Electromagnetic radiation from a rapidly rotating magnetized star in orbit

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A general formula for the electromagnetic energy radiated by a rapidly rotating magnetic dipole in arbitrary motion is obtained. For a pulsar orbiting in a binary system, it is shown that the electromagnetic radiation produced by the orbital motion is usually weaker than the gravitational radiation, but not entirely negligible for general relativistic corrections.

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I. INTRODUCTION

The discovery of the Hulse-Taylor [1] and other binary pulsars [2,3] has been of fundamental importance in testing various predictions of general relativity, the most important one being the loss of energy through gravitational radiation. High precision measurements also permit us to deduce the main parameters of orbiting pulsars at the level of post-Newtonian approximation [4]. More recently, binary white dwarfs in very close orbits have also been proposed for tests of gravitational radiation [5].

A pulsar, due to its rapid rotation and intense magnetic field, emits a large amount of electromagnetic radiation at the expense of its rotational energy [6]. If the pulsar is orbiting in a binary system, there is also electromagnetic radiation produced by its noninertial motion that affects the orbital elements. In most cases of interest, the energy loss due to this latter effect is smaller than that due to gravitational radiation, but it may not be entirely negligible in high precision calculations involving objects with large magnetic fields.

In a previous work related to this subject, Sobacchi and Vietri [7] studied the motion of a pair of dipoles in inspiralling orbit. The aim of the present article is to calculate the electromagnetic energy radiated by a magnetized star in Keplerian orbit, fully taking into account the rapid rotation of its magnetic field, which is the most important source of electromagnetic radiation.

The magnetosphere of a pulsar has a complicated structure due to the presence of relativistic plasma, but the field has a predominantly dipolar structure [8]. The dipole model is sufficient for our purposes. The electromagnetic field produced by a moving (electric or magnetic) dipole has been the subject of many articles (see, e.g., Refs. [9–12]). In the present paper, we follow Ellis's original treatment of the problem [9]. Accordingly, in Sec. II, a general formula for the electromagnetic energy radiated by a (nonprecessing) rotating dipole in arbitrary motion is obtained; the resulting formula is valid for mildly relativistic motion and high rotational frequencies, which is the case for binary pulsars. The formula is applied in Sec. III to the motion of a rotating dipole in Keplerian orbit, and the losses of energy through electromagnetic and gravitational radiation are compared. The results are discussed in Sec. IV, where it is shown that electromagnetic radiation can be a dominant source of orbital energy loss for pulsars with very intense magnetic fields. The particular case of the Hulse-Taylor binary pulsar is considered.

II. CALCULATIONS

Let us define the power dI radiated by a dipole in a solid angle $d\Omega$ as

$$dI = \frac{\mathfrak{m}^2}{4\pi c^3} S d\Omega, \tag{1}$$

where $\mathfrak{m}$ is the dipolar moment. For a dipole moving with velocity $\mathbf{V}$, Ellis [9] has shown that in the far field region (setting $c = 1$ hereafter)

$$S \equiv \frac{1}{\lambda^5} |\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \mathbf{V}) \times \mathbf{X})]|^2, \tag{2}$$

where $\lambda \equiv 1 - \hat{\mathbf{r}} \cdot \mathbf{V}$,

$$\hat{\mathbf{r}} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3}$$

(in standard spherical coordinates), and

$$\mathbf{X} = \mathbf{N}'' + \frac{1}{\lambda}(\hat{\mathbf{r}} \cdot \mathbf{V}')\mathbf{N}' + \frac{2}{\lambda}(\hat{\mathbf{r}} \cdot \mathbf{N}')\mathbf{V}' + \frac{3}{\lambda^2}(\hat{\mathbf{r}} \cdot \mathbf{N})(\hat{\mathbf{r}} \cdot \mathbf{V}')\mathbf{V}' + \frac{1}{\lambda}(\hat{\mathbf{r}} \cdot \mathbf{N})\mathbf{V}''. \tag{4}$$

In this last equation, a prime denotes a derivative with respect to time measured in a fixed system; $\mathbf{N} = |\mathbf{N}|\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector along the direction of the dipole, and

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$$|\mathbf{N}| = \left[\frac{1-|\mathbf{V}|^2}{1-|\hat{\mathbf{n}}\times\mathbf{V}|^2}\right]^{1/2} = 1 - \frac{1}{2}(\hat{\mathbf{n}}\cdot\mathbf{V})^2 + O(V^4). \quad (5)$$

The above formulas can be applied to the problem of an arbitrarily moving dipole spinning around a fixed direction (that is, assuming that there is no axial precession, which is a good approximation for gravitating bodies). The calculation is rather involved, but it simplifies somewhat if only second-order terms in V are kept, and one assumes that the spin frequency is much larger than the orbital frequency, that is, $\omega \gg V'/V$. Thus the last two terms in Eq. (4) can be neglected. Additionally, an average over spin cycles is to be performed.

