PHYSICAL REVIEW D **93,** 044039 (2016)

# Tensor instabilities at the end of the ΛCDM universe

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(Received 3 October 2015; published 16 February 2016)

The unphysical spin-2 massive degrees of freedom in higher-derivative gravity may be either massive unphysical ghosts or tachyonic ghosts. In the last case there is no Planck-scale threshold protecting vacuum cosmological solutions from instabilities. Within the anomaly-induced action formalism the photon-driven IR running of the coefficient of the Weyl-squared term makes the ghost eventually become tachyonic, which should produce a gravitational explosion of vacuum. This effect is stable under higher-loop corrections and takes place also in known versions of perturbative quantum gravity. However, the contribution of massless fields in the far IR are not the same in flat and de Sitter spaces. In the asymptotically de Sitter case one can observe a kind of IR decoupling, which protects the cosmological solution from the future tachyonic instabilities.

DOI: 10.1103/PhysRevD.93.044039

## I. INTRODUCTION

It is well-known that fourth-derivative terms are necessary in the gravitational action, to render the theory consistent at the semiclassical level (see Refs. [1,2] for an introduction to the subject and also Ref. [3] for a recent pedagogical review). In quantum gravity (QG) the same fourth-derivative terms provide renormalizability of the theory [4]. At the same time, adding fourth-derivative terms to the standard Einstein-Hilbert action has an undesirable effect, since the spectrum of the theory in the spin-2 sector gains a massive unphysical ghost in addition to the usual healthy graviton. An attempt to remove these unphysical states from the initial quantum state leads to a nonunitary quantum theory. In the semiclassical theory, when gravity is a classical background for quantum matter fields, there is no problem with the unitarity of the gravitational $S$-matrix, and hence the condition of consistency can be reduced to the requirement of stability of the physically relevant classical solutions with respect to small metric perturbations. In the present work we will discuss a new aspect of this problem, related to the difference between the massive unphysical ghost and the tachyon.

Let us start by fixing the notations. The action of gravity corresponding to the renormalizable semiclassical theory (for an introduction one can see, e.g., Refs. [1–3], further references therein and also recent more formal work [5]) consists of the following terms:

$$S_{\rm vac} = S_{\rm EH} + S_{\rm HD}, \tag{1}$$

where $S_{\rm EH}$ is the Einstein-Hilbert action with a cosmological constant term,

$$S_{\rm EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\{R + 2\Lambda\} \tag{2}$$

and $S_{\rm HD}$ includes a minimal necessary set of higher-derivative terms,

$$S_{\rm HD} = \int d^4x\sqrt{-g}\{a_1C^2 + a_2E_4 + a_3\Box R + a_4R^2\}. \tag{3}$$

In this expression

$$E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2 \tag{4}$$

is the Gauss-Bonnet term (Euler density in $d = 4$) and $C^2$ is the square of the Weyl tensor. Furthermore, we use units corresponding to $c = \hbar = 1$ and denote with $M_p = 1/\sqrt{G}$ the Planck mass. The metric has signature $(+,-,-,-)$. For the sake of simplicity we assume that the absolute value of $a_1$ is of order one.

A general observation about the form of the action and especially the higher-derivative terms (3) is in order. One can deal with these terms in two different (albeit physically equivalent) ways. First, it is possible to consider the classical action of gravity being free of higher-derivative terms. Then one can observe that (different from the flat-space QED, for instance) there are higher-derivative divergences, corresponding to the running of these terms and hence the nonlocal structures such as those we shall discuss later on in Eq. (22). This means that even if the terms (3) are

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not introduced at the classical level, the corresponding quantum contributions emerge anyway. Second, one can follow the standard quantum field theory approach and introduce into the classical action all those terms which will emerge with divergent coefficients in the loop corrections. The main advantage of this procedure is technical simplicity, because one can always deal with a usual renormalization procedure, consider the renormalization group running in a useful conventional way [2] etc. The last observation is that the list of the terms in Eqs. (2) and (3) follows from the fact that only these structures emerge in the semiclassical approach at all loop orders. This fact is known starting from the classical work of Utiyama and DeWitt [6], and can be easily seen from the covariant renormalizability and power counting for the semiclassical theory [5].

The spectrum of the theory (1) linearized around a Minkowski background contains in the spin-2 sector the massless graviton and an additional spin-2 particle with mass of order $M_p$. If $a_1$ is negative this particle has positive mass square, negative kinetic energy and hence is a ghost. For a positive $a_1$ the mass square is negative and the particle is a ghost and a tachyon at the same time [7]. We postpone the detailed description of these two types of particles to the next section and now only make some general observations.

According to the recent discussion in Ref. [8], at least on a cosmological Friedmann-Robertson-Walker (FRW) background massive ghost excitations are not generated far below the Planck threshold of frequencies. The same result was found much earlier [9–12] for a de Sitter (dS) background. Qualitatively similar considerations can be found in Ref. [13] for generic models of massive gravity, where one can also meet massive ghosts or tachyons. The result of Ref. [8] means that a massive unphysical ghost can be destructive for the theory, but only if it appears as a real physical particle. On the other side, the presence of a ghost in the mass spectrum of the theory may not affect stability if such a ghost is just a virtual particle in the vacuum state of the theory. One can note that the problem of a massive ghost is a tree-level problem of quantum theory. It is equivalent to the absence of stability in the classical theory. In practical terms the conclusion of Ref. [8] means that one can choose the initial condition in such a way that the system does not develop instability for a sufficiently long time. According to Ref. [8] this choice of initial conditions is possible for $a_1 < 0$.

On the contrary, for the case $a_1 > 0$ the extra massive particle is a ghost and tachyon at the same time and satisfies an antioscillator equation with exponential solutions. Then instabilities are unavoidable, even without the presence of an external force. Moreover, the intensity of the instability may be even enhanced if the mass of the tachyon increases. There is no possible choice of initial conditions which can help to control the instabilities in this case.

One can naturally ask whether we can define $a_1$ to be zero and thus avoid having both ghost and tachyon states in the spectrum. The answer to this question is negative, because at the quantum level the parameter $a_1$ runs due to the change of the energy scale. In this work we will assume that gravity is classical and we take into account only quantum effects of matter fields. In the last section we will also comment on the possible role of quantum gravity within some of the existing approaches.

Let us stress that the running of $a_1$ is a pretty well-explored theoretical phenomenon. In particular, the full expressions for the "physical" $\beta$ functions in the momentum subtraction scheme of renormalization were calculated in Refs. [14] and [15]. At low energies one can observe quantum decoupling, which is a close analog of the Appelquist and Carazzone theorem in QED [16]. This means that in the present-day Universe, when the typical energy scale of the background is of the order of $H_0 \sim 10^{-42}$ GeV, all massive fields are too massive to affect the running of $a_1$ in the IR. Hence, in the late universe it is sufficient to take into account quantum effects produced by the unique massless particle, which is the photon. Starting from Sec. III it will be done in the framework of the anomaly-induced effective action, which is the most appropriate formalism for classically conformal fields.

