

Paczynski-Wiita-like potential for any static spherical black hole in metric theories of gravity

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The pseudo-Newtonian potential of Paczynski and Wiita for particles orbiting a Schwarzschild black hole is generalized to arbitrary static and spherically symmetric spacetimes, including black hole solutions of alternative theories of gravity. In addition to being more general, our prescription differs substantially from a previous one in the literature, showing that the association of a pseudo-Newtonian potential even with a simple black hole metric is not unique.

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I. INTRODUCTION

Black holes are central to several areas of astrophysics, and supermassive black holes in the cores of galaxies are important factors in galactic evolution. Black holes in astrophysical environments are surrounded by matter and fluids which may form accretion disks orbiting around them. Although black holes were discovered theoretically just after the introduction of general relativity, most (if not all) relativistic theories of gravity are believed to have black hole solutions. There is currently much interest in theories of gravity alternative to Einstein theory for several reasons [1–7]. From the theoretical point of view, attempts to renormalize general relativity and to produce a quantum theory of gravity invariably produce deviations from Einstein theory in the form of higher-order derivatives in the field equations, extra degrees of freedom, or scalar fields coupled nonminimally to gravity and matter [6]. The present acceleration of the Universe discovered with type Ia supernovae [8] can be explained in the context of general relativity by re-introducing the problematic cosmological constant with an incredible amount of fine-tuning, by advocating a completely *ad hoc* dark energy [9], an even more problematic backreaction of cosmological inhomogeneities on the cosmic dynamics [10], or by changing the theory of gravity altogether [4,5,9]. Recently, this last possibility has motivated an enormous interest in alternative theories of gravity.

Going back from cosmology to astrophysics, modifying gravity would have implications for black holes and particles and fluids surrounding them and forming accretion disks. There is, therefore, interest in using observations of black holes in the near future to test deviations from general relativity and possibly detect scalar hair [6,7,11,12].

Due to the complication of relativistic motions around black holes, pseudo-Newtonian potentials have been used for decades to provide an effective simplified description of timelike geodesics for massive particles orbiting black holes [13–17]. These pseudopotentials are surprisingly accurate in determining orbits, given their simplicity: for example, the phase space of massive test particles in the Schwarzschild metric is not too dissimilar from that of the associated pseudopotential [18], a fact that can be understood by noting that the pseudopotential is defined in such a way as to preserve the fixed points of the relevant dynamical system [13,15].

The first pseudo-Newtonian potential introduced in the literature, the Paczynski-Wiita potential [13,15], reproduces exactly the location of the innermost stable circular orbit (ISCO or marginally stable orbit) R_{ISCO} , the location of the marginally bound orbit R_{mb} , and the form of the Keplerian angular momentum $L(R)$. It also reproduces accurately, but not exactly [13,15], the form of the Keplerian angular velocity and the form of the radial epicyclic frequency.

In this article we address the possibility of introducing a pseudo-Newtonian potential in theories of gravity alternative to Einstein gravity. For simplicity, we limit ourselves to spherically symmetric spacetimes (spherically symmetric pseudopotentials are often used even for rotating black holes). Since the field equations of the theories of gravity are used only to provide black hole solutions, it is straightforward to generalize to arbitrary spherically symmetric static metrics the pseudo-Newtonian potential introduced by Paczynski and Wiita [13] for the Schwarzschild spacetime, following the pedagogical derivation of Ref. [15]. A recent derivation of a pseudo-Newtonian potential for certain spherically symmetric black hole metrics in Ref. [17] produced a different pseudo-Newtonian potential. Although the difference becomes irrelevant asymptotically far away from the inner edge of the accretion disk, it persists close to it, showing that the

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association of a pseudo-Newtonian potential with a black hole spacetime is not unique.

We use units in which Newton's constant G and the speed of light in vacuo c are unity, and we follow the notation and conventions of Ref. [19]. The symbol R denotes the areal radius of spherically symmetric geometries.

II. THE PACZYNSKI-WIITA POTENTIAL FOR ANY STATIC SPHERICAL BLACK HOLE

Here we derive the analogue of the Paczynski-Wiita pseudo-Newtonian potential [13] for *any* static and spherically symmetric metric. We follow, step by step, the pedagogical derivation of Ref. [15]. In a restricted class of static spherically symmetric spacetimes, a different pseudo-Newtonian potential was obtained in Ref. [17] by studying the same equations for timelike geodesics (see below).

