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Residual symmetries of the gravitational field

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We develop a geometric criterion that unambiguously characterizes the residual symmetries of a gravitational *Ansatz*. It also provides a systematic and effective computational procedure for finding all the residual symmetries of any gravitational *Ansatz*. We apply the criterion to several examples starting with the Collinson *Ansatz* for circular stationary axisymmetric spacetimes. We reproduce the residual symmetries already known for this *Ansatz* including their conformal symmetry, for which we identify the corresponding infinite generators spanning the two related copies of the Witt algebra. We also consider the noncircular generalization of this *Ansatz* and show how the noncircular contributions on the one hand break the conformal invariance and on the other hand enhance the standard translation symmetries of the circular Killing vectors to supertranslations depending on the direction along which the circularity is lost. As another application of the method, the well-known relation defining conjugate gravitational potentials introduced by Chandrasekhar, which makes possible the derivation of the Kerr black hole from a trivial solution of the Ernst equations, is deduced as a special point of the general residual symmetry of the Papapetrou *Ansatz*. In this derivation we emphasize how the election of Weyl coordinates, which determines the Papapetrou *Ansatz*, breaks also the conformal freedom of the stationary axisymmetric spacetimes. Additionally, we study AdS waves for any dimension generalizing the residual symmetries already known for lower dimensions and exhibiting a very complex infinite-dimensional Lie algebra containing three families: two of them span the semidirect sum of the Witt algebra and scalar supertranslations and the third generates vector supertranslations. Independently of this complexity we manage to comprehend the true meaning of the infinite connected group as the precise diffeomorphisms subgroup allowing to locally deform the AdS background into AdS waves.

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I. INTRODUCTION

A frequent scenario in the now centennial theory of general relativity is that solving Einstein equations can lead to integration constants and even integration functions that turn out to be redundant, i.e. they can be eliminated by appropriate coordinate transformations and are not related to the truly dynamical content of the theory. This is due to the fact that after realizing the original symmetries of the physical configuration under study through a chosen *Ansatz*, a sort of gauge freedom can remain, defining a residual symmetry on the defining variables of the *Ansatz*. Consequently, these eliminations can be expected previous to the integration process by virtue of its kinematical character. In order to illustrate the point, let us consider as an example the Collinson *Ansatz* for a circular stationary axisymmetric spacetime [1–3]

$$\mathbf{d}s^2 = e^{-2Q}\left(-\frac{(\mathbf{d}\tau + a\mathbf{d}\sigma)(\mathbf{d}\tau - b\mathbf{d}\sigma)}{a+b} + e^{-2P}\mathbf{d}z\mathbf{d}\bar{z}\right), \quad (1)$$

where the structural functions a, b, P, and Q only depend on the complex coordinate z and its complex conjugate $\bar{z}$. If one performs the following diffeomorphism,

$$(\tau,\sigma,z) \mapsto (\tilde{\tau} = \tau + \varepsilon\sigma, \tilde{\sigma} = \sigma, \tilde{z} = z), \quad (2)$$

it can be compensated for by the next redefinition of the structural functions,

$$(a,b,P,Q) \mapsto (\tilde{a} = a - \varepsilon, \tilde{b} = b + \varepsilon, \tilde{P} = P, \tilde{Q} = Q), \quad (3)$$

which means that in terms of the new variables the metric is form invariant

$$\mathbf{d}s^2 = e^{-2\tilde{Q}}\left(-\frac{(\mathbf{d}\tilde{\tau} + \tilde{a}\mathbf{d}\tilde{\sigma})(\mathbf{d}\tilde{\tau} - \tilde{b}\mathbf{d}\tilde{\sigma})}{\tilde{a}+\tilde{b}} + e^{-2\tilde{P}}\mathbf{d}\tilde{z}\mathbf{d}\bar{\tilde{z}}\right). \quad (4)$$

This fact implies that if a solution for the functions a and b is found such that both contain homogeneous contributions differing only in sign, as the ones exhibited in the redefinitions (3), these contributions can be eliminated with the previous diffeomorphism (2), which is just a simple rotation in the (τ,σ) plane.

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The above is a concrete example of a residual symmetry of the Collinson *Ansatz*, and it is natural to wonder whether there are more such symmetries and obviously how to find them. Unfortunately, these kinds of symmetries are usually discovered by the large use of experience and having an eye for that is crucial. Hence, it is fundamental to understand if there exists a general way to characterize residual symmetries that additionally provides a systematic way to find all of them. The main objective of this work is precisely to propose a criterion which allows us to accomplish these goals for any metric *Ansatz*.

More concretely, suppose you have a metric whose components $g_{\alpha\beta}$ explicitly depend on some of the coordinates x^μ and its remaining dependence implicitly occurs through some *a priori* unknown functions $u^J = u^J(x^\mu)$, giving rise to a concrete metric *Ansatz*

$$\mathbf{d}s^2 = g_{\alpha\beta}(x^\mu, u^J)\mathbf{d}x^\alpha \mathbf{d}x^\beta. \tag{5}$$

We are interested in the following question: what are the continuous transformations that preserve the form of this *Ansatz*? Namely, coordinate transformations

$$x^\alpha \mapsto \tilde{x}^\alpha = \tilde{x}^\alpha(x^\mu, u^J; \varepsilon), \tag{6a}$$

and redefinitions of the *Ansatz* functions

$$u^I \mapsto \tilde{u}^I = \tilde{u}^I(x^\mu, u^J; \varepsilon), \tag{6b}$$

depending on at least one parameter ε, such that they satisfy

$$\begin{aligned}\mathbf{d}s^2 &= g_{\alpha\beta}(x^\mu, u^J)\mathbf{d}x^\alpha \mathbf{d}x^\beta \\ &= g_{\alpha\beta}(\tilde{x}^\mu, \tilde{u}^J)\mathbf{d}\tilde{x}^\alpha \mathbf{d}\tilde{x}^\beta = \mathbf{d}\tilde{s}^2,\end{aligned} \tag{7}$$

where the latter expression means that the dependence of the gravitational potentials $g_{\alpha\beta}$ in terms of the new quantities must be exactly the same than in the original variables. We emphasize that, in order to cover the more general situation, it is convenient to consider mixed transformations where the coordinates x^μ and the unknown functions u^J are handled as independent variables, i.e. the transformations (6) are just point to point maps in a space built of spacetime coordinates together with the *Ansatz* functions. The advantage of dealing with a parametric family lies in the fact that we can assume that (6) form a one-parameter group of point transformations or, equivalently, a Lie-point transformation and apply the machinery of continuous symmetry groups originally introduced by Sophus Lie for the systematic study of differential equations [4–7].

The Lie-point symmetries of vacuum Einstein equations are well understood [8,9], as well as their local Lie-Bäcklund generalizations [10–12]. Even how to implement nonlocal Lie-Bäcklund generalizations in the presence of Killing vectors is known, see [13] and references therein. However, for the case of configurations enjoying a given symmetry as the ones in which we are interested, also called group invariant solutions, no systematic study is known about their residual symmetries. An exception is Ref. [14] where the Lie symmetry reduction method is reexamined in the context of general relativity in comparison with the standard derivations of exact solution in this field. Residual symmetries play an important role in this construction, they were termed the *automorphism group* in this reference, where the expression *residual group* was reserved for the symmetries of the reduced field equations. Another exception is Ref. [15] which focuses the study of residual symmetries to the case of homogeneous spaces.

The work is organized as follows. In Sec. II we establish the theoretical framework used in order to derive the criterion to study the residual symmetries of a generic metric *Ansatz*. With the criterion at hand, in Sec. III we return to the subject of the residual symmetries of the Collinson *Ansatz* (1). After solving the restrictions imposed by the criterion we consistently found the whole class already exhibited by Collinson himself [1]. In Sec. IV we treat the well-known case of spherically symmetric spacetimes in its static version as well as the time-dependent one and we emphasize its similarities and differences. We investigate the so-called anti–de Sitter (AdS) waves for any dimension in Sec. V, we generalize the residual symmetries of the lower-dimensional cases presented in Refs. [16,17]. The renowned Papapetrou *Ansatz* [18] is studied in Sec. VI. The knowledge of its more general residual symmetry transformation allows us to find as a particular limiting case the conjugate transformation between the Ernst potentials introduced by Chandrasekhar [19]. We remark that this conjugate transformation is the one allowing us to relate a trivial but unphysical solution of the Ernst equations [20] to the celebrated Kerr black hole [21]. Finally, we explore the noncircular generalization of the Collinson *Ansatz* [3] in Sec. VII. In this case, the noncircular contributions change dramatically the nature of the residual symmetries of the *Ansatz* if they are compared to those of the original Collinson metric. The detailed calculations showing how to solve the proposed criterion for each case are included in separate appendixes in order to emphasize the simplicity of the method.

II. THE CRITERION

In this section we will provide a geometrical characterization of residual symmetries. The starting point is to think of coordinates and structural functions *à la* Lie, i.e. all handled as independent variables in an abstract space (we return to this point later). After that, we consider the infinitesimal version of the one-parameter Lie-point transformations defining the residual symmetries (6)

$$\tilde{x}^\alpha = x^\alpha + \varepsilon\xi^\alpha(x^\mu, u^J) + \cdots, \tag{8a}$$

$$\tilde{u}^I = u^I + \varepsilon\eta^I(x^\mu, u^J) + \cdots, \tag{8b}$$

which give rise to the generator

$$X = \xi^\alpha(x^\mu, u^J)\boldsymbol{\partial}_\alpha + \eta^I(x^\mu, u^J)\boldsymbol{\partial}_I, \tag{9}$$

with components defined as usual

$$\xi^\alpha(x^\mu, u^I) \equiv \frac{\partial \tilde{x}^\alpha}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \tag{10a}$$

$$\eta^I(x^\mu, u^J) \equiv \frac{\partial \tilde{u}^I}{\partial \varepsilon}\bigg|_{\varepsilon=0}. \tag{10b}$$

For example, it is easy to see that the residual symmetry of the Collinson *Ansatz* defined by the transformations (2) and (3) coincides with their infinitesimal version, giving as generator

$$X = \sigma\boldsymbol{\partial}_\tau - \boldsymbol{\partial}_a + \boldsymbol{\partial}_b. \tag{11}$$

We are now ready to explore the infinitesimal consequences of the form-invariant condition (7)

$$\begin{aligned}\mathbf{d}\tilde{s}^2 &= g_{\alpha\beta}(x^\mu + \varepsilon\xi^\mu + \cdots, u^J + \varepsilon\eta^J + \cdots)\\ &\quad \times \mathbf{d}(x^\alpha + \varepsilon\xi^\alpha + \cdots)\mathbf{d}(x^\beta + \varepsilon\xi^\beta + \cdots)\\ &= \mathbf{d}s^2 + \varepsilon[(\xi^\mu\partial_\mu g_{\alpha\beta} + 2g_{\mu\alpha}\partial_\beta\xi^\mu + \eta^I\partial_I g_{\alpha\beta})\mathbf{d}x^\alpha\mathbf{d}x^\beta\\ &\quad + 2g_{\alpha\beta}\partial_I\xi^\beta\mathbf{d}x^\alpha\mathbf{d}u^I] + \cdots,\end{aligned} \tag{12}$$

i.e. keeping the form of the metric requires necessarily the following conditions on the infinitesimal generators

$$\xi^\mu\partial_\mu g_{\alpha\beta} + 2g_{\mu(\alpha}\partial_{\beta)}\xi^\mu = -\eta^I\partial_I g_{\alpha\beta}, \tag{13a}$$

$$\partial_I\xi^\beta = 0. \tag{13b}$$

These conditions are interpreted as follows. The condition (13b) establishes that the components of the generator along spacetime depend only on the spacetime coordinates, i.e. these components represent genuine one-parameter diffeomorphisms of spacetime generated by the vector field $\boldsymbol{\xi} = \xi^\mu(x^\alpha)\boldsymbol{\partial}_\mu$. At the same time, a one-parameter diffeomorphism changes the metric by the corresponding Lie derivative, $\pounds_{\boldsymbol{\xi}} g_{\mu\nu}$, which is just what appears in the left-hand side of condition (13a). Hence, the fact that the right-hand side of condition (13a) is minus the directional derivative of the metric along the components of the generator associated with the structural functions, $\boldsymbol{\eta}(g_{\mu\nu}) = \eta^I\boldsymbol{\partial}_I g_{\mu\nu}$, only indicates that the above spacetime diffeomorphisms are compensated for by appropriate redefinitions of the structural functions: $\pounds_{\boldsymbol{\xi}} g_{\mu\nu} = -\boldsymbol{\eta}(g_{\mu\nu})$. These conditions are entirely compatible with the given definition of residual symmetry and are obtained by an elementary infinitesimal reasoning.

For example, it is easy to check that the generator (11) indeed satisfy the conditions (13) once they are evaluated in the Collinson *Ansatz*. Regarding the question posed in the Introduction on whether there are more residual symmetries, we emphasize that for any given fixed gravitational background the conditions (13) are an overdetermined system of linear partial differential equations for the components of the generator as functions of the coordinates and the structural functions; the general solution for such systems can always be found [5].

Before embarking on the expedition to explore the consequence of the obtained conditions on concrete gravitational backgrounds, we would like to highlight that the Lie-point transformation (6) describing residual symmetries, its infinitesimal version (8) with generator (9), and the resulting conditions (13) all have a more geometrical and intuitive interpretation if we consider an abstract space with coordinates

$$z^A = (x^\mu, u^I). \tag{14}$$

In applications to differential equations the use of a prolonged version of this space is common which includes also the derivatives of structural functions as coordinates, $z^A = (x^\mu, u^I, u^I_{\mu_1}, u^I_{\mu_1\mu_2}, \ldots, u^I_{\mu_1\ldots\mu_n})$, and it is formally known in mathematics as the nth *jet space* [5], where $u^I_{\mu_1\ldots\mu_n}$ denotes the nth derivative of the structural function u^I with respect to spacetime coordinates x^μ. In our case, the metric does not depend on derivatives, therefore, it is sufficient to consider the 0th jet space or simply jet space.[1] The residual symmetries (6) are now standard diffeomorphisms in jet space, $z^A \mapsto \tilde{z}^A = \tilde{z}^A(z^B;\varepsilon)$, having as infinitesimal transformations

$$\tilde{z}^A = z^A + \varepsilon X^A(z^B) + \cdots, \tag{15}$$

whose generator is a standard vector field in jet space,

$$X = X^A(z^B)\boldsymbol{\partial}_A, \qquad X^A(z^B) \equiv \frac{\partial \tilde{z}^A}{\partial \varepsilon}\bigg|_{\varepsilon=0}. \tag{16}$$

The spacetime metric (5) can be considered as an object in jet space,

$$\mathbf{d}s^2 = g_{AB}(z^C)\mathbf{d}z^A\mathbf{d}z^B = g_{\alpha\beta}(x^\mu, u^J)\mathbf{d}x^\alpha\mathbf{d}x^\beta, \tag{17}$$

with the trivial choices $g_{\mu I} = 0 = g_{IJ}$ for all μ, and I, J. Since the transformations are now standard one-parameter diffeomorphisms of jet space, the residual symmetries are only standard isometries of the jet space metric (17), i.e. the obvious criterion to find residual symmetries in this geometrical approach is that the generators are just jet space Killing fields,

[1] There are some special examples of metric *Ansatz* which also depend on the derivatives of the structural functions. They deserve separate attention [22].

$$\pounds_X g_{AB} = X^C \partial_C g_{AB} + 2g_{C(A}\partial_{B)}X^C = 0. \qquad (18)$$

Using the fact that the jet space metric (17) has only nontrivial components along spacetime directions we exactly recover the conditions (13) already obtained by an elementary infinitesimal reasoning.

The necessity of conditions (13) is beyond doubt, indeed, it is reinforced by the fact that the previous alternative geometrical viewpoint gives exactly the same result. However, we are obliged to ask whether they are sufficient to characterize the residual symmetries. Fortunately, the geometrical picture sheds some light on how to answer this question. What we have learned so far is that residual symmetries are isometries of the trivial lifting of the metric to jet space. It is very common in many *Ansätze* that some of their structural functions do not depend on all coordinates. Hence, there is a subset of the involved jet space isometries for which the following anomalous scenario is possible: we can compensate for some spacetime diffeomorphisms with redefinitions which keep invariant the form of the metric only formally, because they fail to respect the original dependencies of the structural functions on spacetime coordinates. Consequently, these are not residual symmetries in a strict sense. In order to illustrate this situation we use again the Collinson *Ansatz* (1), where all structural functions are independent of τ and σ. Now, consider generalizing the rotation (2) in the (τ,σ) plane to any other one-parameter reparametrization on the same plane acting exclusively in the time coordinate

$$(\tau,\sigma,z) \mapsto (\tilde\tau = \tilde\tau(\tau,\sigma;\varepsilon), \tilde\sigma = \sigma, \tilde z = z). \qquad (19)$$

If one compensates for the previous diffeomorphism with the following redefinitions,

$$(a,b,P,Q) \mapsto (\tilde a = a\partial_\tau\tilde\tau - \partial_\sigma\tilde\tau, \qquad \tilde b = b\partial_\tau\tilde\tau + \partial_\sigma\tilde\tau, \qquad \tilde P = P - \frac{1}{2}\ln\partial_\tau\tilde\tau, \qquad \tilde Q = Q + \frac{1}{2}\ln\partial_\tau\tilde\tau), \qquad (20)$$

it is easy to verify that the form of the metric remains unchanged as in (4), but the present transformed line element does not represent anymore a manifestly stationary axisymmetric *Ansatz* since the new structural functions depend now on the new coordinates $\tilde\tau$ and $\tilde\sigma$. However, the generator associated with the previous transformation, which according to definitions (10) is

$$\boldsymbol{X} = \xi^\tau(\tau,\sigma)\boldsymbol{\partial_\tau} + (a\partial_\tau\xi^\tau - \partial_\sigma\xi^\tau)\boldsymbol{\partial_a} + (b\partial_\tau\xi^\tau + \partial_\sigma\xi^\tau)\boldsymbol{\partial_b} - \frac{1}{2}\partial_\tau\xi^\tau\boldsymbol{\partial_P} + \frac{1}{2}\partial_\tau\xi^\tau\boldsymbol{\partial_Q}, \qquad (21)$$

perfectly satisfies the vanishing Lie-derivative condition (18), or its dimensional reduction (13), evaluated on the Collinson *Ansatz* (1). Unfortunately, this is not the only example of formal form invariance; an exclusive reparametrization of the angular coordinate σ leads to similar results. The geometrical approach is elegant and concise, but it is blind to the above problem if one does not provide additional information. In summary, the conditions (13) are not sufficient to characterize strict form invariance.

