC}} (F_+ - F_-) \right] \right\} d\Omega, \end{aligned} \quad (40)$$

where all post-Newtonian terms of the higher order with respect to κ have been discarded, the integration is performed over a unit sphere S^2 with respect to the angles λ and θ , and $d\Omega \equiv \sin \theta d\theta d\lambda$ is the element of the solid angle on the unit sphere.

Now, we expand F_{\pm} in a polynomial with respect to R_{\pm} ,

$$\begin{aligned} F_{\pm} = & \alpha_0 + \alpha_1 \cos \theta R_{\pm} + \alpha_2 (\cos \theta R_{\pm})^2 + \alpha_3 (\cos \theta R_{\pm})^3 \\ & + \alpha_4 (\cos \theta R_{\pm})^4, \end{aligned} \quad (41)$$

where the residual term \mathfrak{R}_{\pm} vanishes because it is proportional to $C(\bar{\mathbf{x}}) = AR^2 + 2BR + C = 0 + \mathcal{O}(\kappa)$ and the coefficients

$$\alpha_0 = E_1 + (E_3 - 2E_1) \frac{z^2}{b^2} + (E_1 + E_2 - E_3) \frac{z^4}{b^4}, \quad (42)$$

$$\alpha_1 = \frac{2z}{b^2} \left[E_3 - 2E_1 + 2(E_1 + E_2 - E_3) \frac{z^2}{b^2} \right] \quad (43)$$

$$\alpha_2 = \frac{1}{b^2} \left[E_3 - 2E_1 + 6(E_1 + E_2 - E_3) \frac{z^2}{b^2} \right] \quad (44)$$

$$\alpha_3 = \frac{4z}{b^4} (E_1 + E_2 - E_3), \quad (45)$$

$$\alpha_4 = \frac{1}{b^4} (E_1 + E_2 - E_3), \quad (46)$$

are polynomials of z only. We also notice that on the surface of the PN ellipsoid $F(\bar{\mathbf{x}}) = \alpha_0$, as follows from Eqs. (19) and (20). We can also use $C(\bar{\mathbf{x}}) = 0$ in the post-Newtonian terms.

Replacing Eq. (41) in Eq. (40) transforms it to

$$V = V_N + \kappa V_{pN}, \quad (47)$$

where

$$V_N = G\rho \oint_{S^2} \left(\frac{B^2}{A^2} - \frac{C}{2A} \right) d\Omega \quad (48)$$

$$\begin{aligned} V_{pN} = & G\rho \oint_{S^2} \left[\frac{\alpha_0}{2A} - \alpha_1 \cos \theta \frac{B}{A^2} + \alpha_2 \cos^2 \theta \left(\frac{2B^2}{A^3} - \frac{C}{2A^2} \right) \right. \\ & - 2\alpha_3 \cos^3 \theta \left(\frac{2B^3}{A^4} - \frac{BC}{A^3} \right) \\ & \left. + \alpha_4 \cos^4 \theta \left(\frac{8B^4}{A^5} - \frac{6B^2C}{A^4} + \frac{C^2}{2A^3} \right) \right] d\Omega. \end{aligned} \quad (49)$$

Equations (47)–(49) describe the Newtonian potential exactly, both on the surface of the PN ellipsoid and inside it.

The integrals in Eqs. (48) and (49) are discussed in Appendix A. After evaluating the integrals and reducing similar terms, potentials V_N and V_{pN} take on the form

$$V_N = \pi G\rho a^2 \left[\left(1 - \frac{z^2}{b^2} \right) \lambda_0 - \left(1 - 3 \frac{z^2}{b^2} \right) \lambda_1 - C(\mathbf{x}) \lambda_1 \right], \quad (50)$$

$$V_{pN} = \pi G\rho a^2 [F_1(z) + b^2 F_2(z) C(\mathbf{x}) + b^4 F_3(z) C^2(\mathbf{x})], \quad (51)$$

where

$$F_1(z) = \alpha_0 \lambda_0 - 2\alpha_1 z \lambda_1 + 2\alpha_2 b^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_1 - \left(1 - \frac{3z^2}{b^2}\right) \lambda_2 \right] - 4\alpha_3 b^2 z \left[3 \left(1 - \frac{z^2}{b^2}\right) \lambda_2 - \left(3 - \frac{5z^2}{b^2}\right) \lambda_3 \right] + 6\alpha_4 b^4 \left[\left(1 - \frac{z^2}{b^2}\right)^2 \lambda_2 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4}\right) \lambda_3 + \left(1 - 10 \frac{z^2}{b^2} + \frac{35}{3} \frac{z^4}{b^4}\right) \lambda_4 \right], \quad (52)$$

$$F_2(z) = \alpha_2 (\lambda_1 - 2\lambda_2) - 4\alpha_3 z (2\lambda_2 - 3\lambda_3) + 6\alpha_4 b^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_2 - 3 \left(1 - \frac{3z^2}{b^2}\right) \lambda_3 + 2 \left(1 - \frac{5z^2}{b^2}\right) \lambda_4 \right], \quad (53)$$

$$F_3(z) = \alpha_4 (\lambda_2 - 6\lambda_3 + 6\lambda_4) \quad (54)$$

and the polynomial coefficients $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are given in Eqs. (42)–(46). It is worth noticing that the potential V_N satisfies the Poisson equation (32) exactly. It means that the post-Newtonian function V_{pN} obeys the Laplace equation

$$\Delta V_{pN} = 0, \quad (55)$$

and the right side of Eq. (51) is a harmonic polynomial of the fourth order.

VI. POST-NEWTONIAN POTENTIALS

A. Vector potential V^i

Vector potential V^i obeys the Poisson equation

$$\Delta V^i = -4\pi G \rho(\mathbf{x}) v^i(\mathbf{x}), \quad (56)$$

which has a particular solution,

$$V^i = G \int_{\mathcal{V}} \frac{\rho(\mathbf{x}') v^i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x'. \quad (57)$$

For a rigidly rotating configuration, $v^i(\mathbf{x}) = \varepsilon^{ijk} \omega^j x^k$ so that

$$V^i = \varepsilon^{ijk} \omega^j \mathcal{D}^k, \quad (58)$$

where

$$\mathcal{D}^i = G \int \frac{\rho(\mathbf{x}') x'^i d^3 x'}{|\mathbf{x} - \mathbf{x}'|}. \quad (59)$$

It can be recast to the form

$$\mathcal{D}^i = x^i V_N + G \int \frac{\rho(\mathbf{x}') (x'^i - x^i)}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \quad (60)$$

where V_N is the Newtonian potential given in Eq. (50). For the case of a constant density, $\rho(\mathbf{x}') = \rho = \text{const.}$, the second term in the right-hand side of Eq. (60) can be integrated over the radial coordinate, yielding

$$\int_{\mathcal{V}} \frac{\rho(\mathbf{x}') (x'^i - x^i)}{|\mathbf{x} - \mathbf{x}'|} d^3 x' = \frac{\rho}{6} \oint_{S^2} (R_+^3 + R_-^3) l^i d\Omega. \quad (61)$$

After making use of Eq. (37) to replace R_+ and R_- , we obtain

$$\int_{\mathcal{V}} \frac{\rho(\mathbf{x}') (x'^i - x^i)}{|\mathbf{x} - \mathbf{x}'|} d^3 x' = \rho \oint_{S^2} \left(-\frac{4}{3} \frac{B^3}{A^3} + \frac{BC}{A^2} \right) l^i d\Omega, \quad (62)$$

where we have omitted the post-Newtonian terms being proportional to $\mathcal{O}(\kappa)$ since the vector potential V^i itself appears only in the post-Newtonian terms. Integrals entering Eq. (62) are given in Appendix A. Calculation reveals

$$\int_{\mathcal{V}} \frac{\rho(\mathbf{x}')(x' - x)}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -\pi\rho a^2 x \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_1 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 \right] + x\pi\rho a^2 C(\mathbf{x})(\lambda_1 - \lambda_2), \quad (63)$$

$$\int_{\mathcal{V}} \frac{\rho(\mathbf{x}')(y' - y)}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -\pi\rho a^2 y \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_1 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 \right] + y\pi\rho a^2 C(\mathbf{x})(\lambda_1 - \lambda_2), \quad (64)$$

$$\int_{\mathcal{V}} \frac{\rho(\mathbf{x}')(z' - z)}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -4\pi\rho a^2 z \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_1 - \left(1 - \frac{5z^2}{3b^2}\right) \lambda_2 \right] - 2z\pi\rho a^2 C(\mathbf{x})(\lambda_1 - 2\lambda_2). \quad (65)$$

Substituting this result in Eq. (60) and making use of Eq. (50) yields

$$\mathcal{D}^i \equiv (\mathcal{D}^x, \mathcal{D}^y, \mathcal{D}^z) = (xD_1, yD_1, zD_2), \quad (66)$$

where functions

$$D_1 \equiv \pi G\rho a^2 \left[\left(1 - \frac{3z^2}{b^2}\right) \lambda_1 - \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 - C(\mathbf{x})\lambda_2 \right], \quad (67)$$

$$D_2 \equiv \pi G\rho a^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - \left(5 - 7\frac{z^2}{b^2}\right) \lambda_1 + 4\left(1 - \frac{5z^2}{3b^2}\right) \lambda_2 + C(\mathbf{x})(4\lambda_2 - 3\lambda_1) \right]. \quad (68)$$

B. Scalar potential Φ

Potential Φ is defined by the equation

$$\Delta\Phi = -4\pi G\rho(\mathbf{x})\phi(\mathbf{x}), \quad (69)$$

where the function

$$\phi(\mathbf{x}) \equiv 2\omega^2\sigma^2 + 3\frac{P}{\rho} + 2V_N. \quad (70)$$

In the Newtonian approximation, pressure p inside the massive body with a constant density ρ has an ellipsoidal distribution and is given by the solution of the equation of a hydrostatic equilibrium [53],

$$\frac{P}{\rho} = -\pi G\rho a^2 C(\mathbf{x})(\lambda_0 - 2\lambda_1). \quad (71)$$

Making use of Eqs. (71) and (50), we can write down function $\phi(\mathbf{x})$ as

$$\phi(\mathbf{x}) = a^2[2\omega^2 - \pi G\rho(3\lambda_0 - 4\lambda_1)] \left(\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} \right) - 2a^2[\omega^2 - \pi G\rho(\lambda_0 - 3\lambda_1)] \frac{z^2}{b^2} + \pi G\rho a^2(5\lambda_0 - 6\lambda_1). \quad (72)$$

The particular solution of Eq. (69) can be written, then, as

$$\Phi = 2\omega^2 a^2 (I_1 - I_2) - \pi G\rho a^2 [(3I_1 + 2I_2 - 5V_N)\lambda_0 + 2(2I_1 + 3I_2 - 3V_N)\lambda_1], \quad (73)$$

where we have introduced two new integrals:

$$I_1 = G\rho \int_{\mathcal{V}} \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{\sigma'^2}{a^2} + \frac{z'^2}{b^2} \right), \quad I_2 = \frac{G\rho}{b^2} \int_{\mathcal{V}} \frac{z'^2}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (74)$$

The integrals can be split in several algebraic pieces,

$$I_1 = \left(\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} \right) (2D_1 - V_N) + 2 \frac{z^2}{b^2} (D_2 - D_1) + \frac{G\rho}{8} \oint_{S^2} (R_+^4 + R_-^4) \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} \right) d\Omega, \quad (75)$$

$$I_2 = \frac{z^2}{b^2} (2D_2 - V_N) + \frac{G\rho}{8b^2} \oint_{S^2} (R_+^4 + R_-^4) \cos^2\theta d\Omega, \quad (76)$$

where the integrals

$$\frac{1}{8} \oint_{S^2} (R_+^4 + R_-^4) \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} \right) d\Omega = \oint_{S^2} \left(\frac{2B^4}{A^3} - 2 \frac{B^2C}{A^2} + \frac{C^2}{4A} \right) d\Omega, \quad (77)$$

$$\frac{1}{8} \oint_{S^2} (R_+^4 + R_-^4) \cos^2\theta d\Omega = \oint_{S^2} \left(\frac{2B^4}{A^4} - 2 \frac{B^2C}{A^3} + \frac{C^2}{4A^2} \right) \cos^2\theta d\Omega. \quad (78)$$

We use the results of Appendix A to calculate these integrals and obtain

$$\begin{aligned} \oint_{S^2} \left(\frac{2B^4}{A^3} - 2 \frac{B^2C}{A^2} + \frac{C^2}{4A} \right) d\Omega &= \frac{3}{2} \pi a^2 \left[\left(1 - \frac{z^2}{b^2} \right)^2 \lambda_0 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4} \right) \lambda_1 + \left(1 - 10 \frac{z^2}{b^2} + \frac{35}{3} \frac{z^4}{b^4} \right) \lambda_2 \right] \\ &+ \pi a^2 \left[\left(1 - \frac{z^2}{b^2} \right) \lambda_0 - 4 \left(1 - 3 \frac{z^2}{b^2} \right) \lambda_1 + 3 \left(1 - 5 \frac{z^2}{b^2} \right) \lambda_2 \right] C(\mathbf{x}) \\ &- \pi a^2 \left[\lambda_1 - \frac{3}{2} \lambda_2 \right] C^2(\mathbf{x}), \end{aligned} \quad (79)$$

$$\begin{aligned} \oint_{S^2} \left(\frac{2B^4}{A^4} - 2 \frac{B^2C}{A^3} + \frac{C^2}{4A^2} \right) \frac{\cos^2\theta}{b^2} d\Omega &= \frac{3}{2} \pi a^2 \left[\left(1 - \frac{z^2}{b^2} \right)^2 \lambda_1 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4} \right) \lambda_2 \right. \\ &+ \left. \left(1 - 10 \frac{z^2}{b^2} + \frac{35}{3} \frac{z^4}{b^4} \right) \lambda_3 \right] + \pi a^2 \left[\left(1 - \frac{z^2}{b^2} \right) \lambda_1 - 4 \left(1 - 3 \frac{z^2}{b^2} \right) \lambda_2 \right. \\ &+ \left. 3 \left(1 - 5 \frac{z^2}{b^2} \right) \lambda_3 \right] C(\mathbf{x}) + \pi a^2 \left[\lambda_2 - \frac{3}{2} \lambda_3 \right] C^2(\mathbf{x}). \end{aligned} \quad (80)$$