Accordingly, let $\boldsymbol{\omega}$ be the constant axis of rotation, which we take provisionally as the z axis, and define

$$\hat{\mathbf{n}} = (n_\perp \cos(\omega t), n_\perp \sin(\omega t), n_\parallel), \quad (6)$$

with $\hat{\mathbf{n}}' = \boldsymbol{\omega}\times\hat{\mathbf{n}}$, and $n_\perp^2 + n_\parallel^2 = 1$. Then we can average over spin cycles using formulas (A2) and (A3) in the Appendix. Keeping only the leading terms proportional to ω^4 and ω^3, and neglecting terms of order V^3 and higher, we find after some lengthy but straightforward algebra (see the Appendix),

$$\begin{aligned}\frac{dI}{d\Omega} = \frac{(n_\perp \mathfrak{m})^2}{8\pi}\omega^4\lambda^{-5}\Big\{&1+\cos^2\theta - 2\hat{\mathbf{r}}\cdot\mathbf{V} - 2V_\parallel\cos\theta \\ &+ \left[\left(1+\frac{1}{4}n_\perp^2\right)\sin^2\theta - n_\perp^2\right]|\mathbf{V}|^2 \\ &+ \left[3n_\perp^2 - \left(1+\frac{5}{4}n_\perp^2\right)\sin^2\theta\right]V_\parallel^2 \\ &+ (\hat{\mathbf{r}}\cdot\mathbf{V}_\perp)\left[\frac{1}{2}n_\perp^2(\hat{\mathbf{r}}\cdot\mathbf{V}_\perp) + (4-2n_\perp^2)V_\parallel\cos\theta\right]\Big\} \\ &+ \frac{(n_\perp\mathfrak{m})^2}{2\pi}\omega^2\lambda^{-5}\mathbf{r}\cdot\{\lambda(\mathbf{V}'\times\boldsymbol{\omega}) - (\mathbf{r}\cdot\mathbf{V}')(\mathbf{V}\times\boldsymbol{\omega})\}.\end{aligned} \quad (7)$$

For any vector $\mathbf{V}$, we have defined $V_\parallel = \omega^{-1}\boldsymbol{\omega}\cdot\mathbf{V}$ and $\mathbf{V}_\perp = \mathbf{V} - \omega^{-2}(\boldsymbol{\omega}\cdot\mathbf{V})\boldsymbol{\omega}$ as the components along and perpendicular to $\boldsymbol{\omega}$, respectively.

As the next step, we perform an integration over all solid angles, using formulas (A8) and (A9) in the Appendix. The result is (restoring c)

$$\begin{aligned}I = \frac{2(n_\perp\mathfrak{m})^2}{3c^3}\omega^2\Big\{&\omega^2 + \frac{1}{2}(5-n_\perp^2)\omega^2|\mathbf{V}|^2/c^2 \\ &- \frac{3}{2}n_\parallel^2(\boldsymbol{\omega}\cdot\mathbf{V})^2/c^2 - 3\boldsymbol{\omega}\cdot(\mathbf{V}'\times\mathbf{V})/c^2\Big\},\end{aligned} \quad (8)$$

for the total power radiated.

Notice that for $\mathbf{V}=0$ we recover the textbook formulas [13]

$$\frac{dI_0}{d\Omega} = \frac{(n_\perp\mathfrak{m})^2}{8\pi c^3}\omega^4(1+\cos^2\theta),$$

$$I_0 = \frac{2(n_\perp\mathfrak{m})^2}{3c^3}\omega^4.$$

The last three terms in Eq. (8) are the contributions due to the orbital motion: the first two are of order $I_0(V/c)^2$, and the last one, which can be neglected, is of order $I_0(\Omega/\omega)(V/c)^2$, where Ω is the orbital frequency.