In order to prevent situations as those described above, we need to supplement the Lie-derivative conditions with others including the relevant information on concrete dependencies. This is done by knowing how Lie-point transformations (6) are prolonged to act on the derivatives $u^I_\alpha \equiv \partial u^I/\partial x^\alpha$. In particular at the infinitesimal level (8) we have

$$\tilde u^I_\alpha \equiv \frac{\partial\tilde u^I}{\partial\tilde x^\alpha} = \frac{\mathrm{d}u^I + \varepsilon\mathrm{d}\eta^I + \cdots}{\mathrm{d}x^\alpha + \varepsilon\mathrm{d}\xi^\alpha + \cdots} = u^I_\alpha + \varepsilon\eta^I_\alpha + \cdots, \qquad (22)$$

where the prolongations are given in terms of the components of the original generator as [4]

$$\eta^I_\alpha \equiv \left.\frac{\partial\tilde u^I_\alpha}{\partial\varepsilon}\right|_{\varepsilon=0} = \partial_\alpha\eta^I + u^J_\alpha\partial_J\eta^I - u^I_\beta(\partial_\alpha\xi^\beta + u^J_\alpha\partial_J\xi^\beta). \qquad (23)$$

It is easy to see from here that even when the derivative of a given structural function, let us say $u^{\bar I}$, with respect to a particular coordinate, e.g. $x^{\hat\alpha}$, vanishes, its transformed counterpart is not necessarily zero since it receives many other contributions. Suppose we classify the coordinates and structural functions according to the following: $x^\alpha = (x^{\bar\alpha}, x^{\hat\alpha})$ and $u^I = (u^{\bar I}, u^{\hat I})$, where $u^{\bar I}$ denotes a subset of structural functions which depend exclusively on the subset of coordinates $x^{\bar\alpha}$, i.e. $u^{\bar I} = u^{\bar I}(x^{\bar\alpha})$. The rest of the functions and coordinates are denoted by $u^{\hat I}$ and $x^{\hat\alpha}$, respectively, where in general $u^{\hat I} = u^{\hat I}(x^{\bar\alpha}, x^{\hat\alpha})$. Then, if for the starting *Ansatz* $u^{\bar I}_{\hat\alpha} = 0$, in order to avoid the described problem we need to keep the same condition after the transformation, i.e. $\tilde u^{\bar I}_{\hat\alpha} = 0$, which according to (22) is infinitesimally equivalent to demanding

$$\eta^{\bar I}_{\hat\alpha} = \partial_{\hat\alpha}\eta^{\bar I} + u^{\hat J}_{\hat\alpha}\partial_{\hat J}\eta^{\bar I} - u^{\bar I}_{\bar\beta}\partial_{\hat\alpha}\xi^{\bar\beta} = 0, \qquad (24)$$

where we have made use of conditions (13b). Since this condition must be satisfied for any values of the independent derivatives $u^{\hat J}_{\hat\alpha}$ and $u^{\bar I}_{\bar\beta}$, their companion coefficients must vanish independently, which defines the definitive supplemental conditions

$$\partial_{\hat\alpha}\eta^{\bar I} = \partial_{\hat J}\eta^{\bar I} = \partial_{\hat\alpha}\xi^{\bar\beta} = 0. \qquad (25)$$

These conditions establish that if a structural function $u^{\bar I}$ is independent of some coordinates $x^{\hat\alpha}$, then their corresponding generator component $\eta^{\bar I}$ is also independent of such coordinates as well as of those structural functions that do depend on $x^{\hat{\alpha}}$, and additionally that the components $\xi^{\bar{\beta}}$ along the other coordinates also are independent of $x^{\hat{\alpha}}$.

Summarizing, the full criterion defining the residual symmetries of a gravitational *Ansatz* is given by the next set of equations

$$\xi^\alpha \partial_\alpha g_{\mu\nu} + 2g_{\alpha(\mu}\partial_{\nu)}\xi^\alpha + \eta^I \partial_I g_{\mu\nu} = 0, \tag{26a}$$

$$\partial_I \xi^\alpha = 0, \tag{26b}$$

$$\partial_{\hat{\alpha}}\eta^{\bar{I}} = \partial_{\hat{J}}\eta^{\bar{I}} = \partial_{\hat{\alpha}}\xi^{\bar{\beta}} = 0. \tag{26c}$$

As we will show in detail in the following section and in the appendixes this criterion provides an effective computational procedure for finding all the residual symmetries of almost any gravitational *Ansatz* of interest. In solving this linear system we will find integration constants or functions that parametrize the components of the generator (9). It is an easy task to show that the commutator of two residual symmetry generators also satisfies the above conditions. Hence, the constants or functions appearing in the integration process just span finite- or infinite-dimensional Lie algebras. In other words the solution of the proposed system will be a linear superposition of independent infinitesimal generators of residual symmetries of the metric. As a byproduct, we consequently find the Killing vectors of the corresponding metric as part of the solution; it is done in an easier way than solving the standard Killing equations, where the *Ansatz* functions are considered as dependent variables. Once we know such independent infinitesimal generators we promote them to their corresponding finite one-parameter transformations. For integrating the generators we use the method of differential invariants described in Appendix A. The composition of all the one-parameter transformations gives rise to the most general connected residual symmetry group for any gravitational *Ansatz*. In the following sections we examine the consequences of applying the proposed criterion to several metric *Ansätze*, finding their corresponding residual symmetry algebras and groups. We start with the illustrative example used so far: the Collinson *Ansatz*.

III. THE COLLINSON *ANSATZ*

We now reconsider the Collinson *Ansatz*, whose line element was given in the Introduction. The *Ansatz* (1) was relevant at the beginning of the search of interior solutions for the Kerr exterior spacetime. The so-called Collinson theorem establishes that if a stationary axisymmetric spacetime is also conformally flat then it must be necessarily static [1–3]. Hence, there are no conformally flat Kerr interiors as opposed to the conformally flat interior of the Schwarzschild metric. The virtue of this *Ansatz* is that allows us to write in a simple way the components of the Weyl tensor, producing an almost straightforward integration of the conformally flat conditions.

The jet space parametrization of the Collinson *Ansatz*, $z^A = (x^\mu, u^I)$, consists of the spacetime coordinates $x^\mu = (\tau, \sigma, z, \bar{z})$ together with the structural functions $u^I = (a, b, P, Q)$. Moreover, the structural functions are all independent of the coordinates $x^{\hat{\alpha}} = (\tau, \sigma)$ and using the classification of the previous section $u^{\bar{I}} = u^I$ ($u^{\hat{I}} = \{\emptyset\}$), i.e. all of them only depend on the coordinates $x^{\bar{\alpha}} = (z, \bar{z})$. This implies the complementary residual conditions (26c) become

$$\partial_\tau \eta^I = \partial_\sigma \eta^I = \partial_\tau \xi^z = \partial_\sigma \xi^z = 0. \tag{27}$$

Using also the condition (26b), the general form of the infinitesimal generator of residual symmetries for the Collinson *Ansatz* is the following

$$\begin{aligned} \boldsymbol{X} = {} & \xi^\tau(\tau,\sigma,z,\bar{z})\boldsymbol{\partial_\tau} + \xi^\sigma(\tau,\sigma,z,\bar{z})\boldsymbol{\partial_\sigma} + \xi^z(z,\bar{z})\boldsymbol{\partial_z} \\ & + \bar{\xi}^z(z,\bar{z})\boldsymbol{\partial_{\bar{z}}} + \eta^a(z,\bar{z},a,b,P,Q)\boldsymbol{\partial_a} \\ & + \eta^b(z,\bar{z},a,b,P,Q)\boldsymbol{\partial_b} + \eta^P(z,\bar{z},a,b,P,Q)\boldsymbol{\partial_P} \\ & + \eta^Q(z,\bar{z},a,b,P,Q)\boldsymbol{\partial_Q}, \end{aligned} \tag{28}$$

where all the components are real except ξ^z, and $\bar{\xi}^z$ denotes its complex conjugate. We are now in position to implement the remaining residual conditions (26a), which give the following system of partial differential equations for the components of the previous generator

$$2\eta^Q + \frac{(\eta^a + \eta^b)}{a+b} - 2\partial_\tau\xi^\tau + (b-a)\partial_\tau\xi^\sigma = 0, \tag{29a}$$

$$\begin{aligned} (b-a)\eta^Q + \frac{(b\eta^a - a\eta^b)}{a+b} + \partial_\sigma\xi^\tau - ab\partial_\tau\xi^\sigma \\ - \frac{1}{2}(b-a)(\partial_\tau\xi^\tau + \partial_\sigma\xi^\sigma) = 0, \end{aligned} \tag{29b}$$

$$\begin{aligned} 2ab\eta^Q - \frac{(b^2\eta^a + a^2\eta^b)}{a+b} \\ - (b-a)\partial_\sigma\xi^\tau - 2ab\partial_\sigma\xi^\sigma = 0, \end{aligned} \tag{29c}$$

$$\eta^Q + \eta^P - \frac{1}{2}(\partial_z\xi^z + \partial_{\bar{z}}\bar{\xi}^z) = 0, \tag{29d}$$

$$(b-a)\partial_z\xi^\sigma - 2\partial_z\xi^\tau = 0, \tag{29e}$$

$$(b-a)\partial_z\xi^\tau + 2ab\partial_z\xi^\sigma = 0, \tag{29f}$$

$$\partial_{\bar{z}}\xi^z = 0, \tag{29g}$$

where the complex conjugate of Eqs. (29e)–(29g) must also be included.

In order to integrate the above system we need to remember that in this context the structural functions of the *Ansatz* and the spacetime coordinates are all assumed as independent variables. This turns the integration process into a very easy job; e.g. in Eqs. (29e) and (29f), since the spacetime components of the generator are independent of the structural functions, the only way in which these equations hold is if the coefficients in front of all the functionally independent expressions of the structural functions vanish separately

$$\partial_z \xi^\tau = 0 = \partial_z \xi^\sigma. \tag{30}$$

Since the complex conjugates of these conditions are also valid we conclude that $\xi^\tau = \xi^\tau(\tau, \sigma)$ and $\xi^\sigma = \xi^\sigma(\tau, \sigma)$. Additionally, Eq. (29g) implies the component along the complex plane is holomorphic $\xi^z = \xi^z(z)$. Another consequence of the independency between the structural functions and the coordinates can be obtained when Eqs. (29a)–(29d) are derived with respect to the Killing coordinates τ and σ and the conditions (27) are used; the resulting equations are only satisfied if the following derivatives are constant

$$\begin{aligned} \partial_\tau \xi^\tau &= \hat{C}_1, & \partial_\sigma \xi^\tau &= C_3, \\ \partial_\sigma \xi^\sigma &= \hat{C}_2, & \partial_\tau \xi^\sigma &= C_4. \end{aligned} \tag{31}$$

This system has the trivial solution

$$\xi^\tau = \hat{C}_1 \tau + C_3 \sigma + \kappa_1, \tag{32a}$$

$$\xi^\sigma = \hat{C}_2 \sigma + C_4 \tau + \kappa_2. \tag{32b}$$

If one replaces now the set of conditions (31) or their solutions in Eqs. (29a)–(29d), a linear algebraic system of equations for the η^I is obtained whose solution is given below

$$\eta^a = C_4 a^2 + C_2 a - C_3, \tag{33a}$$

$$\eta^b = -C_4 b^2 + C_2 b + C_3, \tag{33b}$$

$$\eta^P = \frac{1}{2}(\partial_z \xi^z + \partial_{\bar{z}} \bar{\xi}^z - C_1), \tag{33c}$$

$$\eta^Q = \frac{1}{2} C_1, \tag{33d}$$

where we have redefined two of the integration constants as $C_1 = \hat{C}_1 + \hat{C}_2$ and $C_2 = \hat{C}_1 - \hat{C}_2$. This ends the integration of the residual system (29), giving the following generator for the residual symmetries of the Collinson *Ansatz*

$$\begin{aligned} \boldsymbol{X} = {} & C_1(\tau \boldsymbol{\partial_\tau} + \sigma \boldsymbol{\partial_\sigma} + \boldsymbol{\partial_Q} - \boldsymbol{\partial_P}) \\ & + C_2(\tau \boldsymbol{\partial_\tau} - \sigma \boldsymbol{\partial_\sigma} + 2a \boldsymbol{\partial_a} + 2b \boldsymbol{\partial_b}) \\ & + C_3(\sigma \boldsymbol{\partial_\tau} - \boldsymbol{\partial_a} + \boldsymbol{\partial_b}) + C_4(\tau \boldsymbol{\partial_\sigma} + a^2 \boldsymbol{\partial_a} - b^2 \boldsymbol{\partial_b}) \\ & + \xi^z \boldsymbol{\partial_z} + \bar{\xi}^z \boldsymbol{\partial_{\bar{z}}} + \frac{1}{2}(\partial_z \xi^z + \partial_{\bar{z}} \bar{\xi}^z) \boldsymbol{\partial_P} \\ & + \kappa_1 \boldsymbol{\partial_\tau} + \kappa_2 \boldsymbol{\partial_\sigma}. \end{aligned} \tag{34}$$

As we anticipate in the previous section the resulting solution is a linear combination of vector fields, each one generating a one-parameter group of residual symmetries. We label these individual generators as follows

$$\boldsymbol{X}_1 = \tau \boldsymbol{\partial_\tau} + \sigma \boldsymbol{\partial_\sigma} - \boldsymbol{\partial_P} + \boldsymbol{\partial_Q}, \tag{35a}$$

$$\boldsymbol{X}_2 = \tau \boldsymbol{\partial_\tau} - \sigma \boldsymbol{\partial_\sigma} + 2a \boldsymbol{\partial_a} + 2b \boldsymbol{\partial_b}, \tag{35b}$$

$$\boldsymbol{X}_3 = \sigma \boldsymbol{\partial_\tau} - \boldsymbol{\partial_a} + \boldsymbol{\partial_b}, \tag{35c}$$

$$\boldsymbol{X}_4 = \tau \boldsymbol{\partial_\sigma} + a^2 \boldsymbol{\partial_a} - b^2 \boldsymbol{\partial_b}, \tag{35d}$$

$$\hat{\boldsymbol{X}}_{\xi^z} = \xi^z \boldsymbol{\partial_z} + \bar{\xi}^z \boldsymbol{\partial_{\bar{z}}} + \frac{1}{2}(\partial_z \xi^z + \partial_{\bar{z}} \bar{\xi}^z) \boldsymbol{\partial_P}, \tag{35e}$$

$$\boldsymbol{k} = \boldsymbol{\partial_\tau}, \tag{35f}$$

$$\boldsymbol{m} = \boldsymbol{\partial_\sigma}, \tag{35g}$$

where as was pointed out previously, $\xi^z = \xi^z(z)$ is an arbitrary holomorphic function and $\bar{\xi}^z$ is its complex conjugate. The generators (35a)–(35e) are associated with the residual symmetries of the Collinson *Ansatz* while (35f) and (35g) are their corresponding Killing vectors. On the one hand, these Killing vectors together with the first four generators (35a)–(35d) form a six-dimensional Lie subalgebra. On the other hand, the generator (35e) represents an infinite-dimensional Lie subalgebra. The concrete Lie algebra is characterized by Table I, where the entry for a given row and column represents the corresponding commutator.

The infinite-dimensional subalgebra can be characterized as follows. In Table I we denote

$$\Omega^z = \xi_1^z \partial_z \xi_2^z - \xi_2^z \partial_z \xi_1^z. \tag{36}$$

Since ξ^z is a holomorphic function we can expand it by a Laurent series,

$$\xi^z(z) = \sum_{n=-\infty}^{\infty} a_n(-z^{n+1}), \tag{37}$$

and the generator (35e) becomes an infinite expansion

TABLE I. The commutator table for a Collinson *Ansatz*.