Substituting these results in Eqs. (75) and (76) yields

$$\begin{aligned} I_1 &= \frac{1}{2} \pi G \rho a^2 \left[\left(1 - \frac{z^4}{b^4} \right) \lambda_0 - 6 \frac{z^2}{b^2} \left(1 - \frac{5}{3} \frac{z^2}{b^2} \right) \lambda_1 - \left(1 - 10 \frac{z^2}{b^2} + \frac{35}{3} \frac{z^4}{b^4} \right) \lambda_2 \right] \\ &- \pi G \rho a^2 \left[\frac{3z^2}{b^2} \lambda_1 + \left(1 - 5 \frac{z^2}{b^2} \right) \lambda_2 + \frac{1}{2} \lambda_2 C(\mathbf{x}) \right] C(\mathbf{x}), \end{aligned} \quad (81)$$

$$\begin{aligned} I_2 &= \pi G \rho a^2 \left[\left(1 - \frac{z^2}{b^2} \right) \frac{z^2}{b^2} \lambda_0 + \left(\frac{3}{2} - 12 \frac{z^2}{b^2} + \frac{25}{2} \frac{z^4}{b^4} \right) \lambda_1 - \left(3 - 26 \frac{z^2}{b^2} + \frac{85}{3} \frac{z^4}{b^4} \right) \lambda_2 \right. \\ &+ \left. \left(\frac{3}{2} - 15 \frac{z^2}{b^2} + \frac{35}{2} \frac{z^4}{b^4} \right) \lambda_3 \right] + \pi G \rho a^2 \left[\left(1 - 6 \frac{z^2}{b^2} \right) \lambda_1 - 4 \left(1 - 5 \frac{z^2}{b^2} \right) \lambda_2 + 3 \left(1 - 5 \frac{z^2}{b^2} \right) \lambda_3 \right. \\ &+ \left. \left(\lambda_2 - \frac{3}{2} \lambda_3 \right) C(\mathbf{x}) \right] C(\mathbf{x}). \end{aligned} \quad (82)$$

Replacing these expressions in Eq. (73) results in

$$\Phi = \Phi_0 + \Phi_1 C(\mathbf{x}) + \Phi_2 C^2(\mathbf{x}), \quad (83)$$

where

$$\begin{aligned} \Phi_0 = & \frac{1}{2}\pi^2 G^2 \rho^2 a^4 [7\lambda_0^2 - 3\lambda_0(8\lambda_1 - 5\lambda_2 + 2\lambda_3) + 2\lambda_1(15\lambda_1 - 20\lambda_2 + 9\lambda_3)] \\ & - \pi^2 G^2 \rho^2 a^4 [7\lambda_0^2 + \lambda_0(-60\lambda_1 + 67\lambda_2 - 30\lambda_3) + 2\lambda_1(51\lambda_1 - 88\lambda_2 + 45\lambda_3)] \frac{z^2}{b^2} \\ & + \frac{1}{6}\pi^2 G^2 \rho^2 a^4 [21\lambda_0^2 + \lambda_0(-288\lambda_1 + 445\lambda_2 - 210\lambda_3) + 10\lambda_1(57\lambda_1 - 116\lambda_2 + 63\lambda_3)] \frac{z^4}{b^4} \\ & + \pi G \rho a^4 \omega^2 \left[\lambda_0 - 3\lambda_1 + 5\lambda_2 - 3\lambda_3 - 2(\lambda_0 - 9\lambda_1 + 21\lambda_2 - 15\lambda_3) \frac{z^2}{b^2} + (\lambda_0 - 5(3\lambda_1 - 9\lambda_2 + 7\lambda_3)) \frac{z^4}{b^4} \right], \end{aligned} \quad (84)$$

$$\begin{aligned} \Phi_1 = & \pi^2 G^2 \rho^2 a^4 [\lambda_0(-7\lambda_1 + 11\lambda_2 - 6\lambda_3) + 2\lambda_1(6\lambda_1 - 14\lambda_2 + 9\lambda_3)] \\ & + \pi^2 G^2 \rho^2 a^4 [\lambda_0(21\lambda_1 - 55\lambda_2 + 30\lambda_3) - 2\lambda_1(24\lambda_1 - 70\lambda_2 + 45\lambda_3)] \frac{z^2}{b^2} \\ & - 2\pi G \rho a^4 \omega^2 \left[(\lambda_1 - 3\lambda_2 + 3\lambda_3) - 3(\lambda_1 - 5\lambda_2 + 5\lambda_3) \frac{z^2}{b^2} \right], \end{aligned} \quad (85)$$

$$\Phi_2 = \frac{1}{2}\pi G \rho a^4 [(-\lambda_0\lambda_2 + 8\lambda_1\lambda_2 + 6\lambda_0\lambda_3 - 18\lambda_1\lambda_3)\pi G \rho - 6(\lambda_2 - \lambda_3)\omega^2]. \quad (86)$$

This finalizes the calculation of the post-Newtonian potentials inside the rotating fluid body.

VII. CONSERVED QUANTITIES

The post-Newtonian conservation laws have been discussed by a number of researchers, most notably in textbooks [9,42,49]. General relativity predicts that the integrals of energy, linear momentum, angular momentum, and the center of mass of an isolated system are conserved in the post-Newtonian approximation. In the present paper, we are dealing with a single isolated body so that the integrals of the center of mass and the linear momentum are trivial, and we can always chose the origin of the coordinate system at the center of mass of the body with the linear momentum being nil. The integrals of energy and angular momentum are less trivial and require detailed calculations, which are given below.

A. Post-Newtonian mass

The law of the conservation of energy yields the post-Newtonian mass of a rotating fluid ball that is defined as [9,42,49]

$$M = M_N + \frac{1}{c^2} M_{pN}, \quad (87)$$

where

$$M_N = \int_{\mathcal{V}} \rho(\mathbf{x}) d^3x \quad (88)$$

is the Newtonian mass of baryons comprising the body;

$$M_{pN} = \int_{\mathcal{V}} \rho(\mathbf{x}) \left(v^2 + \Pi + \frac{5}{2} V_N \right) d^3x \quad (89)$$

is the post-Newtonian correction taking into account the contribution of the internal kinetic, gravitational, and compressional energies; and \mathcal{V} is the coordinate volume of the PN ellipsoid. In what follows, we shall formally include the compressional energy Π in the density ρ because Π is constant.

Under condition that the density $\rho(\mathbf{x}) = \rho = \text{const}$, the rest mass is reduced to

$$M_N = \rho \mathcal{V}. \quad (90)$$

To calculate the volume, \mathcal{V} , we introduce the normalized spherical coordinates r , θ , λ related to the Cartesian (harmonic) coordinates x , y , z as follows:

$$\begin{aligned} x &= ar \sin \theta \cos \lambda, & y &= ar \sin \theta \sin \lambda, \\ z &= br \cos \theta. \end{aligned} \quad (91)$$

In these coordinates, the volume \mathcal{V} is given by

$$\mathcal{V} = a^2 b \int_0^{r(\theta)} \int_0^\pi \int_0^{2\pi} r^2 \sin \theta dr d\theta d\lambda, \quad (92)$$

where $r(\theta)$ describes the surface of the PN ellipsoid defined above in Eq. (15):

$$r^2(\theta) = 1 + \kappa(E_1 \sin^4 \theta + E_2 \cos^4 \theta + E_3 \sin^2 \cos^2 \theta). \quad (93)$$

Integration in Eq. (92) results in

$$M_N = \frac{4\pi}{3} \rho a^2 b \left[1 + \frac{\kappa}{10} (8E_1 + 3E_2 + 2E_3) \right], \quad (94)$$

which clearly indicates that the Newtonian mass, M_N , depends on the particular choice of the shape of the PN ellipsoid through the linear combination of the coefficients E_1, E_2, E_3 .

The post-Newtonian contribution, M_{pN} , to the rest mass reads

$$\begin{aligned} M_{pN} &= \rho a^4 b \int_0^1 \int_0^\pi \int_0^{2\pi} \left\{ \omega^2 r^2 \sin^2 \theta \right. \\ &\quad \left. + \frac{5}{2} \pi G \rho [\lambda_0 (1 - r^2 \cos^2 \theta) - \lambda_1 r^2 (1 - 3 \cos^2 \theta)] \right\} \\ &\quad \times r^2 \sin \theta dr d\theta d\lambda \\ &= \frac{8\pi}{15} \rho a^4 b (\omega^2 + 5\pi G \rho \lambda_0), \end{aligned} \quad (95)$$

which was obtained from Eq. (89) upon substitution of $v^2 = \omega^2 r^2 \sin^2 \theta$ and V_N from Eq. (50). After adding up formulas (94) and (95), the total mass (87) becomes

$$M = \frac{4\pi}{3} \rho a^2 b \left[1 + \frac{\kappa}{10} \left(\frac{4\omega^2}{\pi G \rho} + 8E_1 + 3E_2 + 2E_3 + 20\lambda_0 \right) \right]. \quad (96)$$

Some clarifications are required at this point to prevent confusion with the residual gauge freedom described by Eqs. (31a)–(31e) and the constancy of the total mass M as the integral of motion of the fluid. If we do calculations in the primed harmonic coordinates, x'^α , related to the original coordinates x^α by Eq. (26), it changes the mathematical expression for the Newtonian mass,

$$M_N = \int_{\mathcal{V}'} \rho'(\mathbf{x}') J(\mathbf{x}') d^3 x', \quad (97)$$

where \mathcal{V}' is the coordinate volume occupied by the same amount of mass in the primed coordinates, $\rho'(\mathbf{x}') = \rho(\mathbf{x}) = \rho$ is the constant mass density, $d^3 x'$ is an element of the coordinate volume in the primed coordinates, and

$$\begin{aligned} J(\mathbf{x}') &= \det \left| \frac{\partial x^i}{\partial x'^j} \right| = 1 - \partial_i \xi^i \\ &= 1 - \kappa \left(\frac{4p}{a^2} + \frac{3q}{b^2} \right) (x'^2 + y'^2 - 2z'^2) \end{aligned} \quad (98)$$

is the Jacobian of the inverse coordinate transformation (26) with $h = k = 0$. Integration in Eq. (97) with the volume bounded by the surface (29) of the PN ellipsoid in the primed coordinates [that is the same as Eq. (29) but with the radial coordinate $r' = (x'^2 + y'^2 + z'^2)^{1/2}$ and the coefficients E'_1, E'_2, E'_3] yields the gauge-invariant expression for the Newtonian mass of the rotating fluid body:

$$\begin{aligned} M_N &= \frac{4\pi}{3} \rho a^2 b \left[1 + \frac{\kappa}{10} \left(8E'_1 + 3E'_2 + 2E'_3 - 16p \right. \right. \\ &\quad \left. \left. + 12q + 16p \frac{b^2}{a^2} - 12q \frac{a^2}{b^2} \right) \right]. \end{aligned} \quad (99)$$

This expression naturally coincides with that given in Eq. (94) after making use of Eqs. (31c)–(31e). The post-Newtonian contribution, M_{pN} , to the total mass is not sensitive to the post-Newtonian coordinate transformation and remains the same as in Eq. (95). Therefore, the total mass is the gauge-invariant quantity under the condition that the coefficients E_1 and E_2 have been fixed in a particular coordinate system. It is also worth mentioning that the combination of the residual gauge parameters $-16p + 12q + 16pb^2/a^2 - 12qa^2/b^2 = 0$, when the residual gauge transformation (26) preserves the coordinate volume of integration, that is, when $\partial_i \xi^i = 0$.

Parameters E_1 and E_2 define the shape of the PN ellipsoid as compared with the shape of the Maclaurin ellipsoid in the chosen coordinate system. Picking up the shape of the PN ellipsoid is equivalent to eliminating the residual gauge freedom. Depending on their choice, we have different options; for example, we can either equate the relativistic mass M of the PN ellipsoid to the Newtonian mass of the Maclaurin ellipsoid (Bardeen-Chandrasekhar's gauge discussed at the end of Sec. VIII) or minimize the deviation of the PN ellipsoid from the surface of the Maclaurin ellipsoid (this option is discussed in Sec. XI) or something else. Comparison of the various gauges is facilitated if we operate with the equatorial, r_e , and polar, r_p , radii of the PN ellipsoid that were introduced earlier in Eqs. (21) and (22).

Making use of r_e and r_p , we can recast the total mass M in Eq. (96) to the form which depends on the linear combination, $E_1 + E_2 - E_3$, that is

$$M = \frac{4\pi}{3} \rho r_e^2 r_p \left[1 - \frac{\kappa}{5} \left(E_1 + E_2 - E_3 - 10\lambda_0 - \frac{2\omega^2}{\pi G \rho} \right) \right]. \quad (100)$$

The inverse relation will be used to convert the density ρ to the total mass,

$$\rho = \frac{3M}{4\pi r_e^2 r_p} \left[1 + \frac{\kappa}{5} \left(E_1 + E_2 - E_3 - 10\lambda_0 - \frac{2\omega^2}{\pi G\rho} \right) \right]. \quad (101)$$

We shall prove in Sec. VIII that the linear combination $E_1 + E_2 - E_3$ is uniquely defined by the physical equation (134) of the equipotential level surface. Thus, Eq. (100) for the total post-Newtonian mass of the rotating fluid depends on the choice of the free parameters E_1 and E_2 solely through the equatorial, r_e , and polar, r_p , radii or, more exactly, on the choice of the ratios: $r_e/a = 1 + \kappa E_1/2$ and $r_p/b = 1 + \kappa E_2/2$.