III. KEPLERIAN ORBITS

The general formula (8) can now be applied to the motion of a rapidly rotating magnetized star in a binary system. Let m_1 be its mass and m_2 that of its companion, and $M = m_1 + m_2$. Taking the average over an orbital period, it follows from basic Newtonian mechanics that

$$|\mathbf{V}|^2 = \frac{Gm_2^2}{Ma}\frac{(1+e^2)}{(1-e^2)}, \quad (9)$$

$$(\boldsymbol{\omega}\cdot\mathbf{V})^2 = \omega^2\frac{Gm_2^2}{Ma}\frac{(1+2e^2\sin^2\beta)}{2(1-e^2)}\sin^2\alpha, \quad (10)$$

where $a = (M/m_2)a_1$ with a_1 the semimajor axis of the star orbit, e its eccentricity, α the angle between $\boldsymbol{\omega}$ and the orbital angular momentum vector, and β the angle between the major axis and the projection of $\boldsymbol{\omega}$ on the plane of the orbit.

Accordingly

$$I = I_0\left[1 + \frac{Gm_2^2}{c^2Ma}g(e,\alpha,\beta,n_\parallel)\right], \quad (11)$$

where

$$\begin{aligned}g(e,\alpha,\beta,n_\parallel) \equiv (1-e^2)^{-1}\Big[&\left(2+\frac{1}{2}n_\parallel^2\right)(1+e^2) \\ &- \frac{3}{4}n_\parallel^2(1+2e^2\sin^2\beta)\sin^2\alpha\Big].\end{aligned} \quad (12)$$

From this last formula it is shown that the radiated electromagnetic energy due to the combined rotational and orbital motion of a neutron star is of the order of magnitude

$$\frac{Gm_2^2}{c^2Ma}I_0$$

(for similar masses, the factor multiplying I_0 is roughly the ratio of the gravitational radius to the orbit of the star). In comparison, the orbital energy of the system is $E_{\rm orb} = -Gm_1m_2/2a$, and the power $I_{\rm grav}$ radiated in the form of gravitational waves is $I_{\rm grav} = h_m(e)a^{-5}$, where

$$h_m(e) \equiv \frac{32G^4 m_1^2 m_2^2 M}{5c^5} f(e),$$

and $f(e) = (1-e^2)^{-7/2}(1+\frac{73}{24}e^2+\frac{37}{96}e^4)$ (see Ref. [14] or Sec. 4.1.2 of Ref. [3]). The important point to notice is that the electromagnetic power is proportional to a^{-1}, while the gravitational power is proportional to a^{-5}. Therefore, the former dominates for large orbits and the latter for close orbits.

Following exactly the same analysis as in Sec. 4.1.2 of Ref. [3], we find that the orbital period T decays as

$$\frac{\dot{T}}{T} = -\frac{3m_2}{c^2 M m_1} I_0 - \frac{3M}{m_1 m_2}(GM)^{-7/3} h_m(e)\left(\frac{T}{2\pi}\right)^{-8/3}; \tag{13}$$

the first and second terms correspond to electromagnetic and gravitational radiation, respectively. If the former dominates, the decay of the period T is exponential. As discussed in the next section, the first term, which is constant, is a small correction in most cases of interest.

IV. DISCUSSION OF RESULTS

For a typical binary pulsar, the loss of energy through electromagnetic radiation can be estimated with the results obtained above. For instance, for the Hulse-Taylor pulsar with a rotational frequency $\omega \simeq 120\ \mathrm{s}^{-1}$, the magnetic field at its surface is estimated to be $\sim 2\times 10^{10}$ G [15], and thus the total electromagnetic power radiated I_0 is of the order of 10^{32}–10^{33} ergs/s. This power is radiated at the expense of the rotational energy of the star, but about 10^{25}–10^{26} ergs/s come from the orbital energy. If we compare this last value with the power emitted as gravitational radiation, which is about 5×10^{31} ergs/s, we see that there is a correction of the order of 10^{-6} to the orbital energy lost by the system. Since the parameters of this binary pulsar have been measured with a precision of 10^{-3} and the observed decay of the period corresponds nicely with the theoretical prediction of gravitational radiation, our present result can be taken as indirect evidence that the magnetic field of this pulsar is weaker than 10^{11} G, since otherwise the effect analyzed in this paper would be noticeable. A magnetic field lower than the average for neutron stars is consistent with the decay of the fields of binary pulsars proposed by some authors (Taam and van den Heuvel [16], Srinivasan *et al.* [8,17]).

For binary pulsars with periods of 10^{-3} s or shorter, or with strong magnetic fields, one can expect $I_0 \sim 10^{40}$ ergs/s or higher. Therefore the electromagnetic radiation due to a combination of spin and orbital motion can be comparable with an important fraction of the gravitational radiation. Moreover, the magnetic dipole model accounts for only a part of the electromagnetic energy emission, since the presence of relativistic plasma in the magnetosphere also produces an additional loss of energy [18].