	X_1	X_2	X_3	X_4	k	m	$\hat{X}_{\xi_2^z}$
X_1	0	0	0	0	$-k$	$-m$	0
X_2	0	0	$-2X_3$	$2X_4$	$-k$	m	0
X_3	0	$2X_3$	0	$-X_2$	0	$-k$	0
X_4	0	$-2X_4$	X_2	0	$-m$	0	0
k	k	k	0	m	0	0	0
m	m	$-m$	k	0	0	0	0
$\hat{X}_{\xi_1^z}$	0	0	0	0	0	0	$\hat{X}_{\Omega^z}$

$$\hat{X}_{\xi^z} = \sum_{n=-\infty}^{\infty} [a_n \boldsymbol{L}_n + \bar{a}_n \bar{\boldsymbol{L}}_n], \tag{38}$$

on the complex generators

$$\boldsymbol{L}_n = -z^{n+1}\boldsymbol{\partial}_z - \frac{n+1}{2} z^n \boldsymbol{\partial}_P, \tag{39}$$

and their complex conjugate $\bar{\boldsymbol{L}}_n$. Both sets of vector fields obey the following algebra

$$[\boldsymbol{L}_n, \boldsymbol{L}_m] = (n-m)\boldsymbol{L}_{n+m}, \tag{40}$$

$$[\bar{\boldsymbol{L}}_n, \bar{\boldsymbol{L}}_m] = (n-m)\bar{\boldsymbol{L}}_{n+m}, \tag{41}$$

i.e. the infinite-dimensional subalgebra of the residual symmetries of the Collinson *Ansatz* is composed of just two copies of the Witt algebra [23], i.e. the so-called conformal algebra without the central extension.

Now we proceed to integrate the various infinitesimal transformations generated by the vector fields (35), in order to find their corresponding one-parameter finite transformations. We made use of the method of differential invariants explained in Appendix A, where it is explicitly shown how to integrate several generators, which repeatedly appears in most of the examples. Hence, we do not include the specific details of each case here. The case of generator (35e) is slightly different since it involves the use of an arbitrary function of the z coordinate, but their integration is also included in Appendix A as an example of how to deal with these kind of generators. Using the method of this appendix, the finite transformation found by integrating the generator (35a) is the following

$$\tilde{\tau} = \lambda\tau, \quad \tilde{\sigma} = \lambda\sigma, \quad \tilde{z} = z,$$
$$\tilde{a} = a, \quad \tilde{b} = b,$$
$$\tilde{P} = P - \ln(\lambda), \quad \tilde{Q} = Q + \ln(\lambda), \tag{42}$$

where $\lambda = \exp\varepsilon$. In other words, if the Killing coordinates τ and σ scale in the same way, this is compensated for by appropriate translations along the functions P and Q. For the generator (35b) we get the transformation

$$\tilde{\tau} = \lambda\tau, \quad \tilde{\sigma} = \lambda^{-1}\sigma, \quad \tilde{z} = z,$$
$$\tilde{a} = \lambda^2 a, \quad \tilde{b} = \lambda^2 b, \quad \tilde{P} = P, \quad \tilde{Q} = Q. \tag{43}$$

This time we see that if τ and σ scale inversely, this must be compensated for with a double scaling in the functions a and b. The transformation arising from the exponentiation of the vector field (35c) is

$$\tilde{\tau} = \tau + \varepsilon\sigma, \quad \tilde{\sigma} = \sigma, \quad \tilde{z} = z,$$
$$\tilde{a} = a - \varepsilon, \quad \tilde{b} = b + \varepsilon, \quad \tilde{P} = P, \quad \tilde{Q} = Q, \tag{44}$$

which is precisely the residual symmetry we use as an example at the beginning. As was already emphasized, this transformation establishes that the effects of rotating the time coordinate τ in the Killing plane (τ, σ) can be eliminated by translations with different signs in the functions a and b. Now, we present the residual symmetry that we get after integrating the generator (35d)

$$\tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma + \varepsilon\tau, \quad \tilde{z} = z,$$
$$\tilde{a} = \frac{a}{1 - \varepsilon a}, \quad \tilde{b} = \frac{b}{1 + \varepsilon b}, \quad \tilde{P} = P, \quad \tilde{Q} = Q, \tag{45}$$

i.e. if one instead rotates the angle σ in the Killing plane it is neutralized by appropriate special conformal transformations on the functions a and b. Finally, the case of the generator (35e) is studied in detail in Appendix A, where the following finite transformation is obtained

$$\tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma, \quad \tilde{z} = \tilde{z}(z),$$
$$\tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{P} = P + \ln\left|\frac{\mathrm{d}\tilde{z}}{\mathrm{d}z}\right|, \quad \tilde{Q} = Q. \tag{46}$$

In this case, any conformal transformation of the complex z plane will be compensated for by a supertranslation of the conformal function P, involving the nonvanishing derivative of the holomorphic transformation in z.

We are ready to write now the most general connected residual transformation that the Collinson *Ansatz* (1) admits. It is defined by the composition of the transformations (42)–(46) and is given by

$$\tilde{\tau} = \alpha\tau + \beta\sigma, \quad \tilde{\sigma} = \delta\sigma + \gamma\tau, \quad \tilde{z} = \tilde{z}(z),$$
$$\tilde{a} = \frac{\alpha a - \beta}{-\gamma a + \delta}, \quad \tilde{b} = \frac{\alpha b + \beta}{\gamma b + \delta},$$
$$\tilde{P} = P + \ln\left|\frac{1}{\sqrt{\alpha\delta - \beta\gamma}}\frac{\mathrm{d}\tilde{z}}{\mathrm{d}z}\right|, \quad \tilde{Q} = Q + \ln\sqrt{\alpha\delta - \beta\gamma}, \tag{47}$$

where $\alpha\delta - \beta\gamma > 0$. This transformation tells us that the Collinson *Ansatz* is form invariant under two-dimensional general linear transformations, with positive determinant, on the Killing (τ, σ) plane and arbitrary conformal transformations on the complex z plane, provided that the structural functions will be redefined according to Eq. (47). These are precisely the transformations actively used in Refs. [1–3].

In the following sections we study many other spacetime examples emphasizing just the main results regarding their residual symmetries. The corresponding details can be found in the appendixes.

IV. SPHERICALLY SYMMETRIC *ANSATZ*

We follow up now by studying the very well-known and studied case of spherically symmetric spacetimes. In their static version they are described by the following metric

$$\mathbf{d}s^2 = -N^2(r)F(r)\mathbf{d}t^2 + \frac{\mathbf{d}r^2}{F(r)} + Y^2(r)(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\varphi^2), \tag{48}$$

where a gauge election remains to be done. Since such an election is done in practice in many different ways, we prefer to keep this freedom in order that our analysis can be adapted to the different choices. The involved jet space coordinates $z^A = (x^\mu, u^I)$ are now the spacetime spherical coordinates $x^\mu = (t, r, \theta, \varphi)$ and the metric functions $u^I = (N, F, Y)$. The infinitesimal generators of the corresponding residual symmetries are studied in detail in Appendix B, being the final result

$$\boldsymbol{X}_1 = t\boldsymbol{\partial_t} - N\boldsymbol{\partial_N}, \tag{49a}$$

$$\hat{\boldsymbol{X}}_{\xi^r} = \xi^r(r)\boldsymbol{\partial_r} + \partial_r\xi^r(2F\boldsymbol{\partial_F} - N\boldsymbol{\partial_N}), \tag{49b}$$

$$\boldsymbol{L}_x = \sin\varphi\boldsymbol{\partial_\theta} + \cot\theta\cos\varphi\boldsymbol{\partial_\varphi}, \tag{49c}$$

$$\boldsymbol{L}_y = \cos\varphi\boldsymbol{\partial_\theta} - \cot\theta\sin\varphi\boldsymbol{\partial_\varphi}, \tag{49d}$$

$$\boldsymbol{L}_z = \boldsymbol{\partial_\varphi}, \tag{49e}$$

$$\boldsymbol{k} = \boldsymbol{\partial_t}. \tag{49f}$$

The generators (49c)–(49f) are just the spherically symmetric and stationary Killing vectors of the metric, therefore there are only two generators strictly corresponding to residual symmetries and one of them spans an infinite-dimensional subalgebra. The generators (49) give rise to the algebra shown in Table II.

The infinite-dimensional subalgebra is obtained from the Lie bracket of the family of vector fields $\hat{\boldsymbol{X}}_{\xi^r}$ among themselves, where we define

$$\Omega^r = \xi^r_1\partial_r\xi^r_2 - \xi^r_2\partial_r\xi^r_1. \tag{50}$$

Expanding the function ξ^r as a Fourier series,

TABLE II. The commutator table for a static spherical *Ansatz*.

	$\boldsymbol{X}_1$	$\boldsymbol{k}$	$\boldsymbol{L}_x$	$\boldsymbol{L}_y$	$\boldsymbol{L}_z$	$\hat{\boldsymbol{X}}_{\xi^r_2}$
$\boldsymbol{X}_1$	0	$-\boldsymbol{k}$	0	0	0	0
$\boldsymbol{k}$	$\boldsymbol{k}$	0	0	0	0	0
$\boldsymbol{L}_x$	0	0	0	$\boldsymbol{L}_z$	$-\boldsymbol{L}_y$	0
$\boldsymbol{L}_y$	0	0	$-\boldsymbol{L}_z$	0	$\boldsymbol{L}_x$	0
$\boldsymbol{L}_z$	0	0	$\boldsymbol{L}_y$	$-\boldsymbol{L}_x$	0	0
$\hat{\boldsymbol{X}}_{\xi^r_1}$	0	0	0	0	0	$\hat{\boldsymbol{X}}_{\Omega^r}$

$$\xi^r(r) = \sum_{n=-\infty}^{\infty} a_n(-i\mathrm{e}^{-inr}), \tag{51}$$

the generator (49b) is written as a linear combination of the infinite number of vector fields

$$\boldsymbol{L}_n = -i\mathrm{e}^{-inr}[\boldsymbol{\partial_r} + in(N\boldsymbol{\partial_N} - 2F\boldsymbol{\partial_F})], \tag{52}$$

whose lie brackets obey the Witt algebra (40).

Let us focus now on the finite transformations associated with the residual symmetries that we find using the methods illustrated in Appendix A. The transformation found by integrating the generator (49a) is

$$\tilde{t} = \lambda t, \qquad \tilde{r} = r, \qquad \tilde{\theta} = \theta, \qquad \tilde{\varphi} = \varphi,$$
$$\tilde{F} = F, \qquad \tilde{N} = \lambda^{-1}N, \qquad \tilde{Y} = Y, \tag{53}$$

which tells us that any rescaling on time is compensated for with the inverse rescaling on the function N; a very well-known and -used property of the spherically symmetric *Ansatz*. The family of generators (49b) involving a general dependence must be treated similarly to the last example of Appendix A. We get the finite transformations

$$\tilde{t} = t, \qquad \tilde{r} = \tilde{r}(r), \qquad \tilde{\theta} = \theta, \qquad \tilde{\varphi} = \varphi,$$
$$\tilde{F} = \left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^2 F, \qquad \tilde{N} = \left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^{-1} N, \qquad \tilde{Y} = Y, \tag{54}$$

encoding that an arbitrary radial reparametrization is compensated for with a local rescaling in the functions N and F involving the derivative of the radial transformation. Consequently, the general residual transformation admitted by the static spherical metric *Ansatz* (48) is given by

$$\tilde{t} = \lambda t, \qquad \tilde{r} = \tilde{r}(r), \qquad \tilde{\theta} = \theta, \qquad \tilde{\varphi} = \varphi,$$
$$\tilde{F} = \left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^2 F, \qquad \tilde{N} = \left(\lambda\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^{-1} N, \qquad \tilde{Y} = Y. \tag{55}$$

It is illustrative to analyze how the above results are generalized when the spherically symmetric metric is allowed to depend on time

$$\mathbf{d}s^2 = -N^2(t,r)F(r,t)\mathbf{d}t^2 + \frac{\mathbf{d}r^2}{F(t,r)} + Y^2(t,r)(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\varphi^2). \tag{56}$$

The coordinates for the jet space are the same as the ones for the static case, but as now the structural functions also depend on time the supplementary conditions (26c) are different, see Appendix B. The criterion gives the following two families of generators for the residual symmetries

$$\check{X}_{\xi^t} = \xi^t(t)\boldsymbol{\partial_t} - \partial_t\xi^t N\boldsymbol{\partial_N}, \tag{57a}$$

$$\hat{X}_{\xi^r} = \xi^r(r)\boldsymbol{\partial_r} - \partial_r\xi^r(N\boldsymbol{\partial_N} - 2F\boldsymbol{\partial_F}), \tag{57b}$$

together with the Killing vectors of the sphere (49c)–(49e). This means that losing the stationary symmetry (49f) enhances the time scaling (49a) to a time reparametrization. The Lie algebra spanned by the involved generators is presented in Table III. This time we have two commuting infinite-dimensional subalgebras, and the quantities defining them are

$$\Omega^k = \xi_1^k\partial_k\xi_2^k - \xi_2^k\partial_k\xi_1^k, \tag{58}$$

where k is a generic label for the coordinates t or r, and no summation over k is understood. The generator (57b) is the same appearing in the static case (49b). Hence, expanding the function $\xi^r(r)$ in a Fourier series (51) we obtain a Witt algebra. A similar expansion for the function $\xi^t(t)$ of the other generator (57a) gives a second copy of the Witt algebra satisfied by the generators

$$\check{L}_n = -i\mathrm{e}^{-\mathrm{int}}(\boldsymbol{\partial_r} + inN\boldsymbol{\partial_N}). \tag{59}$$

These two copies are a manifestation of the conformal algebra without central charge.

The finite symmetry arising from generator (57a) is

$$\tilde{t} = \tilde{t}(t), \qquad \tilde{r} = r, \qquad \tilde{\theta} = \theta, \qquad \tilde{\varphi} = \varphi,$$
$$\tilde{F} = F, \qquad \tilde{N} = \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right)^{-1} N, \qquad \tilde{Y} = Y, \tag{60}$$

i.e. any time reparametrization is compensated for with a local scaling of the function N. The finite residual transformation that results from the exponentiation of the vector field (57b) is exactly the one displayed in the static case (54), with the obvious difference that in the present case F and N depend also on time. The composition of both transformations gives the most general connected residual symmetry allowed by a spherically symmetric gravitational *Ansatz*

$$\tilde{t} = \tilde{t}(t), \qquad \tilde{r} = \tilde{r}(r), \qquad \tilde{\theta} = \theta, \qquad \tilde{\varphi} = \varphi,$$
$$\tilde{F} = \left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^2 F, \qquad \tilde{N} = \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^{-1} N, \qquad \tilde{Y} = Y. \tag{61}$$

As indicated by the related infinite algebras these transformations are the conformal freedom enjoyed by the Lorentzian fibers orthogonal to the spheres.

V. AdS WAVES

We study now the so-called AdS waves [16,17]. They have proved to be a very useful tool to inspect the dynamical and asymptotic properties of highly nontrivial theories since they represent exact realizations of the propagating degrees of freedoms of gravity in several contexts [16,17,24,25]. An AdS wave in D dimensions is given by the metric

$$\mathbf{d}s^2 = \frac{l^2}{y^2}[-F(u,y,\vec{x})\mathbf{d}u^2 - 2\mathbf{d}u\mathbf{d}v + \mathbf{d}y^2 + \mathbf{d}\vec{x}^2], \tag{62}$$

where $\vec{x}$ is a $D-3$ Euclidean vector. The existence of residual symmetries of this *Ansatz* in lower dimensions is already known [16,17], and has been exploited to gauge away the nonpropagating degrees of freedoms of standard gravity in lower dimensions. Nevertheless, the uniqueness of such symmetries has not been established. Additionally, little is known about their general behavior in higher dimensions, except that their famous three-dimensional relation with the Virasoro algebra [26] can be extended to any dimension [27]. It is our interest in this section to apply the derived criterion to the AdS waves to fully investigate these questions. The related jet space coordinates are $z^A = (x^\mu, u^I)$ where $x^\mu = (u, v, y, \vec{x})$, $u^I = (F)$. The infinitesimal generators of residual symmetries are thoroughly studied in Appendix C, and the final result is

$$X_1 = 2v\boldsymbol{\partial_v} + y\boldsymbol{\partial_y} + x^i\boldsymbol{\partial_i} + 2F\boldsymbol{\partial_F}, \tag{63a}$$

$$J_{ij} = x_i\boldsymbol{\partial}_j - x_j\boldsymbol{\partial}_i, \tag{63b}$$

$$\tilde{X}_\alpha = \alpha(u)\boldsymbol{\partial_v} - 2\dot{\alpha}\boldsymbol{\partial_F}, \tag{63c}$$

$$\check{X}_{\xi^u} = \xi^u(u)\boldsymbol{\partial_u} + \frac{\dot{\xi}^u}{2}(y\boldsymbol{\partial_y} + x^i\boldsymbol{\partial_i} - 2F\boldsymbol{\partial_F}) + \frac{1}{4}(y^2 + \vec{x}^2)\tilde{X}_{\ddot{\xi}^u}, \tag{63d}$$

$$X_{\vec{P}} = \vec{P}(u) + \vec{x}\cdot\tilde{X}_{\dot{\vec{P}}}, \qquad \vec{P}(u) = P^i(u)\boldsymbol{\partial_i}. \tag{63e}$$

The null Killing vector of the AdS waves, $\boldsymbol{\partial_v}$, corresponds to the particular case of the generator (63c) when the function α is a constant. It is important to emphasize that the rotations (63b) are not isometries, since in general the

TABLE III. The commutator table for a spherical *Ansatz*.