B. Post-Newtonian angular momentum

The vector of the post-Newtonian angular momentum, $S^i = (S^x, S^y, S^z)$, is defined by [9,42,49]

$$S^i = S_N^i + \frac{1}{c^2} S_{pN}^i, \quad (102)$$

where S_N^i and S_{pN}^i are the Newtonian and post-Newtonian contributions, respectively,

$$S_N^i = \int \rho(\mathbf{x})(\mathbf{x} \times \mathbf{v})^i d^3x, \quad (103)$$

$$S_{pN}^i = \int \rho(\mathbf{x}) \left(v^2 + \Pi + 6V + \frac{p}{\rho} \right) (\mathbf{x} \times \mathbf{v})^i d^3x - 4 \int \rho(\mathbf{x})(\mathbf{x} \times \mathbf{V})^i d^3x, \quad (104)$$

and vector-potential $\mathbf{V} \equiv V^i$ has been given in Eqs. (58) and (66).

It can be checked by inspection that in case of axisymmetric mass distribution with a constant density $\rho(\mathbf{x}) = \rho$ the only nonvanishing component of the angular momentum is $S^3 = S^z \equiv S$. Indeed, $\mathbf{v} = \{v^i\} = (\boldsymbol{\omega} \times \mathbf{x})^i$, and $(\mathbf{x} \times \mathbf{v})^i = (\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x}))^i = \omega^i(x^2 + y^2 + z^2) - x^i \omega z$. Making use of these relations in Eq. (103) results in

$$\begin{aligned} S_N^x &= -\omega\rho \int xz d^3x, & S_N^y &= -\omega\rho \int yz d^3x, \\ S_N^z &= \omega\rho \int (x^2 + y^2) d^3x. \end{aligned} \quad (105)$$

Subsequent calculation of the spin components (105) with the help of the spherical coordinates (91) confirms that the two components, $S_N^x = S_N^y = 0$ and $S_N^z \equiv S_N$, where

$$S_N = a^4 b \rho \omega \int_0^{r(\theta)} \int_0^\pi \int_0^{2\pi} r^4 \sin^3 \theta dr d\theta d\lambda, \quad (106)$$

and the boundary of the integration of the radial coordinate, $r(\theta)$, is defined in Eq. (93). After integration in Eq. (106), we obtain,

$$S_N = \frac{8\pi}{15} a^4 b \rho \omega \left[1 + \frac{\kappa}{14} (24E_1 + 3E_2 + 4E_3) \right]. \quad (107)$$

Replacing the density ρ by the total mass M with the help of Eq. (96) makes the Newtonian part of the angular momentum as follows:

$$S_N = \frac{2}{5} M a^2 \omega \left[1 + \frac{\kappa}{35} \left(32E_1 - 3E_2 + 3E_3 - 70\lambda_0 - \frac{14\omega^2}{\pi G\rho} \right) \right]. \quad (108)$$

The gauge-invariant expression for S can be obtained after making the residual gauge transformation (26) in the defining equation (103). With the repeating calculations being similar to those which led to the gauge-invariant expression for the mass, we obtain

$$S_N = \frac{2}{5} M a^2 \omega \left[1 + \frac{\kappa}{35} \left(32E'_1 - 3E'_2 + 3E'_3 - 70\lambda_0 - \frac{14\omega^2}{\pi G\rho} - 48p + 12q + 32p \frac{b^2}{a^2} - 24q \frac{a^2}{b^2} \right) \right]. \quad (109)$$

The combination of the residual gauge parameters $-48p + 12q + 32pb^2/a^2 - 24qa^2/b^2 \neq 0$ in the general case even if the residual gauge transformation (26) preserves the coordinate volume of integration. This is because in the case of spin we integrate over the volume not simply a local mass density ρ but the local density of the angular momentum, $\rho(\mathbf{x} \times \mathbf{v})^i$, which is not constant.

It is straightforward to prove that x and y components of S_{pN}^i also vanish due to the axial symmetry, and only its z component, $S_{pN}^z \equiv S_{pN}$, remains. We notice that

$$\int \rho(\mathbf{x} \times \mathbf{V})^z d^3x = \int \rho D_1 (\mathbf{x} \times \mathbf{v})^z d^3x, \quad (110)$$

where D_1 is taken from (67). Therefore,

$$S_{pN} = \int \rho \left(v^2 + 6V_N + \frac{p}{\rho} - 4D_1 \right) (\mathbf{x} \times \mathbf{v})^z d^3x, \quad (111)$$

where we have eliminated the compression energy Π by including it in the mass density ρ . Making transformation to the coordinates (91) yields

$$\begin{aligned} S_{pN} &= \omega a^6 b \rho \int_0^1 \int_0^\pi \int_0^{2\pi} \\ &\quad \times \{ \omega^2 r^2 \sin^2 \theta + 7\lambda_0 - 6\lambda_1 - (\lambda_0 + 4\lambda_1 - 4\lambda_2) r^2 \\ &\quad - 2(3\lambda_0 - 15\lambda_1 + 10\lambda_2) r^2 \cos^2 \theta \} r^4 \sin^3 \theta dr d\theta d\lambda \\ &= \frac{4}{35} M a^2 \omega [2\omega^2 + \pi G\rho (19\lambda_0 - 16\lambda_1)]. \end{aligned} \quad (112)$$

After adding up formulas (108) and (112), the total angular momentum becomes

$$S = \frac{2}{5} M a^2 \omega \left\{ 1 + \frac{\kappa}{35} \left[32E_1 - 3E_2 + 3E_3 + 40(3\lambda_0 - 4\lambda_1) + \frac{6\omega^2}{\pi G \rho} \right] \right\}. \quad (113)$$

Making use of the equatorial radii, r_e defined in Eq. (21), we obtain the final expression for the total angular momentum:

$$S = \frac{2}{5} M r_e^2 \omega \left\{ 1 - \frac{\kappa}{35} \left[3(E_1 + E_2 - E_3) - 40(3\lambda_0 - 4\lambda_1) - \frac{6\omega^2}{\pi G \rho} \right] \right\}. \quad (114)$$

This expression depends only on the linear combination of the parameters, $E_1 + E_2 - E_3$, both explicitly and implicitly [through the mass M in Eq. (100)], which is uniquely fixed in the chosen coordinate system by the equation of the level surface (134). The total angular momentum S depends explicitly only on the parameter E_1 through the equatorial radius r_e . It depends implicitly on the parameters E_1 and E_2 through the mass M . The two parameters E_1 and E_2 can be chosen arbitrarily depending on our preferences and the goals we want to reach in relativistic geodesy.

VIII. POST-NEWTONIAN EQUATION OF THE LEVEL SURFACE

The figure of the rotating fluid body is defined by the boundary condition of vanishing pressure, $p = 0$. The boundary surface, $p = 0$, is called the level surface. The relativistic Euler equation derived for the rigidly rotating fluid body tells us [3,64] that the level surface coincides with the equipotential surface of the post-Newtonian gravitational potential W , which is given by [9]

$$W = \frac{1}{2} \omega^2 \sigma^2 + V_N + \kappa V_{pN} + \frac{1}{c^2} \left(\frac{1}{8} \omega^4 \sigma^4 + \frac{3}{2} \omega^2 \sigma^2 V_N - 4 \omega^2 \sigma^2 D_1 - \frac{1}{2} V_N^2 + \Phi \right), \quad (115)$$

where $\kappa \equiv \pi G \rho a^2 / c^2$ and the potentials V_N, V_{pN}, D_1, Φ have been explained in Secs. V and VI. After substituting these potentials in Eq. (115), it can be presented as a quadratic polynomial with respect to the function $C(\mathbf{x})$,

$$W(\mathbf{x}) = W_0 + W_1 C(\mathbf{x}) + W_2 C^2(\mathbf{x}), \quad (116)$$

where the coefficients of the expansion are polynomials of the z coordinate only. In particular, the coefficient W_0 is a polynomial of the fourth order [notice that the coefficients of the expansion K_1 and K_2 have no relation to the gauge-dependent coefficients entering Eq. (16)]

$$W_0 = K_0 + K_1 \frac{z^2}{b^2} + K_2 \frac{z^4}{b^4}, \quad (117)$$

where

$$\begin{aligned} K_0 = & \frac{1}{2} \omega^2 a^2 + \pi G \rho a^2 (\lambda_0 - \lambda_1) + \frac{1}{8c^2} \omega^4 a^4 + \frac{1}{2} \kappa \omega^2 a^2 (5\lambda_0 - 17\lambda_1 + 18\lambda_2 - 6\lambda_3) \\ & + \frac{1}{2} \kappa \pi G \rho a^2 [6\lambda_0^2 - \lambda_0(22\lambda_1 - 15\lambda_2 + 6\lambda_3) + \lambda_1(29\lambda_1 - 40\lambda_2 + 18\lambda_3)] \\ & + \kappa \pi G \rho a^2 [(\lambda_0 - 4\lambda_1 + 10\lambda_2 - 12\lambda_3 + 6\lambda_4)E_1 + 2(\lambda_1 - 4\lambda_2 + 6\lambda_3 - 3\lambda_4)E_3 + 6(\lambda_2 - 2\lambda_3 + \lambda_4)E_2], \end{aligned} \quad (118)$$

$$\begin{aligned} K_1 = & -\frac{1}{2} \omega^2 a^2 - \pi G \rho a^2 (\lambda_0 - 3\lambda_1) - \frac{1}{4c^2} \omega^4 a^4 - \kappa \omega^2 a^2 (5\lambda_0 - 40\lambda_1 + 66\lambda_2 - 30\lambda_3) \\ & - \kappa \pi G \rho a^2 (6\lambda_0^2 - 56\lambda_0\lambda_1 + 99\lambda_1^2 + 67\lambda_0\lambda_2 - 176\lambda_1\lambda_2 - 30\lambda_0\lambda_3 + 90\lambda_1\lambda_3) \\ & - \kappa \pi G \rho a^2 [2(\lambda_0 - 12\lambda_1 + 42\lambda_2 - 60\lambda_3 + 30\lambda_4)E_1 - (\lambda_0 - 18\lambda_1 + 78\lambda_2 - 120\lambda_3 + 60\lambda_4)E_3 \\ & - 12(\lambda_1 - 6\lambda_2 + 10\lambda_3 - 5\lambda_4)E_2], \end{aligned} \quad (119)$$

$$\begin{aligned} K_2 = & \frac{1}{8c^2} \omega^4 a^4 + \frac{1}{2} \kappa \omega^2 a^2 (5\lambda_0 - 63\lambda_1 + 130\lambda_2 - 70\lambda_3) + \frac{1}{6} \kappa \pi G \rho a^2 [18\lambda_0^2 - 5\lambda_0(54\lambda_1 - 89\lambda_2 + 42\lambda_3) \\ & + \lambda_1(543\lambda_1 - 1160\lambda_2 + 630\lambda_3)] + \kappa \pi G \rho a^2 (\lambda_0 - 20\lambda_1 + 90\lambda_2 - 140\lambda_3 + 70\lambda_4)(E_1 + E_2 - E_3). \end{aligned} \quad (120)$$

The coefficient W_1 in Eq. (116) is a polynomial of the second order,

$$W_1 = P + P_1 \frac{z^2}{b^2}, \quad (121)$$

where

$$\begin{aligned} P = & \frac{1}{2} \omega^2 a^2 \left[1 + \frac{1}{2c^2} \omega^2 a^2 + \kappa(3\lambda_0 - 18\lambda_1 + 28\lambda_2 - 12\lambda_3) \right] - \pi G \rho a^2 \lambda_1 \\ & - \kappa \pi G \rho a^2 [2(\lambda_1 - 5\lambda_2 + 9\lambda_3 - 6\lambda_4)E_1 - (\lambda_1 - 8\lambda_2 + 18\lambda_3 - 12\lambda_4)E_3 - 6(\lambda_2 - 3\lambda_3 + 2\lambda_4)E_2] \\ & - \kappa \pi G \rho a^2 (6\lambda_0 \lambda_1 - 11\lambda_1^2 - 11\lambda_0 \lambda_2 + 28\lambda_1 \lambda_2 + 6\lambda_0 \lambda_3 - 18\lambda_1 \lambda_3), \end{aligned} \quad (122)$$

$$\begin{aligned} P_1 = & -\frac{1}{4c^2} \omega^2 a^4 - \frac{3}{2} \kappa \omega^2 a^2 (\lambda_0 - 16\lambda_1 + 36\lambda_2 - 20\lambda_3) + \kappa \pi G \rho a^2 [2(3\lambda_1 - 25\lambda_2 + 51\lambda_3 - 30\lambda_4)(E_1 + E_2 - E_3) \\ & + 20\lambda_0 \lambda_1 - 45\lambda_1^2 - 55\lambda_0 \lambda_2 + 140\lambda_1 \lambda_2 + 30\lambda_0 \lambda_3 - 90\lambda_1 \lambda_3]. \end{aligned} \quad (123)$$