APPENDIX: USEFUL FORMULAS

An averaging over one cycle, defined as

$$\langle F\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} F(\omega t)dt, \tag{A1}$$

gives

$$\begin{aligned}\langle(\hat{\mathbf{n}}\cdot\mathbf{A})(\hat{\mathbf{n}}\cdot\mathbf{B})\rangle &= \frac{1}{2}n_\perp^2 \mathbf{A}_\perp\cdot\mathbf{B}_\perp + n_\parallel^2 A_\parallel B_\parallel \\ \langle(\hat{\mathbf{n}}\cdot\mathbf{A})(\hat{\mathbf{n}}'\cdot\mathbf{B})\rangle &= \frac{1}{2}n_\perp^2 \boldsymbol{\omega}\cdot(\mathbf{A}\times\mathbf{B}),\end{aligned} \tag{A2}$$

etc., and

$$\langle(\hat{\mathbf{n}}\cdot\mathbf{A}_\perp)^2(\hat{\mathbf{n}}\cdot\mathbf{B}_\perp)^2\rangle = \frac{1}{8}n_\perp^4[|\mathbf{A}_\perp|^2|\mathbf{B}_\perp|^2 + 2(\mathbf{A}_\perp\cdot\mathbf{B}_\perp)^2], \tag{A3}$$

where $\hat{\mathbf{n}}$ in all these formulas is given by Eq. (6).

In order to deduce Eq. (7), it is convenient to express the scalar S defined in Eq. (2) in the form

$$S = -(1-V^2)\lambda^{-5}(\hat{\mathbf{r}}\cdot\mathbf{X})^2 + \lambda^{-3}|\mathbf{X}|^2 + 2\lambda^{-4}(\mathbf{V}\cdot\mathbf{X})(\hat{\mathbf{r}}\cdot\mathbf{X}), \tag{A4}$$

and then calculate

$$\begin{aligned}\langle(\hat{\mathbf{r}}\cdot\mathbf{X})^2\rangle = \frac{1}{2}n_\perp^2\omega^4\Bigg[&\left(1-\frac{1}{4}n_\perp^2|\mathbf{V}_\perp|^2 - n_\parallel^2 V_\parallel^2\right)\sin^2\theta \\ &-\frac{1}{2}n_\perp^2(\hat{\mathbf{r}}\cdot\mathbf{V}_\perp)^2 - 2n_\parallel^2 V_\parallel(\hat{\mathbf{r}}\cdot\mathbf{V}_\perp)\cos\theta\Bigg],\end{aligned} \tag{A5}$$

$$\begin{aligned}\langle|\mathbf{X}|^2\rangle = n_\perp^2\omega^4&\left(1-\frac{1}{2}n_\perp^2|\mathbf{V}_\perp|^2 - n_\parallel^2 V_\parallel^2\right) \\ &+ 2n_\perp^2\omega^2\lambda^{-1}\hat{\mathbf{r}}\cdot(\mathbf{V}'_\perp\times\boldsymbol{\omega}),\end{aligned} \tag{A6}$$

$$\begin{aligned}\langle(\mathbf{V}\cdot\mathbf{X})(\hat{\mathbf{r}}\cdot\mathbf{X})\rangle = &\frac{1}{2}n_\perp^2\omega^4(\hat{\mathbf{r}}\cdot\mathbf{V}_\perp) \\ &+ n_\perp^2\omega^2(\hat{\mathbf{r}}\cdot\mathbf{V}'_\perp)\hat{\mathbf{r}}\cdot(\mathbf{V}_\perp\times\boldsymbol{\omega}),\end{aligned} \tag{A7}$$

keeping only terms proportional to ω^4 and ω^3 and of no higher order than V^2. Then Eq. (7) follows.

Integration over all angles can be performed with the formulas

$$\int (\mathbf{A}\cdot\hat{\mathbf{r}})(\mathbf{B}\cdot\hat{\mathbf{r}})d\Omega = \frac{4\pi}{3}\mathbf{A}\cdot\mathbf{B}, \tag{A8}$$

$$\int (\mathbf{A}\cdot\hat{\mathbf{r}})(\mathbf{B}\cdot\hat{\mathbf{r}})(\mathbf{C}\cdot\hat{\mathbf{r}})(\mathbf{D}\cdot\hat{\mathbf{r}})d\Omega = \frac{4\pi}{15}[(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})], \tag{A9}$$

for any vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. Using also the approximation

$$\lambda^{-n} \simeq 1 + n\hat{\mathbf{r}}\cdot\mathbf{V} + \frac{1}{2}n(n+1)(\hat{\mathbf{r}}\cdot\mathbf{V})^2,$$

Eq. (8) follows.

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