	L_x	L_y	L_z	$\check{X}_{\xi_2^t}$	$\hat{X}_{\xi_2^r}$
L_x	0	L_z	$-L_y$	0	0
L_y	$-L_z$	0	L_x	0	0
L_z	L_y	$-L_x$	0	0	0
$\check{X}_{\xi_1^t}$	0	0	0	$\check{X}_{\Omega^t}$	0
$\hat{X}_{\xi_1^r}$	0	0	0	0	$\hat{X}_{\Omega^r}$

structural function F in metric (62) is not rotation invariant, but will change respecting the standard diffeomorphism rule and consequently leaving the *Ansatz* form invariant. The Lie algebra obeyed by these vector fields is displayed in Table IV, where the quantities appearing in the table are defined as follows

$$\Omega^u = \xi_1^u \dot{\xi}_2^u - \xi_2^u \dot{\xi}_1^u, \tag{64}$$

$$\vec{\gamma}_a[\vec{P}] = \xi_a^u \dot{\vec{P}} - \frac{1}{2}\dot{\xi}_a^u \vec{P}, \qquad a = 1, 2, \tag{65}$$

$$\vec{P}_{ij} = (P_i \delta_j^k - P_j \delta_i^k)\boldsymbol{\partial_k}, \tag{66}$$

and the structural constants $c_{ij,kl}{}^{mn}$ are those of standard rotations,

$$c_{ij,kl}{}^{mn} \boldsymbol{J}_{mn} = 2\delta_{i[k}\boldsymbol{J}_{l]j} - 2\delta_{j[k}\boldsymbol{J}_{l]i}. \tag{67}$$

In the present case we have three different infinite-dimensional families which make very complex and nontrivial the algebra of residual symmetries of the AdS waves. The first family of generators (63c), $\tilde{X}_\alpha$, span an infinite-dimensional Abelian subalgebra. We shall see later that it generates scalar supertranslations (78). An expansion in Fourier series of the function labeling these generators,

$$\alpha(u) = \sum_{n=-\infty}^{\infty} a_n(-i\mathrm{e}^{-inu}), \tag{68}$$

shows that they are a linear combination,

TABLE IV. The commutator table for AdS waves.

	X_1	J_{kl}	$\tilde{X}_\beta$	$\check{X}_{\xi_2^u}$	$X_{\vec{Q}}$
X_1	0	0	$-2\tilde{X}_\beta$	0	$-X_{\vec{Q}}$
J_{ij}	0	$c_{ij,kl}{}^{mn}J_{mn}$	0	0	$-X_{\vec{Q}_{ij}}$
$\tilde{X}_\alpha$	$2\tilde{X}_\alpha$	0	0	$-\tilde{X}_{\dot{\alpha}\xi_2^u}$	0
$\check{X}_{\xi_1^u}$	0	0	$\tilde{X}_{\dot{\beta}\xi_1^u}$	$\check{X}_{\Omega^u}$	$X_{\vec{\gamma}_1[\vec{Q}]}$
$X_{\vec{P}}$	$X_{\vec{P}}$	$X_{\vec{P}_{kl}}$	0	$-X_{\vec{\gamma}_2[\vec{P}]}$	$\tilde{X}_{\vec{P}\cdot\dot{\vec{Q}}-\dot{\vec{P}}\cdot\vec{Q}}$

$$\tilde{X}_\alpha = \sum_{n=-\infty}^{\infty} a_n \boldsymbol{K}_n, \tag{69}$$

of the infinite commuting generators

$$\boldsymbol{K}_n = -i\mathrm{e}^{-inu}(\boldsymbol{\partial_v} + 2in\boldsymbol{\partial_F}). \tag{70}$$

The Lie brackets of the family of generators (63d), $\check{X}_{\xi^u}$, among themselves have the same structure as the infinite-dimensional subalgebras we have studied before both in the Collinson *Ansatz* and in the spherically symmetric one, and for which we have obtained the Witt algebra. Following the same lines as in those cases, expanding the function parametrizing the generator in Fourier series

$$\xi^u(u) = \sum_{n=-\infty}^{\infty} b_n(-i\mathrm{e}^{-inu}), \tag{71}$$

the generator $\check{X}_{\xi^u}$ becomes a linear superposition of the infinite generators

$$\boldsymbol{L}_n = -i\mathrm{e}^{-inu}\left[\boldsymbol{\partial_u} - \frac{in}{2}(y\boldsymbol{\partial_y} + x^j\boldsymbol{\partial_j} - 2F\boldsymbol{\partial_F}) - \frac{n^2}{4}(y^2 + \vec{x}^2)(\boldsymbol{\partial_v} + 2in\boldsymbol{\partial_F})\right], \tag{72}$$

which also obey the Witt algebra (40). It has been shown by Bañados, Chamblin, and Gibbons [27] that the related algebra of Noether charges associated to these symmetries acquires a central extension in any dimension, generalizing the famous three-dimensional result of Brown and Henneaux [26]. The last family of generators (63e), $X_{\vec{P}}$, does not form an algebra by itself, but it does in union with the Abelian family $\tilde{X}_\alpha$. This family generates vector supertranslations (80). An expansion in Fourier series of the vector

$$\vec{P} = \sum_{n=-\infty}^{\infty} \vec{c}_n(-i\mathrm{e}^{-inu}), \tag{73}$$

allows us to write the vector field $X_{\vec{P}}$ as a linear combination,

$$X_{\vec{P}} = \sum_{n=-\infty}^{\infty} (\vec{c}_n)^j (\boldsymbol{M}_n)_j, \tag{74}$$

of the infinite generators

$$(\boldsymbol{M}_n)_j = -i\mathrm{e}^{-inu}\boldsymbol{\partial_j} - inx_j\boldsymbol{K}_n. \tag{75}$$

The Lie algebra can be written now in terms of the infinite generators $\boldsymbol{L}_n$, $\boldsymbol{K}_n$, and $(\boldsymbol{M}_n)_j$, which is done in Table V. It is easier to identify in Table V that the subalgebra spanned

TABLE V. The commutator table for AdS waves II: infinite generators.

	$\boldsymbol{X}_1$	$\boldsymbol{J}_{kl}$	$\boldsymbol{K}_m$	$\boldsymbol{L}_m$	$(\boldsymbol{M}_m)_l$
$\boldsymbol{X}_1$	0	0	$-2\boldsymbol{K}_m$	0	$-(\boldsymbol{M}_m)_l$
$\boldsymbol{J}_{ij}$	0	$c_{ij,kl}{}^{mn}\boldsymbol{J}_{mn}$	0	0	$-2\delta_{l[i}(\boldsymbol{M}_{\vert m\vert})_{j]}$
$\boldsymbol{K}_n$	$2\boldsymbol{K}_n$	0	0	$n\boldsymbol{K}_{m+n}$	0
$\boldsymbol{L}_n$	0	0	$-m\boldsymbol{K}_{m+n}$	$(n-m)\boldsymbol{L}_{n+m}$	$(n/2-m)(\boldsymbol{M}_{m+n})_l$
$(\boldsymbol{M}_n)_i$	$(\boldsymbol{M}_n)_i$	$2\delta_{i[k}(\boldsymbol{M}_{\vert n\vert})_{l]}$	0	$-(m/2-n)(\boldsymbol{M}_{n+m})_i$	$\delta_{il}(n-m)\boldsymbol{K}_{n+m}$

by the conformal generators $\boldsymbol{L}_n$ and the scalar supertranslations $\boldsymbol{K}_n$ is the semidirect sum of the Witt algebra and the loop algebra of $u(1)$. This algebra and their central extension appear in the context of the Kerr/CFT and WAdS/CFT correspondences and are used to compute the entropy of the involved black holes [28–31]. More recently, it has been shown they appear also in the near horizon geometry of stationary black holes [32].

We would like to emphasize that regardless of the complexity of the whole infinite-dimensional subalgebra, it is possible to give a precise interpretation for their infinite connected group as a well-defined subgroup of the diffeomorphism group mapping AdS space to the AdS wave *Ansatz* (62). In order to arrive at this interpretation we need to study the finite version of the AdS wave residual symmetries following the methods of Appendix A. We start with the vector field (63a), which gives rise to the anisotropic scaling

$$\tilde{u}=u,\qquad \tilde{v}=\lambda^2 v,\qquad \tilde{y}=\lambda y,\qquad \tilde{\vec{x}}=\lambda\vec{x},$$
$$\tilde{F}=\lambda^2 F. \tag{76}$$

The following vector field (63b) generates rotations

$$\tilde{u}=u,\qquad \tilde{v}=v,\qquad \tilde{y}=y,\qquad \tilde{x}^i=\Lambda^i{}_j x^j,$$
$$\tilde{F}=F, \tag{77}$$

where $\Lambda^i{}_j\in SO(D-3)$, i.e. it is a $(D-3)\times(D-3)$ orthogonal matrix whose determinant is one. We emphasize these rotations are not precisely isometries since the structural function F depends in general on the spatial coordinates and changes according to the diffeomorphism rule keeping invariant the value of the function at each point; this obviously preserves the form of the *Ansatz*. In the case of the generator (63c), the corresponding finite residual symmetry is

$$\tilde{u}=u,\qquad \tilde{v}=v+\alpha(u),\qquad \tilde{y}=y,\qquad \tilde{\vec{x}}=\vec{x},$$
$$\tilde{F}=F-2\dot{\alpha}, \tag{78}$$

we see that any scalar supertranslation along the null rays can be compensated for by another one in the function F involving the derivative of the original supertranslation. The residual symmetry related to generator (63d) is a reparametrization of the retarded time, which is compensated accordingly

$$\tilde{u}=\tilde{u}(u),\qquad \tilde{v}=v+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}u}\left(\ln\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)(y^2+\vec{x}^2),$$
$$\tilde{y}=\sqrt{\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}}y,\qquad \tilde{\vec{x}}=\sqrt{\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}}\vec{x},$$
$$\tilde{F}=\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{-1}\left[F+\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{-1/2}(y^2+\vec{x}^2)\right]. \tag{79}$$

The details of their derivation can be followed step by step in Appendix C. The integration of the generator (63e) establishes the following finite transformation

$$\tilde{u}=u,\qquad \tilde{v}=v+\frac{\dot{\vec{P}}}{2}\cdot(2\vec{x}+\vec{P}),\qquad \tilde{y}=y,\qquad \tilde{\vec{x}}=\vec{x}+\vec{P}(u),$$
$$\tilde{F}=F-\ddot{\vec{P}}\cdot(2\vec{x}+\vec{P}), \tag{80}$$

where this time a vector supertranslation along the spatial coordinates $\vec{x}$ can be suitably compensated for.

The most general connected residual symmetry of AdS waves is the composition of the above transformations and can be written as

$$\tilde{u}=\int\frac{\mathrm{d}u}{f^2},$$
$$\tilde{v}=\lambda^2\left\{v-\frac{1}{2}\frac{\dot{f}}{f}(y^2+\vec{x}^2)+f\frac{\mathrm{d}}{\mathrm{d}u}(f^{-1}\vec{P})\cdot\vec{x}+\frac{1}{2}\int\mathrm{d}u\left[F_0+\dot{\vec{P}}^2-\dot{f}\frac{\mathrm{d}}{\mathrm{d}u}(f^{-1}\vec{P}^2)\right]\right\},$$
$$\tilde{y}=\frac{\lambda}{f}y,\qquad \tilde{\vec{x}}=\frac{\lambda}{f}\overset{\leftrightarrow}{\Lambda}\cdot(\vec{x}+\vec{P}),$$
$$\tilde{F}=(\lambda f)^2[F-F_2(y^2+\vec{x}^2)-\vec{F}_1\cdot\vec{x}-F_0], \tag{81a}$$

where f, F_0, and $\vec{P}$ are arbitrary functions of the retarded time u determining the functions

$$F_2 = -\frac{\ddot{f}}{f}, \qquad \vec{F}_1 = 2(\ddot{\vec{P}} + \vec{P}F_2), \tag{81b}$$

and the involved parameters are the scaling constant λ and the matrix $\overset{\leftrightarrow}{\Lambda} \in SO(D-3)$. Except for the anisotropic scaling, this transformation becomes the one already known in the literature for $D = 4$ [16] and $D = 3$ [17], since no rotation exists for these dimensions. The resulting transformation is the infinite-dimensional subgroup of the connected group of residual symmetries, which can be understood in lower and higher dimensions according to the following lines. It is well known that a vanishing structural constant, $F = 0$, is just the AdS spacetime in Poincaré coordinates. However, as was first noticed by Siklos [16], this is only the minimal way to represent AdS spacetime with the AdS wave *Ansatz*. Imposing that the AdS waves have a constant curvature (Weyl and traceless Ricci tensors vanish), i.e. that they are locally equivalent to AdS, the following expression for the structural function is obtained

$$F_{\mathrm{AdS}} = F_2(y^2 + \vec{x}^2) + \vec{F}_1 \cdot \vec{x} + F_0, \tag{82}$$

where here F_0, $\vec{F}_1$, and F_2 are arbitrary functions of the retarded time u. As is proved at the end of Appendix C, this is the most general way to express AdS spacetime with the AdS wave *Ansatz* (62) in any dimension and it must be locally equivalent to the metric with $F = 0$. It is straightforward to check that the local transformation reducing the above expression for the structural function to the vanishing one is precisely the infinite-dimensional subgroup of the connected group of residual symmetries contained in (81) for $\lambda = 1$ and $\overset{\leftrightarrow}{\Lambda} = \mathbb{1}$. The only requirement is to choose the functions in the transformation to coincide with the ones in (82), which imposes differential equations for the functions f and $\vec{P}$ via the relations (81b). Consequently, for a given AdS wave any quadratic, linear, or homogeneous dependencies on the front-wave coordinates, such as the ones appearing in (82), are reminiscences of the freedom to express AdS spacetime within this *Ansatz* and can be consistently eliminated. This is the interpretation of the infinite-dimensional sector of residual symmetries of the AdS wave *Ansatz*.

VI. THE PAPAPETROU *ANSATZ*

In this section we study the popular *Ansatz* of Papapetrou [18] which also describes circular stationary axisymmetric spacetimes, as the initial Collinson *Ansatz* (1), and whose metric is given by

$$\mathbf{d}s^2 = -\frac{\rho^2}{X}\mathbf{d}t^2 + X(\mathbf{d}\varphi + A\mathbf{d}t)^2 + \frac{e^{2h}}{X}(\mathbf{d}\rho^2 + \mathbf{d}z^2), \tag{83}$$

where X, h, and A are functions of the spatial coordinates ρ and z only. The main difference of the Papapetrou *Ansatz* with respect to the one of Collinson is that the conformal freedom of the latter is fixed in the former choosing the so-called Weyl coordinates [19]. These coordinates exploit the fact that in vacuum and electrovacuum, one of the gravitational potentials becomes harmonic and can be incorporated in the conformal transformation to play the role of the real or imaginary part of the spatial complex coordinates of the Collinson *Ansatz*. Concretely, this potential is identified with the spatial coordinate ρ in (83). As result, the Papapetrou *Ansatz* has one structural function less than in the Collinson case and this is the starting point of the standard approach to study integrable circular stationary axisymmetric systems in general relativity and which results in the so-called Ernst equations [20]. The jet space coordinates for the Papapetrou *Ansatz* are $x^\mu = (t, \varphi, \rho, z)$ and $u^I = (X, A, h)$. The generators of residual symmetries that we found in Appendix D using the criterion (26) are shown below

$$\boldsymbol{X}_1 = t\boldsymbol{\partial}_t + \varphi\boldsymbol{\partial}_\varphi - 2\rho\boldsymbol{\partial}_\rho - 2z\boldsymbol{\partial}_z - 2X\boldsymbol{\partial}_X + \boldsymbol{\partial}_h, \tag{84a}$$

$$\boldsymbol{X}_2 = t\boldsymbol{\partial}_t - \varphi\boldsymbol{\partial}_\varphi + 2X\boldsymbol{\partial}_X - 2A\boldsymbol{\partial}_A + \boldsymbol{\partial}_h, \tag{84b}$$

$$\boldsymbol{X}_3 = \varphi\boldsymbol{\partial}_t - 2AX\boldsymbol{\partial}_X + (A^2 + \rho^2/X^2)\boldsymbol{\partial}_A - A\boldsymbol{\partial}_h, \tag{84c}$$

$$\boldsymbol{X}_4 = t\boldsymbol{\partial}_\varphi - \boldsymbol{\partial}_A, \tag{84d}$$

$$\boldsymbol{X}_5 = \boldsymbol{\partial}_z, \tag{84e}$$

$$\boldsymbol{k} = \boldsymbol{\partial}_t, \tag{84f}$$

$$\boldsymbol{m} = \boldsymbol{\partial}_\varphi. \tag{84g}$$

We recognize in the generators (84f) and (84g) to the stationary and axisymmetric Killing vectors, the rest are generators of genuine residual symmetries and they together form the Lie algebra exhibited in Table VI.

The finite transformations generated by these vector fields are easily obtained using the methods of Appendix A. After integrating the generator (84a) we obtain

$$\begin{aligned} &\tilde{t} = \lambda t, \qquad \tilde{\varphi} = \lambda\varphi, \qquad \tilde{\rho} = \lambda^{-2}\rho, \qquad \tilde{z} = \lambda^{-2}z, \\ &\tilde{X} = \lambda^{-2}X, \qquad \tilde{A} = A, \qquad e^{2\tilde{h}} = \lambda^2 e^{2h}. \end{aligned} \tag{85}$$

We see that scaling Killing coordinates is compensated for by other scalings in both the remaining spatial coordinates and the structural functions. This transformation and their generator (84a) are similar to the Killing scaling of the Collinson *Ansatz*, see Eqs. (35a) and (42). The finite transformation we found in the case of generator (84b) is the following

TABLE VI. The commutator table for the Papapetrou *Ansatz*.