It is well known that the renormalization group in gravity meets a hard problem in identifying the energy scale $\mu$ with some physical quantity related to gravity (see, e.g., the discussion of this issue in Ref. [17] and also the scale-setting procedure suggested in Refs. [18,19]). At the same time, all these difficulties concern only quantum effects of massive fields, while the case of the photon is different. The point is that electromagnetic fields are massless and possess local conformal invariance. Thanks to this symmetry one can integrate the conformal anomaly [20–22] and arrive at the reliable form of the semiclassical corrections to the classical action. The anomaly-induced effective action [23] can be regarded as a local version of the renormalization group [24].

The purpose of the present work is to use the anomaly-induced action to explore the running of $a_1$ in the far IR due to loop effects of virtual photons on a cosmological background. This running predicts that at some point in the distant future the parameter $a_1$ changes sign and becomes positive. At this instant the tachyonic mode with mass parameter unbounded from above $m_2 \sim 1/a_1$ will manifest an explosive growth of gravitational waves within the full range of frequencies starting from zero to the Planck scale. This means that in a very short time interval metric perturbations break down the linear regime.

The paper is organized as follows. Section II mainly serves pedagogical purposes and describes the classification of ghosts and tachyons in free theories with actions of both second and fourth order in derivatives. We also discuss the difference between ghosts and tachyons in the effective field theory approach. In Sec. III we summarize the main results concerning the anomaly-induced effective action of gravity and explore the effective equations of motion on cosmological backgrounds for radiation-, matter- and cosmological-constant-dominated solutions. Section IV includes the final results for the physical running of $a_1$ in the same three cases. In Sec. V we specialize our study to the late de Sitter phase of the evolution of the universe and estimate the time period for the universe to accelerate until the instant when $a_1$ changes sign from negative to positive. Finally, in Sec. VI we critically discuss the list of assumptions which have been introduced in our analysis and draw our conclusions.

## II. GHOSTS AND TACHYONS

In this section we briefly describe the classification of free fields into normal "healthy" ones, ghosts and tachyons. For our purpose it is sufficient to consider flat space-time.

### A. Second-order theories

For the second-order theory the general action of a free field $h(x) = h(t, \mathbf{r})$ is

$$S(h) = \frac{s_1}{2}\int d^4x\{\eta^{\mu\nu}\partial_\mu h \partial_\nu h - s_2 m^2 h^2\} = \frac{s_1}{2}\int d^4x\{\dot{h}^2 - (\nabla h)^2 - s_2 m^2 h^2\}. \tag{5}$$

Both $s_1$ and $s_2$ are sign factors which take values $\pm 1$ for different types of fields. In what follows we consider all four combinations of these signs.

It proves useful to perform the Fourier transform in the space variables,

$$h(t, \mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{r}} h(t, \mathbf{k}), \tag{6}$$

and consider the dynamics of each component $h \equiv h(t, \mathbf{k})$ separately. It is easy to see that this dynamics is defined by the action

$$S_\mathbf{k}(h) = \frac{s_1}{2}\int dt\{\dot{h}^2 - k^2 h - s_2 m^2 h^2\} = \frac{s_1}{2}\int dt\{\dot{h}^2 - m_k^2 h^2\}, \tag{7}$$

where

$$\mathbf{k}^2 = \mathbf{k}\cdot\mathbf{k}, \quad \text{and} \quad m_k^2 = s_2 m^2 + \mathbf{k}^2. \tag{8}$$

The properties of the field are defined by the sign of $s_1$ and $s_2$. The possible options can be classified as follows.

(i) A *normal healthy field* corresponds to $s_1 = s_2 = 1$. The kinetic energy of the field is positive and the minimal action can be achieved for a static configuration. Also, the equation of motion is of the oscillatory type,

$$\ddot{h} + m_k^2 h = 0, \tag{9}$$

with the usual periodic solution.

(ii) A *tachyon* has $s_1 = 1$ and $s_2 = -1$. The classical dynamics of tachyons is described in the literature, e.g., Refs. [25,26], and hence we present just a very brief review.

For relatively small momenta $m_k^2 < 0$ in Eq. (8) and the equation of motion is

$$\ddot{h} - \omega^2 h = 0, \qquad \omega^2 = |m_k^2|. \tag{10}$$

Then we have an antioscillatory equation with exponential solutions,

$$h = h_1 e^{\omega t} + h_2 e^{-\omega t}, \qquad \omega^2 = |m_k^2|. \tag{11}$$

If the particle moves faster than light the solution is of the oscillatory kind, indicating that such a motion is "natural" for this kind of particle [25,26].

An additional observation concerning tachyons is in order. For a field interacting with an external gravitational background, it is not impossible to have a situation where the same wave changes from being a normal healthy state to a tachyonic one in different parts of the space-time manifold. In principle, this situation may produce very strong effects at both quantum [27] and classical [28] levels. In the present paper we are considering a qualitatively similar situation, where the massive component of the gravitational wave changes its fundamental properties due to the change of the energy scale or due to the time evolution in a de Sitter vacuum.

(iii) A *massive ghost* has $s_1 = -1$ and $s_2 = 1$. It is not a tachyon, because $m_k^2 \geq 0$. In this case the kinetic energy of the field is negative, but one can postulate zero variation of the action and arrive at the normal oscillatory equation (9). A particle with negative kinetic energy has the tendency to achieve a maximal speed, but a free particle cannot accelerate, for this would violate energy conservation. Hence a free ghost does not produce any harm to the environment, being isolated from it. However, if we admit an interaction with healthy fields, the tendency of a ghost is to accelerate and transmit a positive energy difference to these healthy fields [29], e.g., in the form of quantum emission of the corresponding particles. There are many detailed discussions of the fundamental problems at both classical and quantum levels which arise in the theories with ghosts (see, e.g., Refs. [30] and [31]). In general, the problem of ghosts and the consequent conflict between renormalizability and unitarity is one of the most important for quantum gravity, and hence has attracted a lot of attention. The list of references on the possible approaches to avoid the problem of higher-derivative ghosts can be found in Ref. [8]. In this work we also suggested a new approach to dealing with ghosts, which is based on the effective field theory ideas, but is technically classical and very simple. There are strong indications that the creation of ghost states from vacuum is strongly suppressed for a relatively weak cosmological background of the late universe. The consideration presented below is based on the same idea, and assumes the same approach.

(iv) A *tachyonic ghost* has $s_1 = s_2 = -1$. For relatively small $\mathbf{k}^2$ we have $m_k^2 < 0$. The kinetic energy is negative and the derivation of the equations of motion requires an additional definition similar to the nontachyonic ghost case. After that, one can notice that the equation of motion is of the anti-oscillatory type [Eq. (10)] and the solutions are exponential [Eq. (11)].