Any static and spherically symmetric metric can be written in the form

$$ds^2 = g_{00}(R)dt^2 + g_{11}(R)dR^2 + R^2d\Omega_{(2)}^2 \quad (1)$$

in polar coordinates (t, R, θ, φ) , where R is the areal radius and $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit 2-sphere. The timelike and spacelike Killing vectors $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial \varphi)^a$ are associated with the time and rotational symmetries, respectively. Let u^a be the 4-tangent to a timelike geodesic followed by a particle of mass m , then $\xi_a u^a = u_0 = -E$ and $\psi_a u^a = u_3 = R^2 u^3$ are constants of motion along these geodesics, corresponding to conservation of energy and angular momentum (per unit mass). Because of spherical symmetry, the orbits of test particles are planar and, without loss of generality, we assume that they take place in the $\theta = \pi/2$ plane, so that $u^2 = 0$. The normalization of the 4-velocity gives

$$g_{00}(u^0)^2 + g_{11}(u^1)^2 + g_{33}(u^3)^2 = -1, \quad (2)$$

or

$$g^{00}(u_0)^2 + g^{11}(u_1)^2 + g^{33}(u_3)^2 = -1. \quad (3)$$

Now, setting [15]

$$g_{11}(u^1)^2 \equiv v^2 \ll 1, \quad (4)$$

where v is the radial velocity, we have

$$\begin{aligned} 1 + v^2 &= -g^{00}(u_0)^2 - g^{33}(u_3)^2 \\ &= (u_0)^2(-g^{00} - g^{33}L^2), \end{aligned} \quad (5)$$

with $L \equiv u_3/u_0 = \text{const.}$ Taking the logarithm gives

$$\ln(1 + v^2) = 2 \ln u_0 + \ln(-g^{00} - g^{33}L^2) \quad (6)$$

and, expanding for small v ,

$$\ln u_0 = \frac{v^2}{2} - \frac{1}{2} \ln(-g^{00} - g^{33}L^2), \quad (7)$$

where $\ln u_0 = \text{const.}$ Equation (7) has the form of an energy conservation equation $v^2/2 + U(R) = E/m$, where

$$U(R) = -\frac{1}{2} \ln\left(-g^{00} - \frac{L^2}{R^2}\right). \quad (8)$$

Stable and unstable circular orbits are located at the extrema of this potential, where

$$\frac{dU}{dR} = \frac{1}{g^{00} + \frac{L^2}{R^2}} \left\{ -\frac{1}{2} \frac{d}{dR}(g^{00}) + \frac{L^2}{R^3} \right\} = 0. \quad (9)$$

The last equation is equivalent to

$$\frac{d\Phi}{dR} - \frac{L^2}{R^3} = 0, \quad (10)$$

where $\Phi(R)$ is the sought-after pseudo-Newtonian potential, which is defined up to an irrelevant additive constant. The choice

$$\Phi(R) = \frac{1}{2}(1 + g^{00}) = \frac{1}{2} \left(1 + \frac{1}{g_{00}(R)} \right) \quad (11)$$

reproduces the Paczynski-Wiita pseudo-Newtonian potential for the Schwarzschild metric, which has $-g_{00} = g_{11}^{-1} = 1 - 2m/R$ [13]:

$$\Phi_{\text{PW}} = \frac{-m}{R - 2m}. \quad (12)$$

Equation (11) gives trivially

$$g_{00} = -\frac{1}{1 - 2\Phi}, \quad (13)$$

which, in the weak-field limit $|\Phi| \ll 1$, would yield $g_{00} = -(1 + 2\Phi)$, a relation familiar from the post-Newtonian limit of general relativity (e.g., [19]), but this limit is not appropriate here because the goal is to investigate strong gravity near black hole horizons. This aspect brings us to the dichotomy inherent in the use of pseudo-Newtonian potentials: one wants to explore strong gravity, but doing this in a Newtonian way, and Newton's theory is intrinsically linear and limited to the weak-field regime. This procedure apparently entails a contradiction.¹ However, pseudo-Newtonian potentials do not attempt to describe *all* aspects of physics in the strong gravity regime,