	X_1	X_2	X_3	X_4	X_5	k	m
X_1	0	0	0	0	$2X_5$	$-k$	$-m$
X_2	0	0	$-2X_3$	$2X_4$	0	$-k$	m
X_3	0	$2X_3$	0	$-X_2$	0	0	$-k$
X_4	0	$-2X_4$	X_2	0	0	$-m$	0
X_5	$-2X_5$	0	0	0	0	0	0
k	k	k	0	m	0	0	0
m	m	$-m$	k	0	0	0	0

$$\begin{aligned}&\tilde{t}=\lambda t, \qquad \tilde{\varphi}=\lambda^{-1}\varphi, \qquad \tilde{\rho}=\rho, \qquad \tilde{z}=z,\\ &\tilde{X}=\lambda^2 X, \qquad \tilde{A}=\lambda^{-2}A, \qquad e^{2\tilde{h}}=\lambda^2 e^{2h}.\end{aligned} \tag{86}$$

Therefore, scaling Killing coordinates inversely can also be compensated for with appropriated scalings, this time from the structural functions only. This is the analog of the inverse Killing scaling of the Collinson *Ansatz* described in Eqs. (35b) and (43). In the case of the generator (84c) we have the following transformation

$$\begin{aligned}&\tilde{t}=t+\varepsilon\varphi, \qquad \tilde{\varphi}=\varphi, \qquad \tilde{\rho}=\rho, \qquad \tilde{z}=z,\\ &\tilde{X}=X[1-2\varepsilon A+\varepsilon^2(A^2-\rho^2/X^2)],\\ &\tilde{A}=\frac{A-\varepsilon(A^2-\rho^2/X^2)}{1-2\varepsilon A+\varepsilon^2(A^2-\rho^2/X^2)},\\ &e^{2\tilde{h}}=e^{2h}[1-2\varepsilon A+\varepsilon^2(A^2-\rho^2/X^2)].\end{aligned} \tag{87}$$

In other words, rotating time in the Killing plane is compensated for by redefining the structural functions with a sort of special conformal transformation. In turn, integration of the vector field (84d) gives a rotation of the angle in the Killing plane,

$$\begin{aligned}&\tilde{t}=t, \qquad \tilde{\varphi}=\varphi+\varepsilon t, \qquad \tilde{\rho}=\rho, \qquad \tilde{z}=z,\\ &\tilde{X}=X, \qquad \tilde{A}=A-\varepsilon, \qquad e^{2\tilde{h}}=e^{2h},\end{aligned} \tag{88}$$

which is compensated for with a translation. These last two Killing rotations are equivalent to those appearing for the Collinson *Ansatz*, Eqs. (44) and (45), with generators (35c) and (35d). The simplest residual symmetry for the Papapetrou *Ansatz* comes from the vector field (84e) and is a translation on the spatial coordinate z. This is not an isometry since the structural functions depend in general on this coordinate, but they will change as functions do under diffeomorphisms which preserve the form of the *Ansatz*. This z translation is the residual symmetry left after breaking the conformal freedom of the Collinson *Ansatz* by fixing Weyl coordinates, choosing one of the gravitational potentials as the coordinate ρ.

The composition of the latter transformations leads to the most general connected residual symmetry admitted by the Papapetrou metric (83),

$$\begin{aligned}&\tilde{t}=\alpha t+\beta\varphi, \qquad \tilde{\varphi}=\gamma t+\delta\varphi,\\ &\tilde{\rho}=\frac{\rho}{\alpha\delta-\beta\gamma}, \qquad \tilde{z}=\frac{z}{\alpha\delta-\beta\gamma}+\epsilon,\\ &\tilde{X}=X\frac{\alpha^2-2\alpha\beta A+\beta^2(A^2-\rho^2/X^2)}{(\alpha\delta-\beta\gamma)^2},\\ &\tilde{A}=\frac{(\alpha\delta+\beta\gamma)A-\beta\delta(A^2-\rho^2/X^2)-\alpha\gamma}{\alpha^2-2\alpha\beta A+\beta^2(A^2-\rho^2/X^2)},\\ &e^{2\tilde{h}}=e^{2h}[\alpha^2-2\alpha\beta A+\beta^2(A^2-\rho^2/X^2)],\end{aligned} \tag{89}$$

where $\alpha\delta-\beta\gamma$ is a positive quantity. A very well-known residual symmetry of the Papapetrou *Ansatz* is the one linking the so-called conjugate potentials introduced by Chandrasekhar. Between other applications it allows us to derive the famous Kerr metric [21] from an unphysical trivial solution to the Ernst equations [19]. It is easy to check that taking the following values for the parameters of transformation (89),

$$\alpha=\delta=\tau=0, \qquad \beta=-\gamma=1, \tag{90}$$

one recovers the relation defining the conjugate potentials [19],

$$\begin{aligned}&\tilde{t}=\varphi, \qquad \tilde{\varphi}=-t, \qquad \tilde{\rho}=\rho, \qquad \tilde{z}=z,\\ &\tilde{X}=X\left(A^2-\frac{\rho^2}{X^2}\right),\\ &\tilde{A}=-A\left(A^2-\frac{\rho^2}{X^2}\right)^{-1},\\ &e^{2\tilde{h}}=e^{2h}\left(A^2-\frac{\rho^2}{X^2}\right).\end{aligned} \tag{91}$$

VII. THE NONCIRCULAR COLLINSON *ANSATZ*

The so-called Collinson theorem we mention at the beginning of Sec. III was first established for circular stationary axisymmetric spacetimes for which the metric is block diagonal, with one block related with the Killing directions and the other with the remaining spatial sector [1,2]. It was later extended to general stationary axisymmetric spacetimes by considering also noncircular contributions [3]. This was possible due to the following generalization of the Collinson *Ansatz*

$$\begin{aligned}\mathbf{ds^2}=e^{-2Q}\Bigg(&-\frac{1}{a+b}(\mathbf{d}\boldsymbol{\tau}+a\mathbf{d}\boldsymbol{\sigma}+M\mathbf{dy})\\ &\times(\mathbf{d}\boldsymbol{\tau}-b\mathbf{d}\boldsymbol{\sigma}-N\mathbf{dy})+e^{-2P}(\mathbf{dx^2}+\mathbf{dy^2})\Bigg),\end{aligned} \tag{92}$$

describing the most general stationary axisymmetric spacetimes and where the structural functions include now the noncircular contributions M and N, and all the functions depend exclusively of the spatial coordinates (x, y) [3]. In this case the jet space coordinates $z^A = (x^\mu, u^I)$ are given by $x^\mu = (\tau, \sigma, x, y)$ and $u^I = (a, b, P, Q, M, N)$. The infinitesimal generators of residual symmetries found in Appendix E for the generalization (92) are listed below

$$\boldsymbol{X}_1 = \tau\boldsymbol{\partial}_\tau + \sigma\boldsymbol{\partial}_\sigma - \boldsymbol{\partial}_P + \boldsymbol{\partial}_Q + M\boldsymbol{\partial}_M + N\boldsymbol{\partial}_N, \tag{93a}$$

$$\boldsymbol{X}_2 = \tau\boldsymbol{\partial}_\tau - \sigma\boldsymbol{\partial}_\sigma + 2a\boldsymbol{\partial}_a + 2b\boldsymbol{\partial}_b + M\boldsymbol{\partial}_M + N\boldsymbol{\partial}_N, \tag{93b}$$

$$\boldsymbol{X}_3 = \sigma\boldsymbol{\partial}_\tau - \boldsymbol{\partial}_a + \boldsymbol{\partial}_b, \tag{93c}$$

$$\boldsymbol{X}_4 = \tau\boldsymbol{\partial}_\sigma + a^2\boldsymbol{\partial}_a - b^2\boldsymbol{\partial}_b + Ma\boldsymbol{\partial}_M - Nb\boldsymbol{\partial}_N, \tag{93d}$$

$$\boldsymbol{X}_5 = x\boldsymbol{\partial}_x + y\boldsymbol{\partial}_y + \boldsymbol{\partial}_P - M\boldsymbol{\partial}_M - N\boldsymbol{\partial}_N, \tag{93e}$$

$$\boldsymbol{X}_6 = \boldsymbol{\partial}_x, \tag{93f}$$

$$\boldsymbol{X}_7 = \boldsymbol{\partial}_y, \tag{93g}$$

$$\boldsymbol{X}_{F_1} = F_1(y)\boldsymbol{\partial}_\sigma - F_1'(a\boldsymbol{\partial}_M + b\boldsymbol{\partial}_N), \tag{93h}$$

$$\hat{\boldsymbol{X}}_{F_2} = F_2(y)\boldsymbol{\partial}_\tau - F_2'(\boldsymbol{\partial}_M - \boldsymbol{\partial}_N). \tag{93i}$$

Note that the Killing vectors of the metric are a particular case of the generators (93h) and (93i) when the involved functions are constant, in other cases they generalize to two commuting infinite-dimensional Abelian Lie subalgebras. The rest of the generators (93a)–(93g) form a seven-dimensional Lie subalgebra, and all them together give the algebra observed in Table VII.

The generators (93a)–(93d) are just those appearing for the circular Collinson *Ansatz* (35a)–(35d), but modified by noncircular contributions. An exception is the generator (93c), which is exactly the same for both spacetimes, as is appreciated in Eq. (35c). The vector fields (93e), (93f), and (93g) correspond to the cases where the conformal generator (35e) becomes $\xi^z(z) = z = x + iy$, $\xi^z(z) = 1$, and $\xi^z(z) = i$, respectively, where the first case is additionally supplemented by noncircular contributions. In other words, the infinite conformal freedom enjoyed by the circular Collinson *Ansatz* breaks in these three generators, which is a consequence of the presence of noncircular terms. Conversely, as we check later, the standard translations symmetries of the circular Killing vectors (35f) and (35g) enhance to supertranslations (93h) and (93i) depending on the direction along which the circularity is lost.

TABLE VII. The commutator table for the noncircular Collinson *Ansatz*.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_{F_1}	$\hat{X}_{F_2}$
X_1	0	0	0	0	0	0	0	$-X_{F_1}$	$-\hat{X}_{F_2}$
X_2	0	0	$-2X_3$	$2X_4$	0	0	0	X_{F_1}	$-\hat{X}_{F_2}$
X_3	0	$2X_3$	0	$-X_2$	0	0	0	$-\hat{X}_{F_1}$	0
X_4	0	$-2X_4$	X_2	0	0	0	0	0	$-X_{F_2}$
X_5	0	0	0	0	0	$-X_6$	$-X_7$	$X_{yF_1'}$	$\check{X}_{yF_2'}$
X_6	0	0	0	0	X_6	0	0	0	0
X_7	0	0	0	0	X_7	0	0	0	0
X_{G_1}	X_{G_1}	$-X_{G_1}$	$\hat{X}_{G_1}$	0	$-X_{yG_1'}$	0	0	0	0
$\hat{X}_{G_2}$	$\hat{X}_{G_2}$	$\hat{X}_{G_2}$	0	X_{G_2}	$-\check{X}_{yG_2'}$	0	0	0	0

We study now how the finite transformations are extended to include noncircular components using again the methods of Appendix A. The finite residual symmetry obtained after integrating generator (93a) is

$$\begin{aligned} &\tilde{\tau} = \lambda\tau, \qquad \tilde{\sigma} = \lambda\sigma, \qquad \tilde{x} = x, \qquad \tilde{y} = y, \\ &\tilde{a} = a, \qquad \tilde{b} = b, \qquad \tilde{P} = P - \ln\lambda, \qquad \tilde{Q} = Q + \ln\lambda, \\ &\tilde{M} = \lambda M, \qquad \tilde{N} = \lambda N, \end{aligned} \tag{94}$$

which is just the Killing scaling (42) trivially extended to the noncircular portions. Something similar happens with generator (93b) with regard to the anisotropic Killing scaling (43)

$$\begin{aligned} &\tilde{\tau} = \lambda\tau, \qquad \tilde{\sigma} = \lambda^{-1}\sigma, \qquad \tilde{x} = x, \qquad \tilde{y} = y, \\ &\tilde{a} = \lambda^2 a, \qquad \tilde{b} = \lambda^2 b, \qquad \tilde{P} = P, \qquad \tilde{Q} = Q, \\ &\tilde{M} = \lambda M, \qquad \tilde{N} = \lambda N. \end{aligned} \tag{95}$$

Since the generator (93c) is the same as in the circular case (35c), their finite residual symmetry is the time rotation (44) used as an example at the beginning of the work and with no effect along the noncircular contributions. In opposition, the angular rotation generated by the vector field (93d),

$$\begin{aligned} &\tilde{\tau} = \tau, \qquad \tilde{\sigma} = \sigma + \varepsilon\tau, \qquad \tilde{x} = x, \qquad \tilde{y} = y, \\ &\tilde{a} = \frac{a}{1 - \varepsilon a}, \qquad \tilde{b} = \frac{b}{1 + \varepsilon b}, \qquad \tilde{P} = P, \qquad \tilde{Q} = Q, \\ &\tilde{M} = \frac{M}{1 - \varepsilon a}, \qquad \tilde{N} = \frac{N}{1 + \varepsilon b}, \end{aligned} \tag{96}$$

nontrivially extends the circular one (45), since it also needs to be compensated for by special conformal transformations of the noncircular structural functions. The residual transformation one gets after exponentiation of the vector field (93e) is the following spatial scaling

$$\begin{aligned} &\tilde{\tau} = \tau, \qquad \tilde{\sigma} = \sigma, \qquad \tilde{x} = \lambda x, \qquad \tilde{y} = \lambda y, \\ &\tilde{a} = a, \qquad \tilde{b} = b, \qquad \tilde{P} = P + \ln\lambda, \qquad \tilde{Q} = Q, \\ &\tilde{M} = \lambda^{-1} M, \qquad \tilde{N} = \lambda^{-1} N, \end{aligned} \tag{97}$$

which is compensated for by a translation in P and inverse scalings in N and M. The conformal symmetry (46) breaks to this spatial scaling and to the spatial translations (93f) and (93g). We emphasize these last two are not precisely isometries, since in general all the structural functions depend on the spatial coordinates and will change according to the diffeomorphism rule for functions, which preserve the form of the *Ansatz*. The generator (93h) leads to the following finite residual symmetry

$$\tilde{\tau} = \tau, \qquad \tilde{\sigma} = \sigma + F_1(y), \qquad \tilde{x} = x, \qquad \tilde{y} = y,$$
$$\tilde{a} = a, \qquad \tilde{b} = b, \qquad \tilde{P} = P, \qquad \tilde{Q} = Q,$$
$$\tilde{M} = M - aF_1', \qquad \tilde{N} = N - bF_1', \tag{98}$$

saying that the circular translation isommetry along the angle is improved to a supertranslation whose local dependence is determined by the spatial direction where the noncircularity appears and which is compensated for by superrotating the noncircular structural functions M and N in the planes (a, M) and (b, N), respectively. The vector field (93i) enhances the time translation isometry of the circular case to a similar supertranslation,

$$\tilde{\tau} = \tau + F_2(y), \qquad \tilde{\sigma} = \sigma, \qquad \tilde{x} = x, \qquad \tilde{y} = y,$$
$$\tilde{a} = a, \qquad \tilde{b} = b, \qquad \tilde{P} = P, \qquad \tilde{Q} = Q,$$
$$\tilde{M} = M - F_2', \qquad \tilde{N} = N + F_2', \tag{99}$$

but compensated for this time with supertranslations along the noncircular structural functions.

Finally, we would like to comment that there is a more symmetrical version of the noncircular *Ansatz* where the diffeomorphism gauge freedom is not enterally fixed and that involves considering M and N arbitrary complex functions [3]. This has the advantage of preserving the conformal invariance of the circular case. However, as is explicitly proved here, this huge and useful symmetry is lost after gauge fixing.

VIII. CONCLUSIONS

In this work we have analyzed how to unambiguously define the residual symmetries of a gravitational *Ansatz*. The intuitive conditions that spacetime diffeomorphisms must be compensated for with redefinitions of the structural functions determining the *Ansatz* have a very nice and geometrical interpretation in terms of general diffeomorphisms on jet space, where both spacetime coordinates and the functions are independent variables. Consequently, these conditions are interpreted such that the generators of residual symmetries are jet space Killing fields of the trivial lifting of the metric. However, it has the disadvantage of including excessively general transformations that only preserve the *Ansatz* formally by changing the original dependencies of the structural functions. Studying the transformations of the structural functions derivatives, we provide the complementary conditions needed to avoid this situation and obtain residual symmetries in a strict sense. The supplemental conditions make the integration of the system of partial differential equations determining the generators easier. In fact, the full criterion provides an effective computational procedure for finding all the residual symmetries of almost any gravitational *Ansatz* of interest. We describe also how to integrate the resulting infinitesimal generators to obtain the related finite transformations by means of the efficient method of differential invariants.

We apply the criterion to study five different gravitational *Ansätze*. We start with the case of the Collinson *Ansatz* describing circular stationary axisymmetric spacetimes and we recover all the residual symmetries previously found by Collinson himself. Apart from the isometries, the finite-dimensional subalgebra includes two Killing scalings and two Killing rotations appropriately compensated for by scalings, translations, and special conformal transformations on the structural functions. There is also an infinite-dimensional subalgebra which is just a conformal symmetry on the spatial plane orthogonal to the Killing vectors. Using a complex Laurent series we characterize the concrete infinite generators spanning the two related copies of the Witt algebra.

Later we study the spherically symmetric *Ansatz* in its static and time-dependent versions and previous to completely fixing the gauge on the radial variable. In the static case in addition to the isometries there is a time scaling and consequently a radial reparametrization as residual symmetries. In the time-dependent case we have the same radial reparametrization and losing the stationary isometry enhances the above time scaling to a time reparametrization. We identify, by means of a Fourier expansion this time, the infinite generators providing also in this example two copies of the Witt algebra characterizing the conformal symmetry now enjoyed by the Lorentzian fibers orthogonal to the spheres.