The coefficient W_2 in Eq. (116) is constant,

$$\begin{aligned} W_2 = & \frac{1}{8c^2} \omega^4 a^4 - \frac{1}{2} \kappa \omega^2 a^2 (3\lambda_1 - 2\lambda_2 - 6\lambda_3) \\ & + \kappa \pi G \rho a^2 \left[(\lambda_2 - 6\lambda_3 + 6\lambda_4)(E_1 + E_2 - E_3) - \frac{1}{2} (\lambda_1^2 + \lambda_0 \lambda_2 - 8\lambda_1 \lambda_2 - 6\lambda_0 \lambda_3 + 18\lambda_1 \lambda_3) \right]. \end{aligned} \quad (124)$$

Let us recall that the coordinates on the surface of the PN ellipsoid are denoted as \bar{x} , \bar{y} , \bar{z} . On the level surface of the PN ellipsoid, we have all three coordinates interconnected by Eq. (18) of the PN ellipsoid, $C(\bar{x}) = \kappa \alpha_0(\bar{z})$, so that Eq. (116) becomes

$$\bar{W} \equiv \bar{W}_0 + \kappa \bar{W}_1 \alpha_0(\bar{z}), \quad (125)$$

and the term with $W_2 \sim \mathcal{O}(\kappa^2)$ is discarded as negligibly small. After reducing similar terms, the potential \bar{W} on the level surface is simplified to the polynomial of the fourth order,

$$\bar{W} = K'_0 + K'_1 \frac{\bar{z}^2}{b^2} + K'_2 \frac{\bar{z}^4}{b^4}, \quad (126)$$

where

$$K'_0 = K_0 + \kappa \left(\frac{1}{2} \omega^2 a^2 - \pi G \rho a^2 \lambda_1 \right) E_1, \quad (127)$$

$$K'_1 = K_1 + \kappa \left(\frac{1}{2} \omega^2 a^2 - \pi G \rho a^2 \lambda_1 \right) (E_3 - 2E_1), \quad (128)$$

$$K'_2 = K_2 + \kappa \left(\frac{1}{2} \omega^2 a^2 - \pi G \rho a^2 \lambda_1 \right) (E_1 + E_2 - E_3). \quad (129)$$

Because the potential \bar{W} is to be constant on the level surface [64], the numerical coefficients K'_1 and K'_2 must vanish. The first condition, $K'_1 = 0$, yields a relation between the angular velocity of rotation, ω , and oblateness, e , of the rotating fluid body,

$$\begin{aligned}
& \frac{\omega^2}{2\pi G\rho} \left[1 + \frac{\omega^2 a^2}{2c^2} + 2\kappa \left(5\lambda_0 - 40\lambda_1 + 66\lambda_2 - 30\lambda_3 - \frac{1}{2}E_3 + E_1 \right) \right] \\
& = 3\lambda_1 - \lambda_0 - \kappa(6\lambda_0^2 - 56\lambda_0\lambda_1 + 99\lambda_1^2 + 67\lambda_0\lambda_2 - 176\lambda_1\lambda_2 - 30\lambda_0\lambda_3 + 90\lambda_1\lambda_3) - \kappa(\lambda_0 - 7\lambda_1 + 6\lambda_2)(E_1 - E_2) \\
& \quad - \kappa(\lambda_0 - 19\lambda_1 + 78\lambda_2 - 120\lambda_3 + 60\lambda_4)(E_1 + E_2 - E_3). \tag{130}
\end{aligned}$$

Equation (130) generalizes the famous result that was first obtained by Colin Maclaurin in 1742, from the Newtonian theory of gravity to the realm of general relativity. The physical meaning of the post-Newtonian Maclaurin relation is that it connects four parameters of the rotating ellipsoid made up of a homogeneous fluid—its eccentricity e , the semimajor axis a , the angular velocity of revolution ω , and the density ρ . In the Newtonian case, the Maclaurin relation connects only three parameters: e , ω , and r . It also gives a rigorous mathematical proof of Newton's original claim that a rotating body must oblate in the direction of rotational axis [6].

Equation (130) can be further simplified by replacing the eccentricity e of the Maclaurin ellipsoid with that ϵ of the PN ellipsoid by making use of Eq. (25). We introduce functions $\lambda_0(\epsilon)$ and $\lambda_1(\epsilon)$ that are given by Eqs. (A9a) and (A9a) after the formal replacement of e in those equations with ϵ . We expand $\lambda_0(\epsilon)$ and $\lambda_1(\epsilon)$ in the Taylor series with respect to the relativistic parameter κ and find out that

$$\lambda_0(\epsilon) = \lambda_0 + \kappa(E_1 - E_2)\lambda_1 + \mathcal{O}(\kappa^2) \tag{131}$$

$$\lambda_1(\epsilon) = \lambda_1 + \kappa \frac{E_1 - E_2}{e^2} (3\lambda_1 - \lambda_0) + \mathcal{O}(\kappa^2), \tag{132}$$

where $\lambda_0 \equiv \lambda_0(e)$ and $\lambda_1 \equiv \lambda_1(e)$ are given by Eqs. (A9a) and (A9a). We express λ_0 and λ_1 in terms of $\lambda_0(\epsilon)$ and $\lambda_1(\epsilon)$ by inverting (131) and (132) and then substitute the expressions having been obtained, to the Newtonian part, $3\lambda_1 - \lambda_0$, in the right side of Eq. (130). It turns out that all terms depending explicitly on E_1 and E_2 cancel each other mutually so that Eq. (130) takes on a more elegant form

$$\begin{aligned}
& \frac{\omega^2}{2\pi G\rho} \left[1 + \frac{\omega^2 a^2}{2c^2} + 2\kappa(5\lambda_0 - 40\lambda_1 + 66\lambda_2 - 30\lambda_3) \right] \\
& = 3\lambda_1(\epsilon) - \lambda_0(\epsilon) - \kappa(6\lambda_0^2 - 56\lambda_0\lambda_1 + 99\lambda_1^2 + 67\lambda_0\lambda_2 - 176\lambda_1\lambda_2 - 30\lambda_0\lambda_3 + 90\lambda_1\lambda_3) \\
& \quad - \kappa \left(\frac{\omega^2}{2\pi G\rho} + \lambda_0 - 19\lambda_1 + 78\lambda_2 - 120\lambda_3 + 60\lambda_4 \right) (E_1 + E_2 - E_3). \tag{133}
\end{aligned}$$

We can see that the Maclaurin relation in the form of Eq. (133) depends explicitly only on the linear combination of the coefficients, $E_1 + E_2 - E_3$, which is fixed by the equation of the level surface (134). Dependence on the free parameters E_1 and E_2 enters Eq. (133) only through the eccentricity ϵ of the PN ellipsoid.

The second condition, $K'_2 = 0$, yields an algebraic equation for the linear combination of three coefficients $E_1 + E_2 - E_3$, namely,

$$\begin{aligned}
& \left(\frac{\omega^2}{2\pi G\rho} + \lambda_0 - 21\lambda_1 + 90\lambda_2 - 140\lambda_3 + 70\lambda_4 \right) (E_1 + E_2 - E_3) \\
& = -\frac{\omega^4}{8\pi^2 G^2 \rho^2} - \frac{\omega^2}{4\pi G\rho} (5\lambda_0 - 63\lambda_1 + 130\lambda_2 - 70\lambda_3) \\
& \quad - \frac{1}{6} [18\lambda_0^2 - 5\lambda_0(54\lambda_1 - 89\lambda_2 + 42\lambda_3) + \lambda_1(543\lambda_1 - 1160\lambda_2 + 630\lambda_3)]. \tag{134}
\end{aligned}$$

Equation (134) imposes one physical constraint on the coefficients E_1, E_2, E_3 defining the shape of the PN ellipsoid (23). Two other algebraic equations are required to fix the numerical coefficients E_1 and E_2 . Because of the residual gauge freedom, explained above in Sec. IV, the two equations can be chosen arbitrarily. This property of the gauge freedom of the post-Newtonian theory of figures of rotating fluid bodies has been noticed by Chandrasekhar [7, 11], who limited the gauge freedom by imposing one condition of the conservation of the volume element of the fluid under the gauge transformation (28a)–(28c). Chandrasekhar believed that the second condition remains free and can be chosen arbitrarily.

On the other hand, Bardeen [8] pointed out that there exist two gauge-fixing conditions arising naturally from the astrophysical point of view and specifying uniquely the shape of the uniformly rotating fluid in the post-Newtonian approximation. Namely, he suggested to equate the total mass and angular momentum of the (Newtonian) Maclaurin ellipsoid to those of the post-Newtonian ellipsoid [66] under condition that they both have equal mass density ρ . Later on, Bardeen's conditions were accepted and implemented by Chandrasekhar [15] and Pyragas *et al.* [20] as well. We can easily impose the Bardeen gauge by making use of our Eqs. (96) and (113). The total mass and the angular momentum of the Maclaurin ellipsoid are given by equations

$$M_{\text{Maclaurin}} = \frac{4\pi}{3}\rho a^2 b, \quad S_{\text{Maclaurin}} = \frac{2}{5}M_{\text{Maclaurin}}a^2\omega. \quad (135)$$

Equating $M = M_{\text{Maclaurin}}$ in Eq. (96) and $S = S_{\text{Maclaurin}}$ in Eq. (113) yields the Bardeen gauge conditions

$$8E_1 + 3E_2 + 2E_3 = -20\lambda_0 - \frac{4\omega^2}{\pi G\rho}, \quad (136)$$

$$32E_1 - 3E_2 + 3E_3 = -40(3\lambda_0 - 4\lambda_1) - \frac{6\omega^2}{\pi G\rho}. \quad (137)$$

It should be emphasized, however, that these constraints suggest that neither equatorial, a , nor polar, b , radii of the (Newtonian) Maclaurin ellipsoid are equal to the equatorial, r_e , and the polar, r_p , radii of the PN ellipsoid, respectively; $r_e \neq a$, $r_p \neq b$ as shown in the left panel of Fig. 1. Indeed, assuming that $r_e = a$, $r_p = b$ imposes two constraints on the parameters E_1 and E_2 , which are simply $E_1 = E_2 = 0$ as follows from Eqs. (21) and (22). This corresponds to the geometrical shape of the PN ellipsoid shown in the right panel of Fig. 1. However, these two constraints makes three equations, Eqs. (134), (136), and (137), for the remaining parameter E_3 incompatible with each other. Thus, in geodesic applications of the relativistic theory of rotating fluids, we have to decide which option has more practical advantages: 1) to keep the same mass and angular momenta but different axes of the Maclaurin and PN ellipsoids or 2) to keep their axes equal but to abandon the equality of their masses and angular momenta.

To understand better the physical meaning of the Bardeen-Chandrasekhar gauge, let us consider three masses introduced earlier: 1) the Newtonian mass $M_{\text{Maclaurin}}$ of the Maclaurin ellipsoid (135), 2) the Newtonian mass M_N of the PN ellipsoid (94), and 3) the total post-Newtonian mass M of the PN ellipsoid (96). The coordinate volumes of the Maclaurin and PN ellipsoids are different, while the density ρ of the constituting matter is the same. Hence, $M_{\text{Maclaurin}}$ cannot be equal to M_N except in the trivial case, $E_1 = E_2 = E_3 = 0$, that is when the surface PN ellipsoid coincides with that of the

Maclaurin ellipsoid. The coordinate volumes occupied by the Newtonian mass M_N , and the post-Newtonian mass M of the PN ellipsoid, are the same. Nonetheless, $M \neq M_N$ because M_N is solely comprised of the rest mass of baryons, while the total post-Newtonian mass M includes an additional (positive) contribution M_{pN} given in Eq. (89) and corresponding to the internal kinetic, compressional, and gravitational energy of the body's matter. By changing the numerical values of the freely adjustable parameters E_1 and E_2 , we can decrease the baryon mass M_N by the amount that compensates the (positive) post-Newtonian contribution M_{pN} and make $M_{\text{Maclaurin}}$ equal to M . This is achieved under the condition that the first gauge-fixing equation (136) is satisfied. The same reasoning is valid with regard to the comparison of the Newtonian and post-Newtonian angular momenta, which leads to the second gauge-fixing condition (137).

It should be understood that the Bardeen-Chandrasekhar gauge imposed on the coordinates to build the post-Newtonian metric of a rotating fluid planet or a star is not the only possible one in the most general case. It is convenient in astrophysics because it facilitates an unambiguous comparison of various physical properties of the rotating Newtonian configurations with respect to relativistic stars having the same mass and angular momenta which are the integrals of the equations of motion. However, the primary goal of geodesy differs from astrophysics and is to build the terrestrial reference frame that is the most precise and adequate for the interpretation of measurements of a baseline's length, motion of geodetic stations, deflections of the plumb line, and variations (anomalies) of Earth's gravitational field. Until recently, these types of measurements have been referred to as a homogeneous reference ellipsoid possessing a rather simple and exact analytic description of the normal gravitational field of the Earth. It seems reasonable to make the post-Newtonian reference configuration in relativistic geodesy as close to the Maclaurin reference ellipsoid as possible to minimize the contribution of relativistic corrections to the coordinates and velocities of the geodetic stations. This can be achieved with the choice of the coefficients $E_1 = E_2 = 0$ in Eq. (15) of the PN ellipsoid which is not the Bardeen-Chandrasekhar gauge. We continue discussion of this question in Sec. XI.

Besides making a decision which gauge is the most appropriate in relativistic geodesy, we have to establish mathematical relations between the parameters of the relativistic PN ellipsoid and the gravimetric measurements of the Earth's gravity force on its topographic surface. These relations are known in classic (Newtonian) geodesy as the theorems of Pizzetti and Clairaut [67], and they connect parameters of the Maclaurin ellipsoid, namely, the semimajor and semiminor axes a and b , mass M , and the angular velocity of rotation ω , with the physically measured values of the gravity force at the pole and equator of the ellipsoid [43,44].