### B. Fourth-order gravity at the linearized level

In the fourth-order gravity (1) the equations for the metric perturbations in flat space can be easily obtained from the more general ones on cosmological backgrounds [12] [see also Eq. (52) in Sec. V],

$$\overset{::}{h} + 2\mathbf{k}^2\ddot{h} + \mathbf{k}^4 h - \frac{1}{32\pi G a_1}(\ddot{h} + \mathbf{k}^2 h) = 0. \tag{12}$$

It proves useful to introduce a new notation

$$\frac{1}{32\pi G a_1} = -s_2 m^2, \tag{13}$$

where $s_2 = -\text{sign} a_1$ and $m^2 > 0$. Then one can recast Eq. (12) into the form

$$\left(\frac{\partial^2}{\partial t^2} + \mathbf{k}^2\right)\left(\frac{\partial^2}{\partial t^2} + m_k^2\right) h = 0, \tag{14}$$

where $m_k^2 = \mathbf{k}^2 + s_2 m^2$. The solutions of the last equation can be different, depending on the sign of $a_1$ and hence of $s_2$. The general formula for the frequencies is

$$\omega_{1,2} \approx \pm i(\mathbf{k}^2)^{1/2} \quad \text{and} \quad \omega_{3,4} \approx \pm(-m_k^2)^{-1/2}. \tag{15}$$

For a negative $a_1$ there are only imaginary frequencies and hence oscillator-type solutions. On the contrary, for a positive $a_1$ we have $s_2 = -1$ and the roots $\omega_{3,4}$ are real, since in this case $-m_k^2 > 0$ for sufficiently small $\mathbf{k}^2$. Indeed, the first couple of roots corresponds to the massless graviton, and the second couple to an extra massive particle. According to our classification, this particle is a ghost for $a_1 < 0$ and, simultaneously, a ghost and tachyon for $a_1 > 0$ (see Ref. [7] for a detailed discussion).

### C. Ghost vs tachyon

The main difference between ghosts and tachyons is that a ghost may cause instabilities only when it couples to some healthy fields or to the background, while with tachyons there is no such protection.

The situation with ghosts can be kept under control in the effective field theory framework, as it was recently discussed in Ref. [32]. In an effective field theory there may be an apparent ghost due to the low-energy expansion in the powers of $\mathcal{E}/M$, where $\mathcal{E}$ is the energy and $M$ is the cutoff. At the same time, there could be no ghost in the underlying fundamental quantum theory. In this situation one can safely use an effective description for energies $\mathcal{E} \ll M$ and not pay much attention to the presence of ghosts. The situation in gravity is similar, but with an important difference that the ghost-free UV completion of the renormalizable theory is not known. Let us note that string theory cannot be regarded as such a UV completion, since the $R_{\mu\nu}^2$-like terms which are the source of ghosts are removed in string theory by a special transformation of the background metric [33] and not by a low-energy expansion. On the other side, there are strong indications that the gravitational theory with massive ghosts can be free of instabilities at low energies, since for $\mathcal{E} \ll M_P$ the ghost may be a virtual particle and effectively it is not created from vacuum [8] (including Erratum of this paper). The reason is that a weak gravitational background does not provide a sufficient density of energy to generate a ghost as a real particle excitation. Only for the typical frequencies of the Planck order of magnitude does a ghost become destructive, while at much lower energies one does not need to worry about its presence in the spectrum. By the end of the day the situation is very close to the one in effective field theories [32].

On the contrary, no low-energy protection can be expected in the theory with tachyons, because they produce instabilities independently of their interaction with normal particles or of the intensity of the background. In other words, for tachyons the exponential behavior (11) occurs at all frequencies, and not only above the Planck threshold [8]. Therefore, the difference between ghosts and tachyons is expected to be critical for the low-energy regimes.

## III. ANOMALY-INDUCED EFFECTIVE ACTION IN THE COSMOLOGICAL SETTING

In what follows we will use the formalism of the anomaly-induced effective action of the vacuum, so let us start by describing an application of this approach to the late-epoch cosmology.

For a general theory including $N_s$ massless conformal scalars, $N_f$ massless Dirac fermions and $N_v$ massless vectors, the anomalous trace of the energy-momentum tensor is [1]

$$\langle T^\mu_\mu \rangle = -(\beta_1 C^2 + \beta_2 E_4 + \beta_3 \Box R), \tag{16}$$

where

$$\begin{aligned}(4\pi)^2\beta_1 &= \frac{1}{120}N_s + \frac{1}{20}N_f + \frac{1}{10}N_v,\\ (4\pi)^2\beta_2 &= -\frac{1}{360}N_s - \frac{11}{360}N_f - \frac{31}{180}N_v,\\ (4\pi)^2\beta_3 &= \frac{1}{180}N_s + \frac{1}{30}N_f - \frac{1}{10}N_v.\end{aligned} \tag{17}$$

It was already mentioned in the Introduction that in the late universe the loops of all massive fields decouple from gravity. The reason is that the typical energy scale of the universe is usually defined by the Hubble parameter, which has the present-day order of magnitude $H_0 \propto 10^{-42}$ GeV. This is a very small value if compared even to the lightest known particles, e.g., the standard estimate for the neutrino is $m_\nu \propto 10^{-12}$ GeV. Therefore, according to the existing results on the gravitational decoupling [14,15] in the present-day (and certainly later) universe one needs to take into account only the contribution of photons, which are massless. Hence one has to set $N_s = N_f = 0$ and $N_v = 1$ in the expressions for the $\beta$ functions (17). Then these $\beta$ functions [Eq. (17)] boil down to

$$\beta^{\mathrm{IR}}_{1,2,3} = \frac{1}{10(4\pi)^2}\left(1, -\frac{31}{18}, -1\right) \equiv (\omega, b, c). \tag{18}$$

In order to find the anomaly-induced action one has to solve the equation

$$\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta\bar{\Gamma}_{\mathrm{ind}}}{\delta g_{\mu\nu}} = -\langle T^\mu_\mu\rangle = \omega C^2 + bE_4 + c\Box R. \tag{19}$$

The solution of this equation has been originally found in Ref. [23] and was discussed, e.g., in Ref. [3,34]. Let us present only the final result for the covariant local form of the solution with two auxiliary fields [34,35],

$$\begin{aligned}\bar{\Gamma}_{\mathrm{ind}} = S_c[g_{\mu\nu}] + \int d^4x\sqrt{-g}\Big\{&\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi\\ &- \frac{3c+2b}{36}R^2 + \frac{1}{2}\varphi\Big[\sqrt{-b}\Big(E_4 - \frac{2}{3}\Box R\Big)\\ &- \frac{1}{\sqrt{-b}}\omega C^2\Big] + \frac{\omega}{2\sqrt{-b}}\psi C^2\Big\},\end{aligned} \tag{20}$$

where

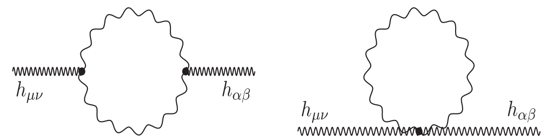

FIG. 1. Photon loops with two external gravitational lines.

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R_{;\mu}\nabla^\mu \tag{21}$$

is a covariant, self-adjoint, fourth-derivative, conformal operator [36].

The relevance of the auxiliary scalars $\varphi$ and $\psi$ is based on the fact that the boundary conditions for these fields are equivalent to the boundary conditions for the two Green functions of the same operator $\Delta_4$ in the nonlocal covariant form of $\Gamma_{\mathrm{ind}}$. The importance of having two auxiliary fields has been addressed in Refs. [34,35,37,38] and Ref. [3], where the discussion of the role of conformal functional $S_c[g_{\mu\nu}]$ can also be found.