We also generalize the residual symmetries already known for lower-dimensional AdS waves to higher dimensions. The resulting infinite-dimensional subalgebra has a very complex structure formed by three infinite-dimensional families. One of them forms a Witt algebra by itself and has been the base to show that the appearance of a central extension associated with their Noether charges is not exclusive of three dimensions. The other two families are scalar and vector supertranslations forming a more complex algebra. Independently of this complexity we manage to understand the precise meaning of the infinite connected group. It is just the well-defined subgroup of the diffeomorphism group encoding the freedom to represent the AdS space by means of the AdS wave *Ansatz*.

We study another famous example of an *Ansatz* describing a circular stationary axisymmetric spacetime, which is the Papapetrou *Ansatz*. It is restricted by the election of the so-called Weyl coordinates, which exist when one of the gravitational potentials is harmonic and can be chosen as one of the spatial coordinates, a situation occurring for vacuum and electrovacuum spacetimes. Obviously, this election breaks the conformal symmetry of the circular stationary axisymmetric spacetimes explicitly manifest in the Collinson *Ansatz*. We found the same Killing scalings and rotations as the Collinson case together with the isometries. We also observe that the previous conformal freedom breaks to a single translation of the other spatial coordinate not identified with one of the gravitational potentials. In other words, Weyl coordinates are undetermined modulo this translation. Additionally, from the general connected group of resulting residual symmetries we identify the precise point defining the relation between conjugate potentials introduced by Chandrasekhar. They are well known by allowing us to derive the famous Kerr metric from an unphysical trivial solution of the Ernst equations.

Finally, we also explore the inclusion of noncircular contributions in the Collinson *Ansatz*. There are two immediate effects of losing circularity, the first is that again the conformal symmetry is broken, in this case to spatial translations and a scaling. The second is that the standard translations symmetries of the circular Killing vectors enhance to supertranslations which depend on the direction along which the circularity is lost.

An immediate generalization of our work is to consider gravitational *Ansätze* that also depend on derivatives of the structural functions. There are interesting examples of this kind in the literature; we will report on these examples in the future. Another extension which can be very useful is to try to include in the unifying perspective of the present approach the study of the asymptotic symmetries of spacetimes, which are nothing but a very particular kind of residual symmetries.

Last but not least, an issue that must be inevitably addressed is the further development of these methods to include matter fields. In the case of a gauge field one should look for diffeomorphisms and gauge transformations that together are compensated for by redefinitions of the structural functions defining the *Ansatz* for the gauge field. However, it is enough to study the residual symmetries of gauge invariant quantities as the field strength in the Abelian case or the energy-momentum tensor in the non-Abelian one, the latter simply being the quantity defining the coupling to the gravitational field. The additional criterion must reduce to the vanishing of the jet space Lie derivative of these gauge invariant quantities along generators prolonged in the new structural directions.

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APPENDIX A: DIFFERENTIAL INVARIANTS METHOD

Any Lie-point transformation (6) defines an infinitesimal generator (16) on jet space $z^A = (x^\mu, u^I, \ldots)$. Conversely, given the infinitesimal generator one can find the associated finite transformations, $\tilde{z}^B = \tilde{z}^B(z^A; \varepsilon)$, which are defined as the solution to the dynamical system

$$\frac{\mathrm{d}\tilde{z}^B(\varepsilon)}{\mathrm{d}\varepsilon} = X^B(\tilde{z}^A(\varepsilon)), \qquad \tilde{z}^B(0) = z^B, \tag{A1}$$

where the starting coordinates play the role of the initial conditions. In this appendix we review how to integrate this system and find the finite transformations from the infinitesimal ones by means of the method of differential invariants.

A local function $\Omega = \Omega(z^A)$ on jet space is a differential invariant of the group of transformations provided that [4–7]

$$\Omega(\tilde{z}^A) = \Omega(z^A), \tag{A2}$$

i.e. they are jet space functions that keep their local functional form under the action of the one-parameter group. Since the infinitesimal action is characterized by the corresponding generator (16), a differential invariant must be constant along the integral curves of the generator

$$\boldsymbol{X}(\Omega) = X^A \boldsymbol{\partial}_A(\Omega) = X^1 \frac{\partial \Omega}{\partial z^1} + \cdots + X^m \frac{\partial \Omega}{\partial z^m} = 0. \tag{A3}$$

This is a linear homogeneous first order partial differential equation which has $m-1$ functionally independent solutions, where m is the jet space dimension. These functionally independent differential invariants are just the $m-1$ integration constants, $\Omega_i(z^A) = c_i$, of the corresponding *characteristic system* of ordinary differential equations

$$\frac{\mathrm{d}z^1}{X^1(z^A)} = \frac{\mathrm{d}z^2}{X^2(z^A)} = \cdots = \frac{\mathrm{d}z^m}{X^m(z^A)}. \tag{A4}$$

The idea of the method is to eliminate $m-1$ coordinates from the differential invariants and rewrite the equation for the remaining coordinate, let us say $z^{\check{B}}$, as an autonomous first order ordinary equation for this single variable,

$$\frac{\mathrm{d}z^{\check{B}}}{X^{\check{B}}(z^{\check{B}}, \Omega_1, \ldots, \Omega_{m-1})} = \mathrm{d}\varepsilon, \tag{A5}$$

which is integrable, at least in quadrature. Integrating this equation with general initial conditions z^A at $\varepsilon = 0$ and denoting the solutions for a generic value of the parameter as $\tilde{z}^A$, we obtain the one-parameter transformation $\tilde{z}^{\tilde{B}} = \tilde{z}^{\tilde{B}}(z^A;\varepsilon)$ after substituting all the differential invariants evaluated on the full initial conditions. The remaining transformations are found by isolating the other $m-1$ coordinates from the definitions of the differential invariants $\Omega_i(\tilde{z}^A) = \Omega_i(z^A)$. In our opinion, this method is more efficient and straightforward than the standard exponentiation of the vector fields. We present some examples which make the involved procedure more clear.

We start with the well-known example of the SO(2) group acting on the plane, which has as generator

$$\boldsymbol{X} = -y\boldsymbol{\partial}_x + x\boldsymbol{\partial}_y. \tag{A6}$$

In this case the characteristic equation

$$-\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x}, \tag{A7}$$

can be rewritten as

$$\mathrm{d}(x^2 + y^2) = 0, \tag{A8}$$

giving rise to the single differential invariant

$$\Omega_1(x, y) = x^2 + y^2. \tag{A9}$$

In order to find the finite transformation for x we must integrate its ordinary equation,

$$\begin{aligned} \mathrm{d}\varepsilon &= -\frac{\mathrm{d}x}{y} \\ &= -\frac{\mathrm{d}x}{\sqrt{\Omega_1 - x^2}}, \end{aligned} \tag{A10}$$

making use of the differential invariant (A9), which is constant along the integral curves of the SO(2) generator. Integrating Eq. (A10) from 0 to ε and denoting the initial conditions by (x, y) and the solutions by $(\tilde{x}, \tilde{y})$ yields

$$\tilde{x} = \cos(\varepsilon)x - \sin(\varepsilon)y, \tag{A11}$$

after substituting back the differential invariant evaluated at the initial conditions. The remaining transformation is found from the definition of the differential invariant since

$$\tilde{x}^2 + \tilde{y}^2 = x^2 + y^2. \tag{A12}$$

Then solving for $\tilde{y}$ and substituting Eq. (A11) gives

$$\tilde{y} = \sin(\varepsilon)x + \cos(\varepsilon)y. \tag{A13}$$

The transformations (A11) and (A13) are the standard rotations in the (x, y) plane as expected.

The next example is the following three-dimensional generator

$$\boldsymbol{X} = -y\boldsymbol{\partial}_x + x\boldsymbol{\partial}_y + (1 + z^2)\boldsymbol{\partial}_z, \tag{A14}$$

having as a characteristic system

$$-\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}z}{1 + z^2}. \tag{A15}$$

In this case we have two differential invariants and the first equality yields the same differential invariant presented in the previous example (A9). Consequently, the planar rotations (A11) and (A13) hold again. The second equality can be written as

$$\begin{aligned} \frac{\mathrm{d}z}{1 + z^2} &= \frac{\mathrm{d}y}{x} \\ &= \frac{\mathrm{d}y}{\sqrt{\Omega_1 - y^2}}, \end{aligned} \tag{A16}$$

by means of the first differential invariant, giving the exact differential

$$\mathrm{d}\left(\arctan z - \arcsin\frac{y}{\sqrt{\Omega_1}}\right) = 0, \tag{A17}$$

whose integration gives us the second differential invariant. Since any function of a differential invariant inherits that property as well, we choose

$$\Omega_2(\vec{x}) = \tan\left(\arctan z - \arcsin\frac{y}{\sqrt{\Omega_1}}\right) = \frac{xz - y}{xz + x}. \tag{A18}$$

Using now the definition for a differential invariant we have the following relation between the solution $\tilde{\vec{x}}$ and the initial condition $\vec{x}$

$$\frac{\tilde{x}\,\tilde{z} - \tilde{y}}{\tilde{y}\,\tilde{z} + \tilde{x}} = \frac{xz - y}{yz + x}, \tag{A19}$$

from which after substituting the planar rotations (A11) and (A13) and isolating $\tilde{z}$ we obtain the SL(2, IR) transformation

$$\tilde{z} = \frac{\cos(\varepsilon)z + \sin(\varepsilon)}{-\sin(\varepsilon)z + \cos(\varepsilon)}. \tag{A20}$$

The transformations (A11), (A13), and (A20) are the finite version of the infinitesimal ones described by the generator (A14).

The last example is one where the generator is characterized by an arbitrary dependence. Those cases involve general subclasses of diffeomorphisms and represent infinite-dimensional algebras. We choose the generator (35e) of the Collinson *Ansatz*,

$$\hat{\boldsymbol{X}}_{\xi^z} = \xi^z \boldsymbol{\partial}_z + \overline{\xi^z} \boldsymbol{\partial}_{\bar{z}} + \frac{1}{2}(\partial_z \xi^z + \partial_{\bar{z}} \bar{\xi}^z) \boldsymbol{\partial}_{\boldsymbol{P}},$$

whose characteristic system is

$$\frac{\mathrm{d}z}{\xi^z(z)} = \frac{\mathrm{d}\bar{z}}{\bar{\xi}^z(\bar{z})} = \frac{2\mathrm{d}P}{\partial_z \xi^z + \partial_{\bar{z}} \bar{\xi}^z}. \tag{A21}$$

Here $\xi^z(z)$ is an arbitrary holomorphic function on the complex z plane. In order for this generator to be uniquely integrable this function must be nonvanishing, hence, this component represents an arbitrary conformal transformation, i.e. a general one-parameter holomorphism

$$\tilde{z} = \tilde{z}(z;\varepsilon), \qquad \xi^z(z) \equiv \frac{\partial \tilde{z}}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \tag{A22a}$$

with nonvanishing derivative

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}z} = \frac{\tilde{\xi}^z(\tilde{z})}{\xi^z(z)} \neq 0, \tag{A22b}$$

and which thus is holomorphically invertible. Now, Eq. (A21) can be rewritten in a way allowing us to identify the corresponding differential invariant,

$$\mathrm{d}(P - \ln|\xi^z|) = 0 \Rightarrow \Omega(z,P) = P - \ln|\xi^z|. \tag{A23}$$

By definition, using again the notation of writing the solutions with a tilde and the initial conditions without it, the differential invariant must comply with

$$\tilde{P} - \ln|\tilde{\xi}^z| = P - \ln|\xi^z|, \tag{A24}$$

from which we derive the remaining transformation,

$$\tilde{P} = P + \ln\left|\frac{\mathrm{d}\tilde{z}}{\mathrm{d}z}\right|. \tag{A25}$$

In summary, the generator (35e) of the Collinson *Ansatz* represents the infinitesimal version of a conformal transformation (A22) compensated for by the supertranslation (A25) on the structural function P.

The three examples studied in this appendix are representative of the procedures needed to find the finite version of the generators appearing throughout the paper, by means of the use of the differential invariant method.

APPENDIX B: SPHERICALLY SYMMETRIC *ANSATZ*, DETAILS

In this appendix and the following ones we provide the details of the calculations concerning the remaining examples we study in the paper. This appendix is devoted to the spherically symmetric *Ansatz*. We start with the static case (48) having jet space coordinates $z^A = (t,r,\theta,\varphi,F,N,Y)$. Since all the structural functions are radial functions then, according to the classification of Sec. II, $u^{\bar{I}} = u^I = (F,N,Y)$, $x^{\bar{\alpha}} = (r)$, and $x^{\hat{\alpha}} = (t,\theta,\varphi)$, which reduce the complementary conditions (26c) to

$$\partial_t \eta^I = \partial_\theta \eta^I = \partial_\varphi \eta^I = \partial_t \xi^r = \partial_\theta \xi^r = \partial_\varphi \xi^r = 0. \tag{B1}$$

Therefore, the generator of infinitesimal residual symmetries must have the following general form

$$\begin{aligned}\boldsymbol{X} = {}& \xi^t(t,r,\theta,\varphi)\boldsymbol{\partial_t} + \xi^r(r)\boldsymbol{\partial_r} + \xi^\theta(t,r,\theta,\varphi)\boldsymbol{\partial_\theta} \\ & + \xi^\varphi(t,r,\theta,\varphi)\boldsymbol{\partial_\varphi} + \eta^F(r,F,N,Y)\boldsymbol{\partial_F} \\ & + \eta^N(r,F,N,Y)\boldsymbol{\partial_N} + \eta^Y(r,F,N,Y)\boldsymbol{\partial_Y}. \end{aligned} \tag{B2}$$

The spherical Lie-derivative criterion (26a) reads

$$N\eta^F + 2F\eta^N + 2NF\partial_t \xi^t = 0, \tag{B3a}$$

$$\eta^F - 2F\partial_r \xi^r = 0, \tag{B3b}$$

$$\eta^Y + Y\partial_\theta \xi^\theta = 0, \tag{B3c}$$

$$\eta^Y + Y(\partial_\varphi \xi^\varphi + \cot(\theta)\xi^\theta) = 0, \tag{B3d}$$

$$\partial_r \xi^t = \partial_r \xi^\theta = \partial_r \xi^\varphi = 0, \tag{B3e}$$

$$-N^2 F\partial_\theta \xi^t + Y^2 \partial_t \xi^\theta = 0, \tag{B3f}$$

$$-N^2 F\partial_\varphi \xi^t + Y^2 \sin^2(\theta)\partial_t \xi^\varphi = 0, \tag{B3g}$$

$$\partial_\varphi \xi^\theta + \sin^2(\theta)\partial_\theta \xi^\varphi = 0. \tag{B3h}$$

From Eq. (B3e) and using in Eqs. (B3f) and (B3g) the fact that the structural functions of the *Ansatz* are free variables from which the generator components along spacetime are independent, we conclude that ξ^t is a function of t only and ξ^θ, ξ^φ are functions of θ and φ only. Using this in Eqs. (B3a), (B3c), and (B3d) and that the generator components along the structural functions are independent of t, θ, and φ, these equations are only satisfied if

$$\partial_t \xi^t = C_1, \tag{B4a}$$

$$\partial_\theta \xi^\theta = C_2 = \partial_\varphi \xi^\varphi + \cot\theta\, \xi^\theta, \tag{B4b}$$

where C_1 and C_2 are constants. This subsystem can be solved together with Eq. (B3h). In fact, deriving Eq. (B3h) with respect to φ and using Eq. (B4b) we obtain that $C_2 = 0$ and $\partial^2_\varphi \xi^\theta = -\xi^\theta$, from which the integration is straightforward,

$$\xi^t = C_1 t + \kappa_4, \tag{B5a}$$

$$\xi^\theta = \kappa_1 \sin\varphi + \kappa_2 \cos\varphi, \tag{B5b}$$

$$\xi^\varphi = (\kappa_1 \cos\varphi - \kappa_2 \sin\varphi)\cot\theta + \kappa_3, \tag{B5c}$$

and the κ's are new integration constants. Now it only remains to solve the consistent algebraic system (B3a) and (B3c) for the generator components along the structural functions, which results after substituting the solutions (B5), that gives us

$$\eta^N = -(\partial_r \xi^r + C_1)N, \tag{B6a}$$

$$\eta^F = 2\partial_r \xi^r F, \tag{B6b}$$

$$\eta^Y = 0. \tag{B6c}$$

This allows us to arrive at the final form of the generator (B2) of residual symmetries on static spherically symmetric spacetimes,

$$\begin{aligned} \boldsymbol{X} = {} & C_1(t\boldsymbol{\partial_t} - N\boldsymbol{\partial_N}) \\ & + \xi^r(r)\boldsymbol{\partial_r} + \partial_r\xi^r(2F\boldsymbol{\partial_F} - N\boldsymbol{\partial_N}) \\ & + \kappa_1(\sin\varphi\boldsymbol{\partial_\theta} + \cot\theta\cos\varphi\boldsymbol{\partial_\varphi}) \\ & + \kappa_2(\cos\varphi\boldsymbol{\partial_\theta} - \cot\theta\sin\varphi\boldsymbol{\partial_\varphi}) \\ & + \kappa_3\boldsymbol{\partial_\varphi} + \kappa_4\boldsymbol{\partial_t}, \end{aligned} \tag{B7}$$

which is a linear combination of the generators (49a)–(49e) exhibit in the main text.