Let us denote the Newtonian force of gravity by $\gamma_i^N(\mathbf{x}) = \{\gamma_x^N, \gamma_y^N, \gamma_z^N\}$, the force of gravity measured at the pole of the ellipsoid by $\gamma_p^N \equiv \gamma_z^N(x=0, y=0, z=b)$, and the force of gravity measured at the equator by $\gamma_e^N \equiv \gamma_y^N(x=0, y=a, z=0)$ [68]. Because of the rotational symmetry of the ellipsoid, the equatorial point can be, in fact, chosen arbitrarily. The classic form of the theorem of Pizzetti is ([43])[Eq. (4.42)]

$$2\frac{\gamma_e^N}{a} + \frac{\gamma_p^N}{b} = \frac{3GM_N}{a^2b} - 2\omega^2, \quad (138)$$

while the theorem of Clairaut states ([43])[Eq. (4.43)]

$$\frac{\gamma_e^N}{a} - \frac{\gamma_p^N}{b} = \frac{3GM_N}{2a^2b} \frac{3e - e^3 - 3\sqrt{1-e^2} \arcsin e}{e^3} + \omega^2. \quad (139)$$

The main value of the theorems of Pizzetti and Clairaut is that they allow us to calculate explicitly the normal gravity field of the reference level ellipsoid in terms of only four ellipsoid's parameters— M_N, ω, a, b —in agreement with the Stokes-Poincaré theorem [69].

Theorems (138) and (139) were crucial in the geodesy of 19th century because they helped scientists to realize that the geometric shape of the Earth's figure can be determined not only from the geometric measurements of the geodetic arcs but, independently, by rendering the intrinsic measurements of the force of gravity on the Earth's surface [45]. The gravity-geometry correspondence expressed in the form of the two theorems (138) and (139) led a number of scientists from Lobachevsky to Einstein to a gradual understanding that the gravity force and geometry of curved spacetime must be interrelated. This geodesy-inspired way of thinking culminated in the 20th century in the development of a general theory of relativity by Einstein.

We derive the post-Newtonian analogs of the Pizzetti and Clairaut theorems in the next two sections. We show that the parameters entering the post-Newtonian formulation of these theorems are still the same four parameters as in the Newtonian approximation with a corresponding replacement of the Newtonian values of the parameters by their relativistic counterparts.

IX. POST-NEWTONIAN THEOREM OF PIZZETTI

We denote the post-Newtonian force of gravity $\gamma_i(\mathbf{x}) = \{\gamma_x, \gamma_y, \gamma_z\}$. The force of gravity measured by a local observer on the equipotential surface of the Earth's gravity field has been derived in Refs. [3,64] and is given by the equation [9]

$$\gamma_i = [\Lambda^j_i \partial_j W]_{\mathbf{x}=\bar{\mathbf{x}}}, \quad (140)$$

where $\partial_i \equiv \partial/\partial x^i$, the post-Newtonian gravity potential W has been defined in Eq. (116),

$$\Lambda^j_i = \delta^{ij} \left(1 - \frac{1}{c^2} V_N \right) - \frac{1}{2c^2} v^i v^j \quad (141)$$

is the matrix of transformation from the global geocentric celestial reference system (GCRS) coordinates to the local inertial (topocentric) coordinates of observer, $v^i = (\boldsymbol{\omega} \times \mathbf{x})^i$ is the velocity of the observer with respect to the global coordinates, and V_N is the Newtonian potential (50). It is worth emphasizing that we first take the partial derivative in Eq. (140) and then take the spatial coordinates, \mathbf{x} , on the equipotential surface, $\mathbf{x} \rightarrow \bar{\mathbf{x}}$.

Velocity $v^i = (\boldsymbol{\omega} \times \mathbf{x})^i$ is orthogonal to the gradient $\partial_i W$ everywhere, that is,

$$v^i \partial_i W = 0. \quad (142)$$

Indeed, it is easy to prove that

$$v^i \partial_i W = \omega(x \partial_y W - y \partial_x W). \quad (143)$$

Partial derivatives of W are calculated from Eq. (116),

$$\begin{aligned} \partial_x W &= \frac{dW}{dC} \partial_x C(\mathbf{x}) = \frac{dW}{dC} \frac{2x}{a^2}, \\ \partial_y W &= \frac{dW}{dC} \partial_y C(\mathbf{x}) = \frac{dW}{dC} \frac{2y}{a^2}. \end{aligned} \quad (144)$$

Substituting the partial derivatives from Eq. (144) to Eq. (143) yields Eq. (142), which was to be demonstrated. After accounting for Eq. (142), Eq. (140) is simplified to

$$\gamma_i(\bar{\mathbf{x}}) = \left[\left(1 - \frac{1}{c^2} V_N \right) \partial_i W \right]_{\mathbf{x}=\bar{\mathbf{x}}}. \quad (145)$$

We take the PN ellipsoid (15) as the equipotential surface enclosing the entire rotating mass and denote the post-Newtonian force of gravity on the pole by $\gamma_p \equiv \gamma_z(x=0, y=0, z=r_p)$ and the force of gravity on the equator by $\gamma_e \equiv \gamma_y(x=0, y=r_e, z=0)$ with the equatorial r_e and polar r_p radii defined in Eqs. (21) and (22), respectively. Taking the partial derivative from W in Eq. (145) yields

$$\begin{aligned}
 \gamma_p = & \frac{2\pi G\rho a^2}{b}(\lambda_0 - 2\lambda_1) + 16\frac{\omega^2 a^2}{b}\kappa(\lambda_1 - 3\lambda_2 + 2\lambda_3) \\
 & + \frac{\pi G\rho a^2}{b}\kappa[2(\lambda_0 - 5\lambda_1 + 4\lambda_2)E_1 - (\lambda_0 - 8\lambda_1 + 8\lambda_2)E_2 - 2(\lambda_0 - 17\lambda_1 + 60\lambda_2 - 76\lambda_3 + 32\lambda_4)(E_1 + E_2 - E_3)] \\
 & + \frac{4\pi G\rho a^2}{3b}\kappa[\lambda_0(27\lambda_1 - 56\lambda_2 + 24\lambda_3) - 2\lambda_1(33\lambda_1 - 74\lambda_2 + 36\lambda_3)], \tag{146}
 \end{aligned}$$

$$\begin{aligned}
 \gamma_e = & a(2\lambda_1 G\rho - \omega^2) - \frac{\omega^4 a^3}{2c^2} + \kappa\left[-3\lambda_0 + 18\lambda_1 - 28\lambda_2 + 12\lambda_3 + \left(\lambda_0 - \lambda_1 - \frac{1}{2}E_1\right)\right]\omega^2 a \\
 & + \kappa\pi G\rho a[(3\lambda_1 - 4\lambda_2)E_1 - 2(\lambda_1 - 2\lambda_2)E_2 + 2(\lambda_1 - 8\lambda_2 + 18\lambda_3 - 12\lambda_4)(E_1 + E_2 - E_3)] \\
 & + 2\kappa\pi G\rho a[\lambda_0(5\lambda_1 - 11\lambda_2 + 6\lambda_3) - 2\lambda_1(5\lambda_1 - 14\lambda_2 + 9\lambda_3)]. \tag{147}
 \end{aligned}$$

The right sides of Eqs. (146) and (147) depend on the semimajor and semiminor axes of the Maclaurin ellipsoid, a and b , but they are not defining parameters of the PN ellipsoid, which are the equatorial and polar radii, r_e and r_p , given in Eqs. (21) and (22), respectively. Moreover, the right sides of Eqs. (146) and (147) depend on the gauge parameters E_1 and E_2 . We replace parameters a and b with r_e and r_p and form a linear combination generalizing the Newtonian theorem of Pizzetti to the post-Newtonian approximation,

$$\begin{aligned}
 2\frac{\gamma_e}{r_e} + \frac{\gamma_p}{r_p} = & 2\pi G\rho\left[2\lambda_1 + \frac{a^2}{b^2}(\lambda_0 - 2\lambda_1)\right] - 2\omega^2 - \frac{1}{c^2}\omega^4 a^2 \\
 & - 2\kappa\left[2\lambda_0 - 17\lambda_1 + 28\lambda_2 - 12\lambda_3 - 8\frac{a^2}{b^2}(\lambda_1 - 3\lambda_2 + 2\lambda_3)\right]\omega^2 \\
 & + 2\kappa\pi G\rho\left[2\lambda_1 - 16\lambda_2 + 36\lambda_3 - 24\lambda_4 - \frac{a^2}{b^2}(\lambda_0 - 17\lambda_1 + 60\lambda_2 - 76\lambda_3 + 32\lambda_4)\right](E_1 + E_2 - E_3) \\
 & + 2\kappa\pi G\rho\left[(2\lambda_1 - 4\lambda_2) + \frac{a^2}{b^2}(\lambda_0 - 5\lambda_1 + 4\lambda_2)\right](E_1 - E_2) \\
 & + 4\kappa\pi G\rho[\lambda_0(5\lambda_1 - 11\lambda_2 + 6\lambda_3) - 2\lambda_1(5\lambda_1 - 14\lambda_2 + 9\lambda_3)] \\
 & + \frac{4a^2}{3b^2}\kappa\pi G\rho[\lambda_0(27\lambda_1 - 56\lambda_2 + 24\lambda_3) - 2\lambda_1(33\lambda_1 - 74\lambda_2 + 36\lambda_3)]. \tag{148}
 \end{aligned}$$

It seems that the right side of Eq. (148) still depends explicitly on the gauge parameters E_1 and E_2 . However, making use of integrals given in Appendix A, we can check that the numerical coefficient standing in front of the difference, $E_1 - E_2$, vanishes identically so that Eq. (148) is simplified to

$$\begin{aligned}
 2\frac{\gamma_e}{r_e} + \frac{\gamma_p}{r_p} = & 4\pi G\rho - 2\omega^2 - \frac{1}{c^2}\omega^4 a^2 \\
 & - \frac{\kappa}{3e^7}[e(1 - e^2)(105 - 104e^2 + 42e^4) - 3\sqrt{1 - e^2}(5 - 4e^2)(7 - 6e^2 + 2e^4)\arcsin e]\omega^2 \\
 & + \frac{\kappa\pi G\rho}{12e^9}(7 - 4e^2)[5e(21 - 31e^2 + 10e^4) - 3\sqrt{1 - e^2}(35 - 40e^2 + 8e^4)\arcsin e](E_1 + E_2 - E_3) \\
 & + \frac{\kappa\pi G\rho}{3e^{10}}(1 - e^2)[-315e^2 + 621e^4 - 250e^6 + 24e^8 + 2e\sqrt{1 - e^2}(315 - 516e^2 + 169e^4 - 18e^6)\arcsin e \\
 & - 3(105 - 242e^2 + 178e^4 - 40e^6)\arcsin^2 e]. \tag{149}
 \end{aligned}$$

Relation (149) depends only on the linear combination $E_1 + E_2 - E_3$ of the parameters, which has been already fixed by Eq. (134) of the level surface, and can be expressed solely as a function of the eccentricity e .

To compare Eq. (149) with its classic counterpart (138), we convert the constant density ρ to the total relativistic mass M of the PN ellipsoid by making use of Eq. (101). It recasts Eq. (149) into

$$\begin{aligned}
 \frac{2\gamma_e}{r_e} + \frac{\gamma_p}{r_p} = & \frac{3GM}{r_e^2 r_p} - 2\omega^2 - \frac{\omega^4 r_e^2}{c^2} \\
 & + \frac{3\sqrt{1-\epsilon^2} G^2 M^2}{16\epsilon^9 r_e^2 r_p^2} \frac{1}{c^2} \left[\epsilon \sqrt{1-\epsilon^2} (-315 + 621\epsilon^2 - 250\epsilon^4 + 24\epsilon^6) + 2(315 - 831\epsilon^2 + 685\epsilon^4 - 187\epsilon^6 + 42\epsilon^8) \arcsin \epsilon \right] \\
 & + \frac{1}{20\epsilon^7 r_p} \frac{GM\omega^2}{c^2} \left[\epsilon (-525 + 1045\epsilon^2 - 730\epsilon^4 + 234\epsilon^6) + 15 \frac{35 - 93\epsilon^2 + 92\epsilon^4 - 42\epsilon^6 + 8\epsilon^8}{\sqrt{1-\epsilon^2}} \arcsin \epsilon \right] \\
 & - \frac{9}{16} \frac{G^2 M^2}{c^2} \frac{105 - 347\epsilon^2 + 420\epsilon^4 - 218\epsilon^6 + 40\epsilon^8}{\epsilon^{10} r_e^2 r_p^2} \arcsin^2 \epsilon + \frac{3}{320\epsilon^8 r_e^2 r_p^2} \frac{G^2 M^2}{c^2} (E_1 + E_2 - E_3) \\
 & \times \left[3675 - 7525\epsilon^2 + 4850\epsilon^4 - 1000\epsilon^6 - 48\epsilon^8 - 15 \frac{\sqrt{1-\epsilon^2}}{\epsilon} (7 - 4\epsilon^2)(35 - 40\epsilon^2 + 8\epsilon^4) \arcsin \epsilon \right], \quad (150)
 \end{aligned}$$

where we have used in the post-Newtonian terms the eccentricity ϵ defined in Eq. (24) instead of e because they differ only in the post-Newtonian terms and where we assume that the combination of the parameters $E_1 + E_2 - E_3 = f(\epsilon)$ with function $f(\epsilon)$ given by formula (134).

Equation (150) represents the Pizzetti theorem generalizing the classic result (138) to the domain of the post-Newtonian approximation. It tells us that similarly to the Newtonian theory, the linear combination of the post-Newtonian forces of gravity measured on the surface of the PN ellipsoid at the pole and equator is a function of only four parameters: the post-Newtonian mass M , the angular velocity of rotation ω , and the equatorial r_e and polar r_p radii of the PN ellipsoid.