Let us make an important general observation. The dependence on the conformal factor in the Weyl-squared term is exactly the same as the one in the well-known logarithmic form factor for the massless quantum fields[1]

$$\Gamma_{Ws} = -\frac{\beta_1}{2}\int \sqrt{-g} C_{\alpha\beta\lambda\tau} \log\left(\frac{\Box}{\mu^2}\right) C^{\alpha\beta\lambda\tau}, \tag{22}$$

where $\Gamma_{Ws}$ means $\Gamma_{\mathrm{Weyl-squared}}$ and $\beta_1 = (160\pi^2)^{-2}$ for the photon. One has to remember that the term (22) comes from quantum corrections (loop of photon, in our case; see Fig. 1 for the illustration) independently of whether the higher-derivative term (3) is included or not into the classical vacuum action. Including this term is useful to make the procedure of renormalization more simple and regular. However, a finite term such as Eq. (22) will show up even if we set $a_1 = 0$. In what follows it will be shown that an arbitrary $a_1$ will also come from fixing the initial condition for the solutions of the effective equations obtained by taking the term (20) into account.[2] The same happens in the case of the quantum term (22), in the instant at which we fix the value of $\mu$. Therefore it is not critically relevant for our considerations that we started from the classical theory with higher-derivative terms [Eq. (3)], since the same terms emerge from quantum corrections in any case. This consideration shows also the deep relation between the anomaly-induced action and renormalization group. This relation will be extensively used in Sec. VI.

Starting from Eq. (20) one can derive the anomaly-induced effective equations for various cosmological epochs. Consider the effective action

[1] At the end of Sec. VI we will discuss the IR limits of this correspondence in the asymptotically dS space.

[2] One can see the details of this procedure in Eqs. (44), (45) and (46).

$$\Gamma = S_{\rm vac} + \Gamma_{\rm ind}, \tag{23}$$

where the classical action $S_{\rm vac}$ was defined in Eq. (1) and the quantum correction $\Gamma_{\rm ind}$ in Eq. (20).

The equations for the auxiliary fields $\psi$ and $\phi$ are

$$\sqrt{-g}\left[\Delta_4\phi + \frac{\sqrt{-b}}{2}\left(E_4 - \frac{2}{3}\Box R\right) - \frac{\omega}{2\sqrt{-b}}C^2\right] = 0, \tag{24}$$

$$\sqrt{-g}\left[\Delta_4\psi - \frac{\omega}{2\sqrt{-b}}C^2\right] = 0. \tag{25}$$

The conformal transformation of the metric

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)e^{2\sigma(x)}, \tag{26}$$

gives the following transformation for the quantities of our interest (the reader can check Ref. [39] for details):

$$\begin{aligned} \sqrt{-g}\Delta_4 &= \sqrt{-\bar{g}}\bar{\Delta}_4, \\ \sqrt{-g}\left(E_4 - \frac{2}{3}\Box R\right) &= \sqrt{-\bar{g}}\left(\bar{E}_4 - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\right). \end{aligned} \tag{27}$$

For the sake of simplicity we consider only the spatially flat metric,

$$g_{\mu\nu} = \eta_{\mu\nu}a^2(\eta) = \eta_{\mu\nu}e^{2\sigma(\eta)}, \tag{28}$$

where $\eta$ is conformal time, and use the identities (27). Then Eqs. (24) and (25) become

$$\Box^2\phi = -2\sqrt{-b}\Box^2\sigma, \tag{29}$$

$$\Box^2\psi = 0, \tag{30}$$

where the d'Alembertian is the flat-space one.

Solutions of Eqs. (29) and (30) can be found for various epochs of the cosmological evolution of our Universe, i.e. separately during radiation, matter and during the de Sitter phase which is the asymptotic future for the ΛCDM universe.

(i) During the radiation epoch, $a(\eta) \propto \eta$ and Eqs. (29) and (30) become

$$\phi^{(iv)} = 12\sqrt{-b}\eta^{-4}, \tag{31}$$

$$\psi^{(iv)} = 0, \tag{32}$$

where primes indicate derivatives with respect to conformal time. The solutions have the form

$$\psi = d_0 + d_1\eta + d_2\eta^2 + d_3\eta^3, \tag{33}$$

$$\phi = c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 - 2\sqrt{-b}\log\left(\frac{\eta}{\eta_0}\right), \tag{34}$$

where $d_0, \ldots d_3$, $c_0, \ldots c_3$, are integration constants and $\eta_0$ is a generic reference time.

(ii) During the matter-dominated period, $a(\eta) \propto \eta^2$ and Eqs. (29) and (30) become

$$\phi^{(iv)} = 24\sqrt{-b}\eta^{-4}, \tag{35}$$

$$\psi^{(iv)} = 0, \tag{36}$$

with solutions

$$\psi = d_0 + d_1\eta + d_2\eta^2 + d_3\eta^3, \tag{37}$$

$$\phi = c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 - 4\sqrt{-b}\log\left(\frac{\eta}{\eta_0}\right), \tag{38}$$

where $d_0, \ldots d_3$, $c_0, \ldots c_3$, are integration constants in general different from the corresponding ones in radiation.

(iii) During the de Sitter phase, we have $a(\eta) \propto |\eta|^{-1}$ and Eqs. (29) and (30) become

$$\phi^{(iv)} = -12\sqrt{-b}\eta^{-4}, \tag{39}$$

$$\psi^{(iv)} = 0, \tag{40}$$

with solutions

$$\psi = d_0 + d_1\eta + d_2\eta^2 + d_3\eta^3, \tag{41}$$

$$\phi = c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 + 2\sqrt{-b}\log\left(\frac{\eta}{\eta_0}\right). \tag{42}$$

Here $d_0, \ldots d_3$, $c_0, \ldots c_3$, are integration constants, in general different from the corresponding ones in the radiation- and matter-dominated periods.

## IV. EFFECTIVE CORRECTIONS TO THE CLASSICAL ACTION

In Refs. [10,12] and [8] tensor perturbations of the classical theory (1) around a FRW background have been investigated in detail. The result is that the stability is fully determined by the sign of the coefficient $a_1$ of the Weyl-squared term and by the frequency of the perturbation $k = |\mathbf{k}|$. It was found that

(i) if $a_1 < 0$ and $k \ll M_p$, the theory is stable under tensor perturbations;

(ii) if $a_1 < 0$ and above some frequency $k$, which is comparable to $M_p$, the theory starts to be unstable under tensor perturbations;

(iii) if $a_1 > 0$, the theory is unstable under tensor perturbations $\forall k$.

This result can be easily understood by taking into account the physical content of the higher-derivative theory linearized around Minkowski space, as it was explained in Sec. II. Our main interest is the dynamics of tensor modes in the late universe. According to Ref. [8], for $a_1 < 0$ gravitational waves start an exponential growth only if their frequencies are close to the Planck order of magnitude. For this reason in what follows we will restrict our analysis to frequencies satisfying $k \ll m_2$; this means that the Planck scale physics is beyond our consideration. One can assume that starting from the Planck scale the high-energy gravitational theory passes through some qualitative change. For instance, in this regime the appropriate description may be (super)string theory, which is free of the problems of ghosts by construction (see Sec. VI B 5).