¹It is also obvious that this pseudo-Newtonian potential cannot be a truly Newtonian description of gravity because, in vacuo (e.g., for the Schwarzschild geometry), it should then be $\nabla^2\Phi = 0$, while in general, this Laplacian does not vanish. Replacing the flat space Laplacian with the Laplace-Beltrami operator $g^{ij}\nabla_i\nabla_j\Phi$ ($i, j = 1, 2, 3$) does not help, either.

but only to catch certain aspects, i.e., the innermost stable and outermost marginally stable circular orbits. Tejada and Rosswog [17] define a pseudo-Newtonian potential for static, spherically symmetric metrics of the form (1) which, in addition, satisfy the condition $g_{00}g_{11} = -1$. Their pseudopotential is

$$\Phi_{\text{TR}}(R) = -\frac{(g_{00} + 1)}{2}, \quad (14)$$

which, using Eq. (13), relates to our potential (11) through

$$\Phi_{\text{TR}} = \frac{\Phi}{1 - 2\Phi}. \quad (15)$$

It is only in the limit $|\Phi| \ll 1$ that the two pseudopotentials coincide, and we have already discussed how this limit is not appropriate in the vicinity of a black hole. That the pseudo-Newtonian potentials Φ and Φ_{TR} do not coincide is best seen at the horizon: when $g_{00} \rightarrow 0^-$, $\Phi \rightarrow -\infty$ while $\Phi_{\text{TR}} \rightarrow -1/2$. Apart from a sign, the difference resides basically in the fact that the inverse metric coefficient g^{00} appears in the generalized Paczynski-Wiita potential (11), while g_{00} appears in the Tejada-Rosswog potential (14). Sarkar, Ghosh, and Bhadra [20] introduce a velocity-dependent potential to include special-relativistic effects in the dynamics, by considering spherically symmetric static metrics with the usual restriction $g_{00}g_{11} = -1$. When the velocity terms are dropped, their potential reduces to (14).

The lesson to draw from Eq. (15) is that, at least in principle if not in practice, the pseudo-Newtonian potential associated with a given spacetime metric is not unique.

At this point, it is appropriate to make explicit the assumptions used to derive Eq. (11):

- (i) The spacetime metric g_{ab} is static and spherically symmetric.
- (ii) The metric is written in the gauge (1) using the areal radius R (which is defined in a geometric, coordinate-independent way).
- (iii) The radial velocities v of the massive test particles defined by $v^2 \equiv g_{11}(u^1)^2$ are small everywhere along the timelike geodesics in comparison with the speed of light (the tangential velocities, by contrast, are not restricted to be small).
- (iv) The pseudo-Newtonian potential is required to produce at its extrema the circular orbits of the spacetime geometry (1).

In particular, the Einstein equations have not been used and the formula (11) is valid in any theory of gravity in which massive test particles follow timelike geodesics. What is more, asymptotic flatness of the metric (1) is not required (see below).

The fact that the result (11) does not depend on the theory of gravity (provided that test particles follow geodesics) will be useful to study particle trajectories and

accretion around black holes in alternative theories of gravity. Before approaching this problem, however, it is useful to restrict to general relativity and give a geometric characterization of the pseudo-Newtonian potential (11) obtained and some examples.

III. GENERAL RELATIVITY

In this section, we restrict ourselves to general relativity, and we assume that the geometry is described by Eq. (1).

A. Relation with the Misner-Sharp-Hernandez mass in general relativity

In this subsection, we make the further assumption that

$$g_{00}g_{11} = -1. \quad (16)$$

This assumption encompasses a wide class of spherically symmetric geometries²: the condition (16) has been studied in Ref. [23], where it is shown that it is equivalent to require that the (double) projection of the Ricci tensor onto radial null vectors l^a vanishes, $R_{ab}l^a l^b = 0$. This condition is also equivalent to require that the restriction of the Ricci tensor to the (t, R) subspace is proportional to the restriction of the metric g_{ab} to this subspace [23]. Equivalently, the areal radius R is an affine parameter along radial null geodesics [23]. These results hold in higher spacetime dimension as well, and the geometries satisfying the condition (16) in general relativity include vacuum, electrovacuum with either Maxwell or nonlinear Born-Infeld electrodynamics, and a spherical global monopole (“string hedgehog” [24]) [23]. Under this assumption, we can use the characterization of the Misner-Sharp-Hernandez mass M_{MSH} [25],

