Physical interpretation of antigravity

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Geodesic incompleteness is a problem in both general relativity and string theory. The Weyl-invariant Standard Model coupled to general relativity (SM + GR), and a similar treatment of string theory, are improved theories that are geodesically complete. A notable prediction of this approach is that there must be antigravity regions of spacetime connected to gravity regions through gravitational singularities such as those that occur in black holes and cosmological bang/crunch. Antigravity regions introduce apparent problems of ghosts that raise several questions of physical interpretation. It was shown that unitarity is not violated, but there may be an instability associated with negative kinetic energies in the antigravity regions. In this paper we show that the apparent problems can be resolved with the interpretation of the theory from the perspective of observers strictly in the gravity region. Such observers cannot experience the negative kinetic energy in antigravity directly, but can only detect in and out signals that interact with the antigravity region. This is no different from a spacetime black box for which the information about its interior is encoded in scattering amplitudes for in/out states at its exterior. Through examples we show that negative kinetic energy in antigravity presents no problems of principles but is an interesting topic for physical investigations of fundamental significance.

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I. WHY ANTIGRAVITY?

The Lagrangian for the geodesically complete version of the Standard Model coupled to General Relativity (SM + GR) is [1]

$$\mathcal{L}(x) = \sqrt{-g}\begin{pmatrix} L_{\rm SM}(A_\mu^{\gamma,W,Z,g},\psi_{q,l},\nu_R,\chi) \\ +g^{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - D_\mu H^\dagger D_\nu H\right) \\ -\left(\frac{\lambda}{4}(H^\dagger H-\omega^2\phi^2)^2+\frac{\lambda'}{4}\phi^4\right) \\ +\frac{1}{12}(\phi^2-2H^\dagger H)R(g) \end{pmatrix}. \quad (1)$$

In the first line, $L_{\rm SM}$ contains all the familiar degrees of freedom in the properly extended conventional Standard Model, including gauge bosons ($A_\mu^{\gamma,W,Z,g}$), quarks and leptons ($\psi_{q,l}$), right-handed neutrinos ν_R, dark matter χ, and their $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge-invariant interactions among themselves and with the spin-0 fields (H,ϕ), where H is an electroweak Higgs doublet, and ϕ is a singlet. In $L_{\rm SM}$ all fields are minimally coupled to gravity. The second and third lines describe the kinetic energy terms and interactions of the scalars among themselves. The last term is the unique nonminimal coupling of conformal scalars to the scalar curvature $R(g)$ that is required by invariance of the full $\mathcal{L}(x)$ under local rescaling (Weyl) with an arbitrary local parameter $\Omega(x)$:

$$g_{\mu\nu}\to\Omega^{-2}g_{\mu\nu},\qquad \phi\to\Omega\phi,\qquad H\to\Omega H,$$
$$\psi_{q,l}\to\Omega^{3/2}\psi_{q,l},\qquad A_\mu^{\gamma,W,Z,g}\ \text{unchanged}. \quad (2)$$

If dark matter χ is a spin-0 field, then lines 2–4 in Eq. (1) should be modified to treat χ as another conformally coupled scalar.

This theory has several pleasing features. There are no dimensionful parameters, so all of those arise from a unique source, namely the gauge fixing of the Weyl symmetry, such as $\phi(x)\to\phi_0$, where ϕ_0 is a dimensionful constant of the order of the Planck scale. Then the gravitational constant is $(16\pi G_N)^{-1}=\phi_0^2/12$; the electroweak scale is $\langle|H|\rangle=\omega\phi_0$; while dark energy and masses for quarks, leptons, gauge bosons, neutrinos and dark matter arise from interactions with the scalars (ϕ,H). The hierarchy of mass scales is put in by hand through a hierarchy of dimensionless parameters. A deeper theory is needed to explain this hierarchy, but in the present effective theory it is at least possible to maintain it under renormalization, since dimensionless constants receive only logarithmic quantum corrections (no need for low-energy supersymmetry for the purpose of "naturalness"). To preserve the local scale symmetry in the quantum theory, one must adopt a Weyl-invariant renormalization scheme in which ϕ is the only renormalization scale, and consequently dimensionless constants receive only Weyl-invariant logarithmic renormalizations of the form $\ln(H/\phi)$, etc. With such a renormalization scheme the scale anomaly of all matter cancels against the scale anomaly of ϕ [2], thus not spoiling the local symmetry. Then the unbroken Weyl symmetry in the renormalized theory plays a central role in explaining the smallness of dark energy as shown in Ref. [3]. This also suggests a definite relation between the electroweak vacuum and dark energy, both of which fill the entire Universe.

The scalar ϕ is compensated by the Weyl symmetry, so ϕ is not a true additional physical degree of freedom but, as a conformally coupled scalar, participates in an important structure of the Weyl symmetry that has further physical consequences involving antigravity spacetime regions in cosmology and black holes as discussed in the following sections. The structure of interest, that leads to the central discussion in the rest of this paper, is the relative minus sign in $(\phi^2 - 2H^\dagger H)R$ and in the scalar kinetic terms in (1). These signs are compulsory and play an important role in the geodesic completeness of the theory. With the given sign patterns, H has the correct sign for its kinetic term, but ϕ has the wrong sign. If ϕ had the same sign of kinetic energy as H, then the conformal coupling to R would become purely negative, which would lead to a negative gravitational constant. So, to generate a positive gravitational constant, ϕ must come with the opposite sign to H. This makes ϕ a ghost, but this is harmless, since the Weyl symmetry can remove this ghost by a gauge fixing.

This scheme has a straightforward generalization to supersymmetry/supergravity and grand unification, but all scalars $\vec{s}$ must be conformally coupled, $(\phi^2 - \vec{s}^2)R$, although some generalization is permitted as long as the geodesically complete feature (related to signs) is maintained [1]. Furthermore, we point out that in all supergravity theories, the curvature term has the form $(1 - K(\varphi_i, \bar{\varphi}_i)/3)R$, where K is the Kähler potential and 1 represents the Einstein-Hilbert term [4]. This is again of the form $(|\phi|^2 - |\vec{s}|^2)R$ with complex $(\phi, \vec{s})$, where a complex version of ϕ has been gauge-fixed to 1 in a Weyl-invariant formulation of supergravity [5] (see also Ref. [6]). Finally, we emphasize that the same relative minus sign occurs also in a Weyl-invariant reformulation of low-energy string theory (ST) but with a different interpretation of s related to the dilaton [7]. Hence, the structure $(\phi^2 - \vec{s}^2)R$ is ubiquitous, but was overlooked because it was commonly assumed that the gravitational constant, or an effective structure that replaces it, could not or should not become negative.

At the outset of this approach in 2008 [8], the immediate question was whether the dynamics would allow $(\phi^2 - s^2)$ to remain always positive. It was eventually determined by Bars, Chen, Steinhardt and Turok, in a series of papers during 2010–2012 (summary in Ref. [9]), that the solutions of the field equations that do not switch sign for this quantity are nongeneric and of measure zero in the phase space of initial conditions for the fields (ϕ, s). So, according to the dynamics, it is untenable to insist on a limited patch $|\phi| > |\vec{s}|$ of field space. By contrast, it was found that the theory becomes geodesically complete when all field configurations are included, thus solving generally the basic problem of geodesic incompleteness.

The other side of the coin is that solving geodesic incompleteness comes with the prediction that there would be antigravity sectors in the theory, since the effective gravitational constant that is proportional to $(\phi^2(x) - s^2(x))^{-1}$ would dynamically become negative in some spacetime regions. In view of the pleasing features of the theory outlined in the second paragraph above, these antigravity sectors must then be taken seriously, and the corresponding new physics must be understood. In our investigations so far, we discovered that the antigravity sectors are geodesically connected to our own gravity sector at gravitational singularities, like the big bang/crunch or black holes, which occur precisely at the same spacetime points where $(\phi^2(x) - s^2(x))$ vanishes or goes to infinity. The related dynamical string tension [7]

$$T(\phi, s) \sim (\phi + s)^{2\frac{1+\sqrt{d-1}}{d-2}} (\phi - s)^{2\frac{1-\sqrt{d-1}}{d-2}} \tag{3}$$

goes to zero or infinity simultaneously. So we need to figure out the physical effects that can be observed in our Universe due to the presence of antigravity sectors behind cosmological [10] and black hole singularities [11]. After overcoming several conceptual as well as technical challenges, we have been able to discuss some new physics problems and have developed new cosmological scenarios that involve an antigravity period in the history of the Universe [12,13]. A remaining conceptual puzzle is an apparent possible instability in the antigravity sector that is addressed and resolved in the remainder of this paper. Our conclusion is that there are no fundamental problems but only interesting physics of crucial significance.

II. GEODESIC COMPLETENESS IN THE EINSTEIN OR STRING FRAMES

The classical or quantum analysis of this theory is best conducted in a Weyl gauge we called the "γ gauge" [1,7,10], which amounts to $\det(-g) = 1$. This allows $\mathrm{Sign}(\phi^2 - s^2)$ to be determined by the dynamics. Note that the sign is gauge invariant, so if the sign switches dynamically in one gauge, it has to also switch in all gauges. If one wishes to use the traditional "Einstein gauge" (E) or the "string gauge" (s), one can err by choosing an illegitimate gauge that corresponds to a geodesically incomplete patch, such as

$$\begin{aligned} &\mathrm{E}^+\text{ gauge:}\ \frac{1}{12}(\phi_{E+}^2 - s_{E+}^2) = \frac{+1}{16\pi G_N},\\ &\mathrm{s}^+\text{ gauge:}\ \frac{d-2}{8(d-1)}(\phi_{s+}^2 - s_{s+}^2) = \frac{+1}{2\kappa_d^2} e^{-2\Phi}, \quad \Phi = \text{dilaton}. \end{aligned} \tag{4}$$

The E or s subscripts on the fields indicate the gauge-fixed form of the corresponding field. If this were all, then there would be nothing new, and the Weyl symmetry could be regarded as "fake" [14]. However, the fact is that conventional general relativity and string theory are geodesically incomplete because the gauge choices just shown are valid only in the field patch in which $|\phi| > |s|$. The dynamics contradict the assumption of gauge fixing to only the positive patch. In the negative regions, one may choose again the Einstein or string gauge, but now with a negative gravitational constant, $\frac{1}{12}(\phi_{E-}^2 - s_{E-}^2) = \frac{-1}{16\pi G_N}$, or $\frac{d-2}{8(d-1)}(\phi_{s-}^2 - s_{s-}^2) = \frac{-1}{2\kappa_d^2} e^{-2\Phi}$. In those spacetime regions, gravity is repulsive (antigravity).

The same situation arises in string theory. In the worldsheet formulation of string theory, the string tension is promoted to a background field $T(\phi, s)$ by connecting it directly to the features of the Weyl-invariant low-energy string theory [7]. Then the string tension $T(\phi, s)$ switches sign together with the corresponding gravitational constant [7]. Thus, the Weyl-symmetric (SM + GR) and ST predict that, in the Einstein or string gauges, one should expect a *sudden sign switch* of the effective Planck mass $\frac{1}{12}(\phi^2 - s^2)$ at certain spacetime points that typically correspond to singularities (e.g. big bang, black holes) encountered in the Einstein or string frames.

One may choose better Weyl gauges (e.g. "γ gauge," choose $\det(-g) \to 1$, or "c gauge," choose $\phi \to$ constant) that cover globally all the positive and negative patches. Then the sign switch of the effective Planck mass $\frac{1}{12}(\phi^2 - s^2)$ is smooth rather than abrupt.