In the late universe the cosmological background is varying quite slowly and one can define an effective coefficient of the Weyl-squared term (3), which takes into account semiclassical anomaly-induced effects (20),

$$a_1^{\mathrm{eff}}(\eta) = a_1 + \frac{\omega}{2\sqrt{-b}}[\psi(\eta) - \phi(\eta)]. \tag{43}$$

Using the results of Sec. III, we obtain, in terms of physical time $t$,

radiation $$a_1^{\mathrm{eff}} = a_1^C + A_1 t^{1/2} + A_2 t + A_3 t^{3/2} + \frac{\omega}{2}\log\left(\frac{t}{t_0}\right), \tag{44}$$

matter $$a_1^{\mathrm{eff}} = a_1^C + A_1 t^{1/3} + A_2 t^{2/3} + A_3 t + \frac{2\omega}{3}\log\left(\frac{t}{t_0}\right), \tag{45}$$

de Sitter $$a_1^{\mathrm{eff}} = a_1^C + A_1 e^{-Ht} + A_2 e^{-2Ht} + A_3 e^{-3Ht} + \omega H t, \tag{46}$$

where $t_0$ is a reference time, the coefficients $A_i$ are arbitrary integration constants and $a_1^C$ contains both the classical $a_1$ and the constant part of the semiclassical contributions. In each epoch the $A_1$ and $a_1^C$ constant coefficients may be different but for the sake of simplicity, we adopted the same notation for all the cases.

The expressions (44), (45) and (46) come from the anomaly-induced action (20), but they can be seen as a local version of the renormalization group running for the parameter $a_1$. Since the quantum electromagnetic field is strictly massless, one can trace this flow to the far IR. In the given theory the UV turns out to be more complicated, because starting from some high-energy scale massive fields start to contribute to the running of $a_1$. The contribution of massive fields breaks down the elegant form of the induced effective action, until the deep UV regime, when all the fields can be treated as massless again [40,41].

Let us additionally comment on the expressions (44), (45) and (46). The complete solution for $a_1^{\mathrm{eff}}$ requires that the integration constants $a_1^C$ and $A_{1,2,3}$ are determined from experimental or observational data and the solutions for different phases are connected by requiring continuity. However, it is difficult to put this program into practice, because of the Planck suppression which makes impossible a direct observation of the effects of $a_1$ or its running. The unique way to have information about this coefficient is related to the evidence that the present universe is stable under tensor perturbations. We shall see in what follows that this is sufficient to obtain some physically interesting information.

Our own existence shows that starting from the beginning of the radiation epoch and along the cosmological evolution up to present time, the sign of $a_1^{\mathrm{eff}}$ has been negative. If not, the tensor perturbations would have been of the tachyonic type with an energy density of the Planck order of magnitude. Therefore, the proper fact of our existence puts strong restrictions on the values of $a_1^C$ and $A_{1,2,3}$ until the present time.

## V. TENSOR PERTURBATIONS IN THE ASYMPTOTICALLY DE SITTER PHASE

According to the recent observations [42,43] our Universe is now entering into a phase of its evolution dominated by the cosmological constant. The analysis of the time-dependent parameter $a_1^{\mathrm{eff}}(t)$ during the last phase of the evolution of the ΛCDM universe deserves a special consideration and will be explored in this section.

Let us parametrize tensor perturbations around the de Sitter background $\hat{g}_{\mu\nu}$ as

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}. \tag{47}$$

Since the variation of $a_1$ in Eq. (46) is quite slow, it is possible to consider a constant $a_1$ in the equations for the tensor perturbations (see also discussions in Refs. [12] and [8]).

The equations of motion in the transverse and traceless (TT) gauge can be cast into the form [8,10,12]

$$\left(\Box + \frac{4\Lambda}{3} + m_2^2\right)\left(\Box + \frac{2\Lambda}{3}\right)h_{\mu\nu} = 0, \tag{48}$$

where $\Box$ is the d'Alembertian operator on a de Sitter background and $m_2$ is the mass of the massive spin-2 mode $m_2^2 = -(32\pi G a_1)^{-1}$. Equation (48) also agrees with Refs. [44,45]. The theory propagates a massless graviton $h_{\mu\nu}^{(m)}$, satisfying

$$\left(\Box + \frac{2\Lambda}{3}\right) h_{\mu\nu}^{(m)} = 0 \tag{49}$$

and a massive spin-2 field $h_{\mu\nu}^{(M)}$, satisfying

$$\left(\Box + \frac{4\Lambda}{3} + m_2^2\right) h_{\mu\nu}^{(M)} = 0. \tag{50}$$

Since we are interested only in the tensor part of the perturbation $h_{\mu\nu}$, let us consider

$$h_{\mu\nu} = h_{\mu\nu}^{\text{tens}} = \begin{pmatrix} 0 & 0 \\ 0 & h_{ij}^{TT} a^2 \end{pmatrix}, \tag{51}$$

where we used TT gauge $\delta^{ij} h_{ij}^{TT} = 0$, $\partial^i h_{ij}^{TT} = 0$ and set $a(t) = a(0)e^{Ht}$, with $a(0) = 1$.

For the analysis of tensor perturbations it is more practical to use physical time $t$. Furthermore, after we separate the $h_{ij}^{TT}$ mode, there is no need to write indices and the field variable can be simply denoted as $h = h(t)$. Then Eq. (48) can be written as

$$\ddot{\ddot{h}} + 6H\dddot{h} + \left(11H^2 + m_2^2 + \frac{2k^2}{a^2}\right)\ddot{h} + \left(6H^3 + 3Hm_2^2 + \frac{2Hk^2}{a^2}\right)\dot{h} + \left(\frac{k^4}{a^4} + \frac{m_2^2 k^2}{a^2}\right) h = 0, \tag{52}$$

where we have used the background constraint $\Lambda = 3H^2$ and where dots indicate time derivatives. For $a_1 = 0$, Eq. (52) reduces to the usual equation for general relativity (GR) tensor perturbations around a de Sitter background.

The numerical investigation of Eq. (52) for both signs of $a_1$ has been performed in Ref. [8]. As we have already mentioned, for frequencies $k \ll M_p$ the case of $a_1 < 0$ shows oscillating solutions without growing amplitudes. This behavior for three different frequencies is shown in Fig. 2. On the contrary, for $a_1 > 0$ one can observe a rapid growth of the metric perturbations for all frequencies; see Fig. 3.[3]

We now want to quantify which is the typical time scale at which anomaly-induced corrections to the classical coefficient of the Weyl-squared term, $a_1$, come into play. In pure de Sitter, we found that the behavior of $a_1^{\text{eff}}$ is described by Eq. (46). We focus our analysis on time scales of order $t \sim 1/M_p$, since this is the expected scale of the instabilities. On these scales, the decaying exponentials in Eq. (46) vary slowly and can be treated as constant functions. Hence, we can parametrize Eq. (46) as

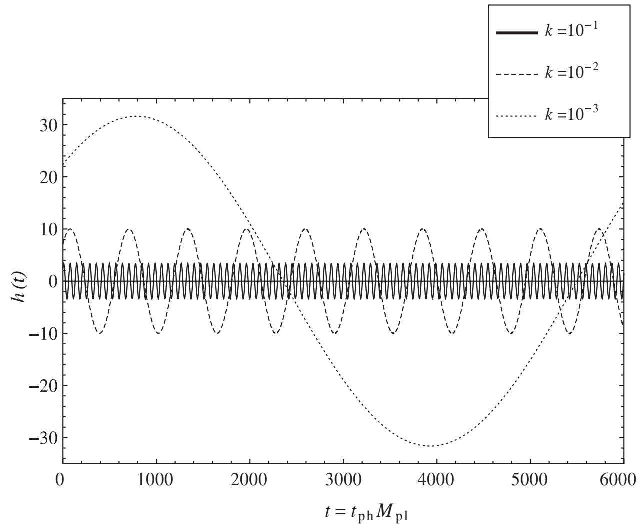


FIG. 2. Behavior of tensor perturbations in the case $a_1 < 0$, for different frequencies. The frequency $k$ is measured in Planck units.

$$a_1^{\text{eff}}(t) = \tilde{a}_1 + \omega H(t - t_{dS}), \tag{53}$$

where $t_{dS}$ is the time at which the density of matter and radiation become effectively negligible and a pure de Sitter phase starts.