X. POST-NEWTONIAN THEOREM OF CLAIRAUT

To derive the post-Newtonian analog of the Clairaut theorem (139), we follow its classic derivation given, for example, in Ref. [44]. To this end, we subtract the ratio of the force of gravity (146) measured at the pole to r_p from that of the force of gravity (147) measured on equator to r_e . We get

$$\begin{aligned}
 \frac{\gamma_e}{r_e} - \frac{\gamma_p}{r_p} = & 2\pi G\rho \left[-\lambda_1 + \frac{a^2}{b^2} (\lambda_0 - 2\lambda_1) \right] + \omega^2 + \frac{1}{2c^2} \omega^4 a^2 \\
 & + \kappa \left[2\lambda_0 - 17\lambda_1 + 28\lambda_2 - 12\lambda_3 + 16 \frac{a^2}{b^2} (\lambda_1 - 3\lambda_2 + 2\lambda_3) \right] \omega^2 \\
 & + \kappa \pi G\rho \left[-2\lambda_1 + 16\lambda_2 - 36\lambda_3 + 24\lambda_4 - \frac{2a^2}{b^2} (\lambda_0 - 17\lambda_1 + 60\lambda_2 - 76\lambda_3 + 32\lambda_4) \right] (E_1 + E_2 - E_3) \\
 & - 2\kappa \pi G\rho \left[(\lambda_1 - 2\lambda_2) - \frac{a^2}{b^2} (\lambda_0 - 5\lambda_1 + 4\lambda_2) \right] (E_1 - E_2) \\
 & + 2\kappa \pi G\rho [\lambda_0 (-5\lambda_1 + 11\lambda_2 - 6\lambda_3) + 2\lambda_1 (5\lambda_1 - 14\lambda_2 + 9\lambda_3)] \\
 & + \frac{4a^2}{3b^2} \kappa \pi G\rho [\lambda_0 (27\lambda_1 - 56\lambda_2 + 24\lambda_3) - 2\lambda_1 (33\lambda_1 - 74\lambda_2 + 36\lambda_3)]. \quad (151)
 \end{aligned}$$

We use the results of Appendix A to replace the integrals entering the right side of Eq. (151), with their explicit expressions given in terms of the eccentricity e of the Maclaurin ellipsoid (13). It yields

$$\begin{aligned}
 \frac{\gamma_e}{r_e} - \frac{\gamma_p}{r_p} = & \frac{6\pi G\rho}{e^3} (e - \sqrt{1-e^2} \arcsin e) - 2\pi G\rho + \omega^2 + \frac{1}{2c^2} \omega^4 a^2 \\
 & + \frac{\kappa}{6e^7} (-75e - 5e^3 + 122e^5 - 42e^7 + 3\sqrt{1-e^2}(25 + 10e^2 - 34e^4 + 8e^6) \arcsin e) \omega^2 \\
 & + \frac{\kappa}{24e^9} \pi G\rho [525e - 1075e^3 + 662e^5 - 112e^7 - 3\sqrt{1-e^2}(175 - 300e^2 + 144e^4 - 16e^6) \arcsin e] (E_1 + E_2 - E_3) \\
 & - \frac{3\kappa}{e^5} \pi G\rho [3e(1-e^2) - \sqrt{1-e^2}(3-2e^2) \arcsin e] (E_1 - E_2) \\
 & - \frac{\kappa}{6e^9} \pi G\rho (1-e^2) [e(225 - 351e^2 - 58e^4 + 24e^6) - 2\sqrt{1-e^2}(225 - 276e^2 - 85e^4 + 18e^6) \arcsin e] \\
 & - \frac{\kappa}{2e^{10}} \pi G\rho (1-e^2) (75 - 142e^2 + 38e^4 + 40e^6) \arcsin^2 e. \tag{152}
 \end{aligned}$$

This form of the Clairaut theorem apparently depends on the gauge parameters E_1 and E_2 and can be used to impose a constraints on one of them in addition to the constraint given by the level surface equation (134). For example, we could demand that all post-Newtonian terms in Eq. (152) vanish. Such a choice of the gauge can be used instead of the constraints (136) and (137) proposed by Bardeen [8].

It is more interesting, however, to bring Eq. (152) to another form which is more physically relevant and does not contain explicitly the gauge parameters E_1 and E_2 . This is achieved after the replacement of the eccentricity e of the Maclaurin ellipsoid with the eccentricity ϵ of the PN ellipsoid with the help of the inversion of Eq. (25),

$$e = \epsilon - \kappa \frac{1 - \epsilon^2}{2\epsilon} (E_1 - E_2). \tag{153}$$

We substitute Eq. (153) in Eq. (152), expand in the Taylor series with respect to κ , and reduce similar terms. This procedure entirely eliminates from Eq. (152) the term being proportional to the difference $E_1 - E_2$ and yields

$$\begin{aligned}
 \frac{\gamma_e}{r_e} - \frac{\gamma_p}{r_p} = & \frac{6\pi G\rho}{\epsilon^3} (\epsilon - \sqrt{1-\epsilon^2} \arcsin \epsilon) - 2\pi G\rho + \omega^2 + \frac{1}{2c^2} \omega^4 a^2 \\
 & + \frac{\kappa}{6\epsilon^7} (-75\epsilon - 5\epsilon^3 + 122\epsilon^5 - 42\epsilon^7 + 3\sqrt{1-\epsilon^2}(25 + 10\epsilon^2 - 34\epsilon^4 + 8\epsilon^6) \arcsin \epsilon) \omega^2 \\
 & + \frac{\kappa}{24\epsilon^9} \pi G\rho [525\epsilon - 1075\epsilon^3 + 662\epsilon^5 - 112\epsilon^7 - 3\sqrt{1-\epsilon^2}(175 - 300\epsilon^2 + 144\epsilon^4 - 16\epsilon^6) \arcsin \epsilon] (E_1 + E_2 - E_3) \\
 & - \frac{\kappa}{6\epsilon^9} \pi G\rho (1-\epsilon^2) [\epsilon(225 - 351\epsilon^2 - 58\epsilon^4 + 24\epsilon^6) - 2\sqrt{1-\epsilon^2}(225 - 276\epsilon^2 - 85\epsilon^4 + 18\epsilon^6) \arcsin \epsilon] \\
 & - \frac{\kappa}{2\epsilon^{10}} \pi G\rho (1-\epsilon^2) (75 - 142\epsilon^2 + 38\epsilon^4 + 40\epsilon^6) \arcsin^2 \epsilon. \tag{154}
 \end{aligned}$$

The last step is to replace the density ρ in Eq. (154) with the total mass of the PN ellipsoid by making use of expression (101). We get

$$\begin{aligned}
 \frac{\gamma_e}{r_e} - \frac{\gamma_p}{r_p} = & \frac{3GM}{2r_e^2 r_p} \frac{3\epsilon - \epsilon^3 - 3\sqrt{1-\epsilon^2} \arcsin \epsilon}{\epsilon^3} + \omega^2 + \frac{a^2 \omega^4}{2c^2} \\
 & - \frac{\epsilon(375 + 25\epsilon^2 - 682\epsilon^4 + 234\epsilon^6) - 3\sqrt{1-\epsilon^2}(125 + 50\epsilon^2 - 194\epsilon^4 + 40\epsilon^6) \arcsin \epsilon}{40b\epsilon^7} \frac{GM}{c^2} \omega^2 \\
 & + \frac{\epsilon(2625 - 5375\epsilon^2 + 3310\epsilon^4 - 704\epsilon^6 + 48\epsilon^8) - \sqrt{1-\epsilon^2}(875 - 1500\epsilon^2 + 720\epsilon^4 - 128\epsilon^6) \arcsin \epsilon}{640a^3 b \epsilon^9} \frac{3G^2 M^2}{c^2} \\
 & \times (E_1 + E_2 - E_3) \\
 & - \frac{\epsilon(1-\epsilon^2)(225 - 351\epsilon^2 - 58\epsilon^4 + 24\epsilon^6) - 2\sqrt{1-\epsilon^2}(225 - 501\epsilon^2 + 191\epsilon^4 + 175\epsilon^6 - 42\epsilon^8) \arcsin \epsilon}{160a^3 b \epsilon^9} \frac{3G^2 M^2}{c^2} \\
 & - \frac{9}{32} \frac{G^2 M^2}{c^2} \frac{75 - 142\epsilon^2 + 38\epsilon^4 + 88\epsilon^6}{a^4 \epsilon^{10}} \arcsin^2 \epsilon. \tag{155}
 \end{aligned}$$

This is the post-Newtonian extension of the classical Clairaut theorem (139). We can see that the right side of Eq. (155) depends on the parameters E_1 , E_2 , and E_3 solely in the form of the linear combination $E_1 + E_2 - E_3$ that is fixed by the physical condition (134). Like the theorem of Pizzetti, the post-Newtonian Clairaut theorem in the form given in Eq. (155) connects the force of gravity on the level surface of the PN ellipsoid with only four parameters—the post-Newtonian mass M , the angular velocity of rotation ω , and the equatorial r_e and polar r_p radii of the PN ellipsoid.

XI. PRACTICAL APPLICATIONS FOR FUNDAMENTAL ASTRONOMY

Fundamental astronomy is an essential branch of modern gravitational physics, which explores the fundamental structure of space and time by studying the dynamics of massive bodies and elementary particles, such as photons, in a gravitational field on time scales from less than 1 sec to the Hubble time. It establishes basic theoretical principles for the high-accuracy calculation and interpretation of various astronomical effects and phenomena observed in gravitationally bounded systems, for example, clusters of galaxies, the Milky Way, stellar clusters, binary and multiple stars, and the Solar System and its subsystems. It also provides definitions and models that describe astronomical constants, time scales, reference systems, and frames used in astronomy and geodesy [9,70].

Fundamental astronomy obtains physical information on celestial objects and investigates physical laws using the methods of astrometry, celestial mechanics, and geodesy which include long-baseline radio and optical interferometry; laser and radio ranging; pulsar timing; Doppler tracking; space astrometry; atomic clocks; GPS; and other experimental tools like absolute gravimeters, gradientometers, etc. [41,47,51,71].

We shall apply the formalism of the previous sections to derive a practically meaningful post-Newtonian equation of the level surface and the other relationships used in geodetic and gravimetric applications on the ground and in space. To this end, we adopt that the classic geodesy operates with the Maclaurin reference ellipsoid defined by Eq. (13) with the semimajor and semiminor axes, a and b , and the eccentricity e defined in Eq. (14). We notice that the Earth's oblateness is about $e^2 \approx 1/149 = 0.0067$ (Ref. [2]) [Sec. I] and can be used as a small parameter for expanding all post-Newtonian terms into the convergent Taylor series. When expanding the post-Newtonian formulas to the Taylor series, we shall keep the Newtonian expressions as they are, without expanding them with respect to the eccentricity e , and take into account only terms of the order of e^2 in the post-Newtonian parts of equations by systematically discarding terms of the order of $\mathcal{O}(e^4)$ and higher. According to Maclaurin's relation (133), the square of the angular velocity $\omega^2 \approx e^2$ (see Refs. [43,44,53] for more

detail), which allows us to discard terms of the order of $\mathcal{O}(\omega^4)$ and $\mathcal{O}(\omega^2 e^2)$ as well.

Now, we have to make a decision about what kind of gauge in the mathematical description of the PN ellipsoid would be more preferable for high-precision geodetic applications. First of all, we expand and solve Eq. (134) of the level surface and find out that the linear combination of the parameters

$$E_1 + E_2 - E_3 = \frac{76}{525} e^4 + \mathcal{O}(e^6), \quad (156)$$

according to our agreement, can be discarded in all post-Newtonian expressions. Then, we look at the Bardeen gauge conditions (136) and (137), which can be solved with respect to the parameters E_1 and E_2 with the help of Eq. (156), that is, $E_3 = E_1 + E_2 + \mathcal{O}(e^4)$. The solution is given by

$$E_1 = -\frac{128}{21}, \quad E_2 = \frac{88}{21}, \quad E_3 = -\frac{40}{21}. \quad (157)$$

Substituting these values in Eq. (15), we get the equation of the PN ellipsoid in terms of the parameters a and b of the Maclaurin ellipsoid,

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{\kappa}{21} \left(128 \frac{\sigma^4}{a^4} - 88 \frac{z^4}{b^4} + 40 \frac{\sigma^2 z^2}{a^2 b^2} \right), \quad (158)$$

where the numerical value $\kappa \approx 5.21 \times 10^{-10}$ for the Earth [2]. Equation (158) can be reduced to the form

$$\frac{\sigma^2}{r_e^2} + \frac{z^2}{r_p^2} = 1, \quad (159)$$

where r_e and r_p are defined in Eqs. (21) and (22),

$$r_e = a \left(1 - \frac{62}{21} \kappa \right), \quad r_p = b \left(1 + \frac{44}{21} \kappa \right). \quad (160)$$

Equation (159) describes the post-Newtonian deviation from the Maclaurin ellipsoid adopted in the Newtonian-based geodesy. It can be viewed as another ellipsoid with the semimajor axis r_e smaller than a by ≈ 1.0 cm and the semiminor axis r_p larger than b by ≈ 0.7 cm (see Fig. 2). The difference (158) between the PN ellipsoid and the Maclaurin ellipsoid can be also interpreted as a long spatial wave,

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{20}{21} \kappa \left(1 - \frac{27}{5} \cos 2\theta \right), \quad (161)$$

with a wavelength equal to one-half of the Earth's radius R_\oplus . The Bardeen-Chandrasekhar gauge condition leads to a noticeable scale difference between the post-Newtonian and Maclaurin ellipsoids, which is not negligibly small for

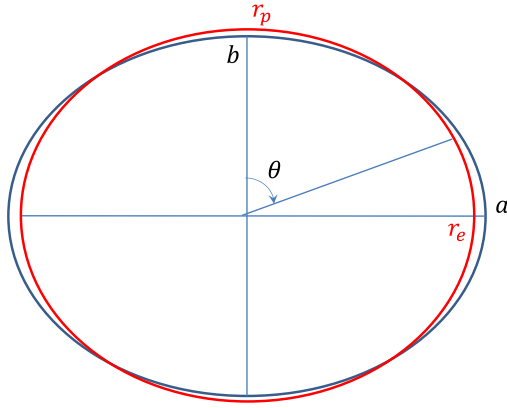


FIG. 2. Meridional cross section of the PN ellipsoid (a red curve in the online version) vs the Maclaurin ellipsoid (a blue curve in the online version) in the Bardeen-Chandrasekhar gauge. The equatorial, r_e , and polar, r_p , radii of the PN ellipsoid differ from the semimajor, a , and semiminor, b , axes of the Maclaurin ellipsoid, $r_e < a$, $r_p > b$. The maximal radial difference (the height difference) between the surfaces of the PN ellipsoid and the Maclaurin ellipsoid amounts to 1 cm, while the difference in the eccentricities ϵ and e is about 8×10^{-7} .

modern geodetic measurements and (in case the Bardeen-Chandrasekhar gauge is adopted by IERS and IUGG) should be carefully taken into account in the near future adjustment of the geodetic parameters of the reference ellipsoid.