$$1 - \frac{2M_{\text{MSH}}}{R} = \nabla^c R \nabla_c R, \quad (17)$$

and the fact that, in the coordinates (t, R, θ, φ) used, $\nabla^c R \nabla_c R = g^{RR}$ to obtain

$$\Phi(R) = -\frac{M_{\text{MSH}}(R)}{R - 2M_{\text{MSH}}(R)}, \quad (18)$$

where $M_{\text{MSH}}(R)$ is the Misner-Sharp-Hernandez mass contained in a 2-sphere of symmetry of radius R . This is a quasilocal mass: the Hawking-Hayward quasilocal energy [26], which seems to be favored by the relativity community among the various notions of quasilocal energy introduced in general relativity since the 1960s (see the review [27]), is well known to reduce to the Misner-Sharp-Hernandez mass in spherical symmetry [28]. Since both the areal radius and the Misner-Sharp-Hernandez mass are geometric quantities defined in a coordinate-independent way, this formula constitutes a geometric characterization

²Early interest in these metrics includes Refs. [21,22].

of the generalized Paczynski-Wiita potential (11). Moreover, Eq. (18) generalizes Eq. (12) valid for the Schwarzschild geometry. The use of the Misner-Sharp-Hernandez mass to express the pseudo-Newtonian potential is particularly appropriate because this construct has a Newtonian character, contrary to other quasilocal energies in the literature [29,30].

The (apparent) horizons in spherical symmetry are located at the roots of the equation $\nabla_c R \nabla^c R = 0$ (see, e.g., [31]) and therefore the pseudopotential Φ diverges at the (apparent) black hole horizon of radius R_{AH} , where $R_{\text{AH}} = 2M_{\text{MSH}}(R_{\text{AH}})$ which is, of course, reminiscent of the behavior of the Paczynski-Wiita potential at the Schwarzschild horizon. If, in addition, asymptotic flatness is imposed, the metric coefficient $g_{00} \rightarrow -1$ as $R \rightarrow +\infty$ and $\Phi \rightarrow \text{const}$ in this limit. However, it is not necessary to impose asymptotic flatness and in certain situations it may even be inappropriate (e.g., in the Schwarzschild–de Sitter–Kottler solution). For example, alternative theories of gravity designed to explain the present acceleration of the Universe without invoking an *ad hoc* dark energy contain an effective time-dependent cosmological “constant” and black hole solutions are not asymptotically flat in these theories, but asymptotically Friedmann-Lemaître-Robertson-Walker [4].

The spherically symmetric metric considered in Ref. [17] satisfies the condition $g_{00}g_{11} = -1$. It is, therefore, possible to express the Tejada-Rosswog potential (14) using the Misner-Sharp-Hernandez mass. The result is

$$\Phi_{\text{TR}} = -\frac{M_{\text{MSH}}}{R}, \quad (19)$$

which shows again the crucial difference between Φ and Φ_{TR} at the horizon $R = 2M_{\text{MSH}}(R)$.

The Misner-Sharp-Hernandez mass [25] (or, more in general, the Hawking-Hayward mass [26]) is only defined in general relativity and in Lovelock gravity [32]; therefore, Eq. (18) does not apply to other theories of gravity, while Eq. (11) does.

B. Examples in Einstein theory

We have already discussed how Eq. (11) reproduces the original Paczynski-Wiita potential for the Schwarzschild black hole. Let us consider now other examples in Einstein theory.

1. Schwarzschild–de Sitter–Kottler black hole

A second example is given by the Schwarzschild–de Sitter–Kottler geometry, which can be written in static coordinates as

$$ds^2 = -\left(1 - \frac{2m}{R} - H^2 R^2\right) dt^2 + \frac{dR^2}{1 - \frac{2m}{R} - H^2 R^2} + R^2 d\Omega_{(2)}^2, \quad (20)$$

where $H_0 = \sqrt{\Lambda/3}$ is the constant Hubble parameter and $\Lambda > 0$ is the cosmological constant. The locally static metric is already in the form (1) with $g_{00}g_{11} = -1$ in the region between the cosmological and black hole horizons, and the corresponding pseudo-Newtonian potential is