Since we are not assisting an overproduction of gravitational waves at present time, the present-day value of $a_1^{\text{eff}}$ has to be negative (i.e., we are in the regime in which tensor perturbations are stable). As soon as $a_1^{\text{eff}}(t)$ crosses zero going to positive values, the gravitational waves enter the unstable phase (the massive ghost becomes tachyonic). Then an exponential instability in the tensor sector instantaneously shows up.

One can ask whether the stable phase, characterized by a negative $a_1^{\text{eff}}$ will last for a long period of time. We can

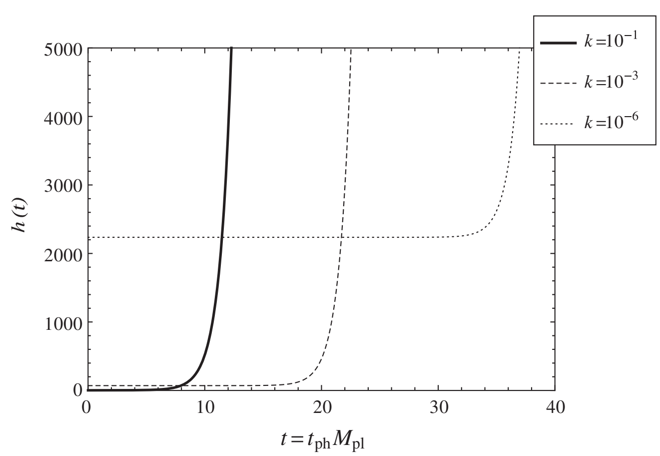


FIG. 3. Dynamics of tensor perturbations for $a_1 > 0$, for three different frequencies $k = |\mathbf{k}|$. The plot for the lowest frequency has been rescaled to fit the figure.

[3] Let us note that the numerical integration cannot be performed for positive $a_1$'s which are too close to zero, since the growth of tensor perturbations is too fast in this case.

assume that the stable phase in which we are living today will last at least until the beginning of the "pure" de Sitter phase of the expansion of our Universe, and hence we take $\tilde{a}_1 = a_1^{\text{eff}}(t_{dS})$ to be negative and of order one. In the ΛCDM model, with $\Omega_\Lambda = 0.7$, the value of the Hubble parameter in the pure de Sitter phase is

$$H_\Lambda = \sqrt{\frac{\Lambda}{3}} \simeq 0.8 H_0 \simeq 1.2 \times 10^{-42}\ \text{GeV}. \qquad (54)$$

Replacing $H = H_\Lambda$ in Eq. (53) we find the time $t_q$ at which tachyonic modes emerge in the spectrum of gravitational waves,

$$t_q = \frac{|\tilde{a}_1|}{H_\Lambda \omega} \simeq 2.4 \times 10^{13}\ \text{yr} = 2.4 \times 10^4 bi. \qquad (55)$$

Hence with the assumption $|\tilde{a}_1| \sim 1$ the remaining time until the gravitational-wave explosion is about 1000 times longer than the time which already passed from the big bang. Of course, since we have no experimental data to define the present-day value of $|\tilde{a}_1|$, the estimate given above has an ambiguity. According to Eq. (55) larger values of $|\tilde{a}_1|$ correspond to a longer stable phase, while smaller values of $|\tilde{a}_1|$ imply a shorter period of stability before the tachyonic explosion.

Regardless of the quantitative side of this prediction, it looks interesting that our knowledge of quantum corrections to gravity is sufficient to know how the ΛCDM universe will end up. According to our analysis, there will be an instant gravitational-wave explosion due to the tachyonic instabilities.

## VI. DISCUSSION OF THE ASSUMPTIONS

We have started from a minimal set of hypotheses and arrived at the conclusion that the final destiny of the ΛCDM universe is not a peaceful infinitely long slightly accelerated expansion in the homogeneous and isotropic phase. On the contrary, the universe will end up in a strong tachyonic explosion of tensor perturbations. At first sight the intensity of this explosion (at least in the first-order approximation) looks unrestricted, since at the instant when $a_1^{\text{eff}}$ changes sign, the mass of the unstable mode is infinite, according to Eq. (11). Then gravitational waves of all frequencies will experience a fast exponential growth. However, since one cannot trust the semiclassical approximation completely, the conservative estimate is that the tachyonic modes will have, at most, an energy density of the Planck order of magnitude. Certainly, this is more than sufficient to provide a dramatic effect on the geometry and matter contents of the universe.

Another natural restriction comes from the fact that the instability which we have found corresponds to linear perturbations. It might happen that next orders in the perturbative expansion in $h_{\mu\nu}$ will restore the stability. However, from the practical side both these restrictions do not matter too much, because even this "restricted" gravitational explosion should be capable of destroying the symmetry (homogeneity and isotropy) of the metric, and thus lead to the strong changes of the properties of space-time.

Since the result of our study looks so dramatic, it is interesting to consider the list of all the approximations which have been introduced in our analysis. Let us formulate various possibilities to avoid the instabilities in the form of questions and answers.

### A. Completeness of the one-loop approximation

Can we expect some qualitative changes in the result by taking higher-order loops into account? Formally, higher-loop contributions do not change the sign of the $\beta_1$ function [40,41], but this is not sufficient to draw the conclusion about the relevance of higher-loop terms. In the UV regime there will be higher logarithmic contributions compared to the one-loop form factor (22) and this could produce a strong effect on the running of $a_1$. However, the situation at low energies is quite different. Let us remember that second- and higher-loop corrections to the one-photon bubble include a loop of electrons or of other massive charged fermions, as shown in Fig. 4. Because of the Appelquist and Carazzone decoupling theorem in QED, the contribution of such a loop is suppressed at least by a factor $(H_0/m_e)^2 \approx 10^{-77}$ for the dynamics of the conformal factor. Since this is the part which governs the running of $a_1^{\text{eff}}$, there are no chances that higher loops can change the result found for the conformal factor dynamics.

At the same time, we have to remember that the goal of our study is not the dynamics of the conformal factor itself, but its interaction with gravitational waves. Indeed, the term in the effective action which we implicitly deal with is Eq. (22). At the two-loop order there will be nonlocal corrections to the photon propagator and photon-graviton vertex. Since we do not quantize gravity, the last type of diagrams can be ruled out. The simplest diagram at higher loops will be the one from Fig. 1 with the polarization operator insertion shown in Fig. 4, due to the electron loop.