However, if one wishes to work in the more familiar Einstein or string frames, to recover the geodesically complete theory one must allow for the gravitational constant to switch sign at singularities, and connect solutions for fields across gravity/antigravity patches. In the $\pm$ Einstein gauges shown above, the last term in Eq. (1) becomes

$$\frac{(\phi_{E\pm}^2 - s_{E\pm}^2)R(g_{E\pm})}{12} = \frac{R(g_{E\pm})}{\pm 16\pi G_N} = \frac{R(\pm g_{E\pm})}{16\pi G_N}, \quad (5)$$

where the $\pm$ for the gravity/antigravity regions can be absorbed into a redefinition of the signature of the metric,

$$\hat{g}_{\mu\nu}^{E} = \pm g_{\mu\nu}^{E\pm}, \quad (6)$$

where the *continuous* $\hat{g}_{\mu\nu}^{E}$ is the geometry in the *union* of the gravity/antigravity patches.

The same $\pm$ gauge choice is applied to every term in the SM + GR action in Eq. (1). Under the replacement $g_{\mu\nu}^{E-} \to -g_{\mu\nu}^{E-}$ in the *antigravity sector*, some terms in the action flip sign and some do not [15]; e.g. $F_{\mu\nu}F^{\mu\nu}$ does not, but $R(g)$ does, as in Eq. (5). One may be concerned that the sign switches of the gravitational constant, or the string tension may lead to problems like unitarity or negative kinetic energy ghosts. We mention that Ref. [7] has already argued that there are no unitarity problems due to sign flips in field/string theories. There remains the question of possible instability due to negative kinetic energy in the antigravity region. We show in this paper that its presence is not a problem of principle for observers in the gravity region and that those observers can detect interesting physical effects related to the geodesically connected regions of antigravity.

III. UNITARITY AND ANTIGRAVITY IN COSMOLOGY

There is a general impression that negative kinetic energy in field or string theory implies ghosts associated with negative norm states. It is not generally appreciated that negative norms (and hence negative probabilities) are automatically avoided by insisting on a strictly unitary quantization of the theory. This has been illustrated in the quantization of the relativistic harmonic oscillator [16] with a timelike direction that appears with the opposite sign to the spacelike directions, just like the ϕ field as compared to the H field in the SM + GR action in Eq. (1). Similar situations occur in the antigravity region where some fields may appear with the wrong sign as described after Eq. (5). The first duty in quantization should be maintaining sanity in the meaning of probability, as in Ref. [16], by avoiding a quantization procedure that introduces negative norm states. Of course, there exist successful cases, such as string theory in the "covariant quantization" procedure, that at first admits negative norms to later kill them by applying constraints that select the positive norm states. In principle, the relativistic oscillators in string theory could also be treated as in Ref. [16] and very likely still recover the same gauge-invariant physical states without ever introducing negative norm states in string theory. It would be preferable to quantize without negative norm states at all from the very beginning.

When there is not enough gauge symmetry to remove a degree of freedom that has the wrong sign of kinetic energy, a unitary quantization procedure like that in Ref. [16] maintains unitarity. However, the effect of the negative kinetic energy is to cause an instability (not unlike a tachyonic mass term, or a bottomless potential, would), so that there may not be a ground state for that degree of freedom while it propagates in the antigravity region. This is the negative kinetic energy issue in the antigravity sector. Perhaps some complete theory as a whole conspires to have a ground state even in antigravity. Although this would be reassuring, it appears that this is not necessary in order to make sense of the physics as detected by observers in the gravity sector. Such observers can verify that the same degree of freedom does have a ground state in the gravity region, while they can never experience directly the negative kinetic energy in the antigravity sector. The only physics questions that make sense for those observers is what can be learned about the existence of antigravity through scattering experiments that involve in/out states as defined in the gravity region. For those questions, the issue of whether there is a ground state in the antigravity region does not matter, but unitarity continues to matter. Therefore, we point out how this works in the case of cosmology that admits an antigravity region.

A. WdW equation and unitarity in mini-superspace

The Wheeler de Witt (WdW) equation is the quantum version of the $\mu = 0$ and $\nu = 0$ component of the Einstein equation, $(G_{00} - T_{00})\psi = 0$. This is a constraint applied on physical states in covariant quantization of general relativity [17]. The "mini-superspace" consists of only time-dependent (homogeneous) scalar fields $[\phi(x^0),\ s(x^0)]$ and the FRW metric, $ds^2 = a^2(x^0)(-(dx^0)^2 + \gamma_{ij}(x^0, \vec{x})dx^i dx^j)$, with γ_{ij} describing spacial curvature and anisotropies, while T_{00} includes the radiation density, $\rho_r(x^0)/a^4(x^0)$. From the action in Eq. (1) we can derive a Wheeler de Witt equation that is invariant under Weyl rescalings $(\phi, s, a) \to (\Omega\phi, \Omega s, \Omega^{-1}a)$ with a time-dependent $\Omega(x^0)$; this allows us to choose a gauge. To allow $(\phi^2 - s^2)$ to have any sign dynamically, we prefer the γ gauge given by

$$(\phi, s, a) \to (\phi_\gamma, s_\gamma, 1), \quad \text{or} \quad a_\gamma(x^0) = 1. \tag{7}$$

We concentrate here on the simplest FRW geometry in the γ gauge,

$$ds_\gamma^2 = -(dx^0)^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2, \tag{8}$$

with no anisotropy or inhomogeneities, but with a positive constant spatial curvature $K > 0$. This is not realistic, but it is the easiest case to illustrate the unitarity properties of the quantum theory that includes antigravity regions (more degrees of freedom, and negative or zero K would be treated in a similar manner). The mini-superspace is just (ϕ_γ, s_γ), while the constraint $(T_{00} - G_{00}) = 0$ derived from (1) is $-\frac{1}{2}\dot{\phi}_\gamma^2 + \frac{1}{2}\dot{s}_\gamma^2 + \frac{1}{2}K(-\phi_\gamma^2 + s_\gamma^2) + \rho_r = 0$. This is recognized as the Hamiltonian for the relativistic harmonic oscillator, $H = \frac{1}{2}(\dot{x}^2 + Kx^2)$, with $x^\mu(\tau) = (\phi_\gamma(\tau), s_\gamma(\tau))$, subject to the constraint $H + \rho_r = 0$, where ρ_r is a constant. Note that this Hamiltonian contains negative energy for the (timelike) ϕ_γ degree of freedom. Recall that we have already used up the Weyl symmetry, so this degree of freedom cannot be removed and its negative energy must be dealt with. The naive quantization of the relativistic harmonic oscillator would introduce negative norm states for the ϕ_γ degree of freedom (as in string theory), so it appears there may be trouble with unitarity. However, this is not the case, because this system (and similar cases) can be quantized by respecting unitarity without ever introducing negative norms as shown in Ref. [16]. This goes as follows: the quantum system obeys the constraint equation $(H + \rho_r)\Psi = 0$. This is the WdW equation that takes the form

$$\left(\frac{1}{2}\partial_{\phi_\gamma}^2 - \frac{1}{2}\partial_{s_\gamma}^2 + \frac{K}{2}(-\phi_\gamma^2 + s_\gamma^2) + \rho_r\right)\Psi(\phi_\gamma, s_\gamma) = 0. \tag{9}$$

This is recognized as the Klein-Gordon equation for the quantized relativistic harmonic oscillator. The eigenstates and eigenvalues of the independent ϕ_γ and s_γ oscillators are

$$\begin{aligned}\frac{1}{2}(-\partial_{\phi_\gamma}^2 + K\phi_\gamma^2)\psi_{n_\phi}(\phi_\gamma) &= \sqrt{K}\left(n_\phi + \frac{1}{2}\right)\psi_{n_\phi}(\phi_\gamma), \\ \frac{1}{2}(-\partial_{s_\gamma}^2 + Ks_\gamma^2)\psi_{n_s}(s_\gamma) &= \sqrt{K}\left(n_s + \frac{1}{2}\right)\psi_{n_s}(s_\gamma),\end{aligned} \tag{10}$$

where (n_ϕ, n_s) are positive integers, $0, 1, 2, 3, \ldots$, and the explicit *positive norm* complete set of off-shell solutions are

$$\begin{aligned}\Psi_{n_\phi n_s}(\phi_\gamma, s_\gamma) &= \psi_{n_\phi}(\phi_\gamma)\psi_{n_s}(s_\gamma), \\ \psi_{n_\phi}(\phi_\gamma) &= A_{n_\phi} e^{-\frac{1}{2}\sqrt{K}\phi_\gamma^2} H_{n_\phi}(\phi_\gamma), \\ \psi_{n_s}(s_\gamma) &= A_{n_s} e^{-\frac{1}{2}\sqrt{K}s_\gamma^2} H_{n_s}(s_\gamma),\end{aligned} \tag{11}$$

where $H_n(z)$ are the Hermite polynomials and A_{n_ϕ}, A_{n_s} are normalization constants. Then the WdW equation (9) is solved by constraining the eigenvalues, $\sqrt{K}(-n_\phi + n_s) + \rho_r = 0$. Hence, the complete on-shell basis that satisfies the constraint is

$$\Psi_n(\phi_\gamma, s_\gamma) = A_{n+r}A_n e^{-\frac{\sqrt{K}}{2}(\phi_\gamma^2 + s_\gamma^2)} H_{n+r}(\phi_\gamma)H_n(s_\gamma), \tag{12}$$

with $n = 0, 1, 2, \ldots$, where we define

$$n_s \equiv n, \qquad n_\phi \equiv n + r, \qquad \text{and} \quad \frac{\rho_r}{\sqrt{K}} \equiv r \text{ a fixed integer.} \tag{13}$$

If $\frac{\rho_r}{\sqrt{K}}$ is not an integer, there is no solution to the constraint; hence radiation must be quantized for this system to be nontrivial at the quantum level. The general on-shell solution of the WdW equation is an arbitrary superposition of this basis:

$$\Psi(\phi_\gamma, s_\gamma) = \sum_{n=0}^{\infty} c_n \Psi_n(\phi_\gamma, s_\gamma). \tag{14}$$

The complex coefficients c_n are chosen to insure that $\Psi(\phi_\gamma, s_\gamma)$ is normalized.

All quantum states have a positive norm and unitarity is satisfied. $\Psi(\phi_\gamma, s_\gamma)$ is the probability amplitude for where the system is in the (ϕ_γ, s_γ) plane. The gravity/antigravity regions are $\phi_\gamma^2 \lessgtr s_\gamma^2$. Evidently there is no way of preventing the generic wave functions from being nonzero in the antigravity region, so the system generically evolves through both the gravity and antigravity regions.

We emphasize that the quantization method in Ref. [16] that we used to maintain unitarity is very different from the quantization of the relativistic oscillator used in string theory. In string theory, one defines relativistic creation/annihilation operators a_μ, $a_\mu^\dagger$ and a vacuum state that satisfies $a_\mu|0\rangle = 0$. Then the quantum states at level l are given by applying l creation operators, $a^\dagger_{\mu_1}a^\dagger_{\mu_2}\cdots a^\dagger_{\mu_l}|0\rangle$. The vacuum state is Lorentz invariant, while the states at level l form a collection of *finite-dimensional* irreducible representations of the Lorentz group. All the states at level l have *positive energy*, $E_l = \sqrt{K}(l+1)$. The constraint $H+\rho_r = 0$ (WdW equation) can be satisfied only for negative quantized ρ_r at only one level, $l = -1 + |\rho_r|/\sqrt{K}$. In position space the vacuum state takes the Lorentz-invariant form $\psi_0(x^\mu) \sim e^{-\sqrt{K}x^2} = e^{-\sqrt{K}(-\phi_\gamma^2+s^2)}$, while the the states at level l are of the form of a polynomial of x^μ of degree l multiplied by the same exponential $e^{-\sqrt{K}x^2}$. A subset of the level-l states have negative norm because finite-dimensional representations are not unitary representations of the Lorentz group, so this method of quantization gets into trouble with unitarity. We contrast this result to ours in Eq. (12), where we have displayed an infinite, rather than finite, number of states and a Gaussian factor $e^{-\sqrt{K}(\phi_\gamma^2+s^2)}$ that converges in all directions, rather than the nonconvergent Lorentz-invariant form $e^{-\sqrt{K}(-\phi_\gamma^2+s^2)}$. There is no Lorentz-invariant vacuum state. As shown in Ref. [16], our states in Eq. (12) form an *infinite-dimensional unitary representation* of the Lorentz group for which all the states have positive norm. Furthermore, those that satisfy the constraint have positive total energy, $H = \rho_r$, as long as ρ_r is positive. However, as seen in Eq. (11), there are *off-shell states* of positive as well as negative energy. These remarks make it clear that the price for maintaining unitarity (which is the first duty in quantization) is the presence of regions of spacetime with negative kinetic energy, which, in our case, amounts to regions of antigravity. Our task in this paper is to explain that negative kinetic energy in the antigravity sector does not necessarily imply a problem by interpreting the physical significance of antigravity.