$$\Phi_{\text{Sds}}(R) = -\frac{\left(\frac{m}{R} + \frac{H^2 R^2}{2}\right)}{1 - \frac{2m}{R} - H^2 R^2}, \quad (21)$$

which coincides with the pseudo-Newtonian potential for Schwarzschild–de Sitter found in Refs. [33,34]. This example shows how asymptotic flatness is not a requirement for the pseudo-Newtonian description (11). This fact has some importance because, as noted above, in theories of modified gravity designed to explain the present acceleration of the Universe without dark energy, black holes are asymptotically Friedmann-Lemaître-Robertson-Walker. Currently, there is much theoretical effort devoted to predicting the effects of scalar (and other) hair around black holes in modified gravity [11], which makes them nonisolated.

2. Kiselev black hole

The Kiselev solution of the Einstein equations [35] describes a static spherical black hole surrounded by n noninteracting quintessence fluids. It can be used as a toy model to study the qualitative modifications induced on a black hole by a dark energy environment. The line element is [35]

$$ds^2 = -\left[1 - \frac{2m}{R} - \sum_{j=1}^n \left(\frac{R_j}{R}\right)^{3w_j+1}\right] dt^2 + \frac{dR^2}{1 - \frac{2m}{R} - \sum_{j=1}^n \left(\frac{R_j}{R}\right)^{3w_j+1}} + R^2 d\Omega_{(2)}^2, \quad (22)$$

where the equation of state parameters of the n fluids satisfy $-1 < w_j < -1/3$ ($j = 1, \dots, n$). This solution has a black hole horizon at the roots of the equation $g^{RR} = 0$ (when these exist).

Using Eq. (11), the pseudo-Newtonian potential for the Kiselev black hole is found to be

$$\Phi_{\text{Kiselev}} = -\frac{2m + \sum_{j=1}^n \left(\frac{R_j}{R}\right)^{3w_j+1}}{2\left[R - 2m - \sum_{j=1}^n \left(\frac{R_j}{R}\right)^{3w_j+1}\right]}, \quad (23)$$

or, for a single fluid,

$$\Phi_{\text{Kiselev}} = -\frac{\left[m + \frac{R}{2} \left(\frac{R_0}{R}\right)^{3w+1}\right]}{R - 2m - R \left(\frac{R_0}{R}\right)^{3w+1}}. \quad (24)$$

Reference [14] derives the pseudo-Newtonian potential for the special case of a single fluid with $w = -2/3$ (although

the generalization to any value of w in the interval $(-1, -1/3)$ is straightforward), and the result coincides with Eq. (24) specialized to this single fluid. The Schwarzschild–de Sitter–Kottler solution (20) and the corresponding pseudo-Newtonian potential (21) are recovered in the parameter limit $w \rightarrow -1$ for a single fluid.

The determination of marginally stable orbits (innermost stable circular orbit and outermost circular orbit) for the Kiselev solution is not trivial because it involves solving a quintic equation for the radii of these orbits [14,36]. The Sturm theorem was applied to this problem, determining the condition for the existence of these orbits, in Ref. [36].

IV. PSEUDO-NEWTONIAN POTENTIAL IN MODIFIED GRAVITIES

Currently, there is significant theoretical research devoted to probing and testing gravity using black holes in view of the Event Horizon Telescope, aiming at resolving the surroundings of black holes up to 1.5 Schwarzschild radii (e.g., [12]). Therefore, it is appropriate to explore solutions of alternative theories of gravity to obtain some insight on the qualitative deviations of black hole properties from those of general relativity. Physical observables are the size of the accretion disks around black holes, which are characterized by the innermost and the outermost stable orbits and the frequency of the radiation emitted near the inner edge of the accretion disk.