We continue by studying the differences with respect to the previous behavior of considering a time-dependent spherically symmetric *Ansatz* (56). Incorporating time dependency changes the jet space classification of Sec. II to $u^{\bar{I}} = u^I = (F, N, Y)$, $x^{\bar{\alpha}} = (t, r)$, and $x^{\hat{\alpha}} = (\theta, \varphi)$, giving now the complementary conditions

$$\partial_\theta \eta^I = \partial_\varphi \eta^I = \partial_\theta \xi^t = \partial_\varphi \xi^t = \partial_\theta \xi^r = \partial_\varphi \xi^r = 0, \tag{B8}$$

which modify the generator (B2) accordingly. In the residual criterion (B3), the Eqs. (B3a)–(B3d), which algebraically determine the generator components along the structural functions, remain unchanged, but the rest is transformed to

$$\partial_t \xi^r - N^2F^2\partial_r\xi^t = 0, \tag{B9a}$$

$$\partial_t \xi^\theta = \partial_r \xi^\theta = \partial_t \xi^\varphi = \partial_r \xi^\varphi = 0, \tag{B9b}$$

$$\partial_\varphi \xi^\theta + \sin^2(\theta)\partial_\theta \xi^\varphi = 0. \tag{B9c}$$

Performing a similar analysis as in the static case we end with the same solution except for the component ξ^t, which is now an arbitrary function of t, and this modifies the component along N as follows

$$\eta^N = -(\partial_r \xi^r + \partial_t \xi^t)N. \tag{B10}$$

Consequently, in the time-dependent case the generator (B7) then changes to

$$\begin{aligned} \boldsymbol{X} = {} & \xi^t(t)\boldsymbol{\partial_t} - \partial_t\xi^t N\boldsymbol{\partial_N} \\ & + \xi^r(r)\boldsymbol{\partial_r} - \partial_r\xi^r(N\boldsymbol{\partial_N} - 2F\boldsymbol{\partial_F}) \\ & + \kappa_1(\sin\varphi\boldsymbol{\partial_\theta} + \cot\theta\cos\varphi\boldsymbol{\partial_\varphi}) \\ & + \kappa_2(\cos\varphi\boldsymbol{\partial_\theta} - \cot\theta\sin\varphi\boldsymbol{\partial_\varphi}) + \kappa_3\boldsymbol{\partial_\varphi}. \end{aligned} \tag{B11}$$

APPENDIX C: AdS WAVES, DETAILS

In the jet space coordinates for the D-dimensional AdS waves (62), $z^A = (u, v, y, \vec{x}, F)$, the structural function $u^{\bar{I}} = u^I = (F)$ depends only on the retarded time and the front-wave coordinates $x^{\bar{\alpha}} = (u, y, \vec{x})$, i.e. it is independent of the null ray parameter $x^{\hat{\alpha}} = (v)$. Hence, the complementary conditions (26c) take the form

$$\partial_v \eta^F = \partial_v \xi^u = \partial_v \xi^y = \partial_v \xi^i = 0, \tag{C1}$$

giving the following general form for the generator of infinitesimal residual symmetries

$$\begin{aligned} \boldsymbol{X} = {} & \xi^u(u, y, \vec{x})\boldsymbol{\partial_u} + \xi^v(u, v, y, \vec{x})\boldsymbol{\partial_v} + \xi^y(u, y, \vec{x})\boldsymbol{\partial_y} \\ & + \vec{\boldsymbol{\xi}}(u, y, \vec{x}) + \eta^F(u, y, \vec{x}, F)\boldsymbol{\partial_F}, \end{aligned} \tag{C2}$$

where $i = 1, \ldots, D-3$ and we use the standard notation for Euclidean vectors $\vec{\boldsymbol{v}} = v^i\boldsymbol{\partial_i}$.

The Lie-derivative criterion (26a) gives the following system of equations

$$\eta^F + 2\left(\partial_u \xi^u - \frac{\xi^y}{y}\right)F + 2\partial_u \xi^v = 0, \tag{C3a}$$

$$\partial_u \xi^u + \partial_v \xi^v - \frac{2}{y}\xi^y = 0, \tag{C3b}$$

$$\partial_y \xi^u = \partial_i \xi^u = 0, \tag{C3c}$$

$$\partial_y \xi^v - \partial_u \xi^y = 0, \tag{C3d}$$

$$\partial_i \xi^v - \partial_u \xi_i = 0, \tag{C3e}$$

$$\partial_y \xi^y - \frac{\xi^y}{y} = 0, \tag{C3f}$$

$$\partial_i \xi^y + \partial_y \xi_i = 0, \tag{C3g}$$

$$\partial_{(i}\xi_{j)} - \frac{\xi^y}{y}\delta_{ij} = 0, \tag{C3h}$$

where we denote $\xi_i = \delta_{ij}\xi^j$. The first obvious conclusion from (C3c) is that the component along the retarded time only depends of the retarded time, $\xi^u = \xi^u(u)$. Second, deriving the system with respect to the null ray parameter v and using the complementary conditions (C1) we obtain

$$\partial^2_{vu}\xi^v = \partial^2_{vv}\xi^v = \partial^2_{vy}\xi^v = \partial^2_{vi}\xi^v = 0, \tag{C4}$$

which means the dependence of ξ^v in v is linear and separable in sum with respect to the rest of the coordinates

$$\xi^v = 2C_1 v + \mathcal{V}(u, y, \vec{x}). \tag{C5}$$

Then, isolating from Eq. (C3b) we obtain

$$\xi^y = \frac{y}{2}(\dot{\xi}^u + 2C_1), \tag{C6}$$

where the dot represents the derivative with respect to the retarded time u. This implies that ξ^y is a function of u and y only, using this fact in Eq. (C3g) to allow us to conclude that the spatial components ξ_i are functions of u and $\vec{x}$ only. Additionally, the symmetric part of their spatial derivatives $\partial_i \xi_j$ is obtained from Eq. (C3h) as

$$\partial_{(i}\xi_{j)} = \frac{1}{2}(\dot{\xi}^u + 2C_1)\delta_{ij}, \tag{C7}$$

and consequently, the derivatives themselves are given by

$$\partial_i \xi_j = \frac{1}{2}(\dot{\xi}^u + 2C_1)\delta_{ij} + \beta_{ij}(u, \vec{x}), \tag{C8}$$

where $\beta_{ij} = -\beta_{ji}$ is the antisymmetric part of the derivatives. Now deriving the symmetric part (C7) with respect to x^l we end with

$$\partial^2_{il}\xi_j + \partial^2_{lj}\xi_i = 0, \tag{C9a}$$

and permuting indices we can also write

$$\partial^2_{lj}\xi_i + \partial^2_{ji}\xi_l = 0, \tag{C9b}$$

$$\partial^2_{ji}\xi_l + \partial^2_{il}\xi_j = 0. \tag{C9c}$$

Adding Eqs. (C9a) and (C9c) and subtracting Eq. (C9b) we arrive at

$$2\partial_l \partial_i \xi_j = 2\partial_l \beta_{ij} = 0. \tag{C10}$$

Then, the β_{ij}'s are functions of u only. This allows us to straightforwardly integrate (C8) as

$$\xi_j = \frac{1}{2}(\dot{\xi}^u + 2C_1)x_j + \beta_{ij}x^i + P_j(u). \tag{C11}$$

Using all this information Eqs. (C3d) and (C3e) reduce to

$$\partial_y \mathcal{V} = \frac{y}{2}\ddot{\xi}^u, \tag{C12a}$$

$$\partial_i \mathcal{V} = \frac{x_i}{2}\ddot{\xi}^u + \dot{\beta}_{ji}x^j + \dot{P}_i. \tag{C12b}$$

Taking the integrability conditions of Eq. (C12b) we get

$$0 = \partial_{[ji]}\mathcal{V} = \dot{\beta}_{ji},$$

concluding that the antisymmetric matrix is in fact a constant one, $\beta_{ij} = C_{ij}$. The system (C12) is then integrable and from (C5) the general solution for the component along the null ray is given by

$$\xi^v = 2C_1 v + \frac{1}{4}(y^2 + \vec{x}^2)\ddot{\xi}^u + \dot{\vec{P}} \cdot \vec{x} + \alpha(u). \tag{C13}$$

Finally, we determine the component along the structural function from Eq. (C3a)

$$\eta^F = 2C_1 F - \dot{\xi}^u - \frac{1}{2}(y^2 + \vec{x}^2)\dddot{\xi}^u - 2\ddot{\vec{P}} \cdot \vec{x} - 2\dot{\alpha}. \tag{C14}$$

When we substitute Eqs. (C6), (C11), (C13), and (C14) in the generator (C2) we obtain

$$\begin{aligned} \boldsymbol{X} = {} & C_1(2v\boldsymbol{\partial_v} + y\boldsymbol{\partial_y} + x^i\boldsymbol{\partial_i} + 2F\boldsymbol{\partial_F}) \\ & + \frac{1}{2}C^{ij}(x_i\boldsymbol{\partial_j} - x_j\boldsymbol{\partial_i}) + \alpha(u)\boldsymbol{\partial_v} - 2\dot{\alpha}\boldsymbol{\partial_F} \\ & + \xi^u(u)\boldsymbol{\partial_u} + \frac{\dot{\xi}^u}{2}(y\boldsymbol{\partial_y} + x^i\boldsymbol{\partial_i} - 2F\boldsymbol{\partial_F}) \\ & + \frac{1}{4}(y^2 + \vec{x}^2)(\ddot{\xi}^u\boldsymbol{\partial_v} - 2\dddot{\xi}^u\boldsymbol{\partial_F}) \\ & + \vec{\boldsymbol{P}}(u) + \vec{x} \cdot (\dot{\vec{P}}\boldsymbol{\partial_v} - 2\ddot{\vec{P}}\boldsymbol{\partial_F}), \end{aligned} \tag{C15}$$

where the spatial indices (i, j) are raised and lowered using the Euclidean metric δ_{ij}.

1. Finite reparametrization of the retarded time

The result of the previous derivation is a superposition of the vector fields (63a)–(63e) we report in Sec. V. Using the method of differential invariants explained in Appendix A, it is very easy to integrate most of these vector fields to show that the corresponding finite transformations are those also reported in the same section. The integration of the generator (63d) is a little more difficult since it comprises an arbitrary function similar to the example related to the Collinson *Ansatz* we study in Appendix A. Since the present case is more involved we found it convenient to give the details of its integration. The characteristic system for the generator (63d) is

$$\frac{\mathrm{d}u}{\xi^u}=\frac{2\mathrm{d}y}{\dot{\xi}^u y}=\frac{2\mathrm{d}x^1}{\dot{\xi}^u x^1}=\cdots=\frac{2\mathrm{d}x^{D-3}}{\dot{\xi}^u x^{D-3}} =\frac{4\mathrm{d}v}{\ddot{\xi}^u(y^2+\vec{x}^2)} =\frac{-2\mathrm{d}F}{2\dot{\xi}^u F+\dddot{\xi}^u(y^2+\vec{x}^2)}, \tag{C16}$$

and is equivalent to the set of equations

$$\frac{\mathrm{d}x^{\hat{\imath}}}{x^{\hat{\imath}}}=\frac{\dot{\xi}^u\mathrm{d}u}{2\xi^u}, \tag{C17a}$$

$$\mathrm{d}v=\frac{1}{4}\frac{x_{\hat{\imath}}x^{\hat{\imath}}}{\xi^u}\ddot{\xi}^u\mathrm{d}u, \tag{C17b}$$

$$\xi^u\mathrm{d}F=-\left(\dot{\xi}^u F+\frac{1}{2}\dddot{\xi}^u x_{\hat{\imath}}x^{\hat{\imath}}\right)\mathrm{d}u, \tag{C17c}$$

where $x^{\hat{\imath}}=(y,x^1,\dots,x^{D-3})$ are the wave-front coordinates and no sum is understood in the first equation. We proceed as we did for the Collinson *Ansatz*, remembering that the arbitrary function $\xi^u=\xi^u(u)$ is the infinitesimal version of a general reparametrization of the retarded time $\tilde{u}=\tilde{u}(u;\varepsilon)$ and

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}=\frac{\tilde{\xi}^u(\tilde{u})}{\xi^u(u)}. \tag{C18}$$

This relation will allow us to eliminate the appearances of the infinitesimal contributions of the reparametrization in favor of the finite one and their derivatives. We begin with the first characteristic equations (C17a) which can be straightforwardly integrated given the first $D-2$ differential invariants,

$$\mathrm{d}\ln\left(\frac{x^{\hat{\imath}}}{\sqrt{\xi^u}}\right)=0,\Rightarrow\Omega^{\hat{\imath}}(u,x^{\hat{\imath}})=\frac{x^{\hat{\imath}}}{\sqrt{\xi^u}}. \tag{C19}$$

Using their invariant property

$$\frac{\tilde{x}^{\hat{\imath}}}{\sqrt{\tilde{\xi}^u(\tilde{u})}}=\frac{x^{\hat{\imath}}}{\sqrt{\xi^u(u)}}, \tag{C20}$$

and the relation (C18) we find the $D-2$ finite transformations for the wave-front coordinates

$$\tilde{x}^{\hat{\imath}}=\sqrt{\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}}x^{\hat{\imath}}. \tag{C21}$$

For solving the second characteristic equation (C17b) we use that any function of a differential invariant is also a differential invariant, hence, from definition (C19) we get that

$$\Omega_{\hat{\imath}}\Omega^{\hat{\imath}}=\frac{x_{\hat{\imath}}x^{\hat{\imath}}}{\xi^u}, \tag{C22}$$

is a differential invariant. Replacing it in Eq. (C17b) we arrive at a new differential invariant

$$\mathrm{d}\left(v-\frac{1}{4}\Omega_{\hat{\imath}}\Omega^{\hat{\imath}}\dot{\xi}^u\right)=0, \Rightarrow\Omega_v(u,v)=v-\frac{1}{4}\Omega_{\hat{\imath}}\Omega^{\hat{\imath}}\dot{\xi}^u. \tag{C23}$$

Using their invariant property $\Omega_v(\tilde{u},\tilde{v})=\Omega_v(u,v)$ and the derivative of the relation (C18) we obtain the finite transformation for the null ray parameter

$$\tilde{v}=v+\frac{1}{4}x_{\hat{\imath}}x^{\hat{\imath}}\frac{\dot{\tilde{\xi}}^u(\tilde{u})-\dot{\xi}^u(u)}{\xi^u(u)} =v+\frac{1}{4}x^{\hat{\imath}}x_{\hat{\imath}}\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right). \tag{C24}$$

The last characteristic equation (C17c) can be rewritten using (C22) as

$$\xi^u\mathrm{d}F+\left(\dot{\xi}^u F+\frac{1}{2}\Omega_{\hat{\imath}}\Omega^{\hat{\imath}}\xi^u\dddot{\xi}^u\right)du=0, \tag{C25}$$

which is an integrable equation since the crossed partial derivatives coincide. Hence, the equation is an exact differential of the last differential invariant,

$$\Omega_F(u,F)=\xi^u F+\frac{\Omega_{\hat{\xi}}\Omega^{\hat{\xi}}}{2}\left(\xi^u\ddot{\xi}^u-\frac{1}{2}(\dot{\xi}^u)^2\right). \tag{C26}$$

From their invariance $\Omega_F(\tilde{u},\tilde{F})=\Omega_F(u,F)$ and using the second derivative of the relation (C18) we get the finite transformation for the structural function

$$\tilde{F}=\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{-1}\left[F+\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\right)^{-1/2}x_{\hat{\imath}}x^{\hat{\imath}}\right]. \tag{C27}$$

Equations (C21), (C24), and (C27) are the way in which the rest of the variables involved in the AdS wave *Ansatz* (62) must change in order to compensate for an arbitrary reparametrization of the retarded time $\tilde{u}=\tilde{u}(u)$. This is precisely the transformation (79) reported in the main text.