B. Feynman propagator in mini-superspace

The Feynman propagator associated with this WdW equation is

$$G(\phi', s'; \phi, s) = \langle \phi', s'| \frac{i}{H+\rho_r + i\varepsilon} |\phi, s\rangle. \qquad (15)$$

We can use the complete basis $|n_\phi, n_s\rangle$ to insert identity in terms of the eigenstates of the off-shell $H = -H_\phi + H_s$ operator without any constraints on the integers (n_ϕ, n_s). Then we compute

$$\begin{aligned} G(\phi', s'; \phi, s) &= i \sum_{n_\phi, n_s \geq 0} \frac{\langle \phi', s'|n_\phi, n_s\rangle \langle n_\phi, n_s|\phi, s\rangle}{-n_\phi + n_s + \rho_r + i\varepsilon} \\ &= i \sum_{n_\phi, n_s \geq 0} \psi_{n_\phi}(\phi')\psi_{n_s}(s')\psi^*_{n_\phi}(\phi)\psi^*_{n_s}(s)(-n_\phi + n_s + \rho_r + i\varepsilon)^{-1} \\ &= i \sum_{n_\phi, n_s \geq 0} \int_0^\infty d\tau \psi_{n_\phi}(\phi')\psi_{n_s}(s')\psi^*_{n_\phi}(\phi)\psi^*_{n_s}(s)\left(-ie^{i\tau(-n_\phi+n_s+\rho_r+i\varepsilon)}\right) \\ &= \int_0^\infty d\tau e^{i\tau(\rho_r+i\varepsilon)} \langle \phi'|e^{-i\tau H_\phi}|\phi\rangle \langle s'|e^{i\tau H_s}|s\rangle \\ &= \int_0^\infty d\tau \frac{\sqrt{K}e^{i\tau(\rho_r+i\varepsilon)}}{2\pi \sin(\sqrt{K}\tau)} \exp\left(\frac{-i\sqrt{K}}{2\sin(\sqrt{K}\tau)}\left[(x^2 + x'^2)\cos\sqrt{K}\tau - 2x\cdot x'\right]\right). \end{aligned} \qquad (16)$$

In the last step, we used the propagator $\langle \phi'|e^{-i\tau H_\phi}|\phi\rangle$ for the one-dimensional harmonic oscillator, and then substituted $x^2 = -\phi^2 + s^2$ and $x\cdot x' = -\phi\phi' + ss'$. This quantum computation in the Hamiltonian formalism agrees with the path integral computation in Ref. [13].

The Feynman propagator is a measure of the probability that the system that starts in some initial state will be found in some final state. For observers outside of the antigravity region, the initial and final states $|\phi, s\rangle$, $|\phi', s'\rangle$ are both in the gravity region, $|\phi| > |s|$ and $|\phi'| > |s'|$, although during the propagation from initial to final state the antigravity region is probed, as seen from the sums over (n_ϕ, n_s) where both positive and negative energy states of the off-shell Hamiltonian $H = -H_\phi + H_s$ enter in the calculation. We see from the last expression in Eq. (16) that $G(\phi', s'; \phi, s)$ is a perfectly reasonable function, indicating that there are no issues with fundamental principles in this calculation which involves an intermediate period of antigravity in the evolution of the Universe.

This was the case of a radiation-dominated spatially curved spacetime, which is far from being a generic configuration in the early Universe close to the singularity. The generic dominant terms in the Einstein frame are the kinetic energy of the scalar and anisotropy (in the spatial metric), and the next nonleading term is radiation. The subdominant terms, including curvature, inhomogeneities, potential energy, dark energy, etc., are negligible near the singularity. The dominant generic behavior near the singularity was computed classically in Ref. [10], where it was discovered that there *must* be an inescapable excursion into the antigravity regime before coming back to the gravity sector, as outlined in the previous paragraph. Hence, a similar computation to Eq. (16), by using the dominant terms in the WdW equation [instead of (9)] should replace our computation here. Unpublished work along these lines dating back to 2011 [9] indicates that the physical picture already obtained through classical solutions in Ref. [10] continues to hold in mini-superspace at the quantum level.

To conclude this section, an important remark is that unitarity is maintained in the WdW treatment throughout gravity and antigravity, while the presence of negative energy during antigravity is not of concern regarding fundamental principles, as already illustrated in this section with the simpler computation based on Eq. (9).

IV. NEGATIVE ENERGY IN ANTIGRAVITY AND OBSERVERS IN GRAVITY

To develop a physical understanding of negative kinetic energy, we will discuss several toy models that will include the analog of a background gravitational field that switches sign between positive and negative kinetic energy. The physical question is, what do observers in the gravity region detect about the presence of a negative kinetic energy sector? Conceptually, this is the analog of a black box being probed by in/out signals detected at the exterior of the box.

In the field theory or particle examples discussed below, a simple sign function that is modeled after the "antigravity loop" in Ref. [10] captures the main effect of antigravity. This sign function is a simple device to answer questions that arose repeatedly on unitarity and possible instability and is not necessarily a solution to the gravitational field equations of some specific model. Rather, it is used here only to capture the main effect of an antigravity sector in a simple and solvable model. In the case of realistic applications, one would need to use a self-consistent solution of matter and gravitational equations (as in Ref. [10]), as long as it captures the main features of antigravity as in the simplified model background discussed here.

A. Particle with time-dependent kinetic energy flips

A free particle with a relativistic (or nonrelativistic) Hamiltonian that switches sign as a function of time provides an example of a system propagating in a background gravitational field that switches sign as in Eq. (6):

$$H = \varepsilon\left(|t| - \frac{\Delta}{2}\right) \times \sqrt{p^2 + m^2} \quad \text{or} \quad H = \varepsilon\left(|t| - \frac{\Delta}{2}\right) \times \frac{p^2}{2m}, \tag{17}$$

TABLE I. Values of the Hamiltonian H and velocity $\dot{x}$, demonstrating how the sign changes as a function of t.

t:	$t < -\frac{\Delta}{2}$	$-\frac{\Delta}{2} < t < \frac{\Delta}{2}$	$t > \frac{\Delta}{2}$
$H_\pm$:	$\sqrt{p^2+m^2}$	$-\sqrt{p^2+m^2}$	$\sqrt{p^2+m^2}$
$\dot{x}$:	$\dot{x} = \frac{p}{\sqrt{p^2+m^2}}$	$\dot{x} = -\frac{p}{\sqrt{p^2+m^2}}$	$\dot{x} = \frac{p}{\sqrt{p^2+m^2}}$

where $\varepsilon(u) \equiv \mathrm{Sign}(u)$. Such a background captures some of the properties of the antigravity loop of Bars-Steinhardt-Turok [10]. The particle's phase space (x, p) can also represent more generally a typical generalized degree of freedom in field theory or string field theory.

The momentum p is conserved, since H is independent of x, but the velocity $\dot{x} = \partial H/\partial p = \varepsilon(|t| - \frac{\Delta}{2}) \frac{p}{\sqrt{p^2+m^2}}$ alternates signs as shown in Table I. The Hamiltonian is time dependent, so it is not conserved. At the $t = \pm\Delta/2$ kinks, the velocity vanishes if we define $\varepsilon(0) = 0$. It is possible to make other models of what happens to the velocity by replacing the sign function $\varepsilon(z)$ with other time-dependent kinky or smooth models; for example, if we replace $\varepsilon(z)$ with $(\varepsilon(z))^{-1}$, then the velocity at the kinks changes sign at an infinite value rather than at zero, while the momentum remains a constant in all cases.

If the initial position before entering antigravity is $x_i(t_i)$, we compute the evolution at any time as follows (see Fig. 1):

$$\begin{aligned} t < -\frac{\Delta}{2} &: x(t) = x_i(t_i) + \frac{p}{\sqrt{p^2+m^2}}(t - t_i), \\ -\frac{\Delta}{2} < t < \frac{\Delta}{2} &: x(t) = x\left(-\frac{\Delta}{2}\right) - \frac{p}{\sqrt{p^2+m^2}}\left(t + \frac{\Delta}{2}\right), \\ t > \frac{\Delta}{2} &: x(t) = x\left(\frac{\Delta}{2}\right) + \frac{p}{\sqrt{p^2+m^2}}\left(t - \frac{\Delta}{2}\right), \end{aligned} \tag{18}$$

where $x(-\frac{\Delta}{2}) = x_i(t_i) + \frac{p}{\sqrt{p^2+m^2}}(-\frac{\Delta}{2} - t_i)$, and $x(\frac{\Delta}{2}) = x_i(t_i) - \frac{p}{\sqrt{p^2+m^2}}(3\Delta/2 + t_i)$. The final position $x_f(t_f)$, at

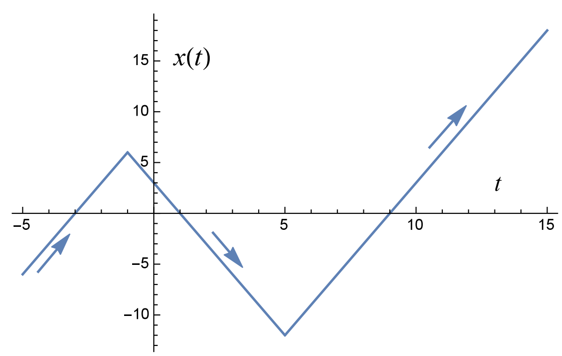

FIG. 1. Propagation through antigravity.

a time t_f after waiting long enough to exit from antigravity, $t_f > \Delta/2$, is

$$x_f(t_f) = x_i(t_i) + \frac{p}{\sqrt{p^2+m^2}}(t_f - t_i - 2\Delta). \quad (19)$$

The effect of antigravity during the interval, $-\frac{\Delta}{2} < t < \frac{\Delta}{2}$, is the backward excursion between the two kinks shown in Fig. 1. For observers waiting for the arrival of the particle at some position $x_f(t_f)$, we see from Eq. (19) that antigravity causes a time delay by the amount of 2Δ as compared to the absence of antigravity. Hence, there is a measurable signal, namely a time delay, as an observable effect in comparing the presence and absence of antigravity.