A. Modified Schwarzschild black holes in quadratic gravity

A formalism incorporating small deviations from a Schwarzschild black hole in a wide class of quadratic theories of gravity was proposed in Ref. [37]. The action includes dynamical Chern-Simons gravity as a special case and is

$$S = \int d^4x \sqrt{-g} \left[\frac{\mathcal{R}}{16\pi G} + \alpha_1 f_1(\phi) \mathcal{R}^2 + \alpha_2 f_2(\phi) R_{ab} R^{ab} + \alpha_3 f_3(\phi) R_{abcd} R^{abcd} + \alpha_4 f_4(\phi) R_{abcd} {}^* R^{abcd} - \frac{\beta}{2} \nabla^c \phi \nabla_c \phi + \mathcal{L}^{(m)} \right], \quad (25)$$

where \mathcal{R} is the Ricci curvature of the metric g_{ab} with determinant g , R_{ab} , R_{abcd} , and ${}^*R_{abcd}$ are the Ricci and Riemann tensor and the dual of the latter, respectively, ϕ is a scalar field, α_i and β are coupling constants, and $f_i(\phi)$ ($i = 1, \dots, 4$) are coupling functions which can be Taylor-expanded around $\phi = 0$. The spherically symmetric, static, and asymptotically flat geometry describing deviations from the Schwarzschild metric is found to be [37]

$$ds^2 = - \left(1 - \frac{2M}{R} \right) (1 + h(R)) dt^2 + \frac{1 + k(R)}{1 - \frac{2M}{R}} dR^2 + R^2 d\Omega_{(2)}^2, \quad (26)$$

where

$$h(R) = \frac{\zeta}{3(1 - 2M/R)} \left(\frac{M}{R} \right)^3 \tilde{h}(R), \quad (27)$$

$$k(R) = \frac{-\zeta}{1 - 2M/R} \left(\frac{M}{R} \right)^2 \tilde{k}(R), \quad (28)$$

$$\zeta = \frac{16\pi\alpha_3^2}{\beta M_0^4}, \quad (29)$$

and where M_0 is a bare mass related to the physical mass M by

$$M = M_0 \left(1 + \frac{49\zeta}{80} \right), \quad (30)$$

which leads to the implicit equation for the dimensionless function $\zeta(M)$ [37]

$$\zeta \left(1 + \frac{49\zeta}{80} \right)^4 = \frac{16\pi\alpha_3^2}{\beta M^4}. \quad (31)$$

The functions $\tilde{h}(R)$ and $\tilde{k}(R)$ are given by [37]

$$\tilde{h}(R) = 1 + \frac{26M}{R} + \frac{66M^2}{5R^2} + \frac{96M^3}{5R^3} - \frac{80M^4}{R^4}, \quad (32)$$

$$\tilde{k}(R) = 1 + \frac{M}{R} + \frac{52M^2}{3R^2} + \frac{2M^3}{R^3} + \frac{16M^4}{5R^4} - \frac{368M^5}{3R^5}. \quad (33)$$

The solution (26) is regular everywhere except at $R = 0$, where it exhibits the usual black hole spacetime singularity. Equation (11) then yields the pseudo-Newtonian potential in the wide class of quadratic theories of gravity (25):

$$\Phi(R) = \frac{-M}{R} \frac{1 - \frac{\zeta}{6(1-2M/R)} \left(\frac{M}{R} \right)^2 (1 - 2M/R) \tilde{h}(R)}{(1 - 2M/R) \left[1 + \frac{3}{(1-2M/R)} \left(\frac{M}{R} \right)^3 \tilde{h}(R) \right]}. \quad (34)$$

The radius of the innermost stable circular orbit corresponding to $\Phi'(R) = 0$ is computed in [37] as

$$R_{\text{ISCO}} = \left(6 - \frac{16297\zeta}{9720} \right) M. \quad (35)$$

In this case, the spacetime metric does not satisfy the condition $g_{00}g_{11} = -1$ and, in the weak-field limit, there are two post-Newtonian potentials since the post-Newtonian line element has the form

$$ds^2 = -(1 - 2\Phi) dt^2 + (1 + 2\Psi) dR^2 + R^2 d\Omega_{(2)}^2. \quad (36)$$

This feature is well known in alternative theories of gravity, also for cosmological perturbations. The fact that the

pseudo-Newtonian potential apparently depends only on g_{00} (and not on g_{11}) seems to be a limitation of attempts to probe alternative theories of gravity near black hole horizons which use such pseudopotentials. However, this is not entirely true since v^2 depends also on g_{11} , as shown by Eq. (4). This aspect is discussed in the next subsection.