In our opinion, another choice of the gauge, which we shall call the Maclaurin gauge, would be more preferable. More specifically, we shall accept that the geodetic coordinate system is chosen in such a way that the shape parameters $E_1 = E_2 = 0$ exactly. This is always possible due to the residual gauge freedom. In this gauge, the equatorial, r_e , and polar, r_p , radii are the same as the semimajor and semiminor axes of the Maclaurin ellipsoid, respectively, with $r_e = a$, $r_p = b$, and the parameter $E_3 = 0$ in the above-adopted approximation with all terms of the order of $\mathcal{O}(e^4)$ having been discarded. The eccentricity of the PN ellipsoid is also equal to that of the Maclaurin ellipsoid, $\epsilon = e$. With this choice of the residual gauge, the surface of the PN ellipsoid in the given approximation coincides with the surface of the Maclaurin ellipsoid in the Newtonian geodesy,

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1, \quad (162)$$

with an error not exceeding $\kappa e^4 R_\oplus \approx 2 \times 10^{-4}$ mm. Such a small error, caused by the relativistic contribution of the Earth's gravitational field, can be safely neglected in all distance/height calculations conducted in the International Terrestrial Reference System defined in Ref. [2]. It also means that the classic ellipsoidal coordinate system, having

been ubiquitously used in the Newtonian geodesy, is not deformed by the relativistic corrections to the gravity field.

We can now easily compare the main post-Newtonian equations in the two different gauges. The Maclaurin relation (133) takes on the approximate form

$$\frac{\omega^2}{2\pi G\rho} = \frac{\sqrt{1-e^2}(3-2e^2)\arcsin e - 3e(1-e^2)}{e^3} + \frac{8}{35} \frac{GM}{ac^2} e^2: \text{ the Maclaurin gauge}, \quad (163)$$

$$\frac{\omega^2}{2\pi G\rho} = \frac{\sqrt{1-\epsilon^2}(3-2\epsilon^2)\arcsin \epsilon - 3\epsilon(1-\epsilon^2)}{\epsilon^3} + \frac{8}{35} \frac{GM}{ac^2} \epsilon^2: \text{ the Bardeen-Chandrasekhar gauge}, \quad (164)$$

which, after replacing ρ with the help of the inverse of Eq. (96) and expanding, can be recast to

$$\omega^2 = \frac{3GM}{2a^3} \left[\frac{(3-2e^2)\arcsin e - 3e\sqrt{1-e^2}}{e^3} - \frac{4}{7} \frac{GM}{ac^2} e^2 \right]: \text{ the Maclaurin gauge}, \quad (165)$$

$$\omega^2 = \frac{3GM}{2r_e^3} \left[\frac{(3-2\epsilon^2)\arcsin \epsilon - 3\epsilon\sqrt{1-\epsilon^2}}{\epsilon^3} - \frac{4}{7} \frac{GM}{ac^2} \epsilon^2 \right]: \text{ the Bardeen-Chandrasekhar gauge}. \quad (166)$$

Numerical values for the geopotential and the semimajor axis of the reference ellipsoid are given in Ref. [2] [Table 1.1]. They yield $GM_\oplus/c^2 R_\oplus \approx 6.7 \times 10^{-10}$. The magnitude of the relativistic correction to the classic Maclaurin relation between the angular velocity ω of the rotating ellipsoid and its oblateness e are given in the Bardeen-Chandrasekhar gauge by the post-Newtonian terms that appear as a consequence of the expansion of the terms with ϵ in the right side of Eq. (166). These post-Newtonian terms amount to 1.3×10^{-10} . On the other hand, the relativistic correction to ω in the Maclaurin gauge is given solely by the last term in the right part of Eq. (165). Its magnitude is reduced by the factor of e^2 and amounts to only 2.6×10^{-12} .

The approximate version of the post-Newtonian theorem of Pizzetti (150) reads

$$2\frac{\gamma_e}{a} + \frac{\gamma_p}{b} = -2\omega^2 + \frac{3GM}{a^2b} + \frac{4GM}{ac^2} \times \left[\left(3 + \frac{47}{25}e^2 \right) \frac{GM}{a^3} + \frac{1}{4}\omega^2 \right]: \text{ the Maclaurin gauge,} \quad (167)$$

$$2\frac{\gamma_e}{r_e} + \frac{\gamma_p}{r_p} = -2\omega^2 + \frac{3GM}{r_e^2r_p} + \frac{4GM}{ac^2} \times \left[\left(3 + \frac{47}{25}e^2 \right) \frac{GM}{a^3} + \frac{1}{4}\omega^2 \right]: \text{ the Bardeen-Chandrasekhar gauge,} \quad (168)$$

where the relations of the radii r_e and r_p to a and b , respectively, are given in Eq. (160). The post-Newtonian corrections in the Clairaut theorem (155), after it is expanded with respect to the eccentricity e , yield

$$\frac{\gamma_e}{a} - \frac{\gamma_p}{b} = \omega^2 + \frac{3GM}{2a^2b} \frac{e(3-e^2) - 3\sqrt{1-e^2}\arcsin e}{e^3} + \frac{GM}{ac^2} \left(\frac{59GM}{25a^3}e^2 + \frac{11}{14}\omega^2 \right): \text{ the Maclaurin gauge,} \quad (169)$$

$$\frac{\gamma_e}{r_e} - \frac{\gamma_p}{r_p} = \omega^2 + \frac{3GM}{2r_e^2r_p} \frac{e(3-e^2) - 3\sqrt{1-e^2}\arcsin e}{e^3} + \frac{GM}{ac^2} \left(\frac{59GM}{25a^3}e^2 + \frac{11}{14}\omega^2 \right): \text{ the Bardeen-Chandrasekhar gauge.} \quad (170)$$

The post-Newtonian corrections to the gravitational field entering the Pizzetti and Clairaut theorems (167)–(170) are not so negligibly small, amount to the magnitude of approximately $3\mu\text{Gal}$ ($1\text{Gal} = 1\text{ cm/s}^2$), and are to be taken into account in the calculation of the parameters of the reference-ellipsoid from astronomical and gravimetric data in the foreseeable future.

The Bardeen-Chandrasekhar gauge has some advantage in the comparison of the masses of the Newtonian and post-Newtonian ellipsoids. According to the Bardeen-Chandrasekhar gauge condition, the masses of the two ellipsoids are exactly the same, $M = M_{\text{Maclaurin}}$. On the other hand, if we chose the Maclaurin gauge, the masses of the two ellipsoids will differ. Post-Newtonian correction to the Newtonian mass of the Earth,

$M_{\text{Maclaurin}} = M_N = 4\pi\rho a^2b/3$, can be evaluated in the Maclaurin gauge from (100), which is given in the approximation under consideration, by

$$M = M_N \left[1 + 2\kappa \left(\lambda_0 + \frac{\omega^2}{5\pi G\rho} \right) \right]. \quad (171)$$

Because the mass couples with the universal gravitational constant G , it contributes to the numerical value of the geocentric gravitational constant $GM_{\oplus} = 3.986004418 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$ (Ref. [2])[Table 1.1]. After expansion of the right side of Eq. (171) with respect to the eccentricity e , the relativistic variation in the value of GM_{\oplus} is

$$\frac{\delta(GM_{\oplus})_{pN}}{GM_{\oplus}} \simeq 2\kappa\lambda_0 \simeq 4\kappa \simeq 2.8 \times 10^{-9}. \quad (172)$$

The current uncertainty in the numerical value of GM_{\oplus} is $8 \times 10^5 \text{ m}^3 \text{ s}^{-2}$ (Ref. [2])[Table 1.1], which gives the fractional uncertainty

$$\frac{\delta(GM_{\oplus})}{GM_{\oplus}} \simeq 2.0 \times 10^{-9}. \quad (173)$$

This is comparable with the relativistic contribution (172), which must be taken into account in the reduction of precise geodetic data processing if the Maclaurin gauge is adopted in the post-Newtonian geodesy.

Similar considerations tells us that in the Bardeen-Chandrasekhar gauge the Newtonian and post-Newtonian values of the angular momentum of the Earth are exactly the same, $S = S_{\text{Maclaurin}}$. However, if we chose the Maclaurin gauge, we should expect the difference between the Newtonian and relativistic angular momenta given by Eq. (114), which, in the approximation where all terms of the order of $\mathcal{O}(e^4)$ are discarded, reads

$$S = S_N \left[1 + \frac{2}{7}\kappa \left(12\lambda_0 - 16\lambda_1 + \frac{3\omega^2}{5\pi G\rho} \right) \right]. \quad (174)$$

The fractional difference between the relativistic and Newtonian angular momenta of the Earth is

$$\frac{\delta S}{S} \simeq \frac{40}{7}\lambda_1 \simeq 2 \times 10^{-9}. \quad (175)$$

It should be noted, however, that the difference (175) is not so important in geodesy because the angular momentum of the Earth is not yet a directly measurable quantity as contrasted with the angular velocity ω , which is measured directly by VLBI of the IERS.

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APPENDIX: INTEGRALS

Integrals entering Eqs. (48), (49), (61), (62), (77), and (78) are solved in two steps:

- (1) integrating with respect to the angle λ from 0 to 2π ,
- (2) integrating with respect to the angle θ from 0 to π by making use of a new variable,

$$u = b^2 \tan^2 \theta, \quad du = 2b^2 \sec^3 \theta \sin \theta d\theta, \quad (\text{A1})$$

which changes from 0 to ∞ .

In terms of the new variable, we have

$$\sin \theta = \sqrt{\frac{u}{b^2 + u}}, \quad \cos \theta = \frac{b}{\sqrt{b^2 + u}}, \quad (\text{A2})$$

and

$$A = \frac{1}{a^2} \frac{a^2 + u}{b^2 + u}, \quad \frac{d\Omega}{A} = \frac{a^2 b}{2} \frac{d\lambda du}{(a^2 + u)\sqrt{b^2 + u}}. \quad (\text{A3})$$

We use these values for transforming integrands in Eqs. (48) and (49), where we also take into account that all functions entering the integrands are even functions of the argument, $f(\cos \theta) = f(-\cos \theta)$. Thus, it makes

$$\int_0^\pi f(\cos \theta) \sin \theta d\theta = 2 \int_0^{\pi/2} f(\cos \theta) \sin \theta d\theta. \quad (\text{A4})$$