A similar problem is analyzed at the quantum level by computing the transition amplitude from an initial state $|x_i, t_i\rangle$ to a final state $|x_f, t_f\rangle$, requiring that the final observation be in the gravity period *after* passing through the antigravity period. This is given by

$$\begin{aligned} A_{fi} &= \langle x_f, t_f| e^{-\frac{i}{\hbar}H_+(t_f - \frac{\Delta}{2})} e^{-\frac{i}{\hbar}H_-(\frac{\Delta}{2} - \frac{-\Delta}{2})} e^{-\frac{i}{\hbar}H_+(\frac{-\Delta}{2} - t_i)} |x_i, t_i\rangle \\ &= \langle x_f, t_f| e^{-\frac{i}{\hbar}H(t_f - t_i - 2\Delta)} |x_i, t_i\rangle \\ &= \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i - 2\Delta)}} \exp\left(\frac{im(x_f - x_i - 2\Delta)^2}{2\hbar(t_f - t_i - 2\Delta)}\right). \end{aligned} \quad (20)$$

The last expression is for the case of a nonrelativistic particle with $H_\pm = \pm H = \pm p^2/2m$. The exponentials involving $H_\pm$ are simplified because $H_\pm$ commute with each other, allowing the combination of the exponentials into a single exponential. Thus, the effect of the intermediate antigravity period is to cause only a time delay, just as in the classical solution above. Note also that there are no unitarity problems; the evolution operator is unitary, and norms of states are positive, at all stages.

B. Particle with space-dependent kinetic energy flips

Consider a nonrelativistic particle with a total energy Hamiltonian that switches sign in different regions of space, for example

$$H = \varepsilon\left(|x| - \frac{\Delta}{2}\right) \times \left(\frac{p^2}{2m} + V(x)\right). \quad (21)$$

This is another example of a system propagating in a background gravitational field that switches sign as in Eq. (6). In this case energy is conserved, since there is no explicit time dependence in H. Therefore, at generic energies, $E = (\frac{p^2}{2m} + V(x))$, the particle cannot cross the boundaries at $|x| = \frac{\Delta}{2}$, since the Hamiltonian would flip sign and this would contradict the energy conservation. Hence, if the particle is in the gravity region, $|x| > \frac{\Delta}{2}$, it stays there, and if it is in the antigravity region, $|x| < \frac{\Delta}{2}$, it stays there. However, the particle can cross from gravity to antigravity and back again to gravity at zero energy, $\frac{p^2}{2m} + V(x) = 0$. This is similar to the geodesics in a black hole that cross from gravity to antigravity [11].

C. Free massless scalar field with sign-flipping kinetic energy

Consider a free massless scalar field in flat space with a time-dependent background field that causes sign flips of the kinetic energy as a function of time:

$$S = -\frac{1}{2}\int d^4x \varepsilon\left(|x^0| - \frac{\Delta}{2}\right)\partial^\mu\phi(x)\partial_\mu\phi(x). \quad (22)$$

The factor $\varepsilon(|x^0| - \frac{\Delta}{2})$ can be viewed as a gravitational background field of the form $\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, with $g^{\mu\nu}(x) = \varepsilon(|x^0| - \frac{\Delta}{2})\eta^{\mu\nu}$ and $\sqrt{-g} = 1$. This sign-flipping metric should be regarded as an example of a geometry that spans the union of the gravity and antigravity regions, as in Eq. (6). We proceed to analyze the time evolution of this system. Let the on-shell initial field configuration at time $x_i^0 < (-\Delta/2)$ be defined by

$$\phi_i(\vec{x}_i, x_i^0) = \int \frac{d^3p}{(2\pi)^{3/2}2|p|}(a(\vec{p})e^{-i|p|x_i^0 + i\vec{p}\cdot\vec{x}_i} + \bar{a}(\vec{p})e^{i|p|x_i^0 - i\vec{p}\cdot\vec{x}_i}). \quad (23)$$

The general solution for $\phi(\vec{x}, x^0)$ evolved up to a final time $x_f^0 > \Delta/2$ is then given by [using the method in Eq. (18)]

$$\phi_f(\vec{x}_f, x_f^0) = \int \frac{d^3p}{(2\pi)^{3/2}2|p|}\begin{pmatrix} a(\vec{p})e^{-i|p|(x_f^0 - x_i^0 - 2\Delta) + i\vec{p}\cdot\vec{x}_f} \\ +\bar{a}(\vec{p})e^{i|p|(x_f^0 - x_i^0 - 2\Delta) - i\vec{p}\cdot\vec{x}_f} \end{pmatrix}. \quad (24)$$

This shows that for initial/final observations that are strictly outside of the antigravity period, the effect of the antigravity period is only a time delay as compared to the complete absence of antigravity. The time evolution of the field in the interim period is just like the time evolution of the particle as shown in Fig. 1. For more details on the classical evolution of the field in the interim period, see the case of the massive field in Sec. IV E, and take the zero mass limit.

An important remark is that the multiparticle Hilbert space $\{|\vec{p}_1, \vec{p}_2 \cdots \vec{p}_n\rangle\}$ is the Fock space constructed from the creation operators applied on the vacuum defined by $a(\vec{p})|0\rangle = 0$, namely $|\vec{p}_1, \vec{p}_2 \cdots \vec{p}_n\rangle \equiv \bar{a}(\vec{p}_1)\bar{a}(\vec{p}_2)\cdots\bar{a}(\vec{p}_n)|0\rangle$. This time-independent Fock space is the complete Hilbert space that can be used during gravity or antigravity. It is clearly unitary, since it is the same Hilbert space that is independent of the existence of an antigravity period (i.e., the same as the $\Delta = 0$ case). This shows that there is no unitarity problem due to the presence of the antigravity period.

However, there is negative kinetic energy during antigravity, seen as follows. The time-dependent Hamiltonian for this system is

$$H(x^0) = \begin{cases} +H, & \text{for } t < -\frac{\Delta}{2} \\ -H, & \text{for } -\frac{\Delta}{2} < t < \frac{\Delta}{2}, \\ +H, & \text{for } t > \frac{\Delta}{2} \end{cases} \tag{25}$$

where H, which is constructed from the quantum creation-annihilation operators as usual, is time independent. So there seems to be a possible source of instability due to negative energy during antigravity. For freely propagating particles there are no transitions that alter the energy, so no questions arise; it is only when there are interactions that an effect may be observed due to transitions created by the negative energy sector. The effect of interactions, as observed by detectors in the gravity sector, is analogous to the case of a time-dependent Hamiltonian as discussed in simple examples below in Sec. IV D. Hence, the presence of a sector with negative kinetic energy is not a fundamental problem in the quantum theory.

Nevertheless, the antigravity sector, with or without interactions, is the source of interesting physical signals for the observers in the gravity sectors. For example, in the absence of additional interactions, consider the quantum propagator that corresponds to initial/final states in the two gravity sectors $|x^0| > \Delta/2$. The transition amplitude from an initial state in gravity ($x_i^0 < -\Delta/2$) to a final state in gravity ($x_f^0 > \Delta/2$), after the field evolves through antigravity, is given by

$$\begin{aligned} A_{fi} &= \langle \phi_f(x_f) | e^{-\frac{i}{\hbar}H_+(t_f - \frac{\Delta}{2})} e^{-\frac{i}{\hbar}H_-(\frac{\Delta}{2} - \frac{-\Delta}{2})} e^{-\frac{i}{\hbar}H_+(\frac{-\Delta}{2} - t_i)} | \phi_i(x_i) \rangle \\ &= \langle \phi_f(\vec{x}_f, x_f^0) | e^{-\frac{i}{\hbar}H(t_f - t_i - 2\Delta)} | \phi_i(\vec{x}_i, x_i^0) \rangle. \end{aligned}$$

Here, $|\phi(x)\rangle$ is defined as the one-particle state in the quantum theory which is created by applying the quantum field $\hat{\phi}(x)$ on the oscillator vacuum $a(\vec{p})|0\rangle = 0$,

$$|\phi(\vec{x}, x^0)\rangle = \hat{\phi}(x)|0\rangle = \int d^3p \frac{e^{i|p|x^0 - i\vec{p}\cdot\vec{x}}}{(2\pi)^{3/2} 2|p|} \bar{a}(\vec{p})|0\rangle. \tag{26}$$

Then we obtain

$$A_{fi} = \int d^3p \frac{e^{i|p|(x_f^0 - x_i^0 - 2\Delta) - i\vec{p}\cdot(\vec{x}_f - \vec{x}_i)}}{(2\pi)^{3/2} 2|p|}. \tag{27}$$

This is the propagator for a free massless particle. From this expression it is clear that the effect of antigravity on the result for the transition amplitude A_{fi} is only a time delay by an amount of 2Δ as compared to the same quantity in the complete absence of antigravity. The same general statement holds true for the transition amplitudes for multiparticle states. Clearly there is no particle production due to antigravity in the case of free massless particles. This will be contrasted with the case of massive particles in Sec. IV E.

Of course, if there are field interactions, there will be additional effects, but none of those are *a priori* problematic from the point of view of fundamental principles.

D. Particle with flipping kinetic energy while interacting in a potential

To learn more about the effects of antigravity, we now add an interaction term that does not flip sign during antigravity. We first investigate the case of a single degree of freedom whose kinetic energy flips sign during antigravity. This phase space (x, p) should be thought of as a generalized coordinate associated with any single degree of freedom within local field theory or string field theory (after integrating out all other degrees of freedom), but in its simplest form it can be regarded as representing a particle moving in one dimension.

We discuss a simple model described by the Hamiltonian

$$H = \varepsilon\left(|t| - \frac{\Delta}{2}\right) \times \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{28}$$

This is a time-dependent Hamiltonian that has two different forms, $H_\pm$, during different periods of time as shown in Table II. During gravity, $|t| > \frac{\Delta}{2}$, the Hamiltonian H_+ is the familiar harmonic oscillator Hamiltonian with a well-defined quantum state, so all energies are positive. But during antigravity, $-\frac{\Delta}{2} < t < \frac{\Delta}{2}$, the Hamiltonian H_- has no bottom, so all positive and negative energies are permitted. Does this pose an instability problem for the entire system? The answer is that, as in the simpler cases already illustrated above, there is no such problem from the perspective of observers in gravity.

A complete basis for a unitary Hilbert space may be defined to be the positive norm complete Fock space associated with the usual harmonic oscillator Hamiltonian H_+ whose energy eigenvalues are strictly positive. The eigenstates of H_- are also positive norm and define another complete unitary basis. Clearly one complete basis may be expanded in terms of another complete basis, so the usual Fock space basis is sufficient to analyze the complete system, including its evolution through antigravity. This shows that the interacting problem that

TABLE II. Values of the Hamiltonian H as a function of time t.

t:	$t < -\frac{\Delta}{2}$	$-\frac{\Delta}{2} < t < \frac{\Delta}{2}$	$t > \frac{\Delta}{2}$
$H_\pm$:	$(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2})$	$(-\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2})$	$(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2})$

includes antigravity is an ordinary time-dependent problem in quantum mechanics. There are no unitarity problems, and the presence of antigravity is analyzed below as a regular problem of a time-dependent Hamiltonian, without encountering any fundamental problems of principles.