Concerning this and further diagrams it is easy to see that the mass which defines the scale of decoupling will be the mass of the quantum fermion, while the typical energy of

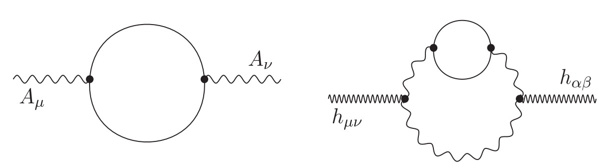


FIG. 4. One-loop diagram with electron loop and two external electromagnetic lines and the diagram representing the two-loop correction to the photon loop from Fig. 1, with the electron-loop insertion. The diagram at the left is subject to decoupling in the IR.

the external particle will be the one of the gravitational wave. Finally, the relevant ratio which defines the effective cutoff for the higher-loop contributions to the running of $a_1$ is $(\mathcal{E}_{\mathrm{GW}}/m_e)^2$ for a tensor mode with energy $\mathcal{E}_{\mathrm{GW}}$. Of course this energy can be much bigger than $H_0$. For instance, taking $\mathcal{E}_{\mathrm{GW}} = 1$ eV we have "only" ten-orders decoupling. Indeed, the gravitational-wave production with its corresponding energy density will have a very strong physical and geometrical effect.

It is worth noting that we are dealing with the diagrams which consist of the loops of matter fields and external lines of gravity. Therefore, the one-loop part is given by the contributions of free fields on a gravitational background. For this reason there is no dependence on the coupling constants at one-loop order, and such dependence can be observed only starting from the second loop [1,2]. Finally, the one-loop effects of photons are *not* suppressed by mass, since the photon is massless. However, all particles to which a photon is coupled, are massive (and their masses are huge compared to the gravitational-wave energy) and that is why the two-loop effects are indeed strongly suppressed.

The consideration presented above shows that the one-loop results (44), (45) and (46) are sufficiently robust in the framework of the semiclassical theory, at least for gravitational waves with a frequency smaller than the electron mass. The case of waves with higher frequencies deserves an additional investigation, but it cannot change the qualitative result which we obtained at the one-loop level.

### B. The possible role of quantum gravity

What about QG effects, which have been neglected so far? It is not easy to give a definite answer due to the variety of existing models of QG. Let us consider a short list of the possibilities which have been better explored up to now.[4]

#### 1. Effective IR quantum gravity

The standard effective framework for the IR effects of QG assumes that GR is a universal theory of IR quantum gravity [49] (see also Ref. [50] for earlier work in this direction and Ref. [51] for an alternative opinion).

The first thing to note is that GR is not a conformal theory, and therefore a compact representation of quantum corrections in the form of Eq. (20) is not possible. In this situation one can make use of the renormalization group and of the perturbative $\beta$ function $\beta_1$ for the parameter $a_1$. The $\beta_1$ function gives, in the case of the massless theory (e.g., quantum GR), the coefficient in the logarithmic form factor (22). Therefore, it can be still considered a good approximation in the present case. In this framework the change of sign of the renormalization group equation (53) looks unclear, at least beyond the one-loop level. The reason is that higher-loop contributions of QG are beyond our control, because QG based on GR is nonrenormalizable. Still we can draw some conclusions about the effective quantum gravity. At the one-loop level, the Weyl-squared quantum counterterm is known to be gauge-fixing dependent [52]. Then one can provide any desirable value of $\beta_1$ by a special choice of the gauge-fixing parameters. This also means that the corresponding contribution vanishes on shell and therefore cannot be regarded as a physical effect. Furthermore, similar gauge-fixing dependence is expected for the leading-logarithmic corrections at higher loops. Therefore, while the subject is not completely clear, one can suppose that the leading contributions of QG in this framework will be sublogarithmic. As we have already seen in Sec. VI A, this means that the change of sign of $\beta_1$ in Eq. (17) due to effective low-energy QG is very unlikely.

#### 2. Weyl conformal gravity

In this case the form (20) of quantum corrections is available and the effect of QG is to increase the positive value of $\beta_1$. The reason is that the contribution of conformal QG is positive [53–55], exactly as the one of all matter fields (17). The expression for $N_v$ copies of massless vector fields has the form

$$\beta_1 = \frac{1}{(4\pi)^2}\left(\frac{N_v}{10} + \frac{199}{30}\right), \tag{56}$$

where the last term is the QG contribution. It is easy to see that no cancellation is possible, and for $N_v = 1$ the QG contribution shortens the period of time until the sign transition for $a_1^{\mathrm{eff}}$ by about 1 order of magnitude.

#### 3. General version of QG with fourth derivatives

In this case, exactly like in the effective QG described in Sec. VI B 1, the form of quantum corrections of the type (20), is not viable, because the original theory is not conformal, and hence there is no anomaly to integrate. At the same time the $\beta$ function for the parameter $a_1$ is well defined, free of ambiguities [35,53,56] and, according to well-verified calculations [53,56,57] has the same positive sign as Eq. (17). The overall $\beta_1$ function for $N_v$ copies of massless vector fields has the form

$$\beta_1 = \frac{1}{(4\pi)^2}\left(\frac{N_v}{10} + \frac{133}{10}\right). \tag{57}$$

In the physically interesting case we have one photon, $N_v = 1$, and no cancellation can be expected. Moreover, in

[4] Let us note that the problem of quantum stability of de Sitter space and the validity of the semiclassical approximation has been extensively discussed in Ref. [46] and more recently in Refs. [47] and [48]. The difference between our conclusions and the ones of these works can be probably explained by the special role of tachyon tensor modes which we consider in the present work.

the far IR the ghost which provides some part of the QG contribution in Eq. (57), is supposed to decouple, and we come back to the situation described in Sec. VI B 1.

#### 4. Super-renormalizable QG

The super-renormalizable models of QG have been first formulated in Ref. [58] on the basis of a polynomial action and shortly after that in Ref. [59] on the basis of an action of gravity that is nonpolynomial in derivatives. In the polynomial case this model of QG has many (tensor and scalar) ghosts. In the nonlocal model of Ref. [59] there are no ghosts at the tree level; however loop corrections lead to the emergence of an infinite amount of massive ghost-like states, corresponding to complex poles [60].

In both cases quantum corrections are well defined, but the $\beta_1$ function for the parameter $a_1$ can be modified by adjusting the terms of third and fourth orders in the curvature tensor [60]. Therefore, it is not difficult to construct the theory with the cancellation of the photon contribution in Eq. (17). However, this cancellation will take place in the UV region, where the quantum effects of ghosts are not suppressed. There is no systematic study of decoupling of massive modes in higher-derivative QG, but in principle one can expect that in the IR the massive ghost states will decouple and the low-energy situation will be covered by the effective QG, as described in Sec. VI B 1. After all, super-renormalizable models of QG cannot be expected to change the sign of $\beta_1$ in the IR limit.

#### 5. String theory

In (super)string theory the terms providing the $\beta$ function for the parameter $a_1$ are usually removed by means of the Zwiebach transformation [33,61,62]). The procedure is ambiguous [63], but, by construction the $\beta$ function for the parameter $a_1$ is zero. At the same time, string theory is not supposed to provide a significant correction to the quantum field theory results at low and very low energies, for otherwise we would observe such corrections in precision experiments, e.g., the ones that test QED and Standard Model calculations. Therefore using string theory to evaluate the $\beta_1$ function in the far IR is not reasonable from a conceptual point of view.

In conclusion, we can see that QG can provide the change of sign of the $\beta$ function for $a_1$, but only at high energies and in the framework of super-renormalizable models of QG of Refs. [58] and [59]. On the other hand, the IR limit of the QG theory with a number of massive states (ghosts and normal particles) should be taken with great care [49,51] and it is expected that the sign of the $\beta_1$ function in the IR would remain positive. Hence, we arrive at the conclusion that the main result concerning the positiveness of the $\beta$ function for the parameter $a_1$ remains robust even if gravity is quantized.