B. Epicyclic frequency

Consider now orbits in the equatorial plane $z = 0$. Switching from spherical to cylindrical coordinates (r, φ, z) , the effective potential and the pseudopotential (11) are cylindrically symmetric, $U = U(r, z)$ and $\Phi = \Phi(r, z)$. Consider an equatorial circular orbit at an extremum of the effective potential U , given by $(r_0, \varphi_0 + \Omega t, 0)$ (where φ_0 is an azimuthal initial condition) and constant angular momentum $L_{(0)}$. The Keplerian frequency $\Omega = \dot{\varphi}$ of this orbit satisfies (e.g., [38])

$$\Omega^2 = \frac{1}{r} \frac{\partial \Phi}{\partial r} \Big|_{(r_0, 0)} = \frac{L_{(0)}}{r_0^2}. \quad (37)$$

In the case of the potential (11), we have

$$\Omega^2 = \frac{1}{2r} \frac{\partial g^{00}}{\partial r} \Big|_{(r_0, 0)}. \quad (38)$$

Now perturb this circular orbit so that

$$r(t) = r_0 + \delta r(t), \quad (39)$$

$$\varphi(t) = \varphi_0 + \Omega t + \delta \varphi(t), \quad (40)$$

$$z(t) = \delta z(t), \quad (41)$$

and the angular momentum per unit mass is $L = L_{(0)} + \delta L$. In the approximation of small perturbations, the horizontal and vertical epicyclic frequencies κ and ν are given by

$$\begin{aligned} \kappa^2 &= \left[\frac{\partial^2 \Phi}{\partial r^2} + \frac{3L_{(0)}^2}{r^4} \right]_{(r_0, 0)} \\ &= \frac{2\Omega}{r} \frac{\partial}{\partial r} (\Omega^2 r) \Big|_{(r_0, 0)} = 4\Omega^2 + r \frac{\partial(\Omega^2)}{\partial r} \Big|_{(r_0, 0)}, \end{aligned} \quad (42)$$

$$\nu^2 = \frac{\partial^2 \Phi}{\partial z^2} \Big|_{(r_0, 0)}, \quad (43)$$

respectively [38]. In our case, these formulas give

$$\kappa^2 = \frac{1}{2} \frac{\partial^2 g^{00}}{\partial r^2} \Big|_{(r_0, 0)}, \quad (44)$$

$$\nu^2 = \frac{1}{2} \frac{\partial^2 g^{00}}{\partial z^2} \Big|_{(r_0, 0)}. \quad (45)$$

In the case of a stable circular orbit of radius r_0 , a test particle will oscillate with $\delta r = \delta x^1 = \delta_0 \cos(\kappa t)$.

As for the radius of the innermost stable circular orbit, it seems that the horizontal and vertical epicyclic frequencies depend only on g_{00} and not on other metric components. However, the quantity v appearing in the conservation equation $v^2/2 + U(R) = E/m$ of the previous section is related with g_{RR} by Eq. (4). This relation can be interpreted as saying that, if *proper* (instead of coordinate) radii and vertical distances are to be used, then one must replace the intervals dr and dz with $dr_{\text{prop}} = \sqrt{g_{rr}} dr$ and $dz_{\text{prop}} = \sqrt{g_{zz}} dz$. This replacement is consistent with the definition of the radial velocity $v^2 \equiv g_{11}(u^1)^2$ in [15]. Then the expressions of the Keplerian and epicyclic frequencies Ω , κ and ν should be replaced by $\Omega/\sqrt{g_{rr}}$, $\kappa/\sqrt{g_{rr}}$, and $\nu/\sqrt{g_{zz}}$, making these physical quantities dependent on metric components other than g_{00} . Then, testing particle orbits would mean doing more than merely testing gravitational shifts. Remember, however, that the Paczynski-Wiita potential does not reproduce the epicyclic frequencies accurately even in the Schwarzschild spacetime [15].