A technical remark may be useful: this model can be treated group-theoretically by using the properties of $\mathrm{SL}(2,R)$ representations. Note that the three Hermitian quantum operators $[x^2, p^2, \frac{1}{2}(xp+px)]$ form the algebra of $\mathrm{SL}(2,R)$ under quantum commutation rules $[x,p]=i\hbar$. The Hamiltonian H_+ is proportional to the compact generator J_0 of $\mathrm{SL}(2,R)$, while H_- is proportional to one of the noncompact generators J_1. The second noncompact generator J_2 appears in the commutator $[H_+,H_-]$. Explicitly,

$$J_0 \equiv \frac{1}{2\hbar\omega} H_+, \qquad J_1 \equiv \frac{1}{2\hbar\omega} H_-,$$
$$J_2 = \frac{1}{4\hbar}(xp+px). \tag{29}$$

The (J_0, J_1, J_2) form the standard Lie algebra of $\mathrm{SL}(2,R)$. Since these $J_{0,1,,2}$ are Hermitian operators, the corresponding quantum states which are labelled as $|j,\mu\rangle$ form a unitary representation of $\mathrm{SL}(2,R)$. The quantum number μ is associated with the eigenvalues of J_0 (which are basically the eigenvalues of H_+) while $j(j+1)$ is associated with the eigenvalues of the Casimir operator C_2 that commutes with all the generators, $C_2 \equiv J_0^2 - J_1^2 - J_2^2$. For the present construction, keeping track of the orders of operators (x,p), one finds that C_2 is a constant, $C_2 = -3/16 = j(j+1)$, which yields two solutions, $j=-\frac{3}{4}$ or $j=-\frac{1}{4}$. Hence, the spectrum of this theory, including the properties of $H_\pm$, can be thought of consisting of two infinite-dimensional irreducible unitary representations of $\mathrm{SL}(2,R)$. For $j=-\frac{3}{4}$ or $j=-\frac{1}{4}$, these are positive discrete series representations. The allowed values of μ are given by $\mu = j+1+k$, where $k=0,1,2,\ldots$ is an integer. We see that the two representations taken together correspond to the spectrum of H_+, which is the spectrum of the harmonic oscillator given by $E_n = \omega(n+\frac{1}{2}) \Leftrightarrow 2\omega\mu$, with even $n=2k$ corresponding to $j=-3/4$ and odd $n=2k+1$ corresponding to $j=-1/4$. Hence, the basis $|j,\mu\rangle$ forms a complete set of eigenstates for the observers in the gravity sector of the theory.

How about the antigravity sector? Since the corresponding Hamiltonian is H_-, a complete set of eigenstates corresponds to diagonalizing the noncompact generator J_1 instead of the compact generator J_0. Either way, the Casimir operator is the same; hence diagonalizing $J_1 \to q$ provides another unitary basis $|j,q\rangle$ for the same unitary representations of $\mathrm{SL}(2,R)$. The spectrum of J_1, $J_1|j,q\rangle = q\,|j,q\rangle$, is continuous q on the real line since this is a noncompact generator of $\mathrm{SL}(2,R)$. This antigravity basis is also a complete unitary basis for this Hamiltonian that includes both sectors $H_\pm$. One basis can be expanded in terms of the other, $|j,q\rangle = \sum_{\mu=j+1}^{\infty} |j,\mu\rangle\langle j,\mu|j,q\rangle$, where the expansion coefficients $\langle j,\mu|j,q\rangle = U^{(j)}_{\mu,q}$ are unitary transformations for each $j=-\frac{3}{4}$ or $-\frac{1}{4}$.

Therefore, it does not matter which basis we use to analyze the quantum properties of this Hamiltonian. Using the discrete basis $|j,\mu\rangle$ which is more convenient to analyze the physics in the gravity sector in no way excludes the antigravity sector from making its effects felt for observers in the gravity sector.

With this understanding of this simple quantum system, we now analyze the transition amplitudes A_{fi} for an initial state $|i\rangle$ to propagate to a final state, both in the gravity sector. We define $|i\rangle, |f\rangle$ at the two edges of the antigravity sector, at times $t_i = -\Delta/2$ and $t_f = \Delta/2$. Moving t_i, t_f to other arbitrary times in the gravity sector is trivial, since we can write $|i\rangle = e^{-iH_+(-\Delta/2-t_i)}|i,t_i\rangle$ and $|f\rangle = e^{-iH_+(t_f-\Delta/2)}|f,t_f\rangle$, and we know how H_+ acts on any linear combination of harmonic oscillator states $|i\rangle, |f\rangle$. Hence, we have

$$A_{fi} = \langle f|e^{-\frac{i}{\hbar}\Delta H_-}|i\rangle, \tag{30}$$

where $|i\rangle, |f\rangle$ are arbitrary states in the gravity sector. If we take any two states in the $\mathrm{SL}(2,R)$ basis $|j,\mu\rangle$, this becomes

$$A_{fi} = \langle j,\mu_f|e^{-i\frac{\Delta}{2\omega}J_1}|j,\mu_i\rangle. \tag{31}$$

This is just the matrix representation of a group element of $\mathrm{SL}(2,R)$ in a unitary representation labeled by $j=-\frac{3}{4}$ or $-\frac{1}{4}$. It must be the same j for both the initial and final states; i.e. there is a selection rule, because there can be no transitions at all from $j=-\frac{3}{4}$ to $j=-\frac{1}{4}$ or vice versa.

This quantity can be computed by using purely group-theoretical means, but it is perhaps more instructive to use the standard harmonic oscillator creation/annihilation operators to evaluate it. Then we can write

$$H_+ = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \qquad H_- = \frac{\hbar\omega}{2}(a^{\dagger 2} + a^2). \tag{32}$$

We have used this form to compute the transition amplitude

$$A_{fi} = \langle f|e^{-\frac{i}{\hbar}\Delta H_-}|i\rangle = \langle f|e^{-i\frac{\omega\Delta}{2}(a^{\dagger 2}+a^2)}|i\rangle \tag{33}$$

by taking initial/final states to be the number states or the coherent states of the harmonic oscillator. To perform the computation, we use the following identity:

$$e^{-i\frac{\omega\Delta}{2}(a^{\dagger 2}+a^2)} = e^{-\frac{i}{2}\tanh(\omega\Delta)a^{\dagger 2}}(\cosh(\omega\Delta))^{-(a^\dagger a+\frac{1}{2})}$$
$$\times e^{-\frac{i}{2}\tanh(\omega\Delta)a^2}. \tag{34}$$

For initial/final coherent states $|z_i\rangle$ and $|z_f\rangle$ for observers in gravity, we define the transition amplitude for normalized states as $A(z_f, z_i) = \langle z_f| e^{-\frac{i}{\hbar}\Delta H_-}|z_i\rangle / \sqrt{\langle z_f|z_f\rangle\langle z_i|z_i\rangle}$, which yields

$$|A(z_f, z_i)|^2 = \frac{e^{-|z_f|^2 - |z_i|^2 + \frac{2\mathrm{Re}(z_i \bar{z}_f)}{\cosh(\omega\Delta)}} e^{\tanh(\omega\Delta)\mathrm{Im}(\bar{z}_f^2 e^{-i\omega\Delta} + z_i^2 e^{i\omega\Delta})}}{\cosh(\omega\Delta)}. \tag{35}$$

This should be compared to the absence of antigravity when $\Delta = 0$, namely $|A(z_f, z_i)|^2 \overrightarrow{\Delta 0} e^{-|z_f - z_i|^2}$.

Similarly, for initial/final number eigenstates $|n\rangle$ and $|m\rangle$ of the Hamiltonian $H_+ = (\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}) = \hbar\omega(a^\dagger a + \frac{1}{2})$ for observers in gravity, we obtain

$$A_{mn} = \sqrt{\frac{m!n!e^{i\omega\Delta(n+m+1)}}{(\cosh(\omega\Delta))^{m+n+1}}} \sum_{k=0}^{\min(m,n)} \frac{(\frac{1}{2i}\sinh(\omega\Delta))^{\frac{m+n}{2}-k}}{k!(\frac{m-k}{2})!(\frac{n-k}{2})!}, \tag{36}$$

where (m, n, k) are all even or all odd. This gives

$$|A_{mn}|^2 = \left(\begin{array}{c} \left({}_2F_1\left(-[\frac{m}{2}], -[\frac{n}{2}]; \left(1 - \frac{(-1)^m}{2}\right); \frac{-1}{\sinh^2(\omega\Delta)}\right)\right)^2 \\ \times \frac{(m!)(n!)(\frac{1}{2}\tanh(\omega\Delta))^{2([\frac{m}{2}]+[\frac{n}{2}])}}{([\frac{m}{2}]![\frac{n}{2}]!)^2(\cosh(\omega\Delta))^{2-(-1)^m}} \end{array} \right),$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function, $[\frac{m}{2}]$ means the integer part of $m/2$, and (m, n) are both even or both odd. Special cases are

$$\begin{aligned} |A_{00}|^2 &= \frac{1}{\cosh(\omega\Delta)}, \\ |A_{2M,0}|^2 &= \frac{(2M)!}{2^{2M}(M!)^2} \frac{(\tanh(\omega\Delta))^{2M}}{\cosh(\omega\Delta)}, \\ |A_{11}|^2 &= \frac{1}{\cosh^3(\omega\Delta)}, \\ A_{2M+1,1} &= \frac{(2M+1)!}{2^{2M}(M!)^2} \frac{(\tanh(\omega\Delta))^{2M}}{\cosh^3(\omega\Delta)}. \end{aligned} \tag{37}$$

As compared to the absence of antigravity, $\Delta = 0$, when there are no transitions, we see that antigravity causes an observable effect. Clearly, these transition amplitudes are well behaved, and do not blow up for large Δ. Unitarity is obeyed: one may verify explicitly that the sum over all states is 100% probability, $\sum_m |A_{mn}|^2 = 1$, for all fixed n, and similarly $\int \frac{d^2 z_f}{\pi} |A(z_f, z_i)|^2 = 1$ for all fixed z_i.

E. Massive scalar field with sign-flipping kinetic energy

This system has some similarities to the interacting particle above, but it is not quite the same. The action is

$$S = \frac{1}{2}\int d^d x \left[-\varepsilon\left(|x^0| - \frac{\Delta}{2}\right)\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\phi^2(x)\right]. \tag{38}$$

As in the case of the massless field in Sec. IV C, the factor $\varepsilon(|x^0| - \frac{\Delta}{2})$ can be viewed as a gravitational background field of the form $\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, with $g^{\mu\nu}(x) = \varepsilon(|x^0| - \frac{\Delta}{2})\eta^{\mu\nu}$ and $\sqrt{-g} = 1$ that spans the union of the gravity and antigravity regions, as explained in Eq. (6). The mass term does not flip sign. Note that, due to the nonzero mass, this is *not* a Weyl-invariant action, but we will investigate it anyway to learn about the properties of such a system.

In momentum space, using the notation $x^0 = t$, we have

$$\phi(\vec{x}, t) = \int \frac{d^{d-1}p}{(2\pi)^{(d-1)/2}} \phi_p(t) e^{i\vec{p}\cdot\vec{x}}. \tag{39}$$

We rewrite the action in momentum space as

$$S = \frac{1}{2}\int dt \int d^{d-1}p \left[\begin{array}{c} \varepsilon\left(|t| - \frac{\Delta}{2}\right)\left[\begin{array}{c} \dot{\phi}_p(t)\dot{\phi}_{-p}(t) \\ -\vec{p}^2\phi_p(t)\phi_{-p}(t) \end{array}\right] \\ -m^2\phi_p(t)\phi_{-p}(t) \end{array} \right]. \tag{40}$$

The equation of motion is

$$\partial_t\left(\varepsilon\left(|t| - \frac{\Delta}{2}\right)\partial_t\phi_p(t)\right) + \left[\varepsilon\left(|t| - \frac{\Delta}{2}\right)\vec{p}^2 + m^2\right]\phi_p(t) = 0. \tag{41}$$

The solutions in separate regions of time are [similar to (18)]

$$\begin{aligned} t < -\frac{\Delta}{2}&: \ \phi_p^A(t) = \begin{pmatrix} A_p^+ e^{-i\sqrt{\vec{p}^2+m^2}(t+\frac{\Delta}{2})} \\ +A_p^- e^{i\sqrt{\vec{p}^2+m^2}(t+\frac{\Delta}{2})} \end{pmatrix}, \\ -\frac{\Delta}{2} < t < \frac{\Delta}{2}&: \ \phi_p^B(t) = \begin{pmatrix} B_p^+ e^{-i\sqrt{\vec{p}^2-m^2}(t+\frac{\Delta}{2})} \\ +B_p^- e^{i\sqrt{\vec{p}^2-m^2}(t+\frac{\Delta}{2})} \end{pmatrix}, \\ t > \frac{\Delta}{2}&: \ \phi_p^C(t) = \begin{pmatrix} C_p^+ e^{-i\sqrt{\vec{p}^2+m^2}(t-\frac{\Delta}{2})} \\ +C_p^- e^{i\sqrt{\vec{p}^2+m^2}(t-\frac{\Delta}{2})} \end{pmatrix}. \end{aligned} \tag{42}$$