This procedure allows us to represent the integrals under discussion in the following form:

$$\mathcal{J}_0 \equiv \oint_{S^2} \frac{d\Omega}{A} = 2\pi a^2 b \int_0^\infty \frac{du}{(a^2 + u)\sqrt{b^2 + u}}, \quad (\text{A5a})$$

$$\mathcal{J}_1 \equiv \oint_{S^2} \frac{\cos^2 \theta}{A^2} d\Omega = 2\pi a^4 b^3 \int_0^\infty \frac{du}{(a^2 + u)^2 \sqrt{b^2 + u}}, \quad (\text{A5b})$$

$$\mathcal{J}_2 \equiv \oint_{S^2} \frac{B}{A^2} \cos \theta d\Omega = 2\pi a^4 b z \int_0^\infty \frac{du}{(a^2 + u)^2 \sqrt{b^2 + u}}, \quad (\text{A5c})$$

$$\mathcal{J}_3 \equiv \oint_{S^2} \frac{B}{A^3} \cos^3 \theta d\Omega = 2\pi a^6 b^3 z \int_0^\infty \frac{du}{(a^2 + u)^3 \sqrt{b^2 + u}}, \quad (\text{A5d})$$

$$\mathcal{J}_4 \equiv \oint_{S^2} \frac{\cos^4 \theta}{A^3} d\Omega = 2\pi a^6 b^5 \int_0^\infty \frac{du}{(a^2 + u)^3 \sqrt{b^2 + u}}, \quad (\text{A5e})$$

$$\mathcal{J}_5 \equiv \oint_{S^2} \frac{B^2}{A^2} d\Omega = \frac{\pi a^2}{b} \left[\int_0^\infty \frac{b^2 u + (2a^2 - u)z^2}{(a^2 + u)^2 \sqrt{b^2 + u}} du + b^2 C(\mathbf{x}) \int_0^\infty \frac{udu}{(a^2 + u)^2 \sqrt{b^2 + u}} \right], \quad (\text{A5f})$$

$$\mathcal{J}_6 \equiv \oint_{S^2} \frac{B^2}{A^4} \cos^4 \theta d\Omega = \pi a^6 b^3 \left[\int_0^\infty \frac{b^2 u + (2a^2 - u)z^2}{(a^2 + u)^4 \sqrt{b^2 + u}} du + b^2 C(\mathbf{x}) \int_0^\infty \frac{udu}{(a^2 + u)^4 \sqrt{b^2 + u}} \right], \quad (\text{A5g})$$

$$\mathcal{J}_7 \equiv \oint_{S^2} \frac{B^2}{A^3} \cos^2 \theta d\Omega = \pi a^4 b \left[\int_0^\infty \frac{b^2 u + (2a^2 - u)z^2}{(a^2 + u)^3 \sqrt{b^2 + u}} du + b^2 C(\mathbf{x}) \int_0^\infty \frac{udu}{(a^2 + u)^3 \sqrt{b^2 + u}} \right], \quad (\text{A5h})$$

$$\mathcal{J}_8 \equiv \oint_{S^2} \frac{B^3}{A^4} \cos^3 \theta d\Omega = \pi a^6 b z \left[\int_0^\infty \frac{3b^2 u + (2a^2 - 3u)z^2}{(a^2 + u)^4 \sqrt{b^2 + u}} du + 3b^2 C(\mathbf{x}) \int_0^\infty \frac{udu}{(a^2 + u)^4 \sqrt{b^2 + u}} \right], \quad (\text{A5i})$$

$$\begin{aligned} \mathcal{J}_9 \equiv \oint_{S^2} \frac{B^4}{A^5} \cos^4 \theta d\Omega &= \frac{\pi a^6 b}{4} \int_0^\infty \frac{3b^4 u^2 + 6b^2 u(4a^2 - u)z^2 + (8a^4 - 24a^2 u + 3u^2)z^4}{(a^2 + u)^5 \sqrt{b^2 + u}} du \\ &+ \frac{3}{2} C(\mathbf{x}) \pi a^6 b^3 \left[\int_0^\infty \frac{(b^2 - z^2)u^2 + 4a^2 u z^2}{(a^2 + u)^5 \sqrt{b^2 + u}} du + \frac{b^2}{2} C(\mathbf{x}) \int_0^\infty \frac{u^2 du}{(a^2 + u)^5 \sqrt{b^2 + u}} \right], \end{aligned} \quad (\text{A5j})$$

$$\mathcal{J}_{10} \equiv \oint_{S^2} \frac{B^3}{A^3} \sin \theta \cos \lambda d\Omega = \frac{3\pi a^2 x}{4b} \left[\int_0^\infty \frac{(b^2 u + 4a^2 z^2 - uz^2)udu}{(a^2 + u)^3 \sqrt{b^2 + u}} + b^2 C(\mathbf{x}) \int_0^\infty \frac{u^2 du}{(a^2 + u)^3 \sqrt{b^2 + u}} \right], \quad (\text{A5k})$$

$$\mathcal{J}_{11} \equiv \oint_{S^2} \frac{B^3}{A^3} \sin \theta \sin \lambda d\Omega = \frac{3\pi a^2 y}{4b} \left[\int_0^\infty \frac{(b^2 u + 4a^2 z^2 - uz^2)udu}{(a^2 + u)^3 \sqrt{b^2 + u}} + b^2 C(\mathbf{x}) \int_0^\infty \frac{u^2 du}{(a^2 + u)^3 \sqrt{b^2 + u}} \right], \quad (\text{A5l})$$

$$\mathcal{J}_{12} \equiv \oint_{S^2} \frac{B^3}{A^3} \cos \theta d\Omega = \frac{\pi a^4 z}{b} \left[\int_0^\infty \frac{(3b^2 u + 2a^2 z^2 - 3uz^2)du}{(a^2 + u)^3 \sqrt{b^2 + u}} + 3b^2 C(\mathbf{x}) \int_0^\infty \frac{udu}{(a^2 + u)^3 \sqrt{b^2 + u}} \right], \quad (\text{A5m})$$

$$\mathcal{J}_{13} \equiv \oint_{S^2} \frac{B}{A^2} \sin \theta \cos \lambda d\Omega = \pi a^2 b x \int_0^\infty \frac{udu}{(a^2 + u)^2 \sqrt{b^2 + u}}, \quad (\text{A5n})$$

$$\mathcal{J}_{14} \equiv \oint_{S^2} \frac{B}{A^2} \sin \theta \sin \lambda d\Omega = \pi a^2 b y \int_0^\infty \frac{udu}{(a^2 + u)^2 \sqrt{b^2 + u}}, \quad (\text{A5o})$$

$$\begin{aligned} \mathcal{J}_{15} \equiv \oint_{S^2} \frac{B^4}{A^3} d\Omega &= \frac{\pi a^2}{4b^3} \int_0^\infty \frac{3b^4 u^2 + 6b^2 u(4a^2 - u)z^2 + (8a^4 - 24a^2 u + 3u^2)z^4}{(a^2 + u)^3 \sqrt{b^2 + u}} du \\ &+ \frac{3}{2} C(\mathbf{x}) \frac{\pi a^2}{b} \left[\int_0^\infty \frac{(b^2 - z^2)u^2 + 4a^2 u z^2}{(a^2 + u)^3 \sqrt{b^2 + u}} du + \frac{b^2}{2} C(\mathbf{x}) \int_0^\infty \frac{u^2 du}{(a^2 + u)^3 \sqrt{b^2 + u}} \right], \end{aligned} \quad (\text{A5p})$$

$$\begin{aligned} \mathcal{J}_{16} \equiv \oint_{S^2} \frac{B^4}{A^4} \cos^2 \theta d\Omega &= \frac{\pi a^4}{4b} \int_0^\infty \frac{3b^4 u^2 + 6b^2 u(4a^2 - u)z^2 + (8a^4 - 24a^2 u + 3u^2)z^4}{(a^2 + u)^4 \sqrt{b^2 + u}} du \\ &+ \frac{3}{2} C(\mathbf{x}) \pi a^4 b \left[\int_0^\infty \frac{(b^2 - z^2)u^2 + 4a^2 u z^2}{(a^2 + u)^4 \sqrt{b^2 + u}} du + \frac{b^2}{2} C(\mathbf{x}) \int_0^\infty \frac{u^2 du}{(a^2 + u)^4 \sqrt{b^2 + u}} \right]. \end{aligned} \quad (\text{A5q})$$

These integrals can be performed analytically,

$$\mathcal{J}_0 \equiv 2\pi a^2 \lambda_0, \quad (\text{A6a})$$

$$\mathcal{J}_1 \equiv 2\pi a^2 b^2 \lambda_1, \quad (\text{A6b})$$

$$\mathcal{J}_2 = 2\pi a^2 z \lambda_1, \quad (\text{A6c})$$

$$\mathcal{J}_3 \equiv 2\pi a^2 b^2 z \lambda_2, \quad (\text{A6d})$$

$$\mathcal{J}_4 \equiv 2\pi a^2 b^4 \lambda_2, \quad (\text{A6e})$$

$$\mathcal{J}_5 = \pi a^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - \left(1 - 3\frac{z^2}{b^2}\right) \lambda_1 + C(\mathbf{x})(\lambda_0 - \lambda_1) \right], \quad (\text{A6f})$$

$$\mathcal{J}_6 \equiv \pi a^2 b^4 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_2 - \left(1 - \frac{3z^2}{b^2}\right) \lambda_3 + C(\mathbf{x})(\lambda_2 - \lambda_3) \right], \quad (\text{A6g})$$

$$\mathcal{J}_7 = \pi a^2 b^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_1 - \left(1 - 3\frac{z^2}{b^2}\right) \lambda_2 + C(\mathbf{x})(\lambda_1 - \lambda_2) \right], \quad (\text{A6h})$$

$$\mathcal{J}_8 = 3\pi a^2 b^2 z \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_2 - \left(1 - \frac{5z^2}{3b^2}\right) \lambda_3 + C(\mathbf{x})(\lambda_2 - \lambda_3) \right], \tag{A6i}$$

$$\begin{aligned} \mathcal{J}_9 = & \frac{3}{4} \pi a^2 b^4 \left[\left(1 - \frac{z^2}{b^2}\right)^2 \lambda_2 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4}\right) \lambda_3 + \left(1 - 10 \frac{z^2}{b^2} + \frac{35z^4}{3b^4}\right) \lambda_4 \right] \\ & + \frac{3}{2} C(\mathbf{x}) \pi a^2 b^4 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_2 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_3 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_4 \right] + \frac{3}{4} C^2(\mathbf{x}) \pi a^2 b^4 (\lambda_2 - 2\lambda_3 + \lambda_4), \end{aligned} \tag{A6j}$$

$$\mathcal{J}_{10} = \frac{3}{4} \pi a^2 x \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_1 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 + C(\mathbf{x})(\lambda_0 - 2\lambda_1 + \lambda_2) \right], \tag{A6k}$$

$$\mathcal{J}_{11} = \frac{3}{4} \pi a^2 y \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_1 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 + C(\mathbf{x})(\lambda_0 - 2\lambda_1 + \lambda_2) \right], \tag{A6l}$$

$$\mathcal{J}_{12} = 3\pi a^2 x \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_1 - \left(1 - \frac{5z^2}{3b^2}\right) \lambda_2 + C(\mathbf{x})(\lambda_1 - \lambda_2) \right], \tag{A6m}$$

$$\mathcal{J}_{13} = \pi a^2 x (\lambda_0 - \lambda_1), \tag{A6n}$$

$$\mathcal{J}_{14} = \pi a^2 y (\lambda_0 - \lambda_1), \tag{A6o}$$

$$\begin{aligned} \mathcal{J}_{15} = & \frac{3}{4} \pi a^2 \left[\left(1 - \frac{z^2}{b^2}\right)^2 \lambda_0 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4}\right) \lambda_1 + \left(1 - 10 \frac{z^2}{b^2} + \frac{35z^4}{3b^4}\right) \lambda_2 \right] \\ & + \frac{3}{2} C(\mathbf{x}) \pi a^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_0 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_1 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_2 \right] + \frac{3}{4} C^2(\mathbf{x}) \pi a^2 (\lambda_0 - 2\lambda_1 + \lambda_2), \end{aligned} \tag{A6p}$$

$$\begin{aligned} \mathcal{J}_{16} = & \frac{3}{4} \pi a^2 b^2 \left[\left(1 - \frac{z^2}{b^2}\right)^2 \lambda_1 - 2 \left(1 - 6 \frac{z^2}{b^2} + 5 \frac{z^4}{b^4}\right) \lambda_2 + \left(1 - 10 \frac{z^2}{b^2} + \frac{35z^4}{3b^4}\right) \lambda_3 \right] \\ & + \frac{3}{2} C(\mathbf{x}) \pi a^2 b^2 \left[\left(1 - \frac{z^2}{b^2}\right) \lambda_1 - 2 \left(1 - \frac{3z^2}{b^2}\right) \lambda_2 + \left(1 - \frac{5z^2}{b^2}\right) \lambda_3 \right] + \frac{3}{4} C^2(\mathbf{x}) \pi a^2 (\lambda_1 - 2\lambda_2 + \lambda_3), \end{aligned} \tag{A6q}$$

where

$$\lambda_n \equiv a^{2n} b \int_0^\infty \frac{du}{(a^2 + u)^{n+1} \sqrt{b^2 + u}} = \frac{2\sqrt{1-e^2}}{e^{2(n+1)}} \int_{\frac{\sqrt{1-e^2}}{e}}^\infty \frac{d\xi}{(1+\xi^2)^{n+1}} \tag{A7}$$

is a table integral given in I.S. Gradshteyn and I.M. Ryzhik's "Table of Integrals, Series, and Products" (Academic Press: London, 1980) as integral 2.148-4,

$$\lambda_n = \frac{(2n-1)!!}{2^{n-1} n!} \left(\frac{\sqrt{1-e^2}}{e^{2n+1}} \arcsin e - (1-e^2) \sum_{k=1}^n \frac{(n-k)!}{(2n-2k+1)!!} \frac{2^{n-k}}{e^{2k}} \right). \tag{A8}$$

In particular,

$$\lambda_0 = 2 \frac{\sqrt{1-e^2}}{e} \arcsin e, \tag{A9a}$$

$$\lambda_1 = \frac{\sqrt{1-e^2}}{e^3} \arcsin e - \frac{1-e^2}{e^2}, \tag{A9b}$$

$$\lambda_2 = \frac{3\sqrt{1-e^2}}{4e^5} \arcsin e - \frac{(1-e^2)(3+2e^2)}{4e^4}, \tag{A9c}$$

$$\lambda_3 = \frac{5\sqrt{1-e^2}}{8e^7} \arcsin e - \frac{(1-e^2)(15+10e^2+8e^4)}{24e^6}, \quad (\text{A9d})$$

$$\lambda_4 = \frac{35\sqrt{1-e^2}}{64e^9} \arcsin e - \frac{(1-e^2)(105+70e^2+56e^4+48e^6)}{192e^8}. \quad (\text{A9e})$$

There is also a recurrent formula that allows us to calculate the integrals λ_n by iterations starting from λ_0 ,

$$\lambda_n = \frac{2n-1}{2n} \frac{\lambda_{n-1}}{e^2} - \frac{1-e^2}{ne^2} \quad (n \geq 1), \quad (\text{A10})$$

where λ_0 is given in Eq. (A9a). Notice that none of these integrals is divergent for small values of the eccentricity because the numerators of the integrals suppress the small value of the eccentricity in the denominators.

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