Let us discuss now the condition (4) in cylindrical coordinates in the equatorial plane. Consider the coordinate transformation

$$\{x^\mu\} = (t, r, \varphi, z) \rightarrow \{x^{\mu'}\} = (t, R, \theta, \varphi) \quad (46)$$

with

$$R = \sqrt{r^2 + z^2}, \quad (47)$$

$$\varphi = \varphi, \quad (48)$$

$$\theta = \tan^{-1} \left(\frac{r}{z} \right), \quad (49)$$

and inverse

$$r = R \sin \theta, \quad (50)$$

$$\varphi = \varphi, \quad (51)$$

$$z = R \cos \theta. \quad (52)$$

Equation (4) states that $v^2 = g_{RR}(u^R)^2$, where $u^R \equiv dR/d\tau$ and τ is the proper time along a timelike geodesic. Using the transformation property of the metric tensor,

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}, \quad (53)$$

one obtains

$$\begin{aligned}
g_{RR} &= \frac{\partial x^\mu}{\partial R} \frac{\partial x^\nu}{\partial R} g_{\mu\nu} \\
&= g_{rr} \left(\frac{\partial r}{\partial R} \right)^2 + g_{zz} \left(\frac{\partial z}{\partial R} \right)^2 + 2g_{rz} \frac{\partial z}{\partial R} \frac{\partial r}{\partial R} \\
&= \sin^2\theta g_{rr} + \cos^2\theta g_{zz} + \sin(2\theta)g_{rz}. \tag{54}
\end{aligned}$$

On the equatorial plane $\theta = \pi/2$, it is $g_{RR} = g_{rr}$. We now have

$$u^R \equiv \frac{dR}{d\tau} = \frac{1}{R}(ru^r + zu^z), \tag{55}$$

using Eq. (47) and, on the equatorial plane, $u^R = u^r$. Therefore, it is also

$$v^2 = g_{RR}(u^R)^2 = g_{rr}(u^r)^2 \tag{56}$$

on the $z = 0$ plane. The radial velocity near an equatorial circular orbit then obeys

$$\delta\dot{r} \equiv \frac{d\delta r}{dt} = \frac{dr}{d\tau} \frac{d\tau}{dt} = \frac{\delta u^r}{u^0} = \frac{v}{\sqrt{g_{rr}u^0}} \approx \frac{v}{\sqrt{g_{rr}}} \tag{57}$$

to first order. Since v depends on g_{rr} , which is in general distinct from g_{00} in alternative theories of gravity, there is hope for tests of black hole metrics which solve these theories.

V. CONCLUSIONS

Given the many reasons to study theories of gravity alternative to general relativity and, as a consequence, black holes in these theories, and given that the future Event Horizon Telescope promises to image the black hole SgrA* at the center of our Galaxy with a resolution of 1.5 Schwarzschild radii [12], it is of interest to study the dynamics of particles around black holes in general spacetimes, not only in those (Schwarzschild or Kerr) which solve the Einstein equations of general relativity. A simplifying tool widely used in numerical simulations of accretion disks around black holes is the pseudo-Newtonian potential. Restricting, for simplicity, to static and spherically

symmetric black hole metrics, we have shown how a pseudo-Newtonian potential can be derived for any such spacetime, following step by step the pedagogical derivation of the Paczynski-Wiita potential [13] given in Ref. [15]. The generalization of the Paczynski-Wiita potential to any static spherically symmetric metric (not necessarily representing a black hole³) is straightforward, but nevertheless it comes with a surprise. A previous generalization of this potential, restricted to the subclass of static spherical symmetric metrics satisfying the condition $g_{00}g_{11} = -1$ [17], produces a different pseudo-Newtonian potential, although essentially the same equations (i.e., the equations for timelike geodesics) were considered in [17]. There is no obvious compelling reason to prefer one of the two potentials over the other. Therefore, this discrepancy means that even the pseudo-Newtonian potential associated with a simple black hole spacetime is not unique. There are now many pseudo-Newtonian potential functions in the literature, for both spherical and cylindrically symmetric (rotating) black holes. Since these potentials are just effective quantities, mere tricks used to simplify complicated equations, one should not regard them as fundamental quantities and attribute to them more importance than they deserve. Nevertheless, these pseudopotentials are used in practical calculations in astrophysics and it would be worth understanding them fully together with their limitations. An interesting aspect is the expression of the (generalized) Paczynski-Wiita potential in terms of the Hawking quasilocal mass (which in spherical symmetry reduces to the Misner-Sharp-Hernandez mass). It would be interesting to relate this potential, and other pseudo-Newtonian potentials, also to the other known quasilocal energy constructs if possible.

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³Reference [20] indeed derives a pseudo-Newtonian potential for naked singularities.

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