We need to match the field $\phi_p(t)$ and its canonical momentum, $\varepsilon(|t| - \frac{\Delta}{2})\partial_t\phi_p(t)$, at each boundary $t = \pm\Delta/2$:

$$\phi_p^A\left(-\frac{\Delta}{2}\right)=\phi_p^B\left(-\frac{\Delta}{2}\right),\quad \text{and}\quad \dot\phi_p^A\left(-\frac{\Delta}{2}\right)=-\dot\phi_p^B\left(-\frac{\Delta}{2}\right),$$
$$\phi_p^C\left(+\frac{\Delta}{2}\right)=\phi_p^B\left(+\frac{\Delta}{2}\right),\quad \text{and}\quad \dot\phi_p^C\left(+\frac{\Delta}{2}\right)=-\dot\phi_p^B\left(+\frac{\Delta}{2}\right), \tag{43}$$

Note the sign flip of $\dot\phi$ at $t=\pm\Delta/2$ although the canonical momentum does not flip. This gives four equations to relate $C_p^\pm$ and $B_p^\pm$ to $A_p^\pm$ as follows:

$$\begin{aligned}
&A_p^+ + A_p^- = B_p^+ + B_p^-,\\
&A_p^+ - A_p^- = -(B_p^+ - B_p^-)\frac{\sqrt{\vec p^2-m^2}}{\sqrt{\vec p^2+m^2}},\\
&C_p^+ + C_p^- = B_p^+ e^{-i\sqrt{\vec p^2-m^2}\Delta} + B_p^- e^{i\sqrt{\vec p^2-m^2}\Delta},\\
&C_p^+ - C_p^- = -\begin{pmatrix} B_p^+ e^{-i\sqrt{\vec p^2-m^2}\Delta}\\ -B_p^- e^{i\sqrt{\vec p^2-m^2}\Delta}\end{pmatrix}\frac{\sqrt{\vec p^2-m^2}}{\sqrt{\vec p^2+m^2}}.
\end{aligned} \tag{44}$$

The solution determines $B_p^\pm$ and $C_p^\pm$ in terms of $A_p^\pm$:

$$\begin{pmatrix} C_p^+\\ C_p^-\end{pmatrix}=\begin{pmatrix}\alpha & \beta^*\\ \beta & \alpha^*\end{pmatrix}\begin{pmatrix} A_p^+\\ A_p^-\end{pmatrix}, \tag{45}$$

$$\begin{pmatrix} B_p^+\\ B_p^-\end{pmatrix}=\begin{pmatrix}\frac12-\frac{\sqrt{\vec p^2+m^2}}{2\sqrt{\vec p^2-m^2}} & \frac12+\frac{\sqrt{\vec p^2+m^2}}{2\sqrt{\vec p^2-m^2}}\\ \frac12+\frac{\sqrt{\vec p^2+m^2}}{2\sqrt{\vec p^2-m^2}} & \frac12-\frac{\sqrt{\vec p^2+m^2}}{2\sqrt{\vec p^2-m^2}}\end{pmatrix}\begin{pmatrix} A_p^+\\ A_p^-\end{pmatrix}, \tag{46}$$

where (α,β) are the parameters of a Bogoliubov transformation [an SU(1,1) group transformation]

$$\begin{aligned}
\alpha &= \cos\left(\Delta\sqrt{\vec p^2-m^2}\right)+i\frac{\vec p^2\sin(\Delta\sqrt{\vec p^2-m^2})}{\sqrt{(\vec p^2)^2-m^4}},\\
\beta &= i\frac{m^2\sin(\Delta\sqrt{\vec p^2-m^2})}{\sqrt{(p^2)^2-m^4}},\\
|\alpha|^2&-|\beta|^2=1.
\end{aligned} \tag{47}$$

Assume the incoming state $\phi_p^A(t)$ has only positive frequency, meaning $A_p^-=0$. Then we see that [unlike the massless case in Sec. IV C] negative frequency fluctuations are produced in the final state $\phi_p^C(t)$, since according to Eq. (45), $C_p^-=\beta A_p^+$. The corresponding probability amplitude for particle production is

$$(C_p^-/A_p^+)=\beta=i\frac{\sin(m\Delta\sqrt{\vec p^2/m^2-1})}{\sqrt{(\vec p^2/m^2)^2-1}}. \tag{48}$$

The produced particle-number density (particles per unit volume) is the integral of $|\beta|^2$ over all momenta:

$$\begin{aligned}
n(m,\Delta) &= \int d^{d-1}p|\beta|^2\\
&= \int d^{d-1}p\frac{\sin^2\left(m\Delta\sqrt{(\vec p^2/m^2)-1}\right)}{|(\vec p^2/m^2)^2-1|},\\
&= m^{d-1}\Omega_{d-1}\int_0^\infty\frac{x^{d-2}\sin^2\left((m\Delta)\sqrt{x^2-1}\right)}{|x^4-1|}dx,
\end{aligned} \tag{49}$$

where $x^2=\vec p^2/m^2$, while Ω_{d-1} is the volume of the solid angle in $d-1$ dimensions, $\Omega_2=2\pi,\Omega_3=4\pi$, etc. This is a convergent integral for $d<(5-\varepsilon)$ dimensions, hence $n(m,\Delta)$ is finite for $d=1,2,3,4$ dimensions. We note that the number density $n(m,\Delta)$ increases monotonically at fixed m as Δ increases. The energy density per unit volume for the produced particles for all momenta is

$$\begin{aligned}
\rho(m,\Delta) &= \int\frac{d^{d-1}p}{(2\pi)^{d-1}}\sqrt{\vec p^2+m^2}|\beta|^2\\
&= \frac{m^d\Omega_{d-1}}{(2\pi)^{d-1}}\int_0^\infty\frac{x^{d-2}\sqrt{x^2+1}\sin^2\left((m\Delta)\sqrt{x^2-1}\right)}{|x^4-1|}dx.
\end{aligned} \tag{50}$$

$\rho(m,\Delta)$ is convergent for $d<(4-\varepsilon)$ dimensions and is logarithmically divergent at $d=4$ despite the rapid oscillations at the ultraviolet limit.

Recall that the massive field is not a scale-invariant model. In the Weyl symmetric limit, $m\to 0$, there is no particle production at all in any dimension. In the scale-invariant theory, masses for fields must come from interactions, such as interactions with the Higgs field. In a cosmological context, the Higgs field is not just a constant, and therefore in the type of investigation above, the parameter m should be replaced by the cosmological behavior of the Higgs field (see Ref. [18] for an example). This very different behavior in a Weyl-invariant theory should be the more serious approach for investigating effectively massive fields to answer the type of questions discussed in this section.

V. CONFORMALLY EXACT SIGN-FLIPPING BACKGROUNDS IN STRING THEORY

We consider the world-sheet formulation of the relativistic string, but we make string theory consistent with target-space Weyl symmetry as suggested in Ref. [7]. This requires promoting the string tension to a dynamical field, $(2\pi\alpha')^{-1}\to T(X^\mu(\tau,\sigma))$. The background field $T(X)$, along with any other additional background fields, must be restricted to satisfy exact world-sheet conformal symmetry at the quantum level. In the world-sheet formalism, typically the tension appears together with the metric

$g_{\mu\nu}(X(\tau,\sigma))$ or antisymmetric tensor $b_{\mu\nu}(X(\tau,\sigma))$ in the Weyl-invariant combination, $Tg_{\mu\nu}$ or $Tb_{\mu\nu}$. The requirement of exact *world-sheet* conformal symmetry constrains these target-space Weyl-invariant combinations. Perturbative world-sheet conformal symmetry (vanishing beta functions) is captured by the properties of the low-energy effective string action. From the study of the Weyl-invariant and geodesically complete formalism of the low-energy string action [7], we have learned that the tension (closely connected to the gravitational constant) switches sign generically near the singularities in the classical solutions of this theory. If we fix the target-space Weyl symmetry by choosing the string gauge as in Eq. (4), then in those generic solutions, the tension becomes $T(X^\mu(\tau,\sigma)) = \pm(2\pi\alpha')^{-1}$ on the two sides of the singularity as it appears *in the string gauge*. Those two sides are identified as the gravity/antigravity sectors of the low-energy theory as discussed in Sec. II. From the perspective of the world-sheet string theory, these observations lead to a simple prescription to capture all these effects in the string gauge—namely, replace the Weyl-invariant structures $(Tg_{\mu\nu}, Tb_{\mu\nu})$ with $(\pm(2\pi\alpha')^{-1}G^{\pm}_{\mu\nu}, \pm(2\pi\alpha')^{-1}B^{\pm}_{\mu\nu})$, where the capital $(G^{\pm}_{\mu\nu}(X), B^{\pm}_{\mu\nu}(X))$ are the background fields on the gravity/antigravity patches that are joined at the singularities *as they appear in the string gauge*. We may absorb the overall $\pm$ due to the signs of the tension into a redefinition of the background fields, and as we did for the Einstein gauge in Eq. (6), define

$$(\hat{G}_{\mu\nu}(X), \hat{B}_{\mu\nu}(X)) = (\pm G^{\pm}_{\mu\nu}(X), \pm B^{\pm}_{\mu\nu}(X)) \tag{51}$$

as the full set of background fields in the union of the gravity/antigravity sectors of the world-sheet string theory. Of course, $(\hat{G}_{\mu\nu}(X), \hat{B}_{\mu\nu}(X))$ are required to satisfy world-sheet conformal invariance at the quantum level as usual. What is new is the geodesic completeness of the background fields $(\hat{G}_{\mu\nu}(X), \hat{B}_{\mu\nu}(X))$, which is achieved by the sign-flipping tension and the union of the corresponding gravity/antigravity sectors.

A. String in flat background with tension that flips sign

A simple example of a conformally exact world-sheet CFT, that includes a dynamical string tension that flips signs, is the flat string background $\eta_{\mu\nu}$ modified only by a time-dependent string tension $T(X) = \frac{1}{2\pi\alpha'}\mathrm{Sign}(|X^0(\tau,\sigma)| - \frac{\Delta}{2})$. This can also be presented in the string gauge by absorbing the sign of the tension into a redefined metric

$$\hat{G}_{\mu\nu}(X) = \eta_{\mu\nu}\mathrm{Sign}\left(|X^0(\tau,\sigma)| - \frac{\Delta}{2}\right),$$
$$\hat{B}_{\mu\nu}(X) = 0, \tag{52}$$

where Δ is a constant. Note the similarity to Eq. (17) or Secs. IV C and IV E. Thus, the tension is positive when $|X^0(\tau,\sigma)| > \frac{\Delta}{2}$ and negative when $-\frac{\Delta}{2} < X^0(\tau,\sigma) < \frac{\Delta}{2}$. This is also similar to the cosmological example with an antigravity loop given in Ref. [7], but we have greatly simplified it here by keeping only the signs but not the magnitude of the tension, thus defining a conformally exact rather than a conformally approximate CFT on the world sheet. The corresponding world-sheet string model is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \times \partial_b X^\nu \eta_{\mu\nu}\mathrm{Sign}\left(|X^0(\tau,\sigma)| - \frac{\Delta}{2}\right). \tag{53}$$

We should mention that it is also possible to consider a model, at least at the classical level, by inserting in the action (53) the inverse of the Sign function $(\mathrm{Sign}(|X^0(\tau,\sigma)| - \frac{\Delta}{2}))^{-1}$. In this case the tension flips sign when it is infinite rather than zero. Both of these possibilities occur smoothly rather than suddenly in cosmological backgrounds in string theory [see Eq. (30) in Ref. [7] or its generalizations]. Both behaviors are significant from the perspective of string theory, because perturbative versus nonperturbative methods would be needed to understand fully the physics in the vicinity of the gravity/antigravity transitions. Namely, when the tension at the transition is large, the string would be close to being pointlike, so the stringy corrections would be small and perturbative in the vicinity of the gravity/antigravity transitions; by contrast, when the tension at the transition is small, the string would be floppy, so stringy corrections could be significant. In the latter case, high-spin fields [19] may be an interesting tool to investigate the gravity/antigravity transition in our setting.