### C. Completeness of the anomaly-induced approximation

As we have seen, there is practically no way to avoid the change of sign of $a_1$ within the anomaly-induced effective action (20). However, we know that this effective action is based on the artificial minimal substraction ($\overline{\mathrm{MS}}$) scheme of renormalization. This scheme is always working perfectly well in the UV. On the other side, it is known that for the theory of massive quantum fields, this scheme is *not* working well in the IR, because of the decoupling theorem [16], which was also obtained for semiclassical gravity in Refs. [14,15] (see also Ref. [3] for the review and further references). In the flat space-time the masslessness guarantees that the main features of the UV will repeat in the IR, due to the UV-IR "duality" of the logarithmic form factor such as the one in Eq. (22). Is it true that the situation is the same in curved space-time? The question is not simple, especially in the asymptotically dS space, which has a natural IR cutoff scale $H$.

For instance, the scalar curvature in the space with $\sigma(t) = H_0 t$ (with $H_0 = \mathrm{const}$) has the global scaling similar to the $\Box$ operator, namely $R = -6e^{-2\sigma} \cdot \sigma''^2$ in the spatially flat case, but the derivatives here are with respect to conformal time $\eta$, where $dt = a(\eta)d\eta$. As a result $R$ does not run with time; instead it is a constant,[5] $R = -12H_0^2$. Similar consideration applies to the operator $\Box$ in the logarithm of the form factor in Eq. (22). For instance, let us consider $\Box$ acting on a scalar field $\varphi$. For the sake of simplicity we assume that this scalar depends only on time, $\varphi = \varphi(t)$. Then a very simple calculation leads to the result

$$\Box\varphi = \ddot{\varphi} + 3H_0\dot{\varphi}. \tag{58}$$

One has to note that in the last equation the dependence $\varphi(t)$ corresponds to the fiducial (flat) metric and, therefore, time derivatives of $\varphi$ are expected to be of the same order of magnitude in different epochs. Thus, Eq. (58) shows that the dependence of time drops out from the coefficients of the equation for perturbations. A similar calculation for the Weyl-squared term provides the result (here we kept the possibility that $H$ is nonconstant for the sake of generality)

$$\begin{aligned}&\sqrt{-g}C^{\alpha\beta\rho\tau}\Box C^{\alpha\beta\rho\tau}\\&\quad= \sqrt{-\bar{g}}\bar{C}^{\alpha\beta\rho\tau}\{\partial_t^2 - 4H\partial_t - 2\dot{H} - 10H^2\}\bar{C}^{\alpha\beta\rho\tau}\\&\qquad+ 8\sqrt{-\bar{g}}\bar{C}^{\alpha\beta\rho t}\{H\partial_t - \dot{H} + 2H^2\}\bar{C}^{\alpha\beta\rho t}.\end{aligned} \tag{59}$$

The dependence of space coordinates does not change the result qualitatively, if the wave vector is taken in the physical space, that is after the rescaling $\mathbf{k} \to \mathbf{k} \cdot e^{\sigma}$.

[5] The authors are very grateful to A.A. Starobinsky for suggesting this example as an argument against IR running in de Sitter space.

The expression (59) shows that in the asymptotic future, where the background becomes very close to de Sitter space, there will not be physical running of $a_1$. The reason is that the extreme IR corresponds to dS and has its own scale $H$, such that the correspondence between the logarithmic asymptotic (22) and anomaly-induced action (20) gets violated. Recently, explicit calculations of the curvature-curvature correlators on a de Sitter background have been performed in Ref. [64]. It is remarkable that these calculations (performed by a completely different method) have also shown the absence of leading-log corrections in the far IR. Our interpretation of this result is a specific IR decoupling in the IR on a de Sitter space background, where $H$ plays the role of the natural minimal mass scale.

Indeed, according to the same logic, the results for the radiation- and dust-dominated epochs, Eqs. (44) and (45) are valid and the same is true for the beginning of the transition period between matter- and cosmological-constant-dominated epochs. However, since we are now in the situation when the Hubble parameter is quite close to the future constant value $H$, this means the universe is entering the epoch of IR decoupling of all quantum effects in the higher-derivative sector of the theory, including the ones of photons. Hence in the real situation the transition to the positive sign of $a_1$ is very improbable.

## VII. CONCLUSIONS

The anomaly-induced effective action of the vacuum is a very efficient tool for a local implementation of the renormalization group running. According to this approach, the semiclassical effects drive the coefficient $a_1$ of the Weyl-squared term in the vacuum (gravitational) action to the positive side. It turns out that the one-loop contribution is exact in this case, and hence the future of the universe should be related to the tachyonic instabilities. Since the time dependence of $a_1$ is linear, one can easily calculate the time which remains until the change of its sign, which would be an instant when the massive ghost transforms into a tachyonic ghost.

The difference between the two types of ghosts is dramatic. For the "usual" ghost there is a mass threshold and hence a possible situation when the ghost is harmless. The situation can be understood in the spirit of considerations presented in Ref. [30]. Let us note that the approach pursued in these papers by Simon and Parker and Simon is effectively equivalent to the one of Ref. [8]. In both cases Planck frequencies are "forbidden" without a clear explanation (we believe that one can find a solution of this problem, but this will be discussed elsewhere). Furthermore, in both cases a physically reasonable choice of initial conditions can be done, and as a result the system does not fall into the runaway solutions. As a result higher-derivative terms produce only tiny, Planck-suppressed corrections and can be simply ignored. For the gravitational waves this means that the dynamics of metric perturbations is essentially the same as in GR, as it was actually shown by Starobinsky in the second paper of Ref. [9].

As far as the ghost also becoming a tachyon, no mass threshold exists and no reasonable choice of initial conditions can be done. Then an instantaneous explosion of gravitational waves starts at all frequencies, independent of initial conditions. Our calculations show that in the framework of the anomaly-induced effective action the transition to a tachyonic ghost is unavoidable in the far future of the ΛCDM universe. It is easy to show that the higher-loops corrections do not change this result, at least for the frequencies below the electron mass scale.

However, in the dS-background case, which is a final stage of the ΛCDM universe, there is a qualitatively new type of IR decoupling which takes place even for massless fields. As a result, the anomaly-induced approximation, which is based on the $\overline{\mathrm{MS}}$ scheme of renormalization becomes inappropriate in the IR, exactly as it happens in the massive theory. Therefore, there are no real chances of the tachyonic explosion of the universe even in the far future, since no change of the sign of $a_1$ can be expected.

## ACKNOWLEDGMENTS

The authors are grateful to M. Maggiore and A. A. Starobinsky for stimulating discussions. The work of G. C. is supported by the Swiss National Science Foundation. F. S. is grateful to FAPEMIG and CAPES for partial support. I. S. is grateful to CNPq, FAPEMIG and ICTP for partial support of his work and especially to the Département de Physique Théorique and Center for Astroparticle Physics at the Université de Genève for partial support and kind hospitality during his sabbatical period. Also, I. S. is grateful to the Mainz Institute for Theoretical Physics (MITP) and to the International Institute of Physics (IIP) in Natal for their hospitality and partial support during the completion of the first and last versions of this work.

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