2. AdS from AdS waves

The last issue we address in this appendix is the derivation of the most general way to locally express the AdS spacetime with the AdS wave *Ansatz* (62), which gives just the structural function (82). This is relevant to the interpretation of the infinite-dimensional subgroup of the connected group of residual symmetries we provide at the end of Sec. V. AdS is defined as the spacetime having constant negative curvature in any dimension D and we need to impose this condition on the AdS waves. Spacetime curvature is in general decomposed in their completely traceless part represented in the Weyl tensor, $W^{\alpha}{}_{\beta\mu\nu}$, and their first trace defining the Ricci tensor, $R_{\alpha\beta}$. In turn, the Ricci tensor has a traceless part, $S_{\alpha\beta}=R_{\alpha\beta}-Rg_{\alpha\beta}/D$, and its trace which is the scalar curvature, R. Hence, the Weyl tensor, the traceless part of the Ricci tensor, and the scalar curvature completely determine the full curvature. Spaces where one or both of the traceless parts vanish have special properties: due to the conformal invariance of the Weyl tensor its vanishing defines the conformally flat spacetimes, a vanishing traceless part of the Ricci tensor defines the so-called Einstein spaces, and when both tensors vanish the full curvature is completely determined by the scalar curvature, which is necessarily constant. The latter are the constant curvature spaces. Since the AdS waves (62) already have constant negative scalar curvature, we only impose they are conformally flat and at the same time Einstein spaces. The traceless curvature tensors for these backgrounds are given by

$$W_{\alpha\beta\mu\nu}=\frac{2l^2}{y^2}\left(\partial^2_{\hat{\imath}\hat{\jmath}}F-\frac{\partial^{\hat{k}}\partial_{\hat{k}}F}{D-2}\delta_{\hat{\imath}\hat{\jmath}}\right)\delta^u_{[\alpha}\delta^{\hat{\imath}}_{\beta]}\delta^u_{[\mu}\delta^{\hat{\jmath}}_{\nu]}, \quad (C28)$$

$$S_{\alpha\beta}=\frac{1}{2}\left(\partial^{\hat{k}}\partial_{\hat{k}}F-\frac{D-2}{y}\partial_yF\right)\delta^u_\alpha\delta^u_\beta. \quad (C29)$$

The vanishing of the Weyl components

$$W_{u\hat{\imath}u\hat{\jmath}}=\frac{l^2}{2y^2}\partial^2_{\hat{\imath}\hat{\jmath}}F=0, \qquad \hat{\imath}\neq\hat{\jmath} \quad (C30)$$

implies the structural function characterizing the profile of the AdS wave is separable in sum in all the front-wave coordinates,

$$F(u,x^{\hat{\imath}})=Y(u,y)+X^1(u,x^1)+\cdots+X^{D-3}(u,x^{D-3}). \quad (C31)$$

From the differences of the Weyl components

$$W_{uiui}-W_{uyuy}=\frac{l^2}{2y^2}(\partial^2_{ii}F-\partial^2_{yy}F)=0, \quad (C32)$$

for each $i=1,\ldots,D-3$ (no sum in i is understood), and using the separability (C31) we obtain

$$\partial^2_{yy}Y=\partial^2_{11}X^1=\cdots=\partial^2_{D-3,D-3}X^{D-3}=2F_2(u), \quad (C33)$$

i.e. the equality between the different dependencies is only possible if there is no dependence at all on the front-wave coordinates for these second derivatives, which in turn defines the function F_2 depending only on the retarded time. Equation (C33) can be easily integrated to give the most general conformally flat AdS wave profile,

$$F_{\rm CF}=F_2(u)x_{\hat{\imath}}x^{\hat{\imath}}+F_{1\hat{\imath}}(u)x^{\hat{\imath}}+F_0(u). \quad (C34)$$

Imposing now that the last backgrounds are additionally Einstein spaces,

$$S_{\alpha\beta}=-\frac{D-2}{2}\frac{F_{1y}}{y}\delta^u_\alpha\delta^u_\beta=0, \quad (C35)$$

implies the vanishing of the function F_{1y}, which reduces the conformally flat profiles (C34) to the constant curvature ones defining AdS spacetime (82).

APPENDIX D: PAPAPETROU *ANSATZ*, DETAILS

The jet space coordinates for the Papapetrou *Ansatz* are $z^A=(t,\varphi,\rho,z,X,A,h)$, where the structural functions $u^{\bar{I}}=u^I=(X,A,h)$ are functions of the spatial coordinates $x^{\bar{\alpha}}=(\rho,z)$ and are independent of the Killing coordinates $x^{\hat{\alpha}}=(t,\varphi)$. In this case the complementary conditions (26c) are

$$\partial_t\eta^I=\partial_\varphi\eta^I=\partial_t\xi^\rho=\partial_t\xi^z=\partial_\varphi\xi^\rho=\partial_\varphi\xi^z=0, \quad (D1)$$

and the generator of residual symmetries has the following general form

$$\begin{aligned}\boldsymbol{X}=\xi^t(t,\varphi,\rho,z)\boldsymbol{\partial_t}+\xi^\varphi(t,\varphi,\rho,z)\boldsymbol{\partial_\varphi}+\xi^\rho(\rho,z)\boldsymbol{\partial_\rho}\\+\xi^z(\rho,z)\boldsymbol{\partial_z}+\eta^I(\rho,z,u^J)\boldsymbol{\partial_I}.\end{aligned} \quad (D2)$$

The criterion (26a) in this case gives the following system of equations for the components of the generator

$$\begin{aligned}(A^2+\rho^2/X^2)\eta^X+2AX\eta^A+2AX\partial_t\xi^\varphi\\+2X(A^2-\rho^2/X^2)\partial_t\xi^t-\frac{2\rho}{X}\xi^\rho=0,\end{aligned} \quad (D3a)$$

$$\begin{aligned}A\eta^X+X\eta^A+(A^2-\rho^2/X^2)X\partial_\varphi\xi^t\\+X\partial_t\xi^\varphi+AX(\partial_t\xi^t+\partial_\varphi\xi^\varphi)=0,\end{aligned} \quad (D3b)$$

$$\eta^X+2AX\partial_\varphi\xi^t+2X\partial_\varphi\xi^\varphi=0, \quad (D3c)$$

$$\eta^X-2X\eta^h-2X\partial_\rho\xi^\rho=0, \quad (D3d)$$

$$\eta^X-2X\eta^h-2X\partial_z\xi^z=0, \quad (D3e)$$

$$\partial_\rho \xi^z + \partial_z \xi^\rho = 0, \quad \text{(D3f)}$$

$$(A^2 - \rho^2/X^2)\partial_\rho \xi^t + A\partial_\rho \xi^\varphi = 0, \quad \text{(D3g)}$$

$$(A^2 - \rho^2/X^2)\partial_z \xi^t + A\partial_z \xi^\varphi = 0, \quad \text{(D3h)}$$

$$\partial_\rho \xi^\varphi + A\partial_\rho \xi^t = \partial_z \xi^\varphi + A\partial_z \xi^t = 0. \quad \text{(D3i)}$$

Since the spacetime components of the generator are independent of the structural functions, Eqs. (D3g)–(D3i) only hold if ξ^t and ξ^φ exclusively depend on t and φ. In Eqs. (D3b) and (D3c) we have dependencies on t and φ through the derivatives of the previous components, and they are compatible with the independence of the η^I on t and φ (D1) only if additionally

$$\left.\begin{array}{ll} \partial_t \xi^t = \hat{C}_1, & \partial_\varphi \xi^t = C_3, \\ \partial_\varphi \xi^\varphi = \hat{C}_2, & \partial_t \xi^\varphi = C_4, \end{array}\right\}$$
$$\Rightarrow \begin{cases} \xi^t = \hat{C}_1 t + C_3 \varphi + \kappa_1, \\ \xi^\varphi = \hat{C}_2 \varphi + C_4 t + \kappa_2. \end{cases}$$

This makes Eqs. (D3b) and (D3c) a compatible linear system for η^X and η^A. Inserting their solution into Eq. (D3a) it is possible to solve also for ξ^ρ giving

$$\xi^\rho = -(\hat{C}_1 + \hat{C}_2)\rho, \quad \text{(D4)}$$

$$\eta^X = -2(\hat{C}_2 + C_3 A)X, \quad \text{(D5)}$$

$$\eta^A = -C_4 - (\hat{C}_1 - \hat{C}_2)A + C_3\left(A^2 + \frac{\rho^2}{X^2}\right). \quad \text{(D6)}$$

Since ξ^ρ is independent of z it follows from Eq. (D3f) that ξ^z is independent of ρ. Moreover, from Eqs. (D3d) and (D3e) we obtain

$$\xi^z = -(\hat{C}_1 + \hat{C}_2)z + C_5. \quad \text{(D7)}$$

Finally, from Eq. (D3d) we find

$$\eta^h = \hat{C}_1 - C_3 A. \quad \text{(D8)}$$

The generator (D2) is fixed as

$$\begin{aligned} \boldsymbol{X} = {} & C_1(t\boldsymbol{\partial_t} + \varphi\boldsymbol{\partial_\varphi} - 2\rho\boldsymbol{\partial_\rho} - 2z\boldsymbol{\partial_z} - 2X\boldsymbol{\partial_X} + \boldsymbol{\partial_h}) \\ & + C_2(t\boldsymbol{\partial_t} - \varphi\boldsymbol{\partial_\varphi} - 2A\boldsymbol{\partial_A} + 2X\boldsymbol{\partial_X} + \boldsymbol{\partial_h}) \\ & + C_3[\varphi\boldsymbol{\partial_t} - 2AX\boldsymbol{\partial_X} + (A^2 + \rho^2/X^2)\boldsymbol{\partial_A} - A\boldsymbol{\partial_h}] \\ & + C_4(t\boldsymbol{\partial_\varphi} - \boldsymbol{\partial_A}) + C_5\boldsymbol{\partial_z} + \kappa_1\boldsymbol{\partial_t} + \kappa_2\boldsymbol{\partial_\varphi}, \end{aligned} \quad \text{(D9)}$$

where we redefined $\hat{C}_1 = C_1 + C_2$ and $\hat{C}_2 = C_1 - C_2$. This is a linear combination of the generators (84) given in the main text.

APPENDIX E: NONCIRCULAR COLLINSON *ANSATZ*, DETAILS

The jet space coordinates of the noncircular Collinson *Ansatz* are extended to include the noncircular contributions as $z^A = (\tau, \sigma, x, y, a, b, P, Q, M, N)$. Since all the structural functions $u^{\bar{I}} = u^I = (a, b, P, Q, M, N)$ are independent of the Killing coordinates $x^{\hat{\alpha}} = (\tau, \sigma)$ and depend only on the spatial coordinates $x^{\bar{\alpha}} = (x, y)$, the complementary conditions (26c) are

$$\partial_\tau \eta^I = \partial_\sigma \eta^I = \partial_\tau \xi^x = \partial_\tau \xi^y = \partial_\sigma \xi^x = \partial_\sigma \xi^y = 0. \quad \text{(E1)}$$

Hence, the generator of residual symmetries for the line element (92) has the general form

$$\begin{aligned} \boldsymbol{X} = {} & \xi^\tau(\tau, \sigma, x, y)\boldsymbol{\partial_\tau} + \xi^\sigma(\tau, \sigma, x, y)\boldsymbol{\partial_\sigma} + \xi^x(x, y)\boldsymbol{\partial_x} \\ & + \xi^y(x, y)\boldsymbol{\partial_y} + \eta^I(x, y, u^J)\boldsymbol{\partial_I}. \end{aligned} \quad \text{(E2)}$$

The Lie-derivative criterion (26a) gives the following system of equations for the components of the previous generator

$$2\eta^Q + \frac{\eta^a + \eta^b}{a+b} - 2\partial_\tau \xi^\tau + (b-a)\partial_\tau \xi^\sigma = 0, \quad \text{(E3a)}$$

$$\begin{aligned} & (b-a)\eta^Q + \frac{b\eta^a - a\eta^b}{a+b} + \partial_\sigma \xi^\tau - ab\partial_\tau \xi^\sigma \\ & \quad - \frac{1}{2}(b-a)(\partial_\tau \xi^\tau + \partial_\sigma \xi^\sigma) = 0, \end{aligned} \quad \text{(E3b)}$$

$$\begin{aligned} & -2ab\eta^Q + \frac{b^2\eta^a + a^2\eta^b}{a+b} + 2ab\partial_\sigma \xi^\sigma \\ & \quad + (b-a)\partial_\sigma \xi^\tau = 0, \end{aligned} \quad \text{(E3c)}$$

$$\begin{aligned} & (M-N)(2\eta^Q - \partial_\tau \xi^\tau - \partial_y \xi^y) + \frac{M-N}{a+b}(\eta^a + \eta^b) - \eta^M + \eta^N \\ & \quad + (b-a)\partial_y \xi^\sigma + (Mb + Na)\partial_\tau \xi^\sigma - 2\partial_y \xi^\tau = 0, \end{aligned} \quad \text{(E3d)}$$

$$\begin{aligned} & (Mb + Na)(2\eta^Q - \partial_\sigma \xi^\sigma - \partial_y \xi^y) + \frac{M-N}{a+b}(b\eta^a - a\eta^b) \\ & \quad - b\eta^M - a\eta^N - 2ab\partial_y \xi^\sigma - (b-a)\partial_y \xi^\tau \\ & \quad + (M-N)\partial_\sigma \xi^\tau = 0, \end{aligned} \quad \text{(E3e)}$$

$$\eta^P + \eta^Q - \partial_x \xi^x = 0, \quad \text{(E3f)}$$

$$\begin{aligned} & -2(a+b)\left(e^{-2P} + \frac{MN}{a+b}\right)(\eta^Q - \partial_y \xi^y) - \frac{MN}{a+b}(\eta^a + \eta^b) \\ & \quad + N\eta^M + M\eta^N - 2(a+b)e^{-2P}\eta^P - (M-N)\partial_y \xi^\tau \\ & \quad + (Mb + Na)\partial_y \xi^\sigma = 0, \end{aligned} \quad \text{(E3g)}$$

$$(b-a)\partial_x \xi^\sigma - (M-N)\partial_x \xi^y - 2\partial_x \xi^\tau = 0, \quad \text{(E3h)}$$

$$(b-a)\partial_x\xi^\tau + (Mb+Na)\partial_x\xi^y + 2ab\partial_x\xi^\sigma = 0, \quad \text{(E3i)}$$

$$-(M-N)\partial_x\xi^\tau + (Mb+Na)\partial_x\xi^\sigma + 2(a+b)e^{-P}(\partial_y\xi^x + \partial_x\xi^y) + 2MN\partial_x\xi^y = 0. \quad \text{(E3j)}$$

Since the structural functions of the metric and the spacetime coordinates are understood as independent variables in jet space, and the spacetime components of the generator only depend on the latter, then Eqs. (E3h)–(E3j) are only fulfilled if

$$\partial_x\xi^\tau = \partial_x\xi^\sigma = \partial_y\xi^x = \partial_x\xi^y = 0. \quad \text{(E4)}$$

Therefore, ξ^τ and ξ^σ are independent of x while ξ^y and ξ^x only are functions of y and x, respectively. Using the same argument together with the complementarity conditions (E1), after taking the derivatives of Eqs. (E3a)–(E3g) with respect to τ and σ, results in the additional constraints

$$\partial^2_{\tau\tau}\xi^\tau = \partial^2_{\sigma\sigma}\xi^\tau = \partial^2_{\tau\sigma}\xi^\tau = \partial^2_{\tau y}\xi^\tau = \partial^2_{\sigma y}\xi^\tau = 0,$$
$$\partial^2_{\tau\tau}\xi^\sigma = \partial^2_{\sigma\sigma}\xi^\sigma = \partial^2_{\tau\sigma}\xi^\sigma = \partial^2_{\tau y}\xi^\sigma = \partial^2_{\sigma y}\xi^\sigma = 0. \quad \text{(E5)}$$

They imply that the components ξ^τ and ξ^σ are separable in sum in all their dependencies and are additionally linear in the Killing coordinates τ and σ

$$\xi^\tau = C_1\tau + C_2\sigma + F_1(y), \quad \text{(E6)}$$

$$\xi^\sigma = C_3\tau + C_4\sigma + F_2(y). \quad \text{(E7)}$$

When we substitute the above solutions into the seventh equations (E3a)–(E3g) we obtain a linear system of equations for the six η^I's plus a constraint. The constraint is just the condition

$$\partial_x\xi^x = \partial_y\xi^y, \quad \text{(E8)}$$

which is solved as

$$\xi^x = C_5x + C_6, \quad \text{(E9)}$$

$$\xi^y = C_5y + C_7. \quad \text{(E10)}$$

Now Eqs. (E3a)–(E3f) are a consistent linear system whose solution is given by

$$\eta^a = C_3a^2 + (C_1 - C_4)a - C_2, \quad \text{(E11)}$$

$$\eta^b = -C_3b^2 + (C_1 - C_4)b + C_2, \quad \text{(E12)}$$

$$\eta^P = C_5 - \frac{1}{2}(C_1 + C_4), \quad \text{(E13)}$$

$$\eta^Q = \frac{1}{2}(C_1 + C_4), \quad \text{(E14)}$$

$$\eta^M = (C_1 - C_5)M + C_3aM - \partial_yF_2a - \partial_yF_1, \quad \text{(E15)}$$

$$\eta^N = (C_1 - C_5)N - C_3bN - \partial_yF_2b + \partial_yF_1. \quad \text{(E16)}$$

Therefore, the generator of residual symmetries for the noncircular Collinson *Ansatz* takes the form

$$\begin{aligned}
\boldsymbol{X} = {} & C_1\left(\tau\boldsymbol{\partial_\tau} + a\boldsymbol{\partial_a} + b\boldsymbol{\partial_b} + M\boldsymbol{\partial_M} + N\boldsymbol{\partial_N} - \frac{1}{2}\boldsymbol{\partial_P} + \frac{1}{2}\boldsymbol{\partial_Q}\right)\\
& + C_2(\sigma\boldsymbol{\partial_\tau} - \boldsymbol{\partial_a} + \boldsymbol{\partial_b})\\
& + C_3(\tau\boldsymbol{\partial_\sigma} + a^2\boldsymbol{\partial_a} - b^2\boldsymbol{\partial_b} + Ma\boldsymbol{\partial_M} - Nb\boldsymbol{\partial_N})\\
& + C_4\left(\sigma\boldsymbol{\partial_\sigma} - a\boldsymbol{\partial_a} - b\boldsymbol{\partial_b} - \frac{1}{2}\boldsymbol{\partial_P} + \frac{1}{2}\boldsymbol{\partial_Q}\right)\\
& + C_5(x\boldsymbol{\partial_x} + y\boldsymbol{\partial_y} - M\boldsymbol{\partial_M} - N\boldsymbol{\partial_N} + \boldsymbol{\partial_P})\\
& + C_6\boldsymbol{\partial_x} + C_7\boldsymbol{\partial_y} + F_1\boldsymbol{\partial_\tau} - \partial_yF_1(\boldsymbol{\partial_M} - \boldsymbol{\partial_N})\\
& + F_2\boldsymbol{\partial_\sigma} - \partial_yF_2(a\boldsymbol{\partial_M} + b\boldsymbol{\partial_N}), \quad \text{(E17)}
\end{aligned}$$

which is a linear combination of the vector fields reported in the main text equations (93).

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