From the form of the action in Eq. (53), it is evident that the string action is invariant under reparametrizations of the world sheet at the classical level. We will use this symmetry to choose a gauge to perform the classical analysis below. But eventually, we also need to know if this symmetry is valid also at the quantum level. The generator of this gauge symmetry is the stress tensor, so the stress tensor vanishes as a constraint to impose the gauge invariance. At the classical level, the stress tensor does vanish as part of the solution of the classical equations and constraints (see below). At the quantum level, in "covariant quantization," the stress tensor does not vanish on all states but only on the gauge-invariant physical states. For consistency of covariant quantization, one must verify that the constraints form a set of first-class constraints that close under quantum operator products. In our case, the stress tensor derived from (53) has the form $T_{\pm\pm} = (\mathrm{Sign}) \times T^0_{\pm\pm}$, where $T^0_{\pm\pm}$ is the usual world-sheet stress tensor in the flat background $\eta_{\mu\nu}$, while the sign factor switches signs at the kinks $|X^0(\tau,\sigma)| = \frac{\Delta}{2}$. In the positive (gravity, Sign $= +$) region, we have symbolically the operator products, $T^0_{\pm\pm} \times T^0_{\pm\pm} \sim T^0_{\pm\pm}$, where the standard CFT result on the right-hand side is computed exactly for the flat string. Similarly, in the negative (antigravity, Sign $= -$) region, we have $(-T^0_{\pm\pm}) \times (-T^0_{\pm\pm}) \sim -(-T^0_{\pm\pm})$. So the algebra is closed like the standard CFT locally in the positive and negative regions away from the kinks. The only regions remaining to be analyzed are the operator products at the kinks $|X^0(\tau,\sigma)| = \frac{\Delta}{2}$ (world-sheet analogs of the kinks in Fig. 1). The operator products involving the Sign factor nontrivially introduce delta functions and derivatives of delta functions multiplied by the sign factor or its derivatives that have support only at the kinks. At one contraction (order-$\hbar$ effects), the coefficient of the delta function includes the flat $T^0_{\pm\pm}$ or its derivatives evaluated at the kinks. Since $T^0_{\pm\pm}$ or its derivatives are in the list of first-class constraints (Virasoro operators), this is still a closed algebra of first-class constraints, all of which vanish on physical states. At two contractions (order $\hbar^2$ in quantum effects), there are again some terms that contain $T^0_{\pm\pm}$ or its derivatives, which again are of no concern, since these still vanish on physical states. However, there are also additional operators of the form of (∂X^0) multiplying products of the sign function, delta function, or its derivatives, all evaluated at the kinks. We have analyzed these complicated distributions and found that they vanish when integrated with (∂X^0), so they do not seem to contribute. Similarly, we can drop several similar terms due to the properties of the distributions. The analysis at the kinks becomes harder at higher contractions ($\hbar^3$ and beyond in quantum effects), and we leave this for future analysis to be reported at a later stage. The main point is that if there are additional constraints that must be imposed at the kinks, they will show up in this type of operator product analysis. So far, we have not found new constraints up to two contractions in the operator products. Thus, the algebra of the operator products is basically the standard algebra of a conformal field theory (CFT) locally in the positive and negative regions away from the kinks. The modification of the CFT algebra at the kinks with terms that are proportional to Virasoro operators does not change the validity of the gauge symmetry at the quantum level, since those terms vanish on physical states anyway. Although we have not yet found other operator modifications of the algebra at the kinks, conceptually it is possible that such terms may arise at higher contractions or in other models that include gravity/antigravity transitions. When and if such terms appear, they must be included in an enlarged list of constraints that should form a closed algebra under operator products; then this will define the proper quantum theory.

In this paper, our aim is to first understand the classical theory of a string described by the action in (53), so we do not need to be concerned here about the subtleties described in the previous paragraph. In fact, the classical analysis that we give below is helpful in further developing the right approach for the quantum theory. Thus, setting aside temporarily the possible stringy corrections, we are at first interested in the classical behavior of strings as they propagate in the union of the gravity/antigravity regions, and later try to figure out the possible additional effects due to interactions at those transitions by using more sophisticated methods, such as string field theory, or others, as outlined in Sec. VI.

1. General string propagating classically through antigravity

In this section, we will discuss the properties of the model in Eq. (53). The main objective is to show that there are no problems due to the negative tension during antigravity from the point of view of fundamental principles, such as unitarity or possible instability due to negative kinetic energy. The unitarity of this string model was already established in Ref. [7] more generally for any time-dependent tension $T(X^0)$ and more general metric, so we will not repeat it here. We will concentrate on the effect of the antigravity period on the propagation of the string and the corresponding signals that observers in gravity may detect. As we will demonstrate, as compared to the complete absence of antigravity, the presence of an antigravity period for a certain amount of time causes only a time delay in the propagation of an open or closed free string of any configuration. This may seem surprising since, at first thought, one may think that string bits would fly apart under an instability caused by a negative string tension. In fact, this does not happen, because a negative tension is simply an overall sign in the action of a free string, and this does not change the equations of motion and constraints of a free string during antigravity.

We work in the conformal gauge at the classical level. There is a remaining reparametrization symmetry that permits the further choice of the following timelike gauge:

$$X^0(\tau,\sigma) = |H|\tau, \tag{54}$$

where H is the total time-dependent Hamiltonian of the string, while $|H|$ is time independent. This is similar to the massless free field in Sec. IV C. In this gauge, the remaining degrees of freedom satisfy the following equations of motion and constraints:

$$(\partial_\tau^2 - \partial_\sigma^2)\vec{X}(\tau,\sigma) = 0, \qquad H^2 = (\partial_\tau \vec{X} \pm \partial_\sigma \vec{X})^2, \tag{55}$$

to be solved in each time region A, B, C defined by

$$A\colon\ \tau|H| < -\Delta/2, \qquad B\colon\ -\Delta/2 < \tau|H| < \Delta/2,$$
$$C\colon\ \tau|H| > \Delta/2. \tag{56}$$

Furthermore, the solutions for $\vec{X}_{A,B,C}(\tau,\sigma)$ and the canonical momenta $\vec{P}_{A,B,C}(\tau,\sigma)=\partial_\tau\vec{X}_{A,B,C}(\tau,\sigma)\times\mathrm{Sign}(|H\tau|-\frac{\Delta}{2})$ should be continuous at the boundaries $\tau|H|=\pm\Delta/2$. The method of solution follows the simple model in Eq. (18) or the massive field in Sec. IV E.

We will discuss the case of an open string; the closed string is treated similarly. The general solution in each region is given in terms of the center of mass $(\vec{q},\vec{p})$ and oscillator $(\vec{\alpha}_n, n=\pm1,\pm2,\ldots)$ degrees of freedom. The general configuration of the string in the positive-tension region A, at a time $\tau<-\Delta/2$, is a general solution $\vec{X}_A(\tau,\sigma)$ given by

$$\vec{X}_A(\tau,\sigma)=\vec{q}_0+\vec{p}\tau+\sum_{n=-\infty,\neq0}^{\infty}\frac{i}{n}\vec{\alpha}_n\cos n\sigma e^{-in\tau}. \quad (57)$$

The time-independent parameters $(\vec{q}_0,\vec{p})$ and $(\vec{\alpha}_n, n=\pm1,\pm2,\ldots)$ determine the initial configuration of the string at the time $\tau=\tau_0$. From the constraint equations, we compute the time-independent $|H|$ and the remaining constraint

$$|H|=\sqrt{\vec{p}^2+\sum_{n=1}^{\infty}\vec{\alpha}_{-n}\cdot\vec{\alpha}_n},$$
$$0=\vec{p}\cdot\vec{\alpha}_n+\frac{1}{2}\sum_{m=-\infty,\neq0}^{\infty}\vec{\alpha}_{-m}\cdot\vec{\alpha}_{n+m}, \quad (58)$$

Thus, the time-dependent Hamiltonian that switches sign is

$$H(\tau)=\mathrm{Sign}\left(|H||\tau|-\frac{\Delta}{2}\right)\sqrt{\vec{p}^2+\sum_{n=1}^{\infty}\vec{\alpha}_{-n}\cdot\vec{\alpha}_n}. \quad (59)$$

Assuming the constraints (58) are satisfied at the classical level by some set of parameters $(\vec{\alpha}_n,\vec{p})$, the momentum, $\vec{P}_A=\dot{\vec{X}}_A$, in region A is

$$\vec{P}_A(\tau,\sigma)=\vec{p}+\sum_{n=-\infty,\neq0}^{\infty}\vec{\alpha}_n\cos n\sigma e^{-in\tau}. \quad (60)$$

In region B, $-\frac{\Delta}{2}<\tau|H|<\frac{\Delta}{2}$, the solution $(\vec{X}_B,\vec{P}_B)$ takes the same form as above, but with a new set of parameters $(\vec{q}_B,\vec{p}_B,\vec{\alpha}_{nB})$. Note that in this region there is a nontrivial minus sign in the relation between momentum and velocity, $\vec{P}_B(\tau,\sigma)=-\partial_\tau\vec{X}_B(\tau,\sigma)$. At the transition time, $\tau_*\equiv-\frac{\Delta}{2|H|}$, we must match the position and momentum, therefore $\vec{X}_A(\tau_*,\sigma)=\vec{X}_B(\tau_*,\sigma)$ and $\partial_\tau\vec{X}_A(\tau_*,\sigma)=-\partial_\tau\vec{X}_B(\tau_*,\sigma)$, noting the negative sign in the case of velocities. Because the matching is for every value of σ, we find that all the parameters $(\vec{q}_B,\vec{p}_B,\vec{\alpha}_{nB})$ are uniquely determined in terms of the initial parameters $(\vec{q}_0,\vec{p},\vec{\alpha}_n)$ in region A. So the solution in region B is

$$\vec{X}_B(\tau,\sigma)=\begin{pmatrix}\vec{q}_0+\vec{p}(-\tau-\frac{\Delta}{|H|})\\+\sum_{n=-\infty,\neq0}^{\infty}\frac{i}{n}\vec{\alpha}_n\cos n\sigma e^{-in(-\tau-\frac{\Delta}{|H|})}\end{pmatrix}, \quad (61)$$

$$\vec{P}_B(\tau,\sigma)=-\dot{\vec{X}}_B(\tau,\sigma)=\vec{p}+\sum_{n=-\infty,\neq0}^{\infty}\vec{\alpha}_{-n}\cos n\sigma e^{-in(-\tau-\frac{\Delta}{|H|})}. \quad (62)$$

There are no new constraints beyond those that are already assumed to have been satisfied in region A by the parameters $(\vec{\alpha}_n,\vec{p})$. Note the structure $(-\tau-\frac{\Delta}{|H|})$ that indicates a backward propagation similar to Fig. 1 as τ increases beyond τ_*.

At the next transition time, $\tau_{**}\equiv+\frac{\Delta}{2|H|}$, we must connect the solution $(\vec{X}_B,\vec{P}_B)$ above to the solution $(\vec{X}_C,\vec{P}_C)$ in region C, $\tau>\tau_{**}$, which is given in terms of a new set of parameters $(\vec{q}_C,\vec{p}_C,\vec{\alpha}_{nC})$. Using the matching conditions $\vec{X}_C(\tau_*,\sigma)=\vec{X}_B(\tau_*,\sigma)$ and $\dot{\vec{X}}_C(\tau_*,\sigma)=-\dot{\vec{X}}_B(\tau_*,\sigma)$ that include the extra minus sign for velocities (as discussed above), we find that $(\vec{q}_C,\vec{p}_C,\vec{\alpha}_{nC})$ are all determined again uniquely in terms of the initial parameters $(\vec{q}_0,\vec{p},\vec{\alpha}_n)$ introduced in region A:

$$\vec{X}_C(\tau,\sigma)=\begin{pmatrix}\vec{q}_0+\vec{p}(\tau-2\frac{\Delta}{|H|})\\+\sum_{n=-\infty,\neq0}^{\infty}\frac{i}{n}\vec{\alpha}_n\cos n\sigma e^{-in(\tau-2\frac{\Delta}{|H|})}\end{pmatrix},$$
$$\vec{P}_C(\tau,\sigma)=\vec{p}+\sum_{n=-\infty,\neq0}^{\infty}\vec{\alpha}_n\cos n\sigma e^{-in(\tau-2\frac{\Delta}{|H|})}. \quad (63)$$

Putting it all together, we see that after the antigravity period, the emergent string experiences only a time delay $2\Delta/|H|$ as compared to the string that propagates in the complete absence of antigravity. This is the same conclusion that was reached for the free particle or the free massless field.

2. Rotating rod propagating through antigravity

As a concrete example of a string configuration that satisfies all the constraints, we present the rotating rod solution that is modified by a tension that flips sign during antigravity as in Eq. (53). We begin with a straight string lying along the $\hat{x}$ axis with its center of mass located at $\vec{q}_0$, as given by $\vec{X}_0(\sigma)=\vec{q}_0+\hat{x}R_0\cos\sigma$. Let this string rotate in the $(\hat{x},\hat{y})$ plane and translate in the $\hat{z}$ direction as follows:

$$\vec{X}_A(\tau,\sigma)=\vec{q}_0+\hat{z}p\tau+R_0\cos\sigma(\hat{x}\cos\tau+\hat{y}\sin\tau). \quad (64)$$

This satisfies the constraints in Eq. (58), since $\partial_\tau \vec{X} \cdot \partial_\sigma \vec{X} = 0$, and gives $|H| = (p^2 + R_0^2)^{1/2}$. Following the steps above, we compute the matching string configuration during the antigravity period $-\frac{\Delta}{2|H|} < \tau < \frac{\Delta}{2|H|}$:

$$\vec{X}_B(\tau,\sigma) = \begin{pmatrix} \vec{q}_0 + \hat{z}p(-\tau-\theta) \\ +R_0\cos\sigma \begin{pmatrix} \hat{x}\cos(-\tau-\theta) \\ +\hat{y}\sin(-\tau-\theta) \end{pmatrix} \end{pmatrix}, \quad (65)$$

where $\theta = \Delta(p^2 + R_0^2)^{-1/2}$, noting that this describes a backward propagation similar to Fig. 1. Finally, the matching string configuration in the time period $\tau > \frac{\Delta}{2|H|}$ is

$$\vec{X}_C(\tau,\sigma) = \begin{pmatrix} \vec{q}_0 + \hat{z}p(\tau - 2\theta) \\ +R_0\cos\sigma(\hat{x}\cos(\tau-2\theta) + \hat{y}\sin(\tau-2\theta)) \end{pmatrix}. \quad (66)$$

As promised, as compared to the complete absence of antigravity, the presence of an antigravity period for a certain amount of time causes only a time delay in the propagation of a string of any configuration. The string bits of a freely propagating string do not fly apart during antigravity when the string tension is negative.

B. 2D black hole including antigravity

Another simple example is the two-dimensional black hole [20] based on the $SL(2,R)/R$ gauged WZW model [21]. The well-known string background metric in this case is $ds^2 = -2(1-uv)^{-1}dudv$, with $uv < 1$, where (u,v) are the string coordinates $X^\mu(\tau,\sigma)$ in the Kruskal-Szekeres basis. This space is geodesically incomplete, similar to the case of the four-dimensional Schwarzschild black hole [11].

The geodesically complete modification consists of allowing the string tension to flip sign precisely at the singularity, namely $T(X) = (2\pi\alpha')^{-1}\mathrm{Sign}(1-uv)$. Then the new geodesically complete 2D black hole action is

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\frac{\partial_a u \partial_b v}{|1-uv|}. \quad (67)$$

This differs from the old 2D black hole action by the absolute value sign, and includes the antigravity region $uv > 1$ just as in the four-dimensional case [11]. Despite the extra sign, this model is an exact CFT on the world sheet, as can be argued in the same way following Eq. (53). Properties of the new 2D black hole, including the related dilaton and all orders of quantum corrections in powers of α', will be investigated in detail in a separate paper [22].

VI. STRING FIELD THEORY WITH ANTIGRAVITY

In the neighborhood of the gravity/antigravity transition, which occurs typically at a gravitational singularity, a proper understanding of the physics would be incomplete without the input of quantum gravity that may possibly contribute large quantum effects. How should we estimate the effects of quantum gravity?

We first point out that attempting to use an effective low-energy field theory that includes higher powers of curvature, such as those computed from string theory, is the wrong approach. Higher powers of curvature capture approximations to quantum gravity that are valid at momenta much smaller than the Planck scale; those cannot be used to investigate the phenomena of interest that are at the Planck scale close to the singularity. For investigating the gravity/antigravity transition more closely, we do not see an alternative to using directly an appropriate theory of quantum gravity that can incorporate the geodesically complete spacetime that includes both gravity and antigravity regions. Hence, we first need to define the proper theory of quantum gravity that is consistent with geodesic completeness. As far as we know, this notion of quantum gravity was first considered in Ref. [7].

Assuming that quantum string theory is a suitable approach to quantum gravity, we outline here how string field theory may be modified to take into account geodesic completeness and the presence of an antigravity sector, so that it can be used as a proper tool to answer the relevant questions.

Open and closed string field theory (SFT) is a formalism for computing string-string interactions, including those that involve stringy gravitons. As in standard field theory, in principle the SFT formalism is suitable for both perturbative and nonperturbative computations. Technically, SFT is hard to compute with, but it has the advantage of being a self-consistent and conceptually complete definition of quantum gravity and the interactions with matter. It is therefore crucial to see how antigravity fits in SFT, and therefore how the pertinent questions involving antigravity can be addressed in a self-consistent manner.

In the context of SFT, gravitational and other backgrounds in which strings propagate are incorporated through the BRST operator Q that appears in the quadratic part of the action [20]

$$S_{\mathrm{open}} = Tr\left[\frac{1}{2}AQA + \frac{g}{3}A\star A\star A\right]. \quad (68)$$

The complete SFT action must also include closed strings, S_{closed}. The supersymmetric versions of these may also be considered. Here $A(X)$ is the string field, the product $\star$ describes string joining or splitting, and the BRST operator Q is given by

$$Q = \int d\sigma \sum_{\pm} \{c_{\pm} T_{\pm\pm}(X) + b_{\pm} c_{\pm} \partial c_{\pm}\}, \quad (69)$$

where $(b_{\pm}, c_{\pm})$ are the Fadeev-Popov ghosts, which is a device of "covariant quantization," while $T_{\pm\pm}(X(\sigma))$ is the stress tensor for left/right-moving strings, associated with any conformal field theory (CFT) on the world sheet that is conformally exact at the quantum level.

The gravitational and other backgrounds, including a dynamical tension that flips signs (i.e. incorporating antigravity) of the type we discussed in the previous sections, are included in the stress tensor $T_{\pm\pm}(X)$. If these backgrounds are not geodesically complete, we expect that the SFT theory is incomplete, since even at the classical level on the world sheet there would be string solutions that would be incomplete, just like particle geodesics that would be incomplete. Thus, for a geodesically complete SFT, we need to make sure that $T_{\pm\pm}(X)$ belongs to a geodesically complete world-sheet string model as described in the previous section. Examples of such string models were provided in Secs. V A and V B. Similarly, one can construct many more geodesically complete backgrounds by allowing the string tension to change sign at singularities (and perhaps more generally) as long as the CFT conditions, that amount to $Q^2 = 0$, are satisfied.

If the interactions in the SFT action (68) are neglected, we do not expect dramatic effects due to the presence of antigravity, since we have seen in the previous section that the effect is only a time delay as compared to the complete absence of antigravity [as in Secs. IV A, IV C and V A]. By including the interactions either perturbatively or nonperturbatively, we can explore the effects of antigravity in the context of the quantum theory. In the previous sections, we have obtained a glimpse of the phenomena that could happen, including particle (or string) production (as in Sec. IV E), excitations of various string states (as in Sec. IV D), and more dramatic phenomena that remain to be explored.

From the discussion in the first part of Sec. V A, one may gather that we are still in the process of addressing some technicalities in the construction of the BRST operator Q for the simple model in that section. So we are not yet in a position to perform explicit computations, but we hope we have provided an outline of how one may formulate an appropriate theory to address and answer the relevant questions.

There may be alternative formalisms that could provide answers more easily than SFT, and of course those should be explored, but the advantage of SFT for being a conceptually complete and self-consistent definition of the system, including the presence of antigravity as outlined above, is likely to remain as an important feature of this approach because of the overall perspective that it provides.

VII. COMMENTS

We have argued that a fundamental theory that could address the physical phenomena close to gravitational singularities, either in the form of field theory or string theory, is unlikely to be complete without incorporating geodesic completeness. The Weyl symmetric approach to the standard model coupled to gravity in Eq. (1), and the similar treatment of string theory [7], generally solves this problem and naturally requires that antigravity regions of spacetime appear on the other side of gravitational singularities as integral parts of the spacetime described by a fundamental theory. There are other views that the notion of spacetime may not even exist at the extremes close to singularities. While acknowledging that there may be other scenarios that are little understood at this time, we believe that our concrete proposal merits further investigation.

While emphasizing that there are nicer Weyl gauges, we have shown how gravitational theories and string theories can be formulated in their traditional Einstein or string frames to include effects of a Weyl symmetry that renders them geodesically complete. A prediction of the Weyl symmetry is to naturally include an antigravity region of field space and spacetime that is geodesically connected to the traditional gravity spacetime at gravitational singularities. Precisely at the singularities that appear in the Einstein or string frames, the gravitational constant or string tension flips sign suddenly (but smoothly in nicer Weyl gauges). As shown in Sec. II, this sign can be absorbed into a redefinition of the metric in the Einstein or string frame, $\hat{g}_{\mu\nu} = \pm g^{\pm}_{\mu\nu}$, where $\hat{g}_{\mu\nu}$ describes the spacetime in the union of the gravity and antigravity regions. This definition of the complete spacetime may then be used to perform computations in the geodesically complete theory.

The appearance of negative kinetic energy terms for some degrees of freedom during antigravity was a source of concern. The arguments presented here show that this was a false alarm. We argued that unitarity is not an issue either in gravity or antigravity and that negative energy does not imply an instability of the theory as seen by observers in the gravity region (namely, observers like us, analyzing the Universe). We made this point by studying many simple examples, and we showed that observers in the gravity sector can deduce the existence and at least some properties of antigravity.

We have thus eliminated the initial concerns regarding unitarity or instability of the complete theory when there is an antigravity sector with negative kinetic energy. We have also demonstrated that there are very interesting physical phenomena associated with antigravity that remain to be explored concerning fundamental physics at the extremities of spacetime. These will have applications in cosmology as in Refs. [1,7,10,12], and in black hole physics as in Refs. [11